# Information Transmission over Two-Way Networks

by

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*This thesis is dedicated to my parents and my wife for their unconditional love.*

# Abstract

Shannon's two-way channels (TWCs) allow two terminals to exchange data streams in a full-duplex manner and enable interactive adaptive coding to improve transmission reliability. However, TWCs are often used in conjunction with orthogonal multiplexing to mitigate the interference incurred from simultaneous transmissions over a shared channel. To date, TWCs with non-orthogonal multiplexing are still not fully explored. This thesis examines channel capacity problems for TWCs and identifies coding methods to facilitate and enhance two-way simultaneous transmission.

We first make use of channel symmetry properties to determine the capacity region of three types of two-way networks: (a) two-terminal discrete-memoryless TWCs (DM-TWCs), (b) two-terminal TWCs with memory, and (c) three-terminal multiaccess/degraded broadcast DM-TWCs. For each network, symmetry conditions under which a Shannon-type random coding inner bound (under independent non-adaptive inputs) is tight are given. The results broaden the class of TWCs whose capacity region can be exactly determined and imply that interactive adaptive coding does not enlarge the capacity region and is hence unnecessary for such channels. Moreover, we generalize Shannon's push-to-talk TWC and analytically derive this generalized channel's capacity region, which is a convex hull of at most four rate pairs. For general two-terminal DM-TWCs that lack channel symmetry properties, a simple outer

bound is further derived to obtain approximation capacity results.

In addition to examining capacity problems, we also study joint source-channel coding (JSCC) for TWCs. Specifically, we propose an adaptive lossy JSCC scheme for sending correlated sources over two-terminal DM-TWCs. Our idea is to couple the independent operations of the terminals via an adaptive coding mechanism which can mitigate cross-interference resulting from simultaneous channel transmissions and concurrently exploit the sources' correlation to reduce the end-to-end reconstruction distortions. Our adaptive JSCC scheme not only subsumes existing lossy coding methods for the same setup, but it also improves on their performance. Several examples are given for illustration. Moreover, we derive outer bounds for our two-way lossy transmission problem and establish complete JSCC theorems in some special settings. In these special cases, a non-adaptive separate source-channel coding scheme achieves the optimal performance, thus significantly simplifying the design of the source-channel communication system.

# Co-Authorship

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# Chapter 1

# Introduction

## 1.1 An Overview

Point-to-point two-way channels (TWCs) first appeared in Shannon's seminal paper [3], where two terminals (or users) desire to communicate with each other as effectively as possible over a shared channel. TWCs allow bidirectional simultaneous transmissions (or in-band full-duplex transmissions) between the two terminals to make full use of channel resources. They also enable the two terminals to generate their channel inputs by adapting to the previously received signals, which may enhance the reliability of the overall communication. Even though TWCs provide several advantages and are building blocks in many communication systems, the major difficulties in the full-duplex implementation are due to user interference and residual self-interference [4]. In the past decades, to avoid such interferences and simplify the system design, TWCs have often been used in a half-duplex mode with the help of orthogonal multiplexing [5]. Recently, non-orthogonal multiplexing techniques [6] have gained popularity to improve the bandwidth efficiency due to advances in digital signal processing. However, how much information can be reliably transmitted in a full-duplex manner is still unknown.

In this thesis, instead of focusing on the full-duplex communication problem under practical constraints, we study the reliable communication problem from an information-theoretic viewpoint. Here, the challenge for optimal reliable two-way communication lies in how each terminal can effectively maximize its individual transmission rate over the shared channel and concurrently provide sufficient feedback to help the other terminal's transmission. These competing objectives impose on each terminal the difficult task of optimally adapting their inputs to the previously received signals of the other terminal. As finding such an optimal coding procedure is still elusive, the exact characterization of the capacity region for general two-terminal TWCs remains open [7]. The design of effective error correction and interference mitigation coding techniques that make the best use of TWCs is also a challenging unsolved problem.

This thesis intends to achieve two objectives: (i) determining the capacity region for a broad class of TWCs (with two or more terminals) and (ii) developing efficient transmission schemes for sending correlated sources over TWCs. Before reviewing related work and defining our research problems, we first present some milestones in the development of TWCs, recent findings, and other subjects surrounding TWCs.

## 1.2 The Development of Two-Way Channels

*Shannon's Inner Bound can be Exceeded*:

In [3], Shannon derived inner and outer bounds for the capacity region of two-terminal discrete-memoryless TWCs (DM-TWCs), where the inner bound is obtained via a standard random coding scheme (where the encoder of each terminal selects its code symbols as a function of the message only, independently of past received signals from the other terminal). He pointed out that his inner bound is tight under certain channel symmetry conditions [3, Sec. 11]; but in general, his inner bound differs from his outer bound. Blackwell's binary-multiplying TWCs (BM-TWCs) [3, Sec. 13] is a classical example for which the two capacity bounds do not coincide. Even though Shannon's inner bound is generally not tight, whether or not his inner bound can be enlarged was still in question at that time.

Jelinek made the first attempt to find DM-TWCs whose capacity region can exceed Shannon's inner bound region [8–10]. For binary-input binary-output DM-TWCs of a certain type, Jelinek claimed (without proof) that an inner bound region larger than Shannon's can be obtained [8, Section 7.7], but later he mentioned in [10, Section VIII] that this assertion remained unsubstantiated. Despite this unsuccessful attempt, his result laid the foundations to study interference channels [11]; a summary of Jelinek's work can be found in [12]. The second attempt [13] was made by Libkind, who claimed that adaptive coding cannot improve Shannon's inner bound. However, his proof of the equality of Shannon's capacity bounds was considered to be incorrect [7]. Twenty-eight years after Shannon's result, the question was finally answered by Dueck [14], who constructed a DM-TWC whose capacity region is strictly larger than Shannon's inner bound. In fact, Dueck's example is not only an answer to the question but it also demonstrates that certain feedback structures in a TWC facilitate adaptive coding.

For the sake of completeness, we remark that Shannon's random coding is further generalized in [15, Section 5] but the scheme given there still uses non-adaptive coding. Whether or not this generalization yields a larger inner bound region is not clear.

*Capacity can be Different under Maximal and Average Error Criteria*:

In [3], Shannon adopted the average error probability to define channel capacity. In the early seventies, Ahlswede showed that the notions of average and maximal error probabilities are different for the capacity of general DM-TWCs [16]. Nevertheless, El Gamal found that at least for "restricted TWCs" [7] both notions result in an identical capacity region [17]. Here, restricted TWCs are TWCs for which interactive adaptive coding is forbidden; the capacity region for such channels is determined by Shannon's inner bound [7].

*Variants of Shannon's TWCs*:

In [15, 16, 18], Ahlswede proposed several variants of Shannon's two-terminal DM-TWC and established capacity results for some cases. Particularly, the variant where two transmitters and two receivers are all located at distinct places and both transmitters simultaneously send independent messages to both receivers [16], also known as a compound multiple-access-channel (MAC) [19], brings a novel viewpoint to Shannon's TWCs. An interesting two-user degraded channel model was also studied in [20].

*Adaptive Coding Inner Bounds and Improved Outer Bounds*:

In addition to Dueck's example, Schalkwijk proposed a coding strategy for BM-TWCs to achieve a rate pair outside Shannon's inner bound region [21]; his team also made progressive refinements [22–27] by using the idea of dividing a unit square. For general DM-TWCs, Han constructed an adaptive coding scheme based on Markov block encoding and illustrated that the resulting achievable rate region contains Shannon's result as a subset [28]. Several improved capacity outer bounds were also derived in various studies in the eighties. For general DM-TWCs, Zhang *et al.* introduced in [29] auxiliary random variables to control the dependency between the terminals' channel inputs, thereby providing a better outer bound than Shannon's result. Hekstra and Willems derived an outer bound specialized for common-output DM-TWCs such as BM-TWCs in [30, 31]. For this class of TWCs, they also gave sufficient conditions for Shannon's inner bound to be tight [31, Corollaries 2 and 3].

*Further Results on the Capacity Region of DM-TWCs*:

In the late nineties, methods to efficiently utilize DM-TWCs were investigated by studying the role of feedback [32]. In [33], Kramer used an idea of concatenated codes to design adaptive coding and extended Han's result. Directed mutual information [34], which is widely used in the study of one-way channels with feedback [35–39], was also employed to characterize the capacity of DM-TWCs. Recently, the capacity of non-binary additive-noise TWCs [40] and more general channel models such as injective semi-deterministic TWCs (ISD-TWCs) [41], Cauchy [41], Poisson [42], and exponential family type TWCs [43] were determined.[1] A new tightness condition of Shannon's inner bound was also derived in [41], which allows us to determine the capacity of a broader class of DM-TWCs. The graph-based coding method [44] for common-output DM-TWCs further generalizes Schalkwijk's 1982 scheme [21]. Zero-error capacity and its bounds for common-output DM-TWCs were studied in [45].

*Two-Way Lossy Source Transmission*:

For noiseless two-terminal TWCs, Kaspi in [46] tackled a two-way lossy source coding problem and established a rate-distortion (RD) region for this system,[2] which characterizes the trade-off between source compression rate and distortion, under an interactive communication protocol. More specifically, the protocol divides the entire

[1] Han has shown that Shannon's inner bound is tight for Gaussian TWCs [28].

[2] Kaspi's original proof relies on tree codes using an intricate approach. A simper proof can be found in [47, Section 20.3.3] based on the Wyzer-Ziv source coding scheme [48].

transmission period into small segments, and only one terminal sends data in each segment. With this protocol, each terminal can decode a coarse description of the other terminal's messages after observing a new segment of channel outputs. All decoded coarse descriptions are then treated as side-information to compress source messages until final reconstructions are obtained. In [49], Maor and Merhav extended Kaspi's result within the application of successive source refinement. Another related two-way source coding problem, where each terminal is only interested in extracting hidden information related to the source messages of the other terminal, is tackled in [50] in the context of the so-called collaborative information bottleneck problem. The rate-relevance trade-off is determined under Kaspi's transmission protocol.

*The Capacity Region for TWCs with Memory and Multi-Terminal DM-TWCs*:

Two-terminal TWCs with memory were first studied by Shannon in [3, Sec. 16]. Assuming that the channel satisfies a so-called recoverable state property, Shannon determined their capacity region by a general formula described in [3, Theorem 5]. Beyond Shannon's two-terminal DM-TWCs, only a few results are available in the literature. Cheng and Devroye in [51] showed that Shannon's random coding scheme is optimal in several deterministic multi-terminal DM-TWC settings (i.e., more than two terminals) such as multiaccess/broadcast (MA/BC), Z, and interference TWCs, hence finding the channel capacity in these cases. The channel capacity for a variant of the multi-terminal TWCs, called three-way channels, was also investigated in different networks such as three-way multi-cast finite-field or phase-fading Gaussian channels [52] and three-way Gaussian channels with multiple unicast sessions [53]. A comprehensive overview of multi-way communication can be found in [54]. We remark that an additional capacity result for deterministic interference TWCs was derived in [55].

*Other Topics Related to TWCs*:

Besides the above developments on channel capacity and source transmission, efforts were also devoted to research different aspects of TWCs such as error exponents [56–58], interactive capacity [59], and channel-optimized scalar quantization [60]. TWCs were also studied in the presence of jammers [61, 62] and relays [63–66] and considered in ad-hoc wireless networks [67], molecular communication [68], and visible light communication [69]. Furthermore, two-way function computation [70] and two-way coding for control systems [71] are related to adaptive coding for TWCs. Apart from the above theoretical perspective of TWCs, practical system designs for two-way simultaneous transmission are now implemented in non-orthogonal multiple-access systems [72] or via reconfigurable intelligent surface [73], which are active areas for future advanced communication systems.

## 1.3 Notation

We next introduce the notation used in the thesis. The symbols $\mathbb{Z}_+$ and $\mathbb{R}_{\geq 0}$ denote the sets of positive integers and non-negative real numbers, respectively. The probability distribution of a random variable $A$ having alphabet $\mathcal{A}$ is denoted by $P_A$, i.e., $P_A(a) = \Pr(A = a)$ for $a \in \mathcal{A}$, and the cardinality of $\mathcal{A}$ is denoted by $|\mathcal{A}|$. The set of all probability distributions on $\mathcal{A}$ is denoted as $\mathcal{P}(\mathcal{A})$, and $P^{\mathrm{U}}_{\mathcal{A}} \in \mathcal{P}(\mathcal{A})$ represents the uniform distribution on $\mathcal{A}$. For $l \geq 1$, let $A^l \triangleq (A_1, A_2, \ldots, A_l)$ denote a length-$l$ sequence of random variables with common alphabet $\mathcal{A}$. A realization of $A^l$ is denoted by $a^l = (a_1, a_2, \ldots, a_l) \in \mathcal{A}^l$, where $\mathcal{A}^l$ is the $l$-fold Cartesian product of $\mathcal{A}$. When the length $l$ is clear from the context, we may write $\boldsymbol{A}$ and $\boldsymbol{a}$ instead of $A^l$

and $a^l$, respectively. For any $1 \leq l_1 \leq l_2 \leq l$, we define $A_{l_1}^{l_2} = (A_{l_1}, A_{l_1+1}, \ldots, A_{l_2})$; we also set $A_{l_1}^{l_2} = \emptyset$, a null sequence, when $l_1 > l_2$. Throughout the thesis, all alphabets are finite, except for the Gaussian case briefly considered in Sections 5.5.1 and 5.6.3.

Furthermore, we delineate each terminal by index $j$ or $j'$. To simplify our presentation, we assume that $j \neq j'$ when the two indices appear together. In capacity problems, the message of terminal $j$ intended for terminal $j'$ is denoted by $M_{jj'}$; when there are only two terminals, we write $M_j$ for the sake of brevity. For two-terminal lossy transmission problems, we use $S_{j,k}$ to denote the $k$th source message of terminal $j$. The reconstructions of $M_j$, $M_{jj'}$, and $S_{j,k}$ is denoted by $\hat{M}_j$, $\hat{M}_{jj'}$, and $\hat{S}_{j,k}$, respectively. The symbols $X_{j,n}$ and $Y_{j,n}$ represent the $n$th channel input and output of terminal $j$, respectively. For channels with erasures, the erasure symbol is given by $\mathbf{E}$. The alphabets of the above system variables will be respectively denoted by $\mathcal{M}_{jj'}$, $\hat{\mathcal{M}}_{jj'}$, $\mathcal{M}_j$, $\hat{\mathcal{M}}_j$, $\mathcal{S}_j$, $\hat{\mathcal{S}}_j$, $\mathcal{X}_j$, and $\mathcal{Y}_j$.

The functions $H(\,\cdot\,)$, $H(\,\cdot\,|\,\cdot\,)$, $I(\,\cdot\,;\,\cdot\,)$, and $I(\,\cdot\,;\,\cdot\,|\,\cdot\,)$ represent entropy, conditional entropy, mutual information, and conditional mutual information, respectively. We define these in the standard way [74] and we hence omit their definitions here. In some cases, we use $H_{\mathrm{b}}(\cdot)$ for the binary entropy function. For a channel with input $X$, output $Y$, and transition probability $P_{Y|X}$, the input-output mutual information $I(X;Y)$ under an input distribution $P_X$ is sometimes written in a functional representation $\mathcal{I}(P_X, P_{Y|X})$ to emphasize its dependence on the probability distributions $P_X$ and $P_{Y|X}$ (the formal definition is given in (2.12)). Moreover, the channel transition probability $P_{Y|X}$ can be also expressed in a matrix form, denoted by $[P_{Y|X}(\,\cdot\,|\,\cdot\,)]$, whose rows and columns are indexed by $x \in \mathcal{X}$ and $y \in \mathcal{Y}$, respectively, and whose $(x, y)$-entry equals to $P_{Y|X}(y|x)$. Given random variables $X$, $Y$, and $Z$ with joint probability distribution $P_{X,Y,Z}$, the notation $X \multimap Y \multimap Z$ indicates a Markov relationship among them; i.e., their joint probability distribution can be decomposed in this way: $P_{X,Y,Z} = P_{X,Y}P_{Z|Y}$.

Finally, for the additive group $(\{0, 1, \ldots, q-1\}, \oplus_q)$ for some $q \geq 2$, where $\oplus_q$ is modulo-$q$ addition, we let $\{0, 1, \ldots, q-1\} \triangleq G_q$ and let $\ominus_q$ denote the modulo-$q$ subtraction. In any derivation involving summation, we will not specify the domain of the summation when it is clear from the context. The standard notation $\mathbb{E}$ stands for the expectation operator and $\mathbf{1}\{\cdot\}$ stands for the indicator function. Other terms not mentioned here will be defined when first introduced.

