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Control and Controllability

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In this short presentation we will try to give a flavour of some ideas in control theory in a general, rather than mechanical, context. For my part, I am especially interested in the so-called controllability problem.

For a general overview of control methodology, see [Sontag 1998].

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1. What do control theoreticians do?

- They consider
$$\dot{x}(t) = f(x(t), u(t))$$
where $x \in M$, $u \in \mathcal{U}$, and $f(x, \cdot)\colon \mathcal{U} \to T_xM$.
- M is the *state space* and $\mathcal{U}$ is the *input space*.
- Design u to accomplish certain tasks, e.g.,
  - steer the system from x_i to x_f,
  - stabilise a point $x_0 \in M$, or
  - follow a reference trajectory $t \mapsto x_{\mathrm{ref}}(t)$.

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Of course, many people who are control theoreticians would not confess to studying $\dot{x} = f(x, u)$...

State space is often a manifold (how else do we get to define $\dot{x}$?) and we shall take it to be of dimension n for concreteness. One can also study infinite-dimensional, or so-called "distributed parameter" systems. The controls may take their values in odd-ball control spaces. We shall deal with systems where $\mathcal{U} = \mathbb{R}^m$.

The list of control problems we give here—the reconfiguration problem, the stabilisation problem, and the trajectory tracking problem—constitute a small sampling of the problems studied by control theoreticians. Other problems include optimal control, disturbance rejection, modelling of uncertainty, etc.

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- The control signal might be designed as
  - $u(t) \implies \dot{x} = f(x, u(t))$ (open-loop control)
  - $u(x) \implies \dot{x} = f(x, u(x))$ (state feedback)
- *Control-affine* systems:

$$\dot{x}(t) = f(x(t)) + u^a(t) g_a(x(t)) \qquad \text{(ACS)}$$

  for vector fields f (the *drift* vector field) and $g_1, \dots, g_m$ (the *control* or *input* vector fields).
- *Linear* systems:

$$\dot{x}(t) = Ax(t) + Bu(t)$$

  for $x \in \mathbb{R}^n$, $u \in \mathbb{R}^m$, $A \in \mathrm{L}(\mathbb{R}^n; \mathbb{R}^n)$, and $B \in \mathrm{L}(\mathbb{R}^m; \mathbb{R}^n)$.

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Open-loop control, i.e., using a *precomputed* control signal, is notoriously bad as it relies on a perfect model to guarantee effectiveness. It is for this reason that the concept of a closed-loop system, employing feedback, is much studied in control theory.

One may also wish to have feedback which is dependent on both state and time. As we shall shortly see, there are things that are not possible with state feedback, so time-dependence is sometimes necessary.

We shall focus on control-affine systems in this little warm-up as they are the systems I shall be dealing with in the mechanical context later.

Not at all unexpectedly, there is an enormous and growing literature on linear control theory. For a "geometric" treatment of linear systems see [Wonham 1985]. Linear control techniques are also by far the most prevalent in "real life" applications. That is to say, much of the control theory employed by professional engineers in industry is linear in nature, and often quite classical at that. Nonetheless, nonlinear methods are starting to make some inroads.

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2. Local accessibility and controllability

- A good first question to ask is, "Can you get there from here?"
- Let us concentrate on local control problems for control-affine systems.
- Let $x_0 \in M$ and let U be a neighbourhood of x_0. Let $\mathcal{R}^U(x_0, T)$ be the set of points x for which there exists a solution $(x(t), u(t))$ defined on $[0, T]$ with the properties
  - $x(t) \in U$ for $t \in [0, T]$,
  - $x(0) = x_0$, and
  - $x(T) = x$.
- Let $\mathcal{R}^U(x_0, \leq T) = \bigcup_{0 \leq t \leq T} \mathcal{R}^U(x_0, t)$.

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If one wishes to stabilise a system to some state, say x_0, one should probably know from which points it is even possible to *reach* x_0. If it is not possible to reach x_0 from every point in M, you can give up hope of generating any sort of general stabilising control scheme. Thus, when someone gives you a control problem, a good first thing to do is try to describe the accessible states.

By "local" we mean that we are interested in describing what we can do when we start at an initial state, and do not allow ourselves large excursions from that state. As one might expect, for analytic systems these local problems are determinable in terms of the problem data and its derivatives [Nagano 1966]. For a non-local approach with analytic systems, see [Sussmann and Jurdjevic 1972].

A *solution* will consist of a pair of curves $t \mapsto x(t)$ and $t \mapsto u(t)$. To do controllability analysis, one normally asks for control signals $t \mapsto u(t)$ which have certain properties. For example, one may wish to consider piecewise constant functions, or bounded, measurable functions. It is unwise to restrict to something as limited as, say, differentiable functions; doing so might render some standard results false or unsolved.

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- We wish to describe $\mathcal{R}^U(x_0, \leq T)$. To do so we use Lie brackets.
- Let us motivate this with an example:

$$\dot{x}^1 = u^1$$
$$\dot{x}^2 = u^2$$
$$\dot{x}^3 = x^2 u^1$$

  with initial state $(0,0,0)$.
- Provide a signal $t \mapsto (1,0)$, $t \in [0, T/4[\,$; $t \mapsto (0,1)$, $t \in [T/4, T/2[\,$; $t \mapsto (-1,0)$, $t \in [T/2, 3T/4[\,$, $t \mapsto (0,-1)$, $t \in [3T/4, T]$, in (u^1, u^2)-space.
- This gives a loop in (x^1, x^2) and a translation by $-T^2/16$ in x^3.
- The resulting motion is exactly in the direction of the Lie bracket of of the input vector fields $\frac{\partial}{\partial x^1} + x^2 \frac{\partial}{\partial x^3}$ and $\frac{\partial}{\partial x^2}$.

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The example is the so-called "Heisenberg system" since the brackets obey relations reminiscent of the Heisenberg Lie algebra. Of course, we do not really need too much convincing that Lie brackets should appear in the controllability problem. After all a possible definition of the Lie bracket is one which generalises what we did in this example (cf. [Nijmeijer and van der Schaft 1990, Proposition 3.6])

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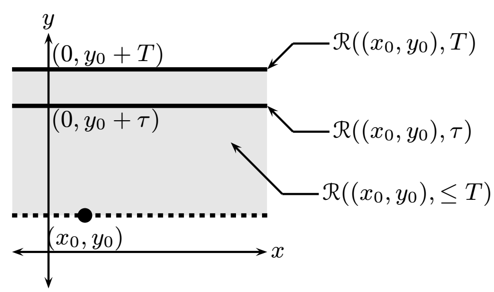
- (ACS) is (*strongly*) *locally accessible* at x if there exists $T > 0$ so that $\mathcal{R}^U(x, \leq T)$ ($\mathcal{R}^U(x, T)$) contains a non-empty open subset of M for each neighbourhood U of x and for each $t \in]0, T]$.
- (ACS) is *locally controllable* at x if there exists $T > 0$ so that $\mathcal{R}^U(q, \leq T)$ contains a neighbourhood of x for each neighbourhood U of x and for each $t \in]0, T]$.
- Accessibility, strong accessibility, and controllability are different:

$$\dot{x} = u$$
$$\dot{y} = 1.$$



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Accessibility and strong accessibility are easy to check. Controllability is genuinely difficult as far as concerns general results. Sussmann [1987] has a fairly general local controllability result. Some known *global* controllability results are topological in nature, e.g., [San Martin and Crouch 1984] and others involve knowledge of the unforced dynamics, e.g., [Manikonda and Krishnaprasad 1997].