## 1.4 Research Problems

Our first research problem concerns finding the capacity region for TWCs. In particular, we tackle the capacity problem for three two-way networks depicted in Fig. 1.1. The two-terminal (point-to-point) memoryless TWC in Fig. 1.1(a) models device-to-device communication [75]. The simplified TWC with memory in Fig. 1.1(b), which is a generalization of additive-noise TWCs in [40], can capture the effect of time-correlated channel noise which commonly arises in wireless communications. The three-terminal memoryless multiaccess/degraded broadcast (MA/DB) TWC [51] in Fig. 1.1(c) models the communication between two mobile users and one base station. For the sake of simplicity, we assume that the shared channel in the users-to-base-station (uplink) direction acts as a MAC while the reverse (downlink) direction acts as a degraded broadcast channel (DBC). For these networks, we investigate when the Shannon-type inner bound is optimal in terms of achieving channel capacity. As a result, we identify TWCs for which interactive adaptive coding is useless in terms of improving the terminals' transmission rates. Such a result has a practical significance since communication without adaptive coding simplifies system design.

Our second research problem is in the scope of joint source-channel coding (JSCC) for two-terminal TWCs. More specifically, we study the scenario where two terminals exchange a block of correlated source messages $(S_1^K, S_2^K)$ of length-$K$ via $N$ uses of a noisy DM-TWC in Fig. 1.1(a). Terminal $j$ only observes $S_j^K$ and intends to reconstruct $S_{j'}^K$ from $S_j^K$ and $Y_j^N$ subject to a distortion constraint. Here, the source pairs $(S_{1,k}, S_{2,k})$, $1 \leq k \leq K$, are assumed to be independent in time having the common joint probability distribution $P_{S_1,S_2}$; i.e., $P_{S_1^K,S_2^K}(s_1^K, s_2^K) = \prod_{k=1}^K P_{S_1,S_2}(s_{1,k}, s_{2,k})$, where $(s_{1,k}, s_{2,k}) \in \mathcal{S}_1 \times \mathcal{S}_2$. The distortion for the reconstruction $\hat{s}_j^K$ of source message $s_j^K$ is assessed via $d_j(s_j^K, \hat{s}_j^K) \triangleq K^{-1} \sum_{k=1}^K d_j(s_{j,k}, \hat{s}_{j,k})$, where $d_j : \mathcal{S}_j \times \hat{\mathcal{S}}_j \to \mathbb{R}_{\geq 0}$ is a single-letter distortion measure for source $S_j$. Furthermore, the noisy DM-TWC is used without adopting any interactive communication protocol such as in [46, 49]. Given channel transition probability $P_{Y_1,Y_2|X_1,X_2}$ of a DM-TWC, the memoryless property of the channel then implies that $P_{Y_{1,n},Y_{2,n}|X_1^n,X_2^n,Y_1^{n-1},Y_2^{n-1}} = P_{Y_{1,n},Y_{2,n}|X_{1,n},X_{2,n}} = P_{Y_1,Y_2|X_1,X_2}$ for all $n$. For this system setup, we seek forward and converse coding theorems for lossy source-channel transmissibility.

## 1.5 Related Work and Our Approach

### 1.5.1 Channel Capacity for Two-Way Channels

The notion of channel symmetry properties, which has been extensively investigated to simplify the computation of the capacity of one-way channels, also plays a key role in determining the capacity region for TWCs. The first channel symmetry property for DM-TWCs was discovered by Shannon [3, Section 12]. Given

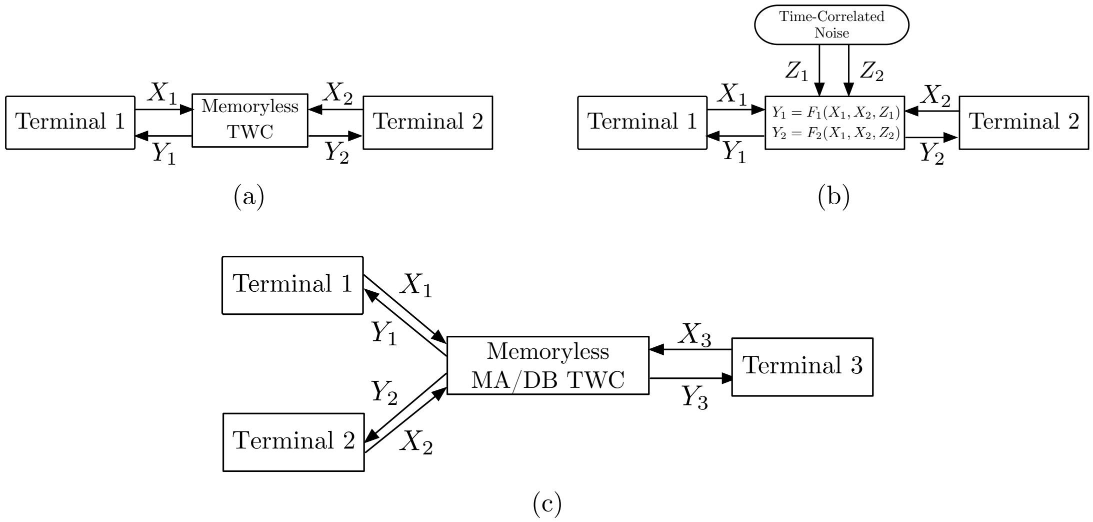


Figure 1.1: Block diagrams of the two-way networks considered: (a) point-to-point memoryless TWC with two channel inputs $X_1$ and $X_2$ and two channel outputs $Y_1$ and $Y_2$; (b) point-to-point TWC with memory, where $F_1$ and $F_2$ are deterministic functions and $(Z_1, Z_2)$ is a time-correlated channel noise pair generated from a two-dimensional random process; (c) three-terminal memoryless MA/DB TWC, where $X_i$ and $Y_i$ respectively denote channel input and output at terminal $j$ for $j = 1, 2, 3$.

$[P_{Y_1,Y_2|X_1,X_2}(\,\cdot\,,\,\cdot\,|\,\cdot\,,\,\cdot\,)]$, the channel transition matrix of a two-terminal DM-TWC. Shannon gave two permutation invariance conditions on $[P_{Y_1,Y_2|X_1,X_2}(\,\cdot\,,\,\cdot\,|\,\cdot\,,\,\cdot\,)]$ which guarantee the equality of his inner and outer bounds (see Propositions 2.1 and 2.2 in Section II for details). A recent work [41] by Chaaban, Varshney, and Alouini (CVA) presented another tightness condition, where the channel symmetry property is given in terms of conditional entropies for the marginal channel distribution $[P_{Y_j|X_1,X_2}(\,\cdot\,|\,\cdot\,,\,\cdot\,)]$ (see Proposition 2.3).

The above conditions delineate classes of two-terminal DM-TWCs for which Shannon's capacity inner bound is tight, hence exactly yielding their capacity region. Examples include Gaussian TWCs [28], $q$-ary additive-noise TWCs [40], and more general channel models such as injective semi-deterministic TWCs (ISD-TWCs) [41], Cauchy [41] and exponential family type TWCs [43]. We remark that Hekstra and Willems [31] also presented conditions under which Shannon's inner bound is tight, but their result is only valid for common-output DM-TWCs.

For three-terminal MA/BC DM-TWCs, Cheng and Devroye [51] studied a class of symmetric TWCs. In particular, they considered deterministic, invertible, and alphabet-restricted MA/BC DM-TWCs, proving that the Shannon-type inner bound is tight for that class of channels. However, to the best of our knowledge, symmetry properties for TWCs beyond these have not been investigated. It is also important to point out that two-terminal TWCs with memory are not well understood either.

In this thesis, we tackle the capacity problem by viewing a TWC as two interactive state-dependent one-way channels [3], [10].[3] Taking the two-terminal network as an example, the state-dependent one-way channel from terminals 1 to 2 has input $X_1$, output $Y_2$, state $X_2$, and transition matrix given by $[P_{Y_2|X_1,X_2}(\cdot\,|\,\cdot,\,\cdot)]$; similarly, the one-way channel $[P_{Y_1|X_1,X_2}(\cdot\,|\,\cdot,\,\cdot)]$ in the reverse direction has input $X_2$, output $Y_1$, and channel state $X_1$. Note that this viewpoint may also be useful for all previously mentioned two-way networks. Another useful tool is the rich set of symmetry concepts for single-user one-way channels.[4] From this viewpoint, the two one-way channels now interact with each other through the channel states. In principle, this interaction could improve bi-directional transmission rates by making use of adaptive coding.

To determine channel capacity, our approach is to study symmetry properties for state-dependent one-way channels that imply that the capacity cannot be increased

[3] Another viewpoint for two-terminal TWCs is based on compound MACs, see [19, Problem 14.11] and [76].

[4] Channel symmetry properties for single-user one-way memoryless channels can be roughly classified into two types. One type focuses on the structure of the channel transition probability such as Gallager symmetric channels [77], weakly symmetric and symmetric channels [74], and quasi-symmetric channels [78]. The other type aims at the invariance of information quantities including $T$-symmetric channels [79] and channels with input-invariance symmetry [80].

with the availability of channel state information at the transmitter (in addition to the receiver).[5] Such properties can potentially render interactive adaptive coding useless in terms of enlarging TWC capacity. Taking two-terminal DM-TWCs as an example, we develop the following two important channel symmetry notions. The *common optimal input distribution* condition identifies a state-dependent one-way channel that has an identical capacity-achieving input distribution for all channel states, while the *invariance of input-output mutual information* condition identifies a state-dependent one-way channel that produces the same input-output mutual information for all channel states under any fixed input distribution. If a DM-TWC satisfies both conditions, one for each direction of the two-way transmission, then the optimal transmission scheme of one terminal is irrelevant to the other terminal's transmission scheme, implying that the interaction between the terminals does not increase their transmission rates and hence channel capacity.

Formally, we can prove that under certain symmetry properties (identified by the derived conditions), any rate pair inside Shannon's outer bound region is always contained in his inner bound region, implying that the latter bound is tight. Furthermore, it should be expected that validating generalized channel symmetry properties can be a very complex procedure. However, we show that such a verification can be greatly simplified for some TWCs. In other words, we not only seek general conditions but also look for conditions which are simple to verify.

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[5] We note that this idea has been adopted to determine capacity for certain one-way channels such as one-way channels with memory [81–84] and compound channels with feedback [85], where feedback information cannot be exploited to enlarge the capacity. More details regarding the inefficacy of feedback in increasing capacity can be found in [86].

### 1.5.2 Lossy Source Transmission over Two-Way Channels

In the literature, there are only few works related to our lossy transmission setup. In [3, Section 14], Shannon implicitly illustrated that perfect matching among the source and channel statistics and alphabets results in error-free communication, with the optimal scheme given by uncoded transmission. In [49], the JSCC problem was studied for DM-TWCs which consist of two independent one-way channels. Together with the protocol mentioned in Section 1.2, Kaspi's source coding result was extended for successive source refinement. Also, a complete JSCC theorem was derived in this particular setting. By contrast, the authors in [76, Section VIII] tackled the two-way transmission problem for general DM-TWCs without deploying any protocol. The correlation-preserving coding scheme of [87] was adopted for almost lossless transmission; i.e., when requiring the block error rate of the source reconstructions to vanish asymptotically. Similar to Shannon's idea, the (non-adaptive) coding scheme of [76] can preserve source correlation in the channel inputs to facilitate two-way transmission; however, it does not apply to the lossy setup. In this thesis, we tackle a transmission problem that is more general in many aspects; e.g., we do not consider a particular type of DM-TWC or assume a given communication protocol. We next sketch the concepts behind our main JSCC achievability result.

As the transmissions of the terminals influence each other on a shared channel and generally cause cross-interference, we propose to design the coding strategies jointly. For this purpose, we construct joint source-channel codes that induce a stationary Markov chain that couples all variables of the communication system. In principle, when the channel inputs are generated by such codes, all system variables will behave according to the stationary distribution of the induced chain, thus coordinating the independent transmissions of the terminals. Specifically, we combine the following coding techniques to build our adaptive codes. First, we adopt the functional form of superposition coding [88] to generate channel inputs, which plays a central role in inducing the desired Markov transmission process. We also modify the analog/digital hybrid coding scheme of [89] to exploit side-information for decoding, in addition to its original source-correlation-preserving mechanism. Moreover, we use past channel inputs and outputs similarly to [28] to enable adaptive coding. We note that although these techniques are not new, combining and integrating them into an adaptive two-way coding framework for our problem setup is challenging.

## 1.6 Contributions of the Thesis

In this section, we summarize the contributions of each chapter. We note that some results have been published in [1, 2, 90–94].

### 1.6.1 Chapter 2

This chapter contains three main contributions concerning two-terminal DM-TWCs. First, we derive several sufficient and necessary conditions for Shannon's random coding inner bound to be tight, and we further show that our tightness conditions strictly generalize prior results [3, 41]. Although the most general form of our conditions, which can be employed to determine the capacity region for a broader class of DM-TWCs, is somewhat complex, we develop some easy-to-apply results. In the simplest scenario, one can verify our conditions by only observing the channel marginal distributions. On the other hand, we also develop a strategy to approximate the capacity region of DM-TWCs that do not satisfy the tightness conditions. The approximation result relies on a simple but non-trivial outer bound, which can be easily obtained within the framework of Shannon's inner bound computation.

We next study the relationships between different tightness conditions, including prior results and our new conditions. These relationships are summarized in an implication diagram; counterexamples are also given to reveal invalid implications. Moreover, we illustrate our conditions via examples. The capacity region for several classes of DM-TWCs, including binary/non-binary additive-noise TWCs with erasures (which subsume several classical TWCs), data-access TWCs, and injective semi-deterministic (ISD) TWCs, are determined in a closed form.

The last part of this chapter presents a generalization of Shannon's push-to-talk channel [3, Table I]. Specifically, we introduce a new DM-TWC model that exhibits a feature similar to Shannon's push-to-talk channel. As this generalized channel does not satisfy our tightness conditions, we develop another method to determine its capacity region. We show that the capacity region can be characterized by at most four extreme rate pairs, and hence its shape for non-trivial cases is either quadrilateral or triangular. Moreover, unlike in the case of the DM-TWCs that satisfy our tightness conditions, we find that one needs to use a time-sharing scheme to achieve capacity for this class of channels. Based on a case study, we further investigate transmission schemes for two terminals to achieve optimal trade-offs between the bidirectional transmission rates.

### 1.6.2 Chapter 3

In this chapter, we establish a Shannon-type inner bound and outer bounds for the capacity of TWCs with memory under certain invertibility, one-to-one mapping, and alphabet size constraints (Theorem 3.1, Corollaries 3.1-3.3, and Lemmas 3.1-3.2). Two sufficient conditions for the tightness of the bounds are given (Theorems 3.2 and 3.3). The first condition is derived for TWCs with strict invertibility and alphabet size constraints, characterizing the channel capacity in single-letter form. The other condition is specialized for injective semi-deterministic TWCs with memory.[6] Furthermore, motivated by a simple example where adaptive coding can enlarge achievable rates, we propose two adaptive coding schemes for two noise processes with memory. The schemes not only show how to make use of time-correlation to achieve our capacity outer bound but also illustrate a combination of the noise/interference cancellation coding and superposition coding.

### 1.6.3 Chapter 4

In this chapter, we establish a Shannon-type inner bound and an outer bound for the capacity region of multiaccess (MA) and degraded broadcast (DB) TWCs (Theorems 4.1 and 4.2) where both bounds admit a common rate expression but have different input distribution requirements. Three sufficient conditions (based on different techniques) for these bounds to coincide are established (Theorems 4.3-4.5). The first condition involves the existence of independent inputs that can achieve the outer bound (similar to the idea of [41]). The second condition is derived from the viewpoint of two interacting state-dependent one-way channels. The last one focuses on the permutation invariance structure of the channel transition matrix (mirroring the Shannon symmetry method [3]). The obtained results extend the results in [51] and readily provide the capacity region for a larger class of MA/DB TWCs. We note

[6] ISD-TWC model with memoryless noise were introduced in [41]. Here, we merely extend this setting by allowing noise processes with memory.

that while the channel model here is admittedly simplified, our intention is to illustrate a potential methodology for determining the capacity regions of multi-terminal TWCs and to motivate future work in this area.

### 1.6.4 Chapter 5

The contributions of this chapter are divided into two parts. The first one establishes a general JSCC result (Theorem 5.1) for two-terminal two-way lossy simultaneous transmission using the concepts of hybrid analog/digital coding, superposition coding and adaptive channel coding, together with a low-complexity sliding-window decoder. Two simplified achievability results (Corollaries 5.1 and 5.2) are derived from the main theorem. Moreover, our coding method is shown to subsume some basic schemes such as uncoded transmission and the concatenation of Wyner-Ziv (WZ) source coding and Shannon's (or Han's) channel coding; it also recovers the almost lossless transmission of [76]. Four illustrated examples (Examples 5.1-5.4) are provided to highlight the difference between the coding schemes. We also investigate the performance of scalar coding.

In the second part, we use standard arguments to obtain two outer bounds (Lemmas 5.1 and 5.2) for the achievable distortion region. The bounds are expressed in terms of the standard RD function and the conditional RD function and are hence easy to compute for many classical models of correlated sources. Furthermore, four complete theorems (Theorems 5.2-5.5) that fully characterize the achievable distortion region for certain system settings are derived. Specifically, for DM-TWCs with symmetry properties (defined in Chapter 2), we show the optimality of SSCC in the following settings:

- lossy transmission of independent sources;
- almost lossless transmission of correlated sources;
- lossy transmission of correlated sources whose WZ and conditional RD functions are equal;
- lossy transmission of correlated sources having a common part in the sense of Gács-Körner-Witsenhausen [47, Section 14.2.2].

Examples for Theorems 5.4 and 5.5 are also provided (Examples 5.5-5.7).

# Chapter 2

# Two-Terminal Discrete-Memoryless Two-Way Channels

This chapter focuses on two-terminal DM-TWCs and its organization is summarized as follows. We first introduce the system model and several definitions regarding the capacity region in Section 2.1.1. The capacity bounds in the literature are reviewed in Section 2.1.2. In Section 2.2, we begin with prior tightness conditions on Shannon's capacity inner bound (in Section 2.2.1). Our new conditions are given in Section 2.2.2, followed by a section of examples (Section 2.2.3) that illustrate our new results. In Section 2.3, we identify the relationships among different tightness conditions. We also derive necessary conditions for our tightness conditions to hold in Section 2.3.3. To give a global picture for our findings in Sections 2.2 and 2.3, we illustrate them in Fig. 2.1. Furthermore, the generalized push-to-talk DM-TWCs are defined and investigated in Section 2.4, and their capacity region is determined. In the last section (Section 2.5), we present a simple yet non-trivial capacity outer bound to approximate the capacity region for general two-terminal DM-TWCs. Numerical examples are also provided at the end of each of Sections 2.4 and 2.5.

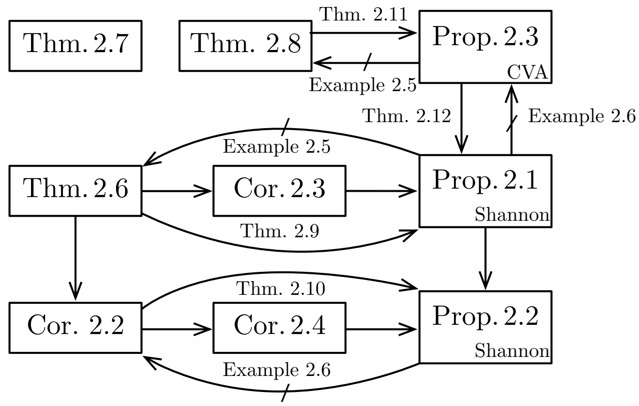



Figure 2.1: The relationships between the results yielding the equality of Shannon's capacity bounds in two-terminal memoryless TWCs. Here, $A \to B$ indicates that result $A$ subsumes result $B$, and $B \nrightarrow A$ indicates that result $B$ does not subsume result $A$. For example, Prop. 2.3 $\to$ Prop. 2.1 and Prop. 2.1 $\nrightarrow$ Prop. 2.3 mean that the CVA result in Prop. 2.3 is more general than the Shannon result in Prop. 2.1.

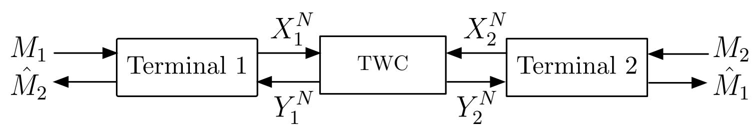



Figure 2.2: The block diagram of two-way transmission

## 2.1 Preliminaries

### 2.1.1 Channel Model and Definitions

Consider the two-terminal (or point-to-point) two-way communication system as shown in Fig. 2.2, where the terminals want to exchange independent messages $M_1$ and $M_2$ via $N$ channel uses. Here, $M_1$ and $M_2$ are assumed to be uniformly distributed on the finite sets $\mathcal{M}_1 \triangleq \{1, 2, ..., 2^{NR_1}\}$ and $\mathcal{M}_2 \triangleq \{1, 2, ..., 2^{NR_2}\}$ for some integers $NR_1 \geq 0$ and $NR_2 \geq 0$, respectively. For $j = 1, 2$ and $n = 1, 2, \ldots, N$, define $X_{j,n} \in \mathcal{X}_j$ and $Y_{j,n} \in \mathcal{Y}_j$ as the channel input and output of terminal $j$ at time $n$, respectively, where $\mathcal{X}_j$ and $\mathcal{Y}_j$ are two finite sets. The joint probability distribution of all random variables that involve the entire $n$ transmissions can be written as

$$P_{M_1,M_2,X_1^N,X_2^N,Y_1^N,Y_2^N} = P_{M_1,M_2} \cdot \left(\prod_{n=1}^{N} P_{X_{1,n}|M_1,Y_1^{n-1}}\right) \left(\prod_{n=1}^{N} P_{X_{2,n}|M_2,Y_2^{n-1}}\right) \cdot \left(\prod_{n=1}^{N} P_{Y_{1,n},Y_{2,n}|X_1^n,X_2^n,Y_1^{n-1},Y_2^{n-1}}\right),$$

where $X_j^n \triangleq (X_{j,1}, X_{j,2}, \ldots, X_{j,n})$ and $Y_j^n \triangleq (Y_{j,1}, Y_{j,2}, \ldots, Y_{j,n})$. Thus, the $N$ channel uses can be described by the sequence of input-output conditional probabilities $\{P_{Y_{1,n},Y_{2,n}|X_1^n,X_2^n,Y_1^{n-1},Y_2^{n-1}}\}_{n=1}^N$.

**Definition 2.1.** *An $(N, R_1, R_2)$ channel code for a two-terminal discrete TWC consists of two message sets $\mathcal{M}_1 = \{1, 2, \ldots, 2^{NR_1}\}$ and $\mathcal{M}_2 = \{1, 2, \ldots, 2^{NR_2}\}$, two sets of encoding functions $\boldsymbol{f}_1 \triangleq \{f_{1,1}, f_{1,2}, \ldots, f_{1,n}\}$ and $\boldsymbol{f}_2 \triangleq \{f_{2,1}, f_{2,2}, \ldots, f_{2,n}\}$:*

$$f_{1,1} : \mathcal{M}_1 \to \mathcal{X}_1, \quad f_{1,n} : \mathcal{M}_1 \times \mathcal{Y}_1^{n-1} \to \mathcal{X}_1,$$
$$f_{2,1} : \mathcal{M}_2 \to \mathcal{X}_2, \quad f_{2,n} : \mathcal{M}_2 \times \mathcal{Y}_2^{n-1} \to \mathcal{X}_2,$$

*for $n = 2, 3, \ldots, N$, and two decoding functions $g_1 : \mathcal{M}_1 \times \mathcal{Y}_1^N \to \mathcal{M}_2$ and $g_2 : \mathcal{M}_2 \times \mathcal{Y}_2^N \to \mathcal{M}_1$.*

When messages $M_1$ and $M_2$ are encoded, the channel inputs at time $n = 1$ are only functions of the messages, i.e., $X_{j,1} = f_{j,1}(M_j)$ for $j = 1, 2$, but all the other channel inputs are generated by also adapting to the previously received signals $Y_j^{n-1}$ via $X_{j,n} = f_{j,n}(M_j, Y_j^{n-1})$ for $j = 1, 2$ and $n = 2, 3, \ldots, N$. Upon receiving $N$ channel outputs, terminal $j$ reconstructs $M_{j'}$ as $\hat{M}_{j'} = g_j(M_j, Y_j^N)$, where $j, j' = 1, 2$ with $j \neq j'$. The probability of decoding error is defined as

$$P_{\mathrm{e}}^{(N)}(\boldsymbol{f}_1, \boldsymbol{f}_2, g_1, g_2) = \Pr(\{\hat{M}_1 \neq M_1\} \cup \{\hat{M}_2 \neq M_2\}).$$

Based on $P_{\mathrm{e}}^{(N)}$, we next define achievable rate pairs and the capacity region.

**Definition 2.2.** *A rate pair* $(R_1, R_2)$ *is said to be achievable for a two-terminal discrete TWC if there exists a sequence of* $(N, R_1, R_2)$ *codes with vanishing error probability, i.e.,* $\lim_{N\to\infty} P_e^{(N)}(\boldsymbol{f}_1, \boldsymbol{f}_2, g_1, g_2) = 0$.

**Definition 2.3.** *The capacity region* $\mathcal{C}(P_{Y_1,Y_2|X_1,X_2})$ *of a two-terminal discrete TWC is defined as the closure of all achievable rate pairs.*

**Definition 2.4.** *A two-terminal discrete TWC is said to be memoryless if the channel input-output probabilities satisfy* $P_{Y_{1,n},Y_{2,n}|X_1^n,X_2^n,Y_1^{n-1},Y_2^{n-1}} = P_{Y_1,Y_2|X_1,X_2}$ *for all* $n \geq 1$.

### 2.1.2 Capacity Bounds

For two-terminal DM-TWCs with channel transition probability $P_{Y_1,Y_2|X_1,X_2}$, let

$$\mathcal{R}(P_{X_1,X_2}, P_{Y_1,Y_2|X_1,X_2}) \triangleq \{(R_1, R_2) \in \mathbb{R}_{\geq 0}^2 : R_1 \leq I(X_1; Y_2|X_2), R_2 \leq I(X_2; Y_1|X_1)\}, \tag{2.1}$$

where the mutual information quantities are evaluated under the joint probability distribution $P_{X_1,X_2,Y_1,Y_2} = P_{X_1,X_2}P_{Y_1,Y_2|X_1,X_2}$. In [3], Shannon derived two bounds for the capacity region of two-terminal DM-TWCs, in which the inner bound is obtained by using the standard random coding (without adaption).

**Theorem 2.1** (**Shannon's Capacity Bounds** [3])**.** *The capacity region of a two-terminal DM-TWC with transition probability* $P_{Y_1,Y_2|X_1,X_2}$ *is inner bounded by*

$$\mathcal{C}_{\mathrm{I}}(P_{Y_1,Y_2|X_1,X_2}) \triangleq \overline{\mathrm{co}}\left( \bigcup_{P_{X_1}\in\mathcal{P}(\mathcal{X}_1), P_{X_2}\in\mathcal{P}(\mathcal{X}_2)} \mathcal{R}(P_{X_1}P_{X_2}, P_{Y_1,Y_2|X_1,X_2}) \right), \tag{2.2}$$

*and outer bounded by*

$$\mathcal{C}_{\mathrm{O}}(P_{Y_1,Y_2|X_1,X_2}) \triangleq \bigcup_{P_{X_1,X_2}\in\mathcal{P}(\mathcal{X}_1\times\mathcal{X}_2)} \mathcal{R}(P_{X_1,X_2}, P_{Y_1,Y_2|X_1,X_2}), \tag{2.3}$$

*where* $\overline{\mathrm{co}}(\cdot)$ *denotes taking the closure of the convex hull.*

Note that Shannon's outer bound region is already convex and hence there is no need to take convex closure. Moreover, Shannon's inner bound can be alternatively expressed by considering a coded time-sharing random variable [47], which results in an inner bound without taking the convex closure operation.

**Theorem 2.2** (**Coded Time-Sharing Inner Bound** [47])**.** *For a two-terminal DM-TWC with transition probability* $P_{Y_1,Y_2|X_1,X_2}$*, any rate pair* $(R_1, R_2)$ *that satisfies*

$$R_1 \leq I(X_1; Y_2|X_2, Q), \tag{2.4a}$$

$$R_2 \leq I(X_2; Y_1|X_1, Q), \tag{2.4b}$$

*is achievable, where the joint probability distribution of all random variables is given by* $P_{Q,X_1,X_2,Y_1,Y_2} = P_Q P_{X_1|Q} P_{X_2|Q} P_{Y_1,Y_2|X_1,X_2}$.

We next summarize improved capacity bounds in the literature.

**Theorem 2.3** (**Block Markov Coding Inner Bound** [28])**.** *For a two-terminal DM-TWC with transition probability* $P_{Y_1,Y_2|X_1,X_2}$*, any rate pair* $(R_1, R_2)$ *that satisfies*

$$R_1 < I(\tilde{U}_1; X_2, Y_2, \tilde{U}_2, \tilde{W}_2), \tag{2.5a}$$

$$R_2 < I(\tilde{U}_2; X_1, Y_1, \tilde{U}_1, \tilde{W}_1), \tag{2.5b}$$

*is achievable, where the joint probability distribution of all random variables is given by* $P_{U_1,U_2,\tilde{U}_1,\tilde{U}_2,\tilde{W}_1,\tilde{W}_2,X_1,X_2,Y_1,Y_2} = P_{U_1} P_{U_2} P_{\tilde{U}_1\tilde{U}_1\tilde{W}_1\tilde{W}_2} P_{X_1|U_1,\tilde{U}_1,\tilde{W}_1} P_{X_2|U_2,\tilde{U}_2,\tilde{W}_2} P_{Y_1,Y_2|X_1,X_2}$, *a distribution that induces a stationary Markov chain for the* $\{Z^{(t)}\}$ *in [28, (4.12)].*

**Theorem 2.4** (**Zhang-Berger-Schalkwijk (ZBS) Outer Bound** [29]). *For a two-terminal DM-TWC with transition probability* $P_{Y_1,Y_2|X_1,X_2}$*, every achievable rate pair* $(R_1, R_2)$ *must satisfy*

$$R_1 \leq \min\left(H(X_1|U_1), I(X_1; Y_2|X_2, U_2)\right), \tag{2.6a}$$

$$R_2 \leq \min\left(H(X_2|U_2), I(X_2; Y_1|X_1, U_1)\right), \tag{2.6b}$$

*for some joint probability distribution* $P_{U_1,U_2,X_1,X_2,Y_1,Y_2} = P_{U_1,U_2}P_{X_1|U_1}P_{X_2|U_2}P_{Y_1,Y_2|X_1,X_2}$.

Note that there is no cardinality bound for auxiliary random variables $U_1$ and $U_2$ in the ZBS outer bound, but weaker results with cardinality bound can be deduced from it [29]. In the literature, there is a so-called dependence balance bound specialized for common-output DM-TWCs, i.e., $Y_1 = Y_2 = Y$, as shown below.

**Theorem 2.5** (**Dependence Balance Bound** [31]). *For a common-output DM-TWC with transition probability* $P_{Y|X_1,X_2}$*, every achievable rate pair* $(R_1, R_2)$ *must satisfy*

$$R_1 \leq I(X_1; Y|X_2, Q),$$

$$R_2 \leq I(X_2; Y|X_1, Q),$$

*for some joint probability distribution* $P_{Q,X_1,X_2,Y} = P_{X_1,X_2,Q}P_{Y|X_1,X_2}$ *and* $|\mathcal{Q}| \leq 3$ *such that* $I(X_1; X_2|Q) \leq I(X_1; X_2|Y, Q)$.

In general, the above inner and outer bounds do not coincide, and hence the capacity region of most DM-TWCs remains unknown. For single-output DM-TWCs, it is known that if $I(X_1; X_2|Y, Q) = 0$ and thus $I(X_1; X_2|Q) = 0$ (i.e., $X_1 \multimap Q \multimap$

$X_2$), then the coded time-sharing inner bound is tight. For general two-terminal DM-TWCs, there are very few results available. Our first objective here is to derive conditions under which the inner and outer bounds can be matched to each other, thereby determining the capacity region. Specifically, we only focus on Shannon's inner and outer bounds and investigate when this inner bound is tight, i.e., when one has $\mathcal{C}_{\mathrm{I}}(P_{Y_1,Y_2|X_1,X_2}) = \mathcal{C}_{\mathrm{O}}(P_{Y_1,Y_2|X_1,X_2})$. For notational simplicity, we use $\mathcal{C}_{\mathrm{I}}$, $\mathcal{C}_{\mathrm{O}}$, and $\mathcal{C}$ in the rest of this chapter.