The simple example is locally accessible, but neither strongly locally accessible nor locally controllable. As we have drawn the reachable sets, we assume unbounded controls. With bounded controls, the reachable sets would have a conical appearance, I suppose. Also, note that if instead of $\mathbb{R}^2$, we worked on $\mathbb{R} \times \mathbb{S}^1$ (with coordinates (x, y)), then the system would be globally controllable.

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- Construct sequences of distributions

$$D_{\mathrm{a}}^{(0)} = \operatorname{span}(f, g_1, \dots, g_m)$$

$$\vdots$$

$$D_{\mathrm{a}}^{(i)} = D_{\mathrm{a}}^{(i-1)} + [D_{\mathrm{a}}^{(0)}, D_{\mathrm{a}}^{(i-1)}]$$

$$\vdots$$

and

$$D_{\mathrm{sa}}^{(0)} = \operatorname{span}(g_1, \dots, g_m)$$

$$\vdots$$

$$D_{\mathrm{sa}}^{(i)} = D_{\mathrm{sa}}^{(i-1)} + [D_{\mathrm{a}}^{(0)}, D_{\mathrm{sa}}^{(i-1)}]$$

$$\vdots$$

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Our strange notation D_{a} and D_{sa} is intended to suggest "accessible" and "strongly accessible" for reasons we shall see shortly. D_{a} is, by definition, the smallest integrable distribution containing $\operatorname{span}(f, g_1, \dots, g_m)$. It is also the case that D_{sa} is the smallest integrable distribution containing $\operatorname{span}(g_1, \dots, g_m)$ which is invariant under f (that is, $[f, D_{\mathrm{sa}}] \subset D_{\mathrm{sa}}$).

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- Suppose these sequences stabilise at some finite index and define constant rank distributions D_{a} and D_{sa}.
- Let Λ_x (resp. $\tilde{\Lambda}_x$) be the maximal integral manifold of D_{a} (resp. D_{sa}) through x.

Theorem 1 *$\mathcal{R}^U(x, \leq T) \subset \Lambda_x$ (resp. $\mathcal{R}^U(x, T) \subset \tilde{\Lambda}_{\Phi^t_f(x)}$) and $\mathcal{R}^U(x, \leq T)$ (resp. $\mathcal{R}^U(x, T)$) contains a non-empty open subset of Λ_x (resp. $\tilde{\Lambda}_x$). In particular, if $\operatorname{rank}(D_{\mathrm{a}}) = \dim(M)$ (resp. $\operatorname{rank}(D_{\mathrm{sa}}) = \dim(M)$) then (ACS) is (resp. strongly) locally accessible.*

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For analytic vector fields, the sequences *do* stabilise on an open dense subset of M. For simplicity we assume this happens on all of M. Some of the results we state below, and I'll try to indicate which ones, are true when D_{a} and D_{sa} are not of constant rank.

The result we state here is a standard result (see, for example, [Nijmeijer and van der Schaft 1990]). Note that it is true that (ACS) is (resp. strongly) locally accessible if the rank of D_{a} (resp. D_{sa}) at x is equal to $\dim(M)$, even if D_{a} (resp. D_{sa}) is not a constant rank distribution.[1] Conversely, if (ACS) is (resp. strongly) locally accessible at *all* points in M, then $\operatorname{rank}(D_{\mathrm{a}})$ (resp. $\operatorname{rank}(D_{\mathrm{sa}})$) must equal the dimension of M on an open dense subset of M. For analytic systems, the rank condition, even when D_{a} (resp. D_{sa}) is not a constant rank distribution, is necessary for (resp. strong) local accessibility [Sussmann and Jurdjevic 1972].

[1] It is then a question how one defines D_{a} and D_{sa}. To define D_{a} one lets $\mathscr{D}_{\mathrm{a}}$ be the smallest Lie subalgebra of the Lie algebra of vector fields which contains $\{f, g_1, \dots, g_m\}$, and then defines $D_{\mathrm{a},x} = \{X(x) \mid X \in \mathscr{D}_{\mathrm{a}}\}$. For D_{sa} one defines $\mathscr{D}_{\mathrm{sa}}$ to be the smallest subalgebra of the Lie algebra of vector fields which (1) contains $\{g_1, \dots, g_m\}$ and (2) is invariant under f, i.e., $[f, \mathscr{D}_{\mathrm{sa}}] \subset \mathscr{D}_{\mathrm{sa}}$, and then defines $D_{\mathrm{sa},x} = \{X(x) \mid X \in \mathscr{D}_{\mathrm{sa}}\}$.

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- What does this look like for linear systems?
- Denote by $D(A,B)$ the subspace which is the column span of the concatenated matrices $[B|AB|\dots,A^{n-1}B]$.
- $D_{\mathrm{a},x} = \mathrm{span}_{\mathbb{R}}(Ax) + D(A,B)$ and $D_{\mathrm{sa},x} = D(A,B)$.
- For linear systems, strong local accessibility is equivalent to local controllability.

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Note for linear systems that D_{a} is not constant rank unless $\mathrm{image}(A) \subset D(A,B)$. The condition that $\dim(D(A,B)) = n$ (recall $x \in \mathbb{R}^n$) is called the *Kalman rank condition* [Kalman, Ho, and Narendra 1963].

Note that local controllability for linear systems follows from strong local accessibility. This will not be true for nonlinear systems in general. Nonlinear local controllability is a difficult question. A quite general result is that of Sussmann [1987]. A fairly sharp result for single-input systems can be found in [Sussmann 1983], but even the single-input case is not resolved. All that is known is that the local controllability problem (for analytic systems) is resolvable in terms of Lie brackets of drift and control vector fields.

3. Decompositions of non-accessible systems

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- If (y,z) are coordinates adapted to the integrable distribution D_{a}, then (ACS) has the local form

$$\begin{aligned}\dot{y}(t) &= \tilde{f}(y(t),z(t)) + u^a(t)\tilde{g}(y(t),z(t))\\ \dot{z}(t) &= 0.\end{aligned}$$

- If (y,z) are coordinates adapted to the integrable distribution D_{sa}, then (ACS) has the local form

$$\begin{aligned}\dot{y}(t) &= \tilde{f}_1(y(t),z(t)) + u^a(t)\tilde{g}(y(t),z(t))\\ \dot{z}(t) &= \tilde{f}_2(z(t)).\end{aligned}$$

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When we say that coordinates (y,z) are *adapted* to an integrable distribution D, we mean that $D_x = \operatorname{span}_{\mathbb{R}}(\frac{\partial}{\partial y}\big|_x, \dots, \frac{\partial}{\partial y^r}\big|_x)$.