## 2.2 Tightness Conditions for Shannon's Inner Bound

### 2.2.1 Prior Results

Before presenting our findings, we first summarize the Shannon [3] and Chaaban-Varshney-Alouini (CVA) [41] conditions that imply the coincidence of $\mathcal{C}_{\mathrm{I}}$ and $\mathcal{C}_{\mathrm{O}}$. In short, the Shannon conditions focus on the symmetry structure of channel transition probability $P_{Y_1,Y_2|X_1,X_2}$, while the CVA condition aims at the existence of independent inputs which can achieve Shannon's outer bound. Some notations are given in order. For a finite set $\mathcal{A}$, let $\pi^{\mathcal{A}} : \mathcal{A} \to \mathcal{A}$ be a permutation (bijection), and for any two symbols $a'$ and $a''$ in $\mathcal{A}$, let $\tau^{\mathcal{A}}_{a',a''} : \mathcal{A} \to \mathcal{A}$ denote the transposition which swaps $a'$ and $a''$ in $\mathcal{A}$, but leaves the other symbols unaffected. Throughout the thesis, we will use $I^{(l)}(X_j; Y_{j'}|X_{j'})$ and $H^{(l)}(Y_j|X_1, X_2)$ to denote the conditional mutual information and the conditional entropy evaluated under the input distribution $P^{(l)}_{X_1,X_2}$ for $j, j' = 1, 2$ with $j \neq j'$. For $P^{(l)}_{X_1,X_2} = P^{(l)}_{X_j} P^{(l)}_{X_{j'}|X_j}$ with $j \neq j'$, the conditional entropy $H^{(l)}(Y_j|X_j)$ is evaluated using the marginal distribution $P^{(l)}_{Y_j|X_j}(y_j|x_j) = \sum_{x_{j'} \in \mathcal{X}_{j'}} P^{(l)}_{X_{j'}|X_j}(x_{j'}|x_j) P_{Y_j|X_j,X_{j'}}(y_j|x_j, x_{j'})$.

**Proposition 2.1** (**Shannon's One-sided Symmetry Condition** [3]). *For a two-terminal DM-TWC with transition probability* $P_{Y_1,Y_2|X_1,X_2}$*, we have* $\mathcal{C} = \mathcal{C}_{\mathrm{I}} = \mathcal{C}_{\mathrm{O}}$ *if for any pair of distinct input symbols* $x_1', x_1'' \in \mathcal{X}_1$*, there exists a pair of permutations* $(\pi^{\mathcal{Y}_1}[x_1', x_1''], \pi^{\mathcal{Y}_2}[x_1', x_1''])$ *on* $\mathcal{Y}_1$ *and* $\mathcal{Y}_2$ *(which depend on* $x_1'$ *and* $x_1''$*) such that*

$$\begin{aligned} &P_{Y_1,Y_2|X_1,X_2}(y_1, y_2|x_1, x_2) \\ &\qquad = P_{Y_1,Y_2|X_1,X_2}(\pi^{\mathcal{Y}_1}[x_1', x_1''](y_1), \pi^{\mathcal{Y}_2}[x_1', x_1''](y_2)|\tau^{\mathcal{X}_1}_{x_1', x_1''}(x_1), x_2) \end{aligned} \tag{2.8}$$

*holds for all* $x_1 \in \mathcal{X}_1$*,* $x_2 \in \mathcal{X}_2$*,* $y_1 \in \mathcal{Y}_1$*, and* $y_2 \in \mathcal{Y}_2$*. Moreover, the capacity region is given by*

$$\mathcal{C} = \bigcup_{P_{X_2} \in \mathcal{P}(\mathcal{X}_2)} \mathcal{R}\Big(P^{\mathrm{U}}_{\mathcal{X}_1} P_{X_2}, P_{Y_1,Y_2|X_1,X_2}\Big). \tag{2.9}$$

Although Proposition 2.1 only describes a channel symmetry property with respect to the channel input of terminal 1, an analogous condition for terminal 2 can be obtained by exchanging the roles of terminals 1 and 2. The next proposition is for a DM-TWC that satisfies one-sided symmetry condition with respect to the channel inputs of both terminals.

**Proposition 2.2** (**Shannon's Two-sided Symmetry Condition** [3]). *For a two-terminal DM-TWC with transition probability* $P_{Y_1,Y_2|X_1,X_2}$*, we have* $\mathcal{C} = \mathcal{C}_I = \mathcal{C}_O$ *if the TWC satisfies one-sided symmetry condition with respect to both channel inputs. Moreover, the capacity region is rectangular and given by*

$$\mathcal{C} = \mathcal{R}(P^{\mathrm{U}}_{\mathcal{X}_1} P^{\mathrm{U}}_{\mathcal{X}_2}, P_{Y_1,Y_2|X_1,X_2}). \tag{2.10}$$

We next summarize the CVA condition in the following proposition; the proof is straightforward and hence omitted here.

**Proposition 2.3** (**CVA Condition** [41])**.** *For a two-terminal DM-TWC with transition probability* $P_{Y_1,Y_2|X_1,X_2}$*, we have that* $\mathcal{C} = \mathcal{C}_{\mathrm{I}} = \mathcal{C}_{\mathrm{O}}$ *if* $H(Y_j|X_1, X_2)$*,* $j = 1, 2$*, does not depend on* $P_{X_1|X_2}$ *for any fixed* $P_{X_2}$ *and* $P_{Y_j|X_1,X_2}$*, and for any* $P^{(1)}_{X_1,X_2} = P^{(1)}_{X_2} P^{(1)}_{X_1|X_2}$ *there exists* $\tilde{P}_{X_1} \in \mathcal{P}(\mathcal{X}_1)$ *such that* $H^{(1)}(Y_j|X_j) \leq H^{(2)}(Y_j|X_j)$ *for* $j = 1, 2$*, where* $P^{(2)}_{X_1,X_2} = \tilde{P}_{X_1} P^{(1)}_{X_2}$*. Also, the capacity region is given by*

$$\mathcal{C} = \bigcup_{P_{X_1} \in \mathcal{P}(\mathcal{X}_1), P_{X_2} \in \mathcal{P}(\mathcal{X}_2)} \mathcal{R}(P_{X_1} P_{X_2}, P_{Y_1,Y_2|X_1,X_2}). \tag{2.11}$$

The capacity region of a two-terminal DM-TWC which satisfies any of the above conditions can be determined by considering independent input distributions, i.e., the input distributions are of a product form $P_{X_1} P_{X_2}$. This result indicates that adaptive coding, where channel inputs are generated by adapting to the previously received signals, cannot improve achievable rates. Memoryless ISD-TWCs [41] are special two-terminal DM-TWCs which satisfy the CVA condition (but not necessarily the Shannon conditions). The DM-TWC with independent $q$-ary additive noise [40] is an example of this class that also satisfies the Shannon's two-sided symmetry condition.

### 2.2.2 New Tightness Conditions

In this section, we establish more general channel symmetry properties regarding the tightness of Shannon's inner and outer bounds and identify the relationship between different tightness conditions. We adopt the viewpoint that a two-way channel consists of two state-dependent one-way channels; for example, the one-way channel from terminal 1 to terminal 2 is governed by the marginal distribution $P_{Y_2|X_1,X_2}$ (derived from the channel probability distribution $P_{Y_1,Y_2|X_1,X_2}$), where $X_1$ and $Y_2$ are respectively the input and the output of the channel with state $X_2$.

Let $P_X$ and $P_{Y|X}$ be probability distributions on finite sets $\mathcal{X}$ and $\mathcal{Y}$. To simplify the presentation, we define

$$\mathcal{I}(P_X, P_{Y|X}) = \sum_{x,y} P_X(x)P_{Y|X}(y|x) \log \frac{P_{Y|X}(y|x)}{\sum_{x'} P_X(x')P_{Y|X}(y|x')}, \tag{2.12}$$

which is the mutual information $I(X;Y)$ between input $X$ (governed by $P_X$) and corresponding output $Y$ of a channel with transition probability $P_{Y|X}$. A useful fact is that $\mathcal{I}(\,\cdot\,,\,\cdot\,)$ is concave in the first argument when the second argument is fixed. The conditional mutual information $I(X_1;Y_2|X_2=x_2)$ and $I(X_2;Y_1|X_1=x_1)$ can be then expressed as $\mathcal{I}(P_{X_1|X_2=x_2}, P_{Y_2|X_1,X_2=x_2})$ and $\mathcal{I}(P_{X_2|X_1=x_1}, P_{Y_1|X_1=x_1,X_2})$, respectively.

Due to the viewpoint of state-dependent one-way channels, each of the following theorems comprises two conditions, one for each direction of the two-way transmission. By symmetry, these theorems are also valid if the roles of terminals 1 and 2 are swapped.

**Theorem 2.6.** *For a two-terminal DM-TWC, if both of the following conditions are satisfied, then* $\mathcal{C}_{\mathrm{I}} = \mathcal{C}_{\mathrm{O}}$*:*

*(i) There exists* $P^*_{X_1} \in \mathcal{P}(\mathcal{X}_1)$ *such that for all* $x_2 \in \mathcal{X}_2$ *we have*

$$\arg\max_{P_{X_1|X_2=x_2}} I(X_1;Y_2|X_2=x_2) = P^*_{X_1};$$

*(ii)* $\mathcal{I}(P_{X_2}, P_{Y_1|X_1=x_1,X_2})$ *does not depend on* $x_1 \in \mathcal{X}_1$ *for any fixed* $P_{X_2} \in \mathcal{P}(\mathcal{X}_2)$*.*

*Moreover, the capacity region is given by*

$$\mathcal{C} = \bigcup_{P_{X_2} \in \mathcal{P}(\mathcal{X}_2)} \mathcal{R}(P^*_{X_1} P_{X_2}, P_{Y_1,Y_2|X_1,X_2}). \tag{2.13}$$

*Proof:* For any $P^{(1)}_{X_1,X_2} = P^{(1)}_{X_2} P^{(1)}_{X_1|X_2}$, let $P^{(2)}_{X_1,X_2} = P^{*}_{X_1} P^{(1)}_{X_2}$, where $P^{*}_{X_1}$ is given by (i). In light of (i), we have

$$
\begin{aligned}
I^{(1)}(X_1;Y_2|X_2) &= \sum_{x_2} P^{(1)}_{X_2}(x_2)\cdot I^{(1)}(X_1;Y_2|X_2=x_2) && (2.14)\\
&\le \sum_{x_2} P^{(1)}_{X_2}(x_2)\cdot \left[\max_{P_{X_1|X_2=x_2}} I(X_1;Y_2|X_2=x_2)\right] && (2.15)\\
&= \sum_{x_2} P^{(1)}_{X_2}(x_2)\cdot \mathcal{I}(P^{*}_{X_1}, P_{Y_2|X_1,X_2=x_2}) && (2.16)\\
&= \sum_{x_2} P^{(1)}_{X_2}(x_2)\cdot I^{(2)}(X_1;Y_2|X_2=x_2) && (2.17)\\
&= I^{(2)}(X_1;Y_2|X_2). && (2.18)
\end{aligned}
$$

Moreover,

$$
\begin{aligned}
I^{(1)}(X_2;Y_1|X_1) &= \sum_{x_1} P^{(1)}_{X_1}(x_1)\cdot I^{(1)}(X_2;Y_1|X_1=x_1) && (2.19)\\
&= \sum_{x_1} P^{(1)}_{X_1}(x_1)\cdot \mathcal{I}(P^{(1)}_{X_2|X_1=x_1}, P_{Y_1|X_1=x_1,X_2}) && (2.20)\\
&= \sum_{x_1} P^{(1)}_{X_1}(x_1)\cdot \mathcal{I}(P^{(1)}_{X_2|X_1=x_1}, P_{Y_1|X_1=x_1',X_2}) && (2.21)\\
&\le \mathcal{I}\left(\sum_{x_1} P^{(1)}_{X_1}(x_1) P^{(1)}_{X_2|X_1=x_1}, P_{Y_1|X_1=x_1',X_2}\right) && (2.22)\\
&= \mathcal{I}(P^{(1)}_{X_2}, P_{Y_1|X_1=x_1',X_2}) && (2.23)\\
&= \sum_{x_1'} P^{*}_{X_1}(x_1')\cdot \mathcal{I}(P^{(1)}_{X_2}, P_{Y_1|X_1=x_1',X_2}) && (2.24)\\
&= I^{(2)}(X_2;Y_1|X_1), && (2.25)
\end{aligned}
$$

where (2.21) holds by the invariance assumption in (ii), (2.22) holds since the functional $\mathcal{I}(\,\cdot\,,\,\cdot\,)$ is concave in the first argument, and (2.24) is obtained from the invariance assumption in (ii). Combining the above yields $\mathcal{R}(P^{(1)}_{X_1,X_2}, P_{Y_1,Y_2|X_1,X_2}) \subseteq \mathcal{R}(P^{*}_{X_1} P^{(1)}_{X_2}, P_{Y_1,Y_2|X_1,X_2})$, which implies that $\mathcal{C}_{\mathrm{O}} \subseteq \mathcal{C}_{\mathrm{I}}$ and hence $\mathcal{C}_{\mathrm{I}} = \mathcal{C}_{\mathrm{O}}$. ■

A special case where condition (i) of Theorem 2.6 trivially holds is when each one-way channel $P_{Y_2|X_1,X_2=x_2}$, $x_2 \in \mathcal{X}_2$, is $T$-symmetric[1] [79]; in this case we have $P^*_{X_1} = P^{\mathrm{U}}_{\mathcal{X}_1}$. This immediately gives the following corollary.

**Corollary 2.1.** *For a given two-terminal DM-TWC, if conditions (i) and (ii) below are satisfied, then $\mathcal{C}_{\mathrm{I}} = \mathcal{C}_{\mathrm{O}} = \mathcal{C}$ with $\mathcal{C}$ given in* (2.9)*:*

*(i) All one-way channels governed by $P_{Y_2|X_1,X_2=x_2}$, $x_2 \in \mathcal{X}_2$, are $T$-symmetric;*

*(ii) $\mathcal{I}(P_{X_2}, P_{Y_1|X_1=x_1,X_2})$ does not depend on $x_1 \in \mathcal{X}_1$ for any fixed $P_{X_2} \in \mathcal{P}(\mathcal{X}_2)$.*

Another sufficient condition for condition (i) of Theorem 2.6 to hold is that $\mathcal{I}(P_{X_1}, P_{Y_2|X_1,X_2=x_2})$ does not depend on $x_2 \in \mathcal{X}_2$ for any fixed $P_{X_1} \in \mathcal{P}(\mathcal{X}_1)$. Using this fact yields the following corollary of Theorem 2.6.

**Corollary 2.2.** *For a given two-terminal DM-TWC, if both of the following conditions are satisfied, then $\mathcal{C}_{\mathrm{I}} = \mathcal{C}_{\mathrm{O}}$:*

*(i) $\mathcal{I}(P_{X_1}, P_{Y_2|X_1,X_2=x_2})$ does not depend on $x_2 \in \mathcal{X}_2$ for any fixed $P_{X_1} \in \mathcal{P}(\mathcal{X}_1)$;*

*(ii) $\mathcal{I}(P_{X_2}, P_{Y_1|X_1=x_1,X_2})$ does not depend on $x_1 \in \mathcal{X}_1$ for any fixed $P_{X_2} \in \mathcal{P}(\mathcal{X}_2)$.*

*Moreover, the capacity region is given by*

$$\mathcal{C} = \mathcal{R}(P^*_{X_1} P^*_{X_2}, P_{Y_1,Y_2|X_1,X_2}), \tag{2.26}$$

*where $P^*_{X_j} = \arg\max_{P_{X_j|X_{j'}=x_{j'}}} I(X_j; Y_{j'}|X_{j'} = x_{j'})$ for $j, j' = 1, 2$ with $j \neq j'$.*

**Remark 2.1.** One can also use the Karush–Kuhn–Tucker (KKT) conditions [95] to verify the optimality of the product input distribution $P^*_{X_1} P^*_{X_2}$. Note that Shannon's

[1] A two-terminal DM one-way channel is called $T$-symmetric if the optimal input distribution (that maximizes the channel's mutual information) is uniform.

outer bound can be found via solving the following convex optimization problem for all $\lambda \in [0, \infty)$:

$$\arg\max_{P_{X_1,X_2} \in \mathcal{P}(\mathcal{X}_1 \times \mathcal{X}_2)} I(X_1; Y_2|X_2) + \lambda I(X_2; Y_1|X_1)$$

since the function to be maximized is a supporting hyperplane (with the normal vector $(1, \lambda)$) to the outer bound region. Based on the KKT conditions, $P_{X_1,X_2} = P^*_{X_1} P^*_{X_2}$ is optimal if and only if for all $\lambda \in [0, \infty)$, we have that

$$\begin{aligned}
&-\sum_{y_2} P_{Y_2|X_1,X_2}(y_2|x'_1, x'_2) \log P_{Y_2|X_2}(y_2|x'_2) - H(Y_2|X_1 = x'_1, X_2 = x'_2) \\
&\quad -\lambda \sum_{y_1} P_{Y_1|X_1,X_2}(y_1|x'_1, x'_2) \log P_{Y_1|X_1}(y_1|x'_1) - \lambda H(Y_1|X_1 = x'_1, X_2 = x'_2) \\
&= \begin{cases} \leq \mu & \text{if } P^*_{X_1}(x'_1) P^*_{X_2}(x'_2) = 0 \\ = \mu & \text{if } P^*_{X_1}(x'_1) P^*_{X_2}(x'_2) > 0 \end{cases}
\end{aligned}$$

where $P_{Y_j|X_j}(y_j|x'_j) = \sum_{x_{j'}} P_{Y_j|X_j,X_{j'}}(y_j|x'_j, x_{j'}) P^*_{X_{j'}}(x_{j'})$ for $j, j' = 1, 2$ with $j \neq j'$ and $\mu$ is the KKT multiplier for the constraint $\sum_{x_1,x_2} P_{X_1,X_2}(x_1, x_2) = 1$.

Different from Theorem 2.6, which can identify DM-TWCs with non-rectangular capacity region, the capacity region of a DM-TWC that satisfies Corollary 2.2 is always rectangular. The capacity region in the next theorem is a case in point, but the theorem is established by relaxing the requirement of the invariance of mutual information with respect to all input distributions.

**Theorem 2.7.** *For a given two-terminal DM-TWC, if both of the following conditions are satisfied, then* $\mathcal{C}_\mathrm{I} = \mathcal{C}_\mathrm{O} = \mathcal{C}$ *with* $\mathcal{C}$ *given by* (2.26)*:*

*(i) There exists* $P^*_{X_1} \in \mathcal{P}(\mathcal{X}_1)$ *such that for all* $x_2 \in \mathcal{X}_2$ *we have a common maximizer* $P^*_{X_1} = \arg\max_{P_{X_1|X_2=x_2}} I(X_1; Y_2|X_2 = x_2)$ *and the mutual information*

$\mathcal{I}(P^*_{X_1}, P_{Y_2|X_1,X_2=x_2})$ *does not depend on* $x_2 \in \mathcal{X}_2$*;*

*(ii) There exists* $P^*_{X_2} \in \mathcal{P}(\mathcal{X}_2)$ *such that for all* $x_1 \in \mathcal{X}_1$ *we have a common maximizer* $P^*_{X_2} = \arg\max_{P_{X_2|X_1=x_1}} I(X_2; Y_1|X_1 = x_1)$ *and the mutual information* $\mathcal{I}(P^*_{X_2}, P_{Y_1|X_1=x_1,X_2})$ *does not depend on* $x_1 \in \mathcal{X}_1$*.*

Unlike the condition (ii) of Theorem 2.6 and conditions in Corollary 2.2, here, we merely need to check the existence of common maximizers and test whether or not $\mathcal{I}(P^*_{X_1}, P_{Y_2|X_1,X_2=x_2})$ is invariant with respect to $x_2 \in \mathcal{X}_2$ and $\mathcal{I}(P^*_{X_2}, P_{Y_1|X_1=x_1,X_2})$ is invariant with respect to $x_1 \in \mathcal{X}_1$. The computational effort is then greatly reduced. The next two corollaries of Theorem 2.6 provide even simpler ways than the above, in which computations can be completely avoided.

Recall that $[P_{Y_2|X_1,X_2}(\cdot\,|\,\cdot\,, x_2)]$ denotes the transition matrix of the channel from terminals 1 to 2 when the input of terminal 2 is fixed to be $x_2$; the entry in the matrix corresponding to $x_1$ (in row) and $y_2$ (in column) is given by $P_{Y_2|X_1,X_2}(y_2|x_1, x_2)$. Similarly, $[P_{Y_1|X_1,X_2}(\cdot\,|x_1,\,\cdot)]$ denotes the transition matrix of the channel from terminals 2 to 1 when the input of terminal 1 is fixed to be $x_1$.

**Corollary 2.3.** *For a given two-terminal DM-TWC, if both of the following conditions are satisfied, then* $\mathcal{C}_{\mathrm{I}} = \mathcal{C}_{\mathrm{O}} = \mathcal{C}$ *with* $\mathcal{C}$ *given by* (2.9)*:*

*(i) The channel with transition matrix* $[P_{Y_2|X_1,X_2}(\cdot\,|\,\cdot\,, x_2)]$ *is quasi-symmetric*[2] *for all* $x_2 \in \mathcal{X}_2$*;*

*(ii) The matrices* $[P_{Y_1|X_1,X_2}(\cdot\,|x_1,\,\cdot)]$, $x_1 \in \mathcal{X}_1$*, are column permutations of each other.*

[2] A discrete memoryless channel with transition matrix $[P_{Y|X}(\cdot\,|\,\cdot)]$ is said to be weakly-symmetric if the rows are permutations of each other and all the column sums are identical [74, 96]. Also a discrete memoryless channel with transition matrix $[P_{Y|X}(\cdot|\cdot)]$ is said to be quasi-symmetric if $[P_{Y|X}(\cdot\,|\,\cdot)]$ can be partitioned along its columns into weakly-symmetric sub-matrices [78].

**Corollary 2.4.** *For a given two-terminal DM-TWC, if both of the following conditions are satisfied, then* $\mathcal{C}_{\mathrm{I}} = \mathcal{C}_{\mathrm{O}} = \mathcal{C}$ *with* $\mathcal{C}$ *given by* (2.26)*:*

*(i) The matrices* $[P_{Y_2|X_1,X_2}(\,\cdot\,|\,\cdot\,, x_2)]$, $x_2 \in \mathcal{X}_2$, *are column permutations of each other;*

*(ii) The matrices* $[P_{Y_1|X_1,X_2}(\,\cdot\,|x_1,\,\cdot\,)]$, $x_1 \in \mathcal{X}_1$, *are column permutations of each other.*

Except for the above new conditions, we found that Shannon's original conditions are too stringent to imply the tightness of his inner bound (from the proof in Appendix A.1). The next two propositions refine Shannon's results.

**Proposition 2.4** (**Extended Shannon's One-Sided Symmetry Condition**). *For a two-terminal DM-TWC with transition probability* $P_{Y_1,Y_2|X_1,X_2}$, *we have that* $\mathcal{C}_{\mathrm{I}} = \mathcal{C}_{\mathrm{O}} = \mathcal{C}$ *with* $\mathcal{C}$ *given by* (2.9) *if for any pair of distinct input symbols* $x_1'$, $x_1'' \in \mathcal{X}_1$, *there exists a pair of permutations* $(\pi^{\mathcal{Y}_1}[x_1', x_1''], \pi^{\mathcal{Y}_2}[x_1', x_1''])$ *on* $\mathcal{Y}_1$ *and* $\mathcal{Y}_2$, *respectively, (which depend on* $x_1'$ *and* $x_1''$*) such that for all* $x_1 \in \mathcal{X}_1$, $x_2 \in \mathcal{X}_2$, $y_1 \in \mathcal{Y}_1$, $y_2 \in \mathcal{Y}_2$,

$$P_{Y_1|X_1,X_2}(y_1|x_1, x_2) = P_{Y_1|X_1,X_2}(\pi^{\mathcal{Y}_1}[x_1', x_1''](y_1)|\tau^{\mathcal{X}_1}_{x_1',x_1''}(x_1), x_2), \tag{2.27a}$$

$$P_{Y_2|X_1,X_2}(y_2|x_1, x_2) = P_{Y_2|X_1,X_2}(\pi^{\mathcal{Y}_2}[x_1', x_1''](y_2)|\tau^{\mathcal{X}_1}_{x_1',x_1''}(x_1), x_2), \tag{2.27b}$$

*where* $P_{Y_1|X_1,X_2}$ *and* $P_{Y_2|X_1,X_2}$ *are the marginals of* $P_{Y_1,Y_2|X_1,X_2}$.

**Proposition 2.5** (**Extended Shannon's Two-Sided Symmetry Condition**). *For a two-terminal DM-TWC with transition probability* $P_{Y_1,Y_2|X_1,X_2}$, *we have that* $\mathcal{C}_{\mathrm{I}} = \mathcal{C}_{\mathrm{O}} = \mathcal{C}$ *with* $\mathcal{C}$ *given by* (2.10) *if the TWC satisfies the extended Shannon's one-sided symmetry condition with respect to both channel inputs.*

The following theorem generalizes the CVA condition.

**Theorem 2.8 (Generalized CVA Condition).** *For a given two-terminal DM-TWC, if both of the following conditions are satisfied, then $\mathcal{C}_\mathrm{I} = \mathcal{C}_\mathrm{O} = \mathcal{C}$ with $\mathcal{C}$ given by* (2.13)*:*

*(i) There exists $P^*_{X_1} \in \mathcal{P}(\mathcal{X}_1)$ such that for all $x_2 \in \mathcal{X}_2$ we have*

$$\arg\max_{P_{X_1|X_2=x_2}} I(X_1; Y_2|X_2 = x_2) = P^*_{X_1};$$

*(ii) $H(Y_1|X_1, X_2)$ does not depend on $P_{X_1|X_2}$ given $P_{X_2}$ and $P_{Y_1|X_1,X_2}$, and $P^*_{X_1}$ given in (i) satisfies $H^{(1)}(Y_1|X_1) \leq H^{(2)}(Y_1|X_1)$ for any $P^{(1)}_{X_1,X_2} = P^{(1)}_{X_2} P^{(1)}_{X_1|X_2}$, where $P^{(2)}_{X_1,X_2} = P^*_{X_1} P^{(1)}_{X_2}$.*

*Proof:* Given any $P^{(1)}_{X_1,X_2} = P^{(1)}_{X_2} P^{(1)}_{X_1|X_2}$, let $P^{(2)}_{X_1,X_2} = P^*_{X_1} P^{(1)}_{X_2}$. By the same argument as in (2.14)-(2.18), we obtain that $I^{(1)}(X_1; Y_2|X_2) \leq I^{(2)}(X_1; Y_2|X_2)$ from the condition (i). Moreover, condition (ii) implies that $I^{(1)}(X_2; Y_1|X_1) = H^{(1)}(Y_1|X_1) - H^{(1)}(Y_1|X_1, X_2) \leq H^{(2)}(Y_1|X_1) - H^{(2)}(Y_1|X_1, X_2) = I^{(2)}(X_2; Y_1|X_1)$. Thus,

$$\mathcal{R}(P^{(1)}_{X_1,X_2}, P_{Y_1,Y_2|X_1,X_2}) \subseteq \mathcal{R}(P^*_{X_1} P^{(1)}_{X_2}, P_{Y_1,Y_2|X_1,X_2}),$$

thereby proving $\mathcal{C}_\mathrm{I} = \mathcal{C}_\mathrm{O}$. ■

We note that a detailed implication chart for the above sufficient conditions is given in Fig. 2.1.

### 2.2.3 Examples

We next illustrate the proposed conditions via examples.

**Example 2.1 (Memoryless Binary Additive-Noise TWCs with Erasures).** Let $\mathcal{X}_1 = \mathcal{X}_2 = G_2 = \{0, 1\}$ (the binary additive group) and $\mathcal{Y}_1 = \mathcal{Y}_2 = \mathcal{Z} = \{0, 1, \mathbf{E}\}$, where $\mathbf{E}$ denotes channel erasure. A binary additive noise DM-TWC with erasures is defined by the channel equations

$$Y_{1,n} = (X_{1,n} \oplus_2 X_{2,n} \oplus_2 Z_{1,n}) \cdot \mathbf{1}\{Z_{1,n} \neq \mathbf{E}\} + \mathbf{E} \cdot \mathbf{1}\{Z_{1,n} = \mathbf{E}\},$$
$$Y_{2,n} = (X_{1,n} \oplus_2 X_{2,n} \oplus_2 Z_{2,n}) \cdot \mathbf{1}\{Z_{2,n} \neq \mathbf{E}\} + \mathbf{E} \cdot \mathbf{1}\{Z_{2,n} = \mathbf{E}\},$$

where $\{(Z_{1,n}, Z_{2,n})\}_{n=1}^{\infty}$ is a memoryless two-dimensional noise-erasure process that is independent of the terminals' messages and has components $Z_{1,n}, Z_{2,n} \in \mathcal{Z}$ such that $\Pr\big(Z_{j,n} = \mathbf{E}\big) = \varepsilon_j$ and $\Pr\big(Z_{j,n} = 1\big) = \alpha_j$, where $0 \leq \varepsilon_j + \alpha_j \leq 1$ for $j = 1, 2$, and $\mathbf{1}\{\cdot\}$ denotes the indicator function. Here, we adopt the convention $\mathbf{E} \cdot 0 = 0$ and $\mathbf{E} \cdot 1 = \mathrm{E}$ to simplify the representation of the channel equations.[3] The channel equations yield the following transition matrices for the one-way channels:

$$[P_{Y_2|X_1,X_2}(\,\cdot\,|\,\cdot\,,0)] = \begin{pmatrix} 1-\varepsilon_2-\alpha_2 & \alpha_2 & \varepsilon_2 \\ \alpha_2 & 1-\varepsilon_2-\alpha_2 & \varepsilon_2 \end{pmatrix},$$
$$[P_{Y_2|X_1,X_2}(\,\cdot\,|\,\cdot\,,1)] = \begin{pmatrix} \alpha_2 & 1-\varepsilon_2-\alpha_2 & \varepsilon_2 \\ 1-\varepsilon_2-\alpha_2 & \alpha_2 & \varepsilon_2 \end{pmatrix},$$
$$[P_{Y_1|X_1,X_2}(\,\cdot\,|0,\,\cdot\,)] = \begin{pmatrix} 1-\varepsilon_1-\alpha_1 & \alpha_1 & \varepsilon_1 \\ \alpha_1 & 1-\varepsilon_1-\alpha_1 & \varepsilon_1 \end{pmatrix},$$
$$[P_{Y_1|X_1,X_2}(\,\cdot\,|1,\,\cdot\,)] = \begin{pmatrix} \alpha_1 & 1-\varepsilon_1-\alpha_1 & \varepsilon_1 \\ 1-\varepsilon_1-\alpha_1 & \alpha_1 & \varepsilon_1 \end{pmatrix},$$

where the rows are indexed by 0 and 1 (from top to bottom) and the columns are indexed by 0, 1, and $\mathbf{E}$ (from left to right). As all our proposed conditions are only based on the marginal transition probabilities, the relationship between $Z_{1,n}$

[3]Strictly speaking, $X_{1,n} \oplus_2 X_{2,n} \oplus_2 Z_{j,n}$ is undefined when $Z_{j,n} = \mathbf{E}$, but we set $(X_{1,n} \oplus_2 X_{2,n} \oplus_2 \mathbf{E}) \cdot 0 = 0$.

and $Z_{2,n}$ can be arbitrary. By Corollary 2.4, we obtain that the optimal channel input distribution is $P^*_{X_1}P^*_{X_2} = P^{\mathrm{U}}_{\mathcal{X}_1}P^{\mathrm{U}}_{\mathcal{X}_2}$ since the marginal channel transition matrices not only exhibit column permutation properties but also are quasi-symmetric. The capacity region is given by

$$\mathcal{C} = \left\{(R_1, R_2) : R_1 \leq (1-\varepsilon_2)\Big(1 - H_{\mathrm{b}}\Big(\frac{\alpha_2}{1-\varepsilon_2}\Big)\Big), R_2 \leq (1-\varepsilon_1)\Big(1 - H_{\mathrm{b}}\Big(\frac{\alpha_1}{1-\varepsilon_1}\Big)\Big)\right\},$$

where $H_{\mathrm{b}}(\cdot)$ denotes the binary entropy function. One can verify that this TWC also satisfies the conditions of Theorems 2.6 and 2.7 and Corollaries 2.2 and 2.3.

**Remark 2.2.** Various TWCs are special cases of this DM-TWC model:

1. If $\alpha_1 = \alpha_2 = 0$, then the memoryless binary additive TWC with erasures is recovered:

$$Y_{1,n} = (X_{1,n} \oplus_2 X_{2,n}) \cdot \mathbf{1}\{Z_{1,n} \neq \mathbf{E}\} + \mathbf{E} \cdot \mathbf{1}\{Z_{1,n} = \mathbf{E}\},$$
$$Y_{2,n} = (X_{1,n} \oplus_2 X_{2,n}) \cdot \mathbf{1}\{Z_{2,n} \neq \mathbf{E}\} + \mathbf{E} \cdot \mathbf{1}\{Z_{2,n} = \mathbf{E}\}.$$

   The capacity region is given by

$$\mathcal{C} = \{(R_1, R_2) : R_1 \leq 1 - \varepsilon_2, R_2 \leq 1 - \varepsilon_1\}.$$

2. If $\varepsilon_1 = \varepsilon_2 = 0$, then the memoryless binary additive-noise TWC is obtained:

$$Y_{1,n} = X_{1,n} \oplus_2 X_{2,n} \oplus_2 Z_{1,n},$$
$$Y_{2,n} = X_{1,n} \oplus_2 X_{2,n} \oplus_2 Z_{2,n}.$$

The capacity region of this channel is given by

$$\mathcal{C} = \{(R_1, R_2) : R_1 \le 1 - H_\mathrm{b}(\alpha_2), R_2 \le 1 - H_\mathrm{b}(\alpha_1)\}.$$

3. If $\varepsilon_1 = \varepsilon_2 = 0$ and $\alpha_1 = \alpha_2 = 0$, then we obtain the memoryless binary additive TWC given by $Y_{1,n} = X_{1,n} \oplus_2 X_{2,n}$ and $Y_{2,n} = X_{1,n} \oplus_2 X_{2,n}$. The capacity region is given by $\mathcal{C} = \{(R_1, R_2) : R_1 \le 1, R_2 \le 1\}$ [3], [51].