For the decomposition associated with D_{a}, the fact that $z(t)$ is constant reflects the fact that the system evolves on the leaves of the foliation associated with D_{a}. The decomposition for D_{sa} is potentially more interesting. Here the z-equation evolves independently, prescribing how the system changes from leaf to leaf of the foliation corresponding to D_{sa}. One may interpret the z-equation as the uncontrolled dynamics induced on some appropriate quotient.

In the event that $D_{\mathrm{a}} = D_{\mathrm{sa}}$ then obviously we have $\tilde{f}_2 = 0$.

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4. "Nonholonomic" control systems

- These are driftless control-affine systems:

$$\dot{x}(t) = u^a(t) g_a(x(t)).$$

- Local accessibility $\implies$ local controllability.
- Stabilisation is non-trivial:
  - Not stabilisable with continuous state feedback (i.e., $u = u(x)$);
  - Not *exponentially* stabilisable with C^∞ time-dependent feedback.

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The monicker "nonholonomic" is common, but I don't like it, so I put it in quotes...

It seems reasonable that local accessibility should imply local controllability since the systems are "symmetric" with respect to time-reversal. So, roughly speaking, if you can reach a non-empty open set in one direction, you can also reach a non-empty open set in the opposite direction.

That nonholonomic systems are not stabilisable by continuous state feedback follows from a result of Brockett [1983]. The result on non-exponential stabilisability via smooth time-dependent feedback is given, for example, by M'Closkey and Murray [1997]. It *is* possible to *asymptotically* stabilise nonholonomic systems using smooth time-dependent feedback [Teel, Murray, and Walsh 1992]. Exponential stabilisation may be accomplished by discontinuous state feedback, or non-differentiable, time-dependent feedback. There is a small industry concerned with topics such as this... Bob M'Closkey is my friend, and Richard Murray was my PhD supervisor, so let me cite M'Closkey and Murray [1997] as an example of many papers in this area. That paper will also contain some additional references.

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Controllability of simple mechanical control systems

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The work presented in this talk was initiated by some work which went into my PhD dissertation [Lewis 1995], and has been ongoing, to some extent, ever since. The initial aim of the work was to address some of the basic nonlinear control questions in the specific context of *mechanical* systems. Although there is some fairly general work in the Hamiltonian control framework (see [Nijmeijer and van der Schaft 1990, Chapter 12]), existing work in the Lagrangian framework was ad hoc and example based. Examples commonly studied were robotic systems and satellite control. Since these systems are "simple" (i.e., their Lagrangians are kinetic minus potential energy), as are many mechanical control systems which arise in applications, it seems reasonable to focus on this class of systems.

5. What are we after?

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- Consider a linear control system:

$$\dot{x}(t) = Ax(t) + Bu(t) \tag{1}$$

$x \in \mathbb{R}^n$, $u \in \mathbb{R}^m$, $A \in \mathrm{L}(\mathbb{R}^n; \mathbb{R}^n)$, and $B \in \mathrm{L}(\mathbb{R}^m; \mathbb{R}^n)$.

- Starting at $x = 0$, where can we go?
- Linear system $\implies$ basic questions have answers.
- $\mathcal{R}(0) = \mathrm{span}_{\mathbb{R}}([B|AB|\dots|A^{n-1}B])$ (computable).
- $\mathcal{R}(0)$ is the smallest A-invariant subspace containing image(B) ("geometric" meaning).
- We want to do something similar for a class of mechanical systems.

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Since the idea of control may not be all that familiar, let me make sure we understand what the linear system (1) represents. One should think of $t \mapsto u(t)$ as being a specified signal, i.e., a function on the time interval $[0, T]$ (say). The job of a control theoretician is to design the signal to make the "state" $t \mapsto x(t)$ do what we want. What this is may vary, depending on the situation at hand. For example, one may want to steer from an initial state x_i to a final state x_f, perhaps in an optimal way. Or, one may wish to design $u\colon \mathbb{R}^n \to \mathbb{R}^m$ so that some state, perhaps $x = 0$, is stable for the dynamical system $\dot{x} = Ax + Bu(x)$. This latter is called *state feedback* (often one asks that u be linear). One could also design u to be a function of both x and t. I think we get the idea…

One of the basic control questions is controllability, which comes in many guises, some of which we shall take some care with later. For now, let us just say we are asking for "reachable" points. In particular, $\mathcal{R}(0)$ denotes the set of points reachable from $0 \in \mathbb{R}^n$. For linear systems we provide two equivalent answers which have different flavours. The first answer is nice because with it one can compute the set of reachable points. However, it presents a somewhat "non-obvious appearance." The second answer is nice because it sounds "believable," and it gives one some insight into how the components of the control system (here the matrices A and B) interact to provide the set of reachable points.

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6. Simple mechanical control systems

- Our systems are characterised by:
  - an n-dimensional configuration manifold Q;
  - a Riemannian metric g on Q (kinetic energy);
  - a potential energy function V on Q;
  - linearly independent one-forms $F^1, \dots, F^m$ on Q (input forces).
- To make life easier, let us suppose $V = 0$ unless otherwise stated.
- We shall always consider initial conditions with zero velocity, and we are interested in the reachable configurations.
- Such systems are not amenable to linearisation-based methods.

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Some of our results require problem data to be analytic. So, to be safe, let us suppose this to be the case. That is, suppose Q, g, V, and $\{F^1, \dots, F^m\}$ to be analytic.

Of course, the Lagrangian we use given the above problem data is $L(v_q) = \frac{1}{2}g(v_q, v_q) - V(q)$ where $v_q \in T_qQ$.

Some of the results which we state have analogues when $V \neq 0$, but they are somewhat awkward to state. Therefore we shall simply say when these analogous results exist without being too specific about them. Besides, the results when $V \neq 0$ are interesting and beautiful (particularly the latter, in my opinion) in their own right.

When we say these systems are not amenable to linearisation-based methods, we mean that their linearisations at zero velocity are not controllable, and that they are not feedback linearisable. This makes simple mechanical control systems a non-trivial class of nonlinear control systems, especially from the point of view of control design.

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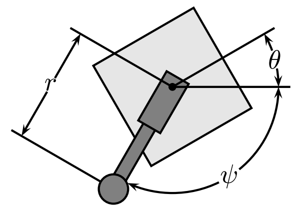
7. Examples of simple mechanical control systems

- Robotic leg:



- Inputs are (1) an internal torque moving the leg relative to the body and (2) a force extending the leg, i.e., $F^1 = \mathrm{d}\theta - \mathrm{d}\psi$ and $F^2 = \mathrm{d}r$.

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The robotic leg has been studied as a nonholonomic system (i.e., one of the form $\dot{q}(t) = u^a X_a(q(t))$ for vector fields $X_1, \dots, X_m$ on Q) by Li, Montgomery, and Raibert [1989] and Murray and Sastry [1993]. Such a treatment differs somewhat from ours, but the two approaches are ultimately equivalent [Lewis 1999].