**Remark 2.3.** Example 2.1 can be generalized to a non-binary setting: for some integer $q > 2$, $\mathcal{X}_1 = \mathcal{X}_2 = \{0, 1, \ldots, q-1\} (= G_q)$ and $\mathcal{Y}_1 = \mathcal{Y}_2 = \mathcal{Z} = \{0, 1, \ldots, q-1, \mathbf{E}\}$, the $q$-ary channel model obeys the same equations as in Example 2.1 with modulo-2 addition replaced with the modulo-$q$ operation $\oplus_q$. Furthermore, the channel noise-erasure variables have marginal distributions given by $\Pr\big(Z_{j,n} = \mathbf{E}\big) = \varepsilon_j$ and $\Pr\big(Z_{j,n} = z\big) = \alpha_j/(q-1)$ for $z = 1, 2, \ldots, q-1$, where $0 \le \alpha_j + \varepsilon_j \le 1$ for $j = 1, 2$. By Corollary 2.4, we directly have that $\mathcal{C}_\mathrm{I} = \mathcal{C}_\mathrm{O}$, and

$$\mathcal{C} = \Bigg\{(R_1, R_2) : R_1 \le (1-\varepsilon_2)\left(\log_2 q - H_q\left(\frac{\alpha_2}{(q-1)(1-\varepsilon_2)}\right)\right), \\ R_2 \le (1-\varepsilon_1)\left(\log_2 q - H_q\left(\frac{\alpha_1}{(q-1)(1-\varepsilon_1)}\right)\right)\Bigg\},$$

where $H_q(x) \triangleq x \cdot \log_2(q-1) - x \cdot \log_2 x - (1-x) \cdot \log_2(1-x)$.

**Example 2.2** (**Data Access DM-TWCs**)**.** Let $q = 2^m$ for some integer $m \ge 1$ and consider the alphabets $\mathcal{X}_1 = \mathcal{X}_2 = \mathcal{X} = \{0, 1, \ldots, q-1\}$, $\mathcal{Y}_1 = \mathcal{Y}_2 = \{0, 1, \ldots, q-1, \mathbf{E}\}$, and $\mathcal{Z} = \{0, 1, 2\}$. A data access TWC linking two storage devices is described by

$$Y_{1,n} = (X_{1,n} \boxplus_q X_{2,n}) \cdot \mathbf{1}\{Z_{1,n} = 0\} + ((q-1) \boxplus_q X_{1,n} \boxplus_q X_{2,n}) \cdot \mathbf{1}\{Z_{1,n} = 1\} + \mathbf{E} \cdot \mathbf{1}\{Z_{1,n} = 2\},$$

$$Y_{2,n} = (X_{1,n} \boxplus_q X_{2,n}) \cdot \mathbf{1}\{Z_{2,n} = 0\} + ((q-1) \boxplus_q X_{1,n} \boxplus_q X_{2,n}) \cdot \mathbf{1}\{Z_{2,n} = 1\} + \mathbf{E} \cdot \mathbf{1}\{Z_{2,n} = 2\},$$

where $a \boxplus_q b$ denotes bit-wise addition for the length-$q$ standard binary representation of $a, b \in \mathcal{X}$, and $\{(Z_{1,n}, Z_{2,n})\}_{n=1}^{\infty}$ is a memoryless two-dimensional noise-erasure process that is independent of the stored messages and has components $Z_{1,n}, Z_{2,n} \in \mathcal{Z}$ such that $\Pr\big(Z_{j,n} = 1\big) = \alpha_j$ and $\Pr\big(Z_{j,n} = \mathbf{E}\big) = \varepsilon_j$, where $0 \le \alpha_j + \varepsilon_j \le 1$ for $j = 1, 2$. This channel model can capture the effect of terminal signal superpositions (when $Z_{j,n} = 0$), bit-level burst errors which flip all bits of $X_{1,n} \boxplus_q X_{2,n}$ (when $Z_{j,n} = 1$), and data package losses (when $Z_{j,n} = 2$).

For this channel, an application of Corollary 2.4 immediately gives the capacity region:

$$\mathcal{C} = \Bigg\{ (R_1, R_2) : R_1 \le (1 - \varepsilon_2) \left( m - H_{\mathrm{b}} \left( \frac{\alpha_2}{1 - \varepsilon_2} \right) \right), \\ R_2 \le (1 - \varepsilon_1) \left( m - H_{\mathrm{b}} \left( \frac{\alpha_1}{1 - \varepsilon_1} \right) \right) \Bigg\}.$$

The next example rederives a known result in [41] based on our approach.

**Example 2.3** (**Memoryless Injective Semi-Deterministic TWCs** [41]). Let $\mathcal{T}_j$ and $\mathcal{Z}_j$ denote finite sets. A memoryless ISD-TWC is defined in [41] by the channel equations

$$Y_{j,n} = h_j(X_{j,n}, T_{j,n}) \text{ and } T_{j,n} = \tilde{h}_j(X_{j',n}, Z_{j,n}) \tag{2.28}$$

for $j, j' = 1, 2$ with $j \neq j'$, where $h_j : \mathcal{X}_j \times \mathcal{T}_j \to \mathcal{Y}_j$ is injective in $\mathcal{T}_j$ and $\tilde{h}_j : \mathcal{X}_{j'} \times \mathcal{Z}_j \to \mathcal{T}_j$ is injective in $\mathcal{Z}_j$, i.e., for every $x_j \in \mathcal{X}_j$, $h_j(x_j, t_j)$ is one-to-one in $t_j \in \mathcal{T}_j$ and for every $x_{j'} \in \mathcal{X}_{j'}$, $\tilde{h}_j(x_{j'}, z_j)$ is one-to-one in $z_j \in \mathcal{Z}_j$. Here, $\{(Z_{1,n}, Z_{2,n})\}_{n=1}^{\infty}$ is a memoryless two-dimensional noise process that is independent of terminals' messages.

For this channel, we have [41]

$$I(X_1;Y_2|X_2=x_2) \le \max_{P_{X_1}} H(\tilde{h}_2(X_1,Z_2)) - H(Z_2).$$

This upper bound does not depend on $X_2$, and hence a common maximizer exists, i.e., $P^*_{X_1} = \arg\max_{P_{X_1}} H(\tilde{h}_2(X_1,Z_2))$. Moreover, the value of $\max_{P_{X_1}} I(X_1;Y_2|X_2=x_2)$ is identical for all $x_2 \in \mathcal{X}_2$. We immediately observe that condition (i) in Theorem 2.7 holds. By a similar argument, condition (ii) in Theorem 2.7 also holds, implying that Shannon's inner and outer bounds coincide. The capacity region is given by

$$\mathcal{C} = \Big\{ (R_1,R_2) : R_1 \le \max_{P_{X_1}} H(\tilde{h}_2(X_1,Z_2)) - H(Z_2), \\ R_2 \le \max_{P_{X_2}} H(\tilde{h}_1(X_2,Z_1)) - H(Z_1) \Big\}.$$

**Example 2.4.** Consider the DM-TWC with

$$[P_{Y_1,Y_2|X_1,X_2}(\cdot,\cdot|\cdot,\cdot)] = \begin{array}{c|cccc} & 00 & 01 & 10 & 11 \\ \hline 00 & 0.783 & 0.087 & 0.117 & 0.013 \\ 01 & 0.0417 & 0.3753 & 0.0583 & 0.5247 \\ 10 & 0.261 & 0.609 & 0.039 & 0.091 \\ 11 & 0.2919 & 0.1251 & 0.4081 & 0.1749 \end{array}$$

The corresponding one-way channel marginal distributions are given by

$$[P_{Y_2|X_1,X_2}(\,\cdot\,|\,\cdot\,,0)] = \begin{pmatrix} 0.9 & 0.1 \\ 0.3 & 0.7 \end{pmatrix}, \quad [P_{Y_1|X_1,X_2}(\,\cdot\,|0,\,\cdot\,)] = \begin{pmatrix} 0.87 & 0.13 \\ 0.417 & 0.583 \end{pmatrix},$$

$$[P_{Y_2|X_1,X_2}(\,\cdot\,|\,\cdot\,,1)] = \begin{pmatrix} 0.1 & 0.9 \\ 0.7 & 0.3 \end{pmatrix}, \quad [P_{Y_1|X_1,X_2}(\,\cdot\,|1,\,\cdot\,)] = \begin{pmatrix} 0.87 & 0.13 \\ 0.417 & 0.583 \end{pmatrix}.$$

For this DM-TWC, Shannon's symmetry condition in Theorem 2.1 does not hold since there are no permutations on $\mathcal{Y}_1$ and $\mathcal{Y}_2$ which can result in (2.8). Furthermore, since $H(Y_2|X_1 = 0, X_2 = 0) = H_{\mathrm{b}}(0.1)$ and $H(Y_2|X_1 = 1, X_2 = 0) = H_{\mathrm{b}}(0.3)$, where $H_{\mathrm{b}}(\cdot)$ denotes the binary entropy function, $\mathcal{H}(P_{X_2}\tilde{P}_{X_1|X_2}, P_{Y_2|X_1,X_2})$ depends on $\tilde{P}_{X_1|X_2}$ for given $P_{X_2}$. Thus, the CVA condition in Theorem 2.3 does not hold, either. However by Corollary 2.4, Shannon's inner and outer bounds coincide. Moreover, the optimal input distribution for this TWC can be obtained by searching for the common maximizer for each of the two one-way channels via the Blahut-Arimoto algorithm [74, Section 10.8] yielding $P^*_{X_1}(0) = P^*_{X_2}(0) = 0.471$. Thus, the capacity region is achieved by $P^*_{X_1,X_2} = P^*_{X_1}P^*_{X_2}$, i.e., $\mathcal{C} = \{(R_1, R_2) : 0 \leq R_1 \leq 0.2967, 0 \leq R_2 \leq 0.1715\}$.

**Example 2.5.** Consider the DM-TWC with

$$[P_{Y_1,Y_2|X_1,X_2}(\cdot\,,\cdot\,|\,\cdot\,,\cdot\,)] = \begin{array}{c} \\ 00 \\ 01 \\ 10 \\ 11 \end{array}\!\!\begin{array}{c} \begin{array}{cccc} 00 & 01 & 10 & 11 \end{array} \\ \begin{pmatrix} 0.783 & 0.087 & 0.117 & 0.013 \\ 0.36279 & 0.05421 & 0.50721 & 0.07579 \\ 0.261 & 0.609 & 0.039 & 0.091 \\ 0.173889 & 0.243111 & 0.243111 & 0.339889 \end{pmatrix} \end{array}$$

where two one-way channel marginal distributions are

$$[P_{Y_2|X_1,X_2}(\cdot\,|\,\cdot\,,0)] = \begin{pmatrix} 0.9 & 0.1 \\ 0.3 & 0.7 \end{pmatrix}, \quad [P_{Y_2|X_1,X_2}(\cdot\,|\,\cdot\,,1)] = \begin{pmatrix} 0.87 & 0.13 \\ 0.417 & 0.583 \end{pmatrix},$$

and $[P_{Y_1|X_1,X_2}(\cdot\,|0,\cdot\,)] = [P_{Y_1|X_1,X_2}(\cdot\,|1,\cdot\,)] = [P_{Y_2|X_1,X_2}(\cdot\,|\,\cdot\,,1)]$. Using the same arguments as in the previous example, one can easily see that this TWC satisfies neither the Shannon nor the CVA conditions. However, it satisfies the conditions in Theorem 2.6 since a common maximizer exists for the one-way channel from terminals 1

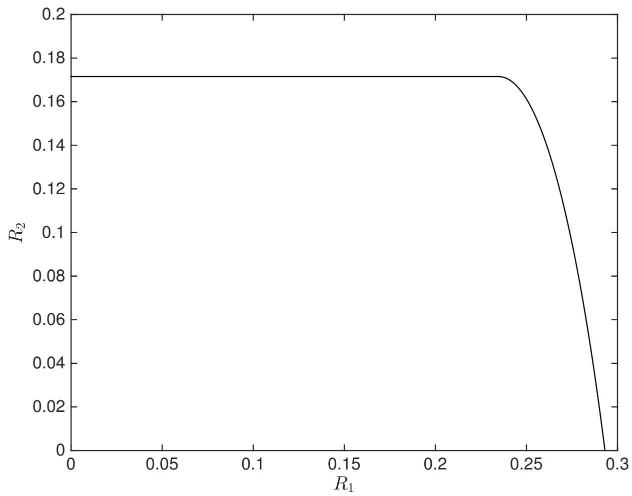



Figure 2.3: The capacity region of the DM-TWC in Example 2.5.

to 2, i.e., $P^*_{X_1}(0) = 0.471$, and condition (ii) trivially holds. By considering all input distributions of the form $P_{X_1,X_2} = P^*_{X_1} P_{X_2}$, the capacity region of this channel is determined as shown in Fig. 2.3.

## 2.3 Further Exploration on the Tightness Conditions

### 2.3.1 Comparison with Prior Results

In this section, we show that Theorem 2.6 and Corollary 2.2 generalize the Shannon results in Propositions 2.1 and 2.2, respectively, and that Theorem 2.8 subsumes the CVA result in Proposition 2.3 as a special case.

**Theorem 2.9.** *A DM-TWC that satisfies the Shannon's one-sided symmetry condition of Proposition 2.1 must satisfy the conditions of Theorem 2.6.*

*Proof:* If a DM-TWC satisfies the Shannon condition in Proposition 2.1, the capacity-achieving input distribution is of the form $P_{X_1,X_2} = P^{\mathrm{U}}_{\mathcal{X}_1} P_{X_2}$ for some $P_{X_2} \in \mathcal{P}(\mathcal{X}_2)$ [3].

This implies that condition (i) of Theorem 2.6 is satisfied because a common maximizer exists for all $x_2 \in \mathcal{X}$ and is given by $P^*_{X_1} = P^{\mathrm{U}}_{\mathcal{X}_1}$. To prove that condition (ii) is also satisfied, we consider the transition matrices $[P_{Y_1|X_1,X_2}(\,\cdot\,|x_1',\,\cdot\,)]$ and $[P_{Y_1|X_1,X_2}(\,\cdot\,|x_1'',\,\cdot\,)]$ for arbitrary $x_1', x_1'' \in \mathcal{X}_1$ and show that these are column permutations of each other and hence $\mathcal{I}(P_{X_2}, P_{Y_1|X_1=x_1',X_2}) = \mathcal{I}(P_{X_2}, P_{Y_1|X_1=x_1'',X_2})$. The first claim is true because

$$\begin{aligned} P_{Y_1|X_1,X_2}(y_1|x_1',x_2) &= P_{Y_1|X_1,X_2}(\pi^{\mathcal{Y}_1}[x_1',x_1''](y_1)|\tau^{\mathcal{X}_1}_{x_1',x_1''}(x_1'),x_2) \qquad (2.29)\\ &= P_{Y_1|X_1,X_2}(\pi^{\mathcal{Y}_1}[x_1',x_1''](y_1)|x_1'',x_2), \end{aligned}$$

where (2.29) is obtained by marginalizing over $Y_2$ on both sides of (2.8). For the second claim, we have

$$\begin{aligned} &\mathcal{I}(P_{X_2}, P_{Y_1|X_1=x_1',X_2})\\ &= \sum_{x_2,y_1} P_{X_2}(x_2)\cdot P_{Y_1|X_1,X_2}(y_1|x_1',x_2)\cdot \log \frac{P_{Y_1|X_1,X_2}(y_1|x_1',x_2)}{\sum_{\tilde{x}_2} P_{X_2}(\tilde{x}_2)\cdot P_{Y_1|X_1,X_2}(y_1|x_1',\tilde{x}_2)}\\ &= \sum_{x_2,y_1} P_{X_2}(x_2)\cdot P_{Y_1|X_1,X_2}(\pi^{\mathcal{Y}_1}[x_1',x_1''](y_1)|x_1'',x_2)\\ &\qquad\qquad \cdot\log \frac{P_{Y_1|X_1,X_2}(\pi^{\mathcal{Y}_1}[x_1',x_1''](y_1)|x_1'',x_2)}{\sum_{\tilde{x}_2} P_{X_2}(\tilde{x}_2)\cdot P_{Y_1|X_1,X_2}(\pi^{\mathcal{Y}_1}[x_1',x_1''](y_1)|x_1'',\tilde{x}_2)} \qquad (2.30)\\ &= \sum_{x_2,\tilde{y}_1} P_{X_2}(x_2)\cdot P_{Y_1|X_1,X_2}(\tilde{y}_1|x_1'',x_2)\cdot \log \frac{P_{Y_1|X_1,X_2}(\tilde{y}_1|x_1'',x_2)}{\sum_{\tilde{x}_2} P_{X_2}(\tilde{x}_2)\cdot P_{Y_1|X_1,X_2}(\tilde{y}_1|x_1'',\tilde{x}_2)}\\ &= \mathcal{I}(P_{X_2}, P_{Y_1|X_1=x_1'',X_2}), \end{aligned}$$

where (2.30) holds by the first claim. ■

**Remark 2.4.** Since the optimal input distribution of terminal 1 in Theorem 2.6 is not necessarily uniform as illustrated in Example 2.5, Theorem 2.6 is more general than Proposition 2.1.

**Theorem 2.10.** *A DM-TWC that satisfies the Shannon two-sided symmetry condition of Proposition 2 must satisfy the conditions of Corollary 2.2.*

This theorem is immediate, and hence the proof is omitted. Together with Example 2.6 given in the next section, Theorem 2.2 is shown to be more general than Proposition 2.2. We next show that the symmetry properties identified by the conditions of Theorem 2.8 are more general than those in the CVA condition.

**Theorem 2.11.** *A DM-TWC that satisfies the CVA condition in Proposition 2.3 must satisfy the conditions in Theorem 2.8.*

*Proof:* Suppose that the condition of Proposition 2.3 is satisfied. To prove the theorem, we show that for $j = 1, 2$, $H(Y_j|X_1 = x_1', X_2 = x_2) = H(Y_j|X_1 = x_1'', X_2 = x_2)$ for all $x_1', x_1'' \in \mathcal{X}_1$ and $x_2 \in \mathcal{X}_2$. Given arbitrary pairs $(x_1', x_2)$ and $(x_1'', x_2)$, consider the probability distributions

$$P^{(1)}_{X_1,X_2}(a, b) = \begin{cases} 1, & \text{if } a = x_1' \text{ and } b = x_2, \\ 0, & \text{otherwise,} \end{cases}$$

and

$$P^{(2)}_{X_1,X_2}(a, b) = \begin{cases} 1, & \text{if } a = x_1'' \text{ and } b = x_2, \\ 0, & \text{otherwise.} \end{cases}$$

Noting that $P^{(1)}_{X_2} = P^{(2)}_{X_2}$, we have $H(Y_j|X_1 = x_1', X_2 = x_2) = H^{(1)}(Y_j|X_1, X_2) = H^{(2)}(Y_j|X_1, X_2) = H(Y_j|X_1 = x_1'', X_2 = x_2)$, where the first and last equality are due to the definitions of $P^{(1)}_{X_1,X_2}$ and $P^{(2)}_{X_1,X_2}$, respectively, and the second equality follows from the CVA condition since $P^{(1)}_{X_2} = P^{(2)}_{X_2}$. Thus $H(Y_j|X_1 = x_1, X_2 = x_2)$ does not depend on $x_1$ for fixed $x_2$ as claimed. Also, since $H(Y_j|X_1, X_2 = x_2) =$

$\sum_{x_1} P_{X_1|X_2}(x_1|x_2) \cdot H(Y_j|X_1 = x_1, X_2 = x_2)$, the conditional entropy $H(Y_j|X_1, X_2 = x_2)$ does not depend on $P_{X_1|X_2=x_2}$.

Next, we show that condition (i) of Theorem 2.8 holds by constructing a common maximizer from the CVA condition. For fixed $x_2 \in \mathcal{X}_2$, let

$$\begin{aligned} P^*_{X_1|X_2=x_2} &= \arg\max_{P_{X_1|X_2=x_2}} I(X_1; Y_2|X_2 = x_2) \\ &= \arg\max_{P_{X_1|X_2=x_2}} \Big( H(Y_2|X_2 = x_2) - H(Y_2|X_1, X_2 = x_2) \Big), \end{aligned}$$

and define $P^{(1)}_{X_1,X_2} = P^{(1)}_{X_2} P^*_{X_1|X_2}$ for some $P^{(1)}_{X_2} \in \mathcal{P}(\mathcal{X}_2)$. Since $H(Y_j|X_1, X_2 = x_2)$ does not depend on $P_{X_1|X_2=x_2}$, $P^*_{X_1|X_2=x_2}$ is in fact a maximizer of $H(Y_2|X_2 = x_2)$. Note that the maximizer $P^*_{X_1|X_2=x_2}$ is not necessarily unique, but any choice works for our purposes. Now for $P^{(1)}_{X_1,X_2}$, by the CVA condition, there exists $\tilde{P}_{X_1} \in \mathcal{P}(\mathcal{X}_1)$ such that $H^{(1)}(Y_2|X_2) \leq H^{(2)}(Y_2|X_2)$, where $P^{(2)}_{X_1,X_2} = \tilde{P}_{X_1} P^{(1)}_{X_2}$. Since $P^*_{X_1|X_2=x_2}$ is the maximizer for $H(Y_2|X_2 = x_2)$, we have

$$\begin{aligned} H^{(1)}(Y_2|X_2) &= \sum_{x_2} P^{(1)}_{X_2}(x_2) \cdot H^{(1)}(Y_2|X_2 = x_2) \\ &= \sum_{x_2} P^{(1)}_{X_2}(x_2) \cdot \left[ \max_{P_{X_1|X_2=x_2}} H(Y_2|X_2 = x_2) \right] \\ &\geq \sum_{x_2} P^{(1)}_{X_2}(x_2) \cdot H^{(2)}(Y_2|X_2 = x_2) \\ &= H^{(2)}(Y_2|X_2) \end{aligned}$$

Thus, $H^{(1)}(Y_2|X_2) = H^{(2)}(Y_2|X_2)$, i.e.,

$$\sum_{x_2} P^{(1)}_{X_2}(x_2) \cdot H^{(1)}(Y_2|X_2 = x_2) = \sum_{x_2} P^{(1)}_{X_2}(x_2) \cdot H^{(2)}(Y_2|X_2 = x_2).$$

Since $H^{(2)}(Y_2|X_2 = x_2) \leq H^{(1)}(Y_2|X_2 = x_2)$ for each $x_2 \in \mathcal{X}_2$, we obtain $H^{(1)}(Y_2|X_2 =$

$x_2) = H^{(2)}(Y_2|X_2 = x_2)$, i.e., $\tilde{P}_{X_1}$ achieves the same value for $H(Y_2|X_2 = x_2)$ as $P^*_{X_1|X_2=x_2}$ for all $x_2 \in \mathcal{X}_2$. Consequently, $\tilde{P}_{X_1}$ is a common maximizer and thus condition (i) of Theorem 2.8 is satisfied. Moreover, since the common maximizer $\tilde{P}_{X_1}$ is from the CVA condition, we have that $H^{(1)}(Y_1|X_1) \le H^{(2)}(Y_1|X_1)$, which together with the fact that $H(Y_1|X_1, X_2)$ does not depend on $P_{X_1|X_2}$ given $P_{X_2}$ and $P_{Y_1|X_1,X_2}$ (guaranteed by the CVA condition) implies that condition (ii) of Theorem 2.8 holds.

■

**Remark 2.5.** As illustrated by Example 2.5, a DM-TWC that satisfies the conditions of Theorem 2.8 does not necessarily satisfy the CVA condition in Proposition 2.3. Therefore, Theorem 2.8 is a more general result than Proposition 2.3. We note that the main difference between Theorem 2.8 and Proposition 2.3 lies in the fact that we allow $H(Y_2|X_1, X_2)$ to depend on $P_{X_1|X_2}$, given $P_{X_2}$.

### 2.3.2 Connection Between the Shannon and CVA Conditions

In this section, we connect Shannon's results to the CVA condition through our refinement of Shannon's results in Propositions 2.4 and 2.5. We first give an example showing that the extended Shannon's symmetry conditions in Propositions 2.4 and 2.5 are more general than their original versions since (2.1) implies (2.27) but the reverse implication is not true as follows.

**Example 2.6.** Consider the DM-TWC with $\mathcal{X}_1 = \mathcal{X}_2 = \mathcal{Y}_1 = \mathcal{Y}_2 = \{0, 1\}$ and transition probability

$$[P_{Y_1,Y_2|X_1,X_2}(\,\cdot\,,\,\cdot\,|\,\cdot\,,\,\cdot\,)] = \begin{array}{c} \\ 00 \\ 01 \\ 10 \\ 11 \end{array}\!\!\begin{array}{c} \begin{array}{cccc} 00 & 01 & 10 & 11 \end{array} \\ \begin{pmatrix} 0.25 & 0.5 & 0.25 & 0 \\ 0.375 & 0375 & 0.125 & 0.125 \\ 0.125 & 0.125 & 0.375 & 0.375 \\ 0.125 & 0.125 & 0.375 & 0.375 \end{pmatrix} \end{array}$$

The marginal distributions are

$$[P_{Y_1|X_1,X_2}(\,\cdot\,|\,\cdot\,,\,\cdot\,)] = \begin{array}{c} \\ 00 \\ 01 \\ 10 \\ 11 \end{array}\!\!\begin{array}{c} \begin{array}{cc} 0 & 1 \end{array} \\ \begin{pmatrix} 0.75 & 0.25 \\ 0.75 & 0.25 \\ 0.25 & 0.75 \\ 0.25 & 0.75 \end{pmatrix} \end{array} \quad \text{and} \quad [P_{Y_2|X_1,X_2}(\,\cdot\,|\,\cdot\,,\,\cdot\,)] = \begin{array}{c} \\ 00 \\ 01 \\ 10 \\ 11 \end{array}\!\!\begin{array}{c} \begin{array}{cc} 0 & 1 \end{array} \\ \begin{pmatrix} 0.5 & 0.5 \\ 0.5 & 0.5 \\ 0.5 & 0.5 \\ 0.5 & 0.5 \end{pmatrix} \end{array}$$

Clearly, neither of the Shannon conditions in Proposition 2.1 or 2.2 holds, but the extended condition in (2.27) holds.

We now show that the above extended symmetry condition implies the CVA condition.

**Theorem 2.12.** *A DM-TWC that satisfies the conditions in Proposition 2.4 must satisfy the CVA condition of Proposition 2.3.*

*Proof:* If the marginal channels $P_{Y_1|X_1,X_2}$ and $P_{Y_2|X_1,X_2}$ satisfy the extended one-sided symmetry condition in (2.27), then $H(Y_j|X_1 = x_1, X_2 = x_2)$ does not depend on $x_1 \in \mathcal{X}_1$ for any fixed $x_2 \in \mathcal{X}_2$ since the rows of $[P_{Y_j|X_1,X_2}(\,\cdot\,|\,\cdot\,, x_2)]$ are permutations of each other. Hence, $H(Y_j|X_1, X_2)$ does not depend on $P_{X_1|X_2}$ given $P_{X_2} \in \mathcal{P}(\mathcal{X}_2)$ as required by the CVA condition.

Next, for any given joint distribution $P^{(1)}_{X_1,X_2} = P^{(1)}_{X_2} P^{(1)}_{X_1|X_2}$, we show that $P^{(2)}_{X_1,X_2} = \tilde{P}_{X_1} P^{(1)}_{X_2}$ with the choice $\tilde{P}_{X_1} = P^{\mathrm{U}}_{\mathcal{X}_1}$ meets the remaining requirements of the CVA condition in Proposition 2.3. Since the DM-TWC satisfies the extended Shannon condition, Lemma A.1.3 in Appendix A.1 gives the two inequalities: $I^{(1)}(X_1;Y_2|X_2) \leq I^{(2)}(X_1;Y_2|X_2)$ and $I^{(1)}(X_2;Y_1|X_1) \leq I^{(2)}(X_2;Y_1|X_1)$. Observing that

$$I^{(1)}(X_1;Y_2|X_2) = H^{(1)}(Y_2|X_2) - H^{(1)}(Y_2|X_1,X_2) = H^{(1)}(Y_2|X_2) - H^{(2)}(Y_2|X_1,X_2),$$

we immediately obtain that $H^{(1)}(Y_2|X_2) \leq H^{(2)}(Y_2|X_2)$ since

$$I^{(1)}(X_1;Y_2|X_2) \leq I^{(2)}(X_1;Y_2|X_2).$$

Moreover, as $H^{(1)}(Y_1|X_1,X_2) = H^{(2)}(Y_1|X_1,X_2)$ and $I^{(1)}(X_2;Y_1|X_1) \leq I^{(2)}(X_2;Y_1|X_1)$, we have that $H^{(1)}(Y_1|X_1) \leq H^{(2)}(Y_1|X_1)$. Thus, the CVA condition is fulfilled. ■

**Remark 2.6.** In [41], the existence of examples showing that the Shannon and CVA results are not equivalent was posed as an open question. The example below shows that the CVA condition is more general than the extended (one-sided) Shannon's symmetry condition in (2.3). Together with Example 2.6, we conclude that the CVA result is more general than the Shannon result.

**Example 2.7.** Consider the DM-TWC with $\mathcal{X}_1 = \mathcal{Y}_1 = \mathcal{Y}_2 = \{0,1,2\}$ and $\mathcal{X}_2 = \{0,1\}$ and marginal distributions given by

$$[P_{Y_1|X_1,X_2}(\,\cdot\,|\,\cdot\,,0)] = \begin{array}{c} \\ 0 \\ 1 \\ 2 \end{array}\!\!\begin{array}{c} \begin{array}{ccc} 0 & 1 & 2 \end{array} \\ \begin{pmatrix} 0.3 & 0.2 & 0.5 \\ 0.5 & 0.3 & 0.2 \\ 0.2 & 0.5 & 0.3 \end{pmatrix} \end{array}$$

with $[P_{Y_1|X_1,X_2}(\,\cdot\,|\,\cdot\,,1)] = [P_{Y_2|X_1,X_2}(\,\cdot\,|\,\cdot\,,0)] = [P_{Y_2|X_1,X_2}(\,\cdot\,|\,\cdot\,,1)] = [P_{Y_1|X_1,X_2}(\,\cdot\,|\,\cdot\,,0)]$.

Clearly, there are no relabeling functions for $\mathcal{Y}_1$ and $\mathcal{Y}_2$ which recover $[P_{Y_1|X_1,X_2}(\,\cdot\,|\,\cdot\,,0)]$ after exchanging the labels of $X_1 = 0$ and $X_1 = 1$, so that the extended one-sided symmetry condition does not hold. To check the CVA condition, we first observe that $H(Y_j|X_1 = x_1, X_2 = x_2)$ does not depend on $x_1 \in \mathcal{X}_1$ and $x_2 \in \mathcal{X}_2$; thus $H(Y_j|X_1, X_2)$ does not depend on $P_{X_1,X_2}$ for $j = 1, 2$. Furthermore, for any given $P^{(1)}_{X_1,X_2} = P^{(1)}_{X_2} P^{(1)}_{X_1|X_2}$, consider $P^{(2)}_{X_1,X_2} = \tilde{P}_{X_1} P^{(1)}_{X_2}$ with $\tilde{P}_{X_1} = P^{\mathrm{U}}_{\mathcal{X}_1}$. Then, we have

$$\begin{aligned} I^{(1)}(X_1; Y_2|X_2) &= \sum_{x_2} P^{(1)}_{X_2}(x_2) \cdot I^{(1)}(X_1; Y_2|X_2 = x_2) \\ &\le \sum_{x_2} P^{(1)}_{X_2}(x_2) \cdot I^{(2)}(X_1; Y_2|X_2 = x_2) \\ &= I^{(2)}(X_1; Y_2|X_2), \end{aligned}$$

where the inequality follows from the fact that $P^{\mathrm{U}}_{\mathcal{X}_1}$ is the capacity-achieving input distribution for all one-way channels from terminals 1 to 2. On the other hand, since the matrices $[P_{Y_1|X_1,X_2}(\,\cdot\,|x_1,\,\cdot\,)]$, $x_1 \in \mathcal{X}_1$, are column permutations of each other, $\mathcal{I}(P_{X_2}, P_{Y_1|X_1=x_1,X_2})$ does not depend on $x_1 \in \mathcal{X}_1$ for any fixed $P_{X_2} \in \mathcal{P}(\mathcal{X}_2)$. One can then follow the proof of Theorem 2.6 to obtain that $I^{(1)}(X_2; Y_1|X_1) \le I^{(2)}(X_2; Y_1|X_1)$. Now, since $H(Y_j|X_1, X_2)$ does not depend on the input distribution, we conclude that $H^{(1)}(Y_j|X_j) \le H^{(2)}(Y_j|X_j)$ for $j = 1, 2$, and thus the CVA condition is satisfied.