Interestingly, if one asks for the *states* (i.e., configurations and velocities) reachable from configurations with zero initial velocity, one finds that not all states are reachable. This is a consequence of the fact that angular momentum is conserved, even with inputs. Thus if one starts with zero momentum, the momentum will remain zero (this is what enables one to treat the system as nonholonomic). Nevertheless, all *configurations* are accessible. This suggests that the question of controllability is different depending on whether one is interested in configurations or states. We have formally declared our interest in reachable configurations.

Considering the system with just one of the two possible input forces is also interesting. In the case where we are just allowed to use F^2, the possible motions are quite simple; one can only move the ball on the leg back and forth. With just the force F^1 available, things are a bit more complicated. But, for example, one can still say that no matter how you apply the force, the ball with never move "inwards."

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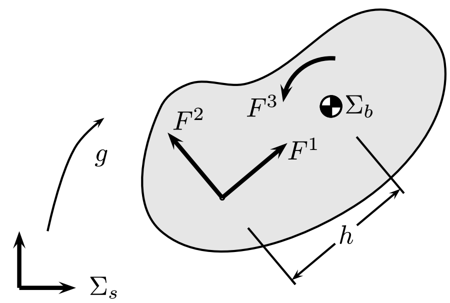
- The planar rigid body:



- Use coordinates (x, y, θ).
- Inputs are (1) force pointing towards centre of mass, $F^1 = \cos\theta \mathrm{d}x + \sin\theta \mathrm{d}y$, (2) force orthogonal to line to centre of mass, $F^2 = -\sin\theta \mathrm{d}x + \cos\theta \mathrm{d}y - h\mathrm{d}\theta$, and (3) torque at centre of mass $F^3 = \mathrm{d}\theta$.

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The planar rigid body, although seemingly quite simple, is actually somewhat interesting. Of course, if one uses all three inputs, the system is fully actuated, and so boring for what we are doing (investigating reachable configurations, that is). But if one takes various combinations of one or two inputs, one gets a pretty nice sampling of what can happen for these systems. For example, all possible combinations of two inputs allow one to reach all configurations. Using F^1 or F^3 alone give simple, one-dimensional reachable sets (similar to using F^2 for the robotic leg). Remember we are always starting with zero initial velocity! However, if one is allowed to only use F^2, then it is not quite clear what to expect, at least just on the basis of intuition.

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8. Recap of our objectives

- Note: All problem data is on $Q \implies$ expect answers to be describable using data on Q.
- We expect computations to simplify because of zero initial velocity assumption, and because we are interested in reachable configurations.
- We want a "computable" description of the reachable configurations.
- How do the input one-forms $F^1, \dots, F^m$ interact with the unforced mechanics of the system as described by the kinetic energy Riemannian metric?

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It turns out that our simplifying assumptions, i.e., zero initial velocity and restriction of our interest to configurations, makes our task *much* simpler. In fact, the computations without these assumptions have been attempted, but have yet to yield coherent answers.

In some sense, we wish to emulate the results we gave for linear systems at the beginning of the talk. And we shall in fact be able to do exactly this, inasmuch as it is possible. Without knowing the answer, it is worth thinking about the question of how the inputs interact with the Riemannian metric. That is, what is the analogue of "the smallest A-invariant subspace containing image(B)" for simple mechanical control systems?

9. The formal setting up of the problem

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- We let $\overset{g}{\nabla}$ denote the Levi-Civita affine connection for the Riemannian metric g.
- The equations of motion are then $\overset{g}{\nabla}_{c'(t)}c'(t) = u^a(t)Y_a(c(t))$ where $Y_a = (F^a)^\sharp$, $a = 1, \ldots, m$.
- There is nothing to be gained by using a Levi-Civita connection, or by assuming that the vector fields come from one-forms... So we study the control system

$$\nabla_{c'(t)}c'(t) = u^a(t)Y_a(c(t)) \quad \Big(+Y_0(c(t))\Big) \tag{CS}$$

with ∇ a general affine connection on Q, and $Y_1, \ldots, Y_m$ linearly independent vector fields on Q.

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Let us briefly recall how the Levi-Civita affine connection comes up in the problem. If we let $L(q, v) = g_{ij}\dot{q}^i\dot{q}^j$, then the Euler-Lagrange equations are

$$g_{ij}\ddot{q}^j + \left(\frac{\partial g_{ij}}{\partial q^k} - \frac{1}{2}\frac{\partial g_{jk}}{\partial q^i}\right)\dot{q}^j\dot{q}^k = u_a F_i^a, \quad i = 1, \ldots, n.$$

Now multiply this by g^{li} and take the symmetric part of the coefficient of $\dot{q}^j\dot{q}^k$ to get $\ddot{q}^l + \Gamma^l_{jk}\dot{q}^j\dot{q}^k = u^a Y_a^l$, $l = 1, \ldots, n$, where $\Gamma^i_{jk} = \frac{1}{2}g^{il}\left(\frac{\partial g_{lj}}{\partial q^k} + \frac{\partial g_{lk}}{\partial q^j} - \frac{\partial g_{jk}}{\partial q^l}\right)$, $i, j, k = 1, \ldots, n$, are exactly the Christoffel symbols for the Levi-Civita connection.

Here $\sharp\colon T^*Q \to TQ$ is the musical isomorphism associated with the Riemannian metric g.

At this point, perhaps the generalisation to an arbitrary affine connection seems like a senseless abstraction. However, as we shall see, this abstraction allows us to include, for "free," another large class of mechanical control systems.

The "optional" term Y_0 in (CS) indicates how potential energy may be added. In this case $Y_0 = -\operatorname{grad} V$. However, one looses nothing by considering a general vector field instead of a gradient. But I want to emphasise that one should *always* take $Y_0 = 0$ below, unless it is otherwise stated.

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- A *solution* to (CS) is a pair (c, u) satisfying (CS) where $c\colon [0, T] \to Q$ is a curve and $u\colon [0, T] \to \mathbb{R}^m$ is (say) bounded and measurable.
- Let U be a neighbourhood of $q_0 \in Q$ and denote by $\mathcal{R}_Q^U(q_0, T)$ those points in Q for which there exists a solution (c, u) with the properties
  1. $c(t) \in U$ for $t \in [0, T]$,
  2. $c'(0) = 0_q$, and
  3. $c(T) \in T_q Q$.
- Also $\mathcal{R}_Q^U(q_0, \leq T) = \bigcup_{0 \leq t \leq T} \mathcal{R}_Q^U(q_0, t)$.

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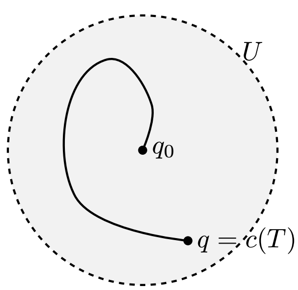
The following picture



gives an idea of what is meant by $\mathcal{R}_Q^U(q_0, T)$. It is the precise description of reachable sets as we shall need them. Note that we do not ask for the final velocity to be zero.

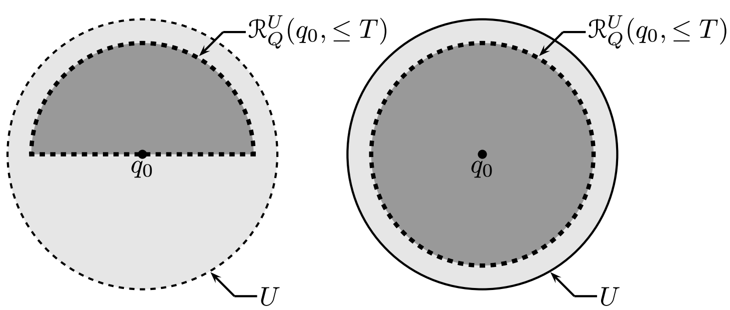
As you can see, we are only interested in points which can be reached without taking "large excursions." Control problems which are local in this way have the advantage that they can be characterised (at least for analytic systems) by Lie brackets. We do not address global issues, but they generally fall into two classes: (1) those of a topological nature (e.g., compactness in [San Martin and Crouch 1984]) and (2) those exploiting the unforced dynamics (e.g., Poisson stable systems in [Manikonda and Krishnaprasad 1997]).

Slide 8

- We want to "describe" $\mathcal{R}_Q^U(q, \leq T)$.
- (CS) is *locally configuration accessible* (*LCA*) at q if there exists $T > 0$ so that $\mathcal{R}_Q^U(q, \leq t)$ contains a non-empty open subset of Q for each neighbourhood U of q and each $t \in]0, T]$.
- (CS) is *locally configuration controllable* (*LCC*) at q if there exists $T > 0$ so that $\mathcal{R}_Q^U(q, \leq t)$ contains a neighbourhood of q for each neighbourhood U of q and each $t \in]0, T]$.

Notes for Slide 8

The notions of local configuration accessibility (on the left in the picture below) and local configuration controllability (on the right in the picture below) are genuinely different.



Indeed, one need only look at the example of the robotic leg with the F^1 input. In this example one may show that the system is LCA, but is not LCC. To show the former is something we will get to momentarily. The latter is clear for the reasons we have already mentioned: the ball cannot move "inward" no matter what kind of inputs you use.

10. Local configuration accessibility

Slide 9

- The accessibility problem is solved by looking at Lie brackets.
- Write (CS) in first order form: $\dot{v} = Z(v) + u^a \operatorname{vlft}(Y_a(v))$ where Z is the geodesic spray for ∇.
- We evaluate all brackets at 0_q—recall $T_{0_q}TQ \simeq T_qQ \oplus T_qQ$.
- We need the *symmetric product*: $\langle X : Y \rangle = \nabla_X Y + \nabla_Y X$.
- Here are some sample brackets:
  1. $[Z, \operatorname{vlft}(Y_a)](0_q) = (-Y_a(q), 0)$;
  2. $[\operatorname{vlft}(Y_a), [Z, \operatorname{vlft}(Y_b)]](0_q) = (0, \langle Y_a : Y_b \rangle(q))$;
  3. $[[Z, \operatorname{vlft}(Y_a)], [Z, \operatorname{vlft}(Y_b)]](0_q) = ([Y_a, Y_b](q), 0)$.

Notes for Slide 9

Recall the vertical lift:

$$\operatorname{vlft}(Y)(v_q) = \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} (v_q + tY(q)).$$

In coordinates, if $Y = Y^i \frac{\partial}{\partial q^i}$, then $\operatorname{vlft}(Y) = Y^i \frac{\partial}{\partial v^i}$.

When we write $T_{0_q}TQ = T_qQ \oplus T_qQ$, the first component we think of as being the "horizontal" bit which is tangent to the zero section in TQ, and we think of the second component as being the "vertical" bit which is the tangent space to the fibre of $\pi_{TQ} : TQ \to Q$.

To get an answer to the local configuration accessibility problem, we employ standard nonlinear control techniques involving Lie brackets. Doing so gives us our first look at the symmetric product. Our sample brackets suggest that perhaps the only things which appear in the bracket computations are symmetric products and Lie brackets of the input vector fields $Y_1, \dots, Y_m$. This is, in fact, the case, and the way they appear is also interesting as we shall see.

Slide 10

- Let C_{ver} be the closure of $\mathrm{span}(Y_1, \ldots, Y_m)$ under symmetric product.
- Let C_{hor} be the closure of C_{ver} under Lie bracket.
- The closure of $\mathrm{span}(Z, \mathrm{vlft}(Y_1), \ldots, \mathrm{vlft}(Y_m))$ under Lie bracket, when evaluated at 0_q, is then the distribution $q \mapsto C_{\mathrm{hor}}(q) \oplus C_{\mathrm{ver}}(q) \subset T_qQ \oplus T_qQ$.
- C_{hor} is integrable—let Λ_q be the maximal integral manifold through $q \in Q$.

Theorem 2 *$\mathcal{R}^U_Q(q, \leq T)$ is contained in Λ_q, and $\mathcal{R}^U_Q(q, \leq T)$ contains a non-empty open subset of Λ_q. In particular, if* $\mathrm{rank}(C_{\mathrm{hor}}) = n$ *then (CS) is LCA.*

Notes for Slide 10

We tacitly assume C_{ver} and C_{hor} to be distributions (i.e., of constant rank) on Q. With our underlying analyticity assumption, this will be true on an open dense subset of Q. Proving that the involutive closure of $\mathrm{span}(Z, \mathrm{vlft}(Y_1), \ldots, \mathrm{vlft}(Y_m))$ is equal at 0_q to $C_{\mathrm{hor}}(q) \oplus C_{\mathrm{ver}}(q)$ is a matter of computing brackets, samples of which are given on the previous slide, and seeing the patterns to suggest an inductive proof. The brackets for these systems are *very* structured. For example, the brackets of input vector fields are identically zero. Many other brackets vanish identically, and many more vanish when evaluated at 0_q. For details on the bracket computations, including systems with potential energy, we refer to [Lewis and Murray 1997a]. With potential energy included, the computations get rather messy.

11. The geometry of the reachable configurations

- What is the geometric meaning of C_{ver} and C_{hor}?
- Recall: A submanifold M of Q is *totally geodesic* if every geodesic with initial velocity tangent to M remains on M.
- Weaken to distributions: a distribution D on Q is *geodesically invariant* if for every geodesic $c\colon [0,T] \to Q$, $c'(0) \in D_{c(0)}$ implies $c'(t) \in D_{c(t)}$ for $t \in]0,T]$.

Theorem 3 *D is geodesically invariant iff it is closed under symmetric product.*

Slide 11

Notes for Slide 11

Theorem 2 gives a "computable" description of the reachable sets (in the sense that you can compute Λ_q by solving some over-determined nonlinear pde's). But it does not give the kind of insight that we had with the "smallest A-invariant subspace containing image(B)." It is this which we now describe.

Note that Theorem 3 says that the symmetric product plays for geodesically invariant distributions the same rôle the Lie bracket plays for integrable distributions. The result is proved by [Lewis 1998]. This result was key in providing the geometric description of the reachable configurations.

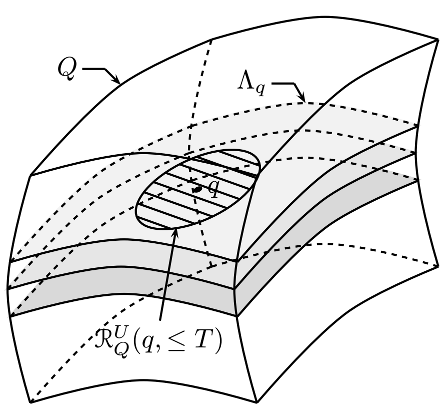
Slide 12

- An integrable distribution is *geodesically generated* if it is the involutive closure of a geodesically invariant distribution.
- Clearly C_{ver} is the smallest geodesically invariant distribution containing $\operatorname{span}(Y_1, \dots, Y_m)$.
- Also, C_{hor} is "geodesically generated" by $\operatorname{span}(Y_1, \dots, Y_m)$.
- Thus $\mathcal{R}_Q^U$ is contained in, and contains a non-empty open subset of, the distribution geodesically generated by $\operatorname{span}(Y_1, \dots, Y_m)$.

Notes for Slide 12

To be geodesically generated basically means that one may reach all points on a leaf with geodesics lying in some subdistribution.

The picture one should have in mind with the geometry of the reachable sets is a foliation of Q by geodesically generated (immersed) submanifolds onto which the control system restricts if the initial velocity is zero.



The idea is that *when you start with zero velocity* you remain on leaves of the foliation defined by C_{hor}. This decomposition is described by Lewis and Murray [1997b]. Note that for cases when the affine connection possesses no geodesically invariant distributions, the system (CS) is automatically LCA. This is true, for example, of $\mathbb{S}^2$ with the affine connection associated with its round metric.

We should also mention that the pretty decomposition we have for systems with no potential energy does not exist at this point for systems with potential energy.

Slide 13

12. Local configuration controllability

- This is harder...
- Call a symmetric product in $\{Y_1, \dots, Y_m\}$ *bad* if it contains an even number of each of the input vector fields. Otherwise call it *good*. The *degree* is the total number of vector fields.
- For example, $\langle\langle Y_a : Y_b\rangle : \langle Y_a : Y_b\rangle\rangle$ is bad and of degree 4, and $\langle Y_a : \langle Y_b : Y_b\rangle\rangle$ is good and of degree 3.

Theorem 4 *If each bad symmetric product at q is a linear combination of good symmetric products of lower degree, then (CS) is LCC at q.*

Notes for Slide 13

This business of good and bad symmetric products comes from an adaptation of work of Hermes [1982] and [Sussmann 1983, 1987]. Theorem 4 was proven in [Lewis and Murray 1997a]. To properly state the result, one should use free Lie and symmetric algebras.

Slide 14

- The single-input case can be solved completely:

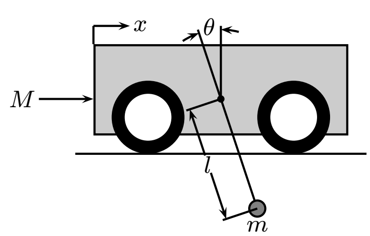
Theorem 5 *The system (CS) with $m = 1$ is LCC if and only if* $\dim(Q) = 1$.

- The local controllability question for general single-input control systems has not been answered $\implies$ our systems are special $\overset{?}{\implies}$ the controllability question may be solvable for arbitrary numbers of inputs!

Notes for Slide 14

The single-input result we state here follows (with some modification) from a result of Sussmann [1983]. It is presented in [Lewis 1997]. Although it seems an innocuous enough result, it is actually quite important for the reasons stated: it suggests that perhaps the general problem of local configuration controllability for these systems is solvable. This would be quite interesting as there are not many classes of systems for which this is the case, never mind that the mechanical systems we are looking at come up often in applications.

The result we state is not generally true for systems with potential energy. In particular, there are single-input systems with potential energy which are LCC at certain configurations. For example, the "crane,"

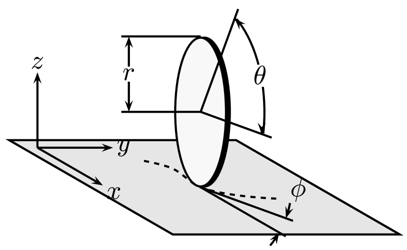


with the single input being a horizontal force applied to the base, is LCC about both vertical configurations of the arm.

Slide 15

13. Systems with nonholonomic constraints

- Let us now add to the data a distribution D defining nonholonomic constraints.
- Rolling disk:



- We consider two inputs: (1) a "rolling" torque ($F^1 = \mathrm{d}\theta$) and (2) a "spinning" torque ($F^2 = \mathrm{d}\phi$).

Notes for Slide 15

One of the interesting things about this affine connection approach is that we can integrate into our framework systems with nonholonomic constraints "for free."

The rolling disk we present here can be analysed as a nonholonomic system [Lewis 1999].

Slide 16

- The control equations for a simple mechanical control system with constraints are

$$\overset{g}{\nabla}_{c'(t)}c'(t) = \lambda(t) + u^a(t)Y_a(c(t)) \quad \Big(-\operatorname{grad} V(c(t))\Big)$$
$$c'(t) \in D_{c(t)}$$

  where $\lambda(t) \in D^{\perp}_{c(t)}$ are *Lagrange multipliers*.
- Let $P\colon TQ \to TQ$ and $P'\colon TQ \to TQ$ be orthogonal projections onto D and $D^{\perp}$, respectively
- Define an affine connection $\overset{D}{\nabla}$ by

$$\overset{D}{\nabla}_X Y = \overset{g}{\nabla}_X Y + (\overset{g}{\nabla}_X P')(Y)$$

Notes for Slide 16

The idea of writing constrained equations for simple mechanical systems with an affine connection seems to date to Synge [1928]. It has been rediscovered many times since then. For example, Bloch and Crouch [1995] use a variant of what we do here to investigate integrability of nonholonomic systems.

The properties of the affine connection $\overset{D}{\nabla}$ are discussed, along with other topics involving affine connections and distributions, in [Lewis 1998].

Slide 17

- The control equations are then equivalent to

$$\overset{D}{\nabla}_{c'(t)} c'(t) = u^a(t) P(Y_a)(c(t)) \quad \Big(-P(\operatorname{grad} V)(c(t))\Big)$$

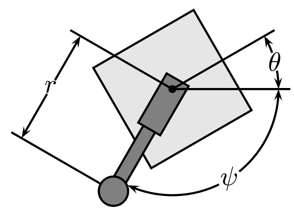
  which is of the form (CS).
- All the above analysis applies verbatim.
- Examples are somewhat unpleasant computationally…

Notes for Slide 17

The derivation of the constrained control equations in affine connection form is given by [Lewis 2000]. In that paper, a sometimes useful computational simplification is also presented. I have written a *Mathematica* package to do some of these computations, but they are still pretty horrific for non-trivial examples.

Slide 18

14. Examples (some revisited)



- Recall Y_1 was internal torque and Y_2 was extension force.
  - *Both inputs:* LCA and LCC (satisfies sufficient condition).
  - Y_1 *only:* LCA but not LCC.
  - Y_2 *only:* not LCA.

Notes for Slide 18

In the three cases, C_{hor} is generated by the following linearly independent vector fields:

1. *Both inputs:* $\{Y_1, Y_2, [Y_1, Y_2]\}$;
2. Y_1 *only:* $\{Y_1, \langle Y_1 : Y_1\rangle, \langle Y_1 : \langle Y_1 : Y_1\rangle\rangle\}$;
3. Y_2 *only:* $\{Y_2\}$.

Of course, these generators are not unique.

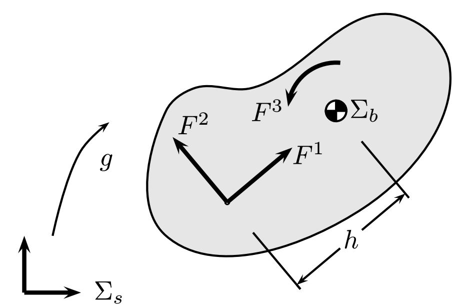
The sufficient condition we refer to here is the good/bad symmetric product result, Theorem 4.

Recall that with both inputs the system (we claimed) was not accessible in TQ as a consequence of conservation of angular momentum.

With the input Y_2 only, the control system behaves very simply when given zero initial velocity. The ball on the end of the leg just gets moved back and forth. This reflects the foliation of Q by the maximal integral manifolds of C_{hor}, which are evidently one-dimensional in this case.

With the Y_1 input, recall that the ball will always fly "outwards" no matter what one does with the input. Thus the system is not LCC. But apparently (since $\operatorname{rank}(C_{\text{hor}}) = \dim(Q)$) one can reach a non-empty open subset of Q. The behaviour exhibited in this case is typical of what one can expect for single-input systems with no potential energy.

Slide 19



- Y_1 *and* Y_2*:* LCA and LCC (satisfies sufficient condition).
- Y_1 *and* Y_3*:* LCA and LCC (satisfies sufficient condition).
- Y_1 *only or* Y_3 *only:* not LCA.
- Y_2 *only:* LCA but not LCC.

Notes for Slide 19

With the inputs Y_1 or Y_3 alone, the motion of the system is simple. In the first case the body moves along the line connecting the point of application of the force and the centre of mass, and in the other case the body simply rotates. The equations in (x, y, θ) coordinates are

$$\begin{aligned}\ddot{x} &= \frac{\cos\theta}{m}u^1 - \frac{\sin\theta}{m}u^2 \\ \ddot{y} &= \frac{\sin\theta}{m}u^1 + \frac{\cos\theta}{m}u^2 \\ \ddot{\theta} &= -\frac{h}{J}u^2 + \frac{1}{J}u^3\end{aligned}$$

which illustrates that the θ-equation decouples when only Y_3 is applied. We make a change of coordinates for the case where we have only Y_1: $(\xi, \eta, \psi) = (x\cos\theta + y\sin\theta, -x\sin\theta + y\cos\theta, \theta)$. In these coordinates we have

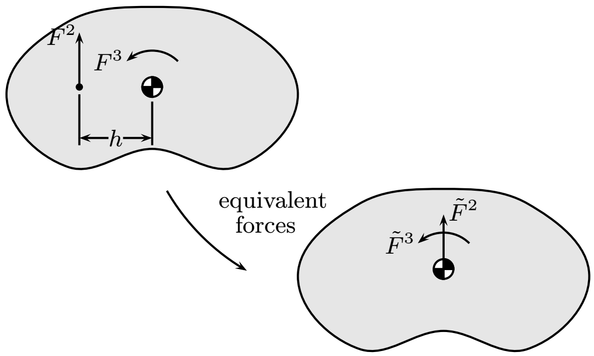
$$\begin{aligned}&\ddot{\xi} - 2\dot{\eta}\dot{\psi} - \xi\dot{\psi}^2 = \frac{1}{m}u^1 \\ &\ddot{\eta} + 2\dot{\xi}\dot{\psi} - \eta\dot{\psi}^2 = 0 \\ &\ddot{\psi} = 0\end{aligned}$$

which illustrates the decoupling of the ξ-equation in this case.

The case with Y_2 only stood unresolved, simple though it seems, until the single-input result, Theorem 5, was proved. Global controllability, i.e., not necessarily asking that motions remain small, is still unresolved for the system, although it is suspected to be globally controllable. Some attention is given to this problem by Manikonda and Krishnaprasad [1997].

Slide 20

- *Y_2 and Y_3:* LCA and LCC (fails sufficient condition).



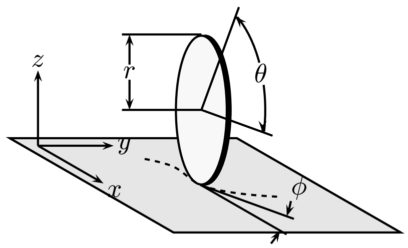
Notes for Slide 20

C_{hor} has the following generators:

1. *Y_1 and Y_2:* $\{Y_1, Y_2, [Y_1, Y_2]\}$;
2. *Y_1 and Y_3:* $\{Y_1, Y_3, [Y_1, Y_3]\}$;
3. *Y_1 only or Y_3 only:* $\{Y_1\}$ or $\{Y_3\}$;
4. *Y_2 only:* $\{Y_2, \langle Y_2 : Y_2\rangle, \langle Y_2 : \langle Y_2 : Y_2\rangle\rangle\}$.
5. *Y_2 and Y_3:* $\{Y_2, Y_3, [Y_2, Y_3]\}$.

The inputs we deal with in this slide *do* give a system which is LCC, but which fails our good/bad symmetric product test. However, a simple change of input, as illustrated in the figure, suggests that the system ought to be LCC. In fact, with these modified inputs, the system now satisfies the conditions of Theorem 4. A possible conjecture is that this is *always* possible. That is, perhaps (CS) is LCC if and only if there exists a basis of input vector fields which satisfy the hypotheses of Theorem 4. But this is speculation at this point…

Slide 21



- Recall Y_1 was "rolling" input and Y_2 was "spinning" input.
  - Y_1 *and* Y_2*:* LCA and LCC (satisfies sufficient condition).
  - Y_1 *only:* not LCA.
  - Y_2 *only:* not LCA.

Notes for Slide 21

C_{hor} has generators

1. Y_1 *and* Y_2*:* $\{Y_1, Y_2, [Y_1, Y_2], [Y_2, [Y_1, Y_2]]\}$,
2. Y_1 *only:* $\{Y_1\}$, and
3. Y_2 *only:* $\{Y_2\}$.

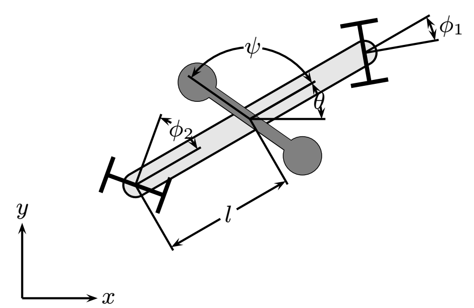
The rolling disk passes our good/bad symmetric product test. Another way to show that it is LCC is to show that the inputs allow one to follow any curve which is admitted by the constraints. Local configuration controllability then follows as the constraint distribution for the rolling disk has an involutive closure of maximal rank. This is the gist of the approach described in [Lewis 1999].

The decomposition corresponding to the input Y_2 only may be seen in the standard coordinates (x, y, θ, ϕ). To obtain the decomposition for the input Y_1 alone, we make the change of coordinates

$$(\xi, \eta, \zeta, \psi) = (x\cos\phi + y\sin\phi, -x\sin\phi + y\cos\phi, x\cos\phi + y\sin\phi - r\theta, \phi).$$

To see the form of the equations in this case, we refer to [Lewis 2000].

Slide 22



- Take $\phi_1 = \phi_2 = \phi$.
- Inputs are a synchronised torque to rotate the wheels ($F^1 = \mathrm{d}\phi$) and a torque to rotate the "rider" ($F^2 = \mathrm{d}\psi$).

Notes for Slide 22

The snakeboard example we look at here was first investigated by Lewis, Ostrowski, Murray, and Burdick [1994]. As a nonholonomic system with symmetry it was further studied by Ostrowski [1995] and Bloch, Krishnaprasad, Marsden, and Murray [1996]. A different control treatment is given by Ostrowski and Burdick [1997].

Slide 23

- Caressing *Mathematica* gives:
  - Y_1 *and* Y_2*:* LCA and LCC (satisfies sufficient condition).
  - Y_1 *only:* not LCA.
  - Y_2 *only:* not LCA.

Notes for Slide 23

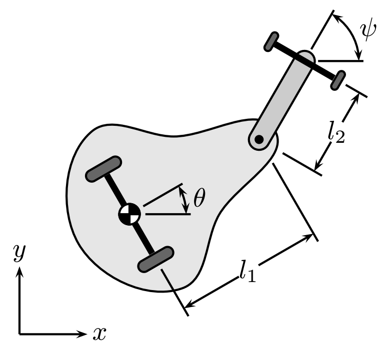
Generators for C_{hor} are

1. Y_1 *and* Y_2*:* $\{Y_1, Y_2, [Y_1, Y_2], [Y_2, [Y_1, Y_2]], [Y_2, [Y_1, [Y_2, [Y_1, Y_2]]]]\}$,
2. Y_1 *only:* $\{Y_1\}$, and
3. Y_2 *only:* $\{Y_2\}$.

The computations for the snakeboard are somewhat unpleasant, and are given in detail by Lewis [2000]. Part of the reason the snakeboard equations are so awkward in affine connection form is that the $SE(2)$-symmetry of the system is not taken into account as it is, for example, by Ostrowski and Burdick [1997]. It is probably interesting to see how symmetry figures into our whole picture. The "fully symmetric" case where Q is a Lie group G is presented by [Bullo and Lewis 1996]. The case where $Q \to Q/G$ is a principal bundle is next in line.

The motion with only the input Y_1 is easy to describe: the wheels rotate and nothing else happens. With only Y_2, the maximal integral manifolds of C_{hor} are still one-dimensional, but are not so easy to describe (i.e., I don't know what they look like).

Slide 24



- Single input: $F = \mathrm{d}\theta - \mathrm{d}\psi$.
- Probably LCA, but (definitely) not LCC.

Notes for Slide 24

I was not able to obtain an expression for generators for C_{hor} for the roller racer. However, C_{hor} does *not* have full rank at the "standard" configuration $(x = 0, y = 0, \theta = 0, \psi = 0)$.

The roller racer pictured here was studied by Krishnaprasad and Tsakiris [2001] as a system with $SE(2)$-symmetry. Because we have not incorporated this into our framework, the roller racer computations put even the snakeboard computations to shame... Nevertheless, Theorem 5 allows one to immediately say that local configuration controllability is impossible. Global controllability is unresolved, but it seems likely that the roller racer is globally controllable.

Slide 25

15. Control design (F. Bullo)

- Up to now, consider case when $Q = G$ and data is left-invariant.
- Use low amplitude, periodic inputs and averaging methods based on controllability analysis.
  - Steer the system from state $(q_1, 0)$ to $(q_2, 0)$ (with small final error).
  - Exponentially stabilise the system to $(q_0, 0)$ in a neighbourhood.
  - Steer the system along a path connecting points $q_1, \dots, q_N$ (with small error terms).

Notes for Slide 25

Control design for these systems, especially in the absence of potential energy, is a bit challenging. This is a direct consequence of the system's not being amenable to linearisation-based control design methods.

Averaging on Lie groups was used by Leonard and Krishnaprasad [1995] to study kinematic systems, i.e., systems with no drift. It is possible to modify these methods to the systems we consider, and this is done by Bullo, Leonard, and Lewis [2000], at least in some cases.

It should be mentioned that the exponential stabilisation of these systems is somewhat non-trivial. For example, they cannot be stabilised by continuous state feedback [Brockett 1983], and cannot be exponentially stabilised by smooth, time-varying feedback [M'Closkey and Murray 1997]. The controllers defined by Bullo, Leonard, and Lewis [2000] are continuous and time-varying.

Slide 26

16. Stuff to do

- Figure out local accessibility at non-zero velocity.
- Get sharper controllability results.
- Refine understanding of how potential energy enters the picture.
- Control design for general systems.
- Optimal control.
- ***Punchline:*** The affine connection formalism can be useful.

Notes for Slide 26

Some preliminary bracket computations at non-zero initial velocity have been done. They are quite complicated. However, curvature and its derivatives appear, so one expects the infinitesimal holonomy algebra for ∇ to come into the picture. This is not altogether surprising.

As was suggested above, it might be possible to get very crisp controllability results for these systems, and this would be interesting. But almost nothing has been done in this area.

At this point, design methodology for controllers for the systems we have been talking about do not really exist. The averaging methods as discussed above for Lie groups ought to be able to be applied to general systems in some manner, but the details here have yet to be worked out. For some preliminary results see [Bullo 1999].

It would appear that these systems offer a very nice framework within which to study optimal control. Nothing really has been done here, however.

If there were to be a point of this talk, it would be that the framework we provide here using affine connections to describe certain classes of mechanical control systems can be valuable, especially for obtaining general results. As another example of this, see [Lewis 1999] where a concise recipe is given for deciding when a mechanical system is kinematic. I was forced to consider such a condition by the propensity of some in the control community to *equate* mechanical systems with "nonholonomic" systems. For example, I once saw someone talk about the snakeboard as a nonholonomic system, which it most certainly is not! For work which is not mine (!) and which employs the affine connection, see [Rathinam and Murray 1998].

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