**Remark 2.7.** The channel in the above example in fact also satisfies the conditions of Theorem 2.6. Nevertheless, the connection between the conditions of Theorem 2.6 and the CVA condition is still unclear.

### 2.3.3 Necessary Conditions for the Tightness Results

Known numerical methods to obtain Shannon's capacity bounds involve (step 1) uniformly quantizing the probability simplex of channel inputs; (step 2) evaluating the region $\mathcal{R}$ for each quantized input distribution; (step 3) taking the convex hull. By definition, the quantized input distributions for $\mathcal{C}_{\mathrm{I}}$ and $\mathcal{C}_{\mathrm{O}}$ take values in the spaces $\mathcal{P}(\mathcal{X}_1) \times \mathcal{P}(\mathcal{X}_2)$ and $\mathcal{P}(\mathcal{X}_1 \times \mathcal{X}_2)$, with dimensions $(|\mathcal{X}_1| - 1)(|\mathcal{X}_2| - 1)$ and $|\mathcal{X}_1||\mathcal{X}_2| - 1$, respectively. Let $\Delta \in (0, 1)$ denote the step size of the quantization and suppose that $\Delta^{-1} \in \mathbb{N}$. We thus have $\binom{\Delta^{-1}+|\mathcal{X}_1|-1}{\Delta^{-1}}\binom{\Delta^{-1}+|\mathcal{X}_2|-1}{\Delta^{-1}}$ and $\binom{\Delta^{-1}+|\mathcal{X}_1||\mathcal{X}_2|-1}{\Delta^{-1}}$ quantized input distributions to compute for $\mathcal{C}_{\mathrm{I}}$ and $\mathcal{C}_{\mathrm{O}}$, respectively. For example, when all channel alphabet sizes are not larger than 3, setting $\Delta = 0.025$ is enough to have visually indistinguishable region estimates of $\mathcal{C}_{\mathrm{I}}$ and $\mathcal{C}_{\mathrm{O}}$. The number of quantized input distributions for $\mathcal{C}_{\mathrm{I}}$ is roughly $7 \times 10^6$, but it is about $3 \times 10^9$ for $\mathcal{C}_{\mathrm{O}}$; evaluating $\mathcal{C}_{\mathrm{O}}$ clearly involves significantly more calculations. Although one can apply the Lagrange multiplier method [3, Section 11] to find $\mathcal{C}_{\mathrm{O}}$, the implementation cost is still considerable.

On the other hand, even though the validation of individual symmetry conditions is usually easy, the accumulated computational complexity can be significant. For instance, validating condition (i) of Theorem 2.6 can be efficiently done via the Blahut-Arimoto algorithm. The verification of condition (ii), though slightly complex, only involves checking input distributions in $\mathcal{P}(\mathcal{X}_2)$. Clearly, the overall computational complexity for Theorem 2.6 is lower than the one needed for evaluating $\mathcal{C}_{\mathrm{O}}$. Nevertheless, if this validation fails, then we need to swap the roles of terminals 1 and 2 and verify conditions (i) and (ii) of Theorem 2.6 again. If this process is still unsuccessful, then one may switch to Proposition 2.8. Hence, the entire validation process can be lengthy and requires significant computational resources.

To reduce the computational complexity of validating symmetry conditions, we provide three simple conditions that can be used to rule out certain symmetry properties. Two of these conditions appeared in the proof of Theorem 2.11. As the conditions are useful in practice, we present them here as standalone results without proof. The first one is derived for condition (ii) of Theorem 2.6, which can be efficiently validated via the Blahut-Arimoto algorithm (as done for condition (i) of Theorem 2.6). The second one is for condition (ii) of Theorem 2.8, which sometimes can be verified by directly observing the channels' marginal transition matrices.

**Theorem 2.13.** *If a DM-TWC satisfies condition (ii) of Theorem 2.6, then there exists $P^*_{X_2} \in \mathcal{P}(\mathcal{X}_2)$ such that for all $x_1 \in \mathcal{X}_1$,*

$$\arg\max_{P_{X_2|X_1=x_1}} I(X_2; Y_1|X_1 = x_1) = P^*_{X_2}.$$

*Proof:* Since the same channel input distribution yields the same input-output mutual information for all sub-channels $[P_{Y_1|X_1,X_2}(\,\cdot\,|x_1,\,\cdot\,)]$'s, the capacity-achieving input distribution for one sub-channel must also be the capacity-achieving input distribution for any other sub-channel. ■

**Theorem 2.14.** *If a DM-TWC satisfies condition (ii) of Theorem 2.8, then $H(Y_1|X_1 = x_1', X_2 = x_2) = H(Y_1|X_1 = x_1'', X_2 = x_2)$ for any $x_1', x_1'' \in \mathcal{X}_1$ and fixed $x_2 \in \mathcal{X}_2$.*

Furthermore, for DM-TWCs that satisfy the conditions of Theorem 2.8, the inequality $I^{(1)}(X_{j'}; Y_j|X_j) \leq I^{(2)}(X_{j'}; Y_j|X_j)$, $j \neq j'$, holds for the specific input distributions $P^{(1)}_{X_1,X_2} = P^*_{X_1} P^{(1)}_{X_2|X_1} = P^{(1)}_{X_2} P^{(1)}_{X_1|X_2}$ and $P^{(2)}_{X_1,X_2} = P^*_{X_1} P^{(1)}_{X_2}$. For $j$=2 and $j'$=1, expanding the inequality, we have that

$$\sum_{x_1} P^*_{X_1}(x_1) \cdot \mathcal{I}\big(P^{(1)}_{X_2|X_1=x_1}, P_{Y_1|X_1=x_1,X_2}\big) \leq \sum_{x_1} P^*_{X_1}(x_1) \cdot \mathcal{I}\big(P^{(1)}_{X_2}, P_{Y_1|X_1=x_1,X_2}\big),$$

which indicates that using an *average input* $P^{(1)}_{X_2}$ at terminal 2 does not incur any information loss when terminal 1 uses the common optimal input $P^*_{X_1}$. As one can choose $P^{(1)}_{X_2|X_1=x_1} = \arg\max_{P_{X_2|X_1=x_1}} \mathcal{I}(P_{X_2|X_1=x_1}, P_{Y_1|X_1=x_1,X_2})$, the inequality suggests another common maximizer property, which we use to derive a necessary condition for conditions (i) and (ii) of Theorem 2.8 to hold.

**Theorem 2.15.** *If a DM-TWC satisfies the conditions of Theorem 2.8 with* $P^*_X(x_1) > 0$ *for all* $x_1 \in \mathcal{X}_1$*, then there exists a common output conditional entropy maximizer* $P^*_{X_2}$ *such that for all* $x_1 \in \mathcal{X}_1$, $\arg\max_{P_{X_2|X_1=x_1}} H(Y_1|X_1 = x_1) = P^*_{X_2}$.

*Proof:* For any $P^{(1)}_{X_1,X_2} = P^{(1)}_{X_2}P^{(1)}_{X_1|X_2} = P^{(1)}_{X_1}P^{(1)}_{X_2|X_1}$, let $P^{(2)}_{X_1,X_2} = P^*_{X_1}P^{(1)}_{X_2}$; the symmetry condition (ii) of Theorem 2.8 gives the inequality $H^{(1)}(Y_1|X_1) \leq H^{(2)}(Y_1|X_1)$. Consider the particular choice $P^{(1)}_{X_1} = P^*_{X_1}$ and

$$P^{(1)}_{X_2|X_1=x_1} = \arg\max_{P_{X_2|X_1=x_1}} H(Y_1|X_1 = x_1)$$

for all $x_1 \in \mathcal{X}_1$. Together with the assumption that $P^*_{X_1}(x_1) > 0$ for all $x_1 \in \mathcal{X}_1$ and the non-negativity of entropy, we obtain that $H^{(1)}(Y_1|X_1 = x_1) \leq H^{(2)}(Y_1|X_1 = x_1)$ for every $x_1 \in \mathcal{X}_1$. Since $P^{(1)}_{X_2|X_1=x_1}$ is the maximizer of $H(Y_1|X_1 = x_1)$, we further have that $H^{(1)}(Y_1|X_1 = x_1) = H^{(2)}(Y_1|X_1 = x_1)$ for any $x_1 \in \mathcal{X}_1$. In other words, the conditional entropies $H(Y_1|X_1 = x_1)$, $x_1 \in \mathcal{X}_1$, have a common maximizer (and $P^{(1)}_{X_2}$ is the common maximizer). ■

As $H(Y_1|X_1 = x_1)$ is a concave function of $P_{X_2|X_1=x_1}$ for fixed $P_{Y_1|X_1=x_1,X_2}$, one can use a standard convex optimization program to check this necessary condition. This result is useful since the inequality $H^{(1)}(Y_1|X_1) \leq H^{(2)}(Y_1|X_1)$ in condition (ii) of Theorem 2.8 is often difficult to verify.

## 2.4 Generalized Push-to-Talk Two-Way Channels

### 2.4.1 Shannon's Push-to-Talk Channels

Let $X_j \in \{0, 1, 2\}$ and $Y_j \in \{0, 1\}$ denote terminal-$j$'s channel input and output for $j = 1, 2$, respectively. Shannon's discrete-memoryless PTT-TWC (DM-PTT-TWC) [3] as shown in Table 2.1(a) is a classic example where two-way simultaneous (i.e., full-duplex) transmission is completely unreliable and time-sharing between two one-way transmissions (i.e., half-duplex communication) is necessary to achieve capacity. As observed from the channel's marginal transition matrices in Tables 2.1(b) and 2.1(c), terminal 1 can perfectly transmit a one-bit message to terminal 2 only when the channel input of terminal 2 is '0', and vice versa. A simple time-sharing argument then gives the set of reliable transmission rate pairs $(R_1, R_2) = (\alpha, 1 - \alpha)$, where $0 \leq \alpha \leq 1$. Since there is no other way to transmit information reliably, that set of rate pairs clearly constitutes the boundary of the capacity region and thus determines capacity.[4]

Inspired by Shannon's TWC setup, the PTT idea was extended to other multi-terminal channels such as PTT multiaccess channels [19, Problem 14.7], [97], switch-to-talk broadcast channels, and incompatible broadcast channels [88, Section V]. In [98], a capacity result was established for a DM-PTT network with more than two terminals.

[4] A formal proof of this statement via the Lagrange multiplier method can be found in [32, Section 2.5.3].

Table 2.1: The full and marginal transition matrices of Shannon's PTT-TWC, where the rows and columns are indexed by the channel inputs and outputs, respectively.

(a) $P_{Y_1,Y_2|X_1,X_2}$ [3, Table I]

| $(X_1, X_2)$ | $(0,0)$ | $(0,1)$ | $(1,0)$ | $(1,1)$ |
|---|---|---|---|---|
| $(0,0)$ | $\frac{1}{4}$ | $\frac{1}{4}$ | $\frac{1}{4}$ | $\frac{1}{4}$ |
| $(0,1)$ | $\frac{1}{2}$ | $\frac{1}{2}$ | $0$ | $0$ |
| $(0,2)$ | $0$ | $0$ | $\frac{1}{2}$ | $\frac{1}{2}$ |
| $(1,0)$ | $\frac{1}{2}$ | $0$ | $\frac{1}{2}$ | $0$ |
| $(1,1)$ | $\frac{1}{4}$ | $\frac{1}{4}$ | $\frac{1}{4}$ | $\frac{1}{4}$ |
| $(1,2)$ | $\frac{1}{4}$ | $\frac{1}{4}$ | $\frac{1}{4}$ | $\frac{1}{4}$ |
| $(2,0)$ | $0$ | $\frac{1}{2}$ | $0$ | $\frac{1}{2}$ |
| $(2,1)$ | $\frac{1}{4}$ | $\frac{1}{4}$ | $\frac{1}{4}$ | $\frac{1}{4}$ |
| $(2,2)$ | $\frac{1}{4}$ | $\frac{1}{4}$ | $\frac{1}{4}$ | $\frac{1}{4}$ |

(b) $P_{Y_2|X_1,X_2}$

| $(X_1, X_2)$ | 0 | 1 |
|---|---|---|
| $(0,0)$ | $\frac{1}{2}$ | $\frac{1}{2}$ |
| $(1,0)$ | $1$ | $0$ |
| $(2,0)$ | $0$ | $1$ |
| $(0,1)$ | $\frac{1}{2}$ | $\frac{1}{2}$ |
| $(1,1)$ | $\frac{1}{2}$ | $\frac{1}{2}$ |
| $(2,1)$ | $\frac{1}{2}$ | $\frac{1}{2}$ |
| $(0,2)$ | $\frac{1}{2}$ | $\frac{1}{2}$ |
| $(1,2)$ | $\frac{1}{2}$ | $\frac{1}{2}$ |
| $(2,2)$ | $\frac{1}{2}$ | $\frac{1}{2}$ |

(c) $P_{Y_1|X_1,X_2}$

| $(X_1, X_2)$ | 0 | 1 |
|---|---|---|
| $(0,0)$ | $\frac{1}{2}$ | $\frac{1}{2}$ |
| $(0,1)$ | $1$ | $0$ |
| $(0,2)$ | $0$ | $1$ |
| $(1,0)$ | $\frac{1}{2}$ | $\frac{1}{2}$ |
| $(1,1)$ | $\frac{1}{2}$ | $\frac{1}{2}$ |
| $(1,2)$ | $\frac{1}{2}$ | $\frac{1}{2}$ |
| $(2,0)$ | $\frac{1}{2}$ | $\frac{1}{2}$ |
| $(2,1)$ | $\frac{1}{2}$ | $\frac{1}{2}$ |
| $(2,2)$ | $\frac{1}{2}$ | $\frac{1}{2}$ |

### 2.4.2 A Generalization of Shannon's Model

For $j = 1, 2$, let $\mathcal{X}_j \triangleq \{0, 1, \ldots, r_j - 1\}$ and $\mathcal{Y}_j \triangleq \{0, 1, \ldots, s_j - 1\}$, where $r_j \geq 3$ and $s_j \geq 2$ (to avoid trivial cases). Without loss of generality, we set $X_1 = 0$ and $X_2 = 0$ as the signals for the "PTT mode." For $j = 1, 2$, let $\boldsymbol{v}_j$ denote the length-$s_j$ row vector with all entries equal to $\frac{1}{s_j}$. Also, let $\boldsymbol{Q}_{j,x_{j'}}$ denote a $(r_j - 1) \times s_{j'}$ channel transition matrix with capacity $C_{j,x_{j'}}$ for $j, j' = 1, 2$ with $j \neq j'$ and $x_{j'} \in \mathcal{X}_{j'}$. An $(r_1, r_2, s_1, s_2)$ generalized DM-PTT-TWC with transition probability $P_{Y_1,Y_2|X_1,X_2}$ is defined by imposing the following structure for the marginal channel transition matrices $[P_{Y_j|X_1,X_2}(\cdot\,|\,\cdot\,,\,\cdot)]$ (where the rows and columns are indexed by the channel inputs and outputs, respectively): for all $x_2 \in \mathcal{X}_2$,

$$[P_{Y_2|X_1,X_2}(\cdot\,|\,\cdot\,, x_2)] = \begin{pmatrix} \boldsymbol{v}_2 \\ \boldsymbol{Q}_{1,x_2} \end{pmatrix},$$

and for all $x_1 \in \mathcal{X}_1$,

$$[P_{Y_1|X_1,X_2}(\cdot\,|x_1,\,\cdot)] = \begin{pmatrix} \boldsymbol{v}_1 \\ \boldsymbol{Q}_{2,x_1} \end{pmatrix}.$$

We note that the above structures do not imply that $P_{Y_1,Y_2|X_1,X_2} = P_{Y_1|X_1,X_2}P_{Y_2|X_1,X_2}$.

Unlike Shannon's original PTT-TWC, our proposed model considers both perfect and noisy reception in the PTT mode and allows reliable full-duplex transmission. Shannon's PTT-TWC can be recovered by setting $(r_1, r_2, s_1, s_2) = (3, 3, 2, 2)$, $\boldsymbol{Q}_{j,0} = \boldsymbol{I}_2$, and $\boldsymbol{Q}_{j,1} = \boldsymbol{Q}_{j,2} = \frac{1}{2} \cdot \mathbf{1}_{2\times 2}$ for $j = 1, 2$, where $\boldsymbol{I}_2$ and $\mathbf{1}_{2\times 2}$ denote the $2 \times 2$ identity and all-one matrices, respectively, and the overall channel transition probability can be obtained as $P_{Y_1,Y_2|X_1,X_2} = P_{Y_1|X_1,X_2}P_{Y_2|X_1,X_2}$. In fact, the capacity region of an $(r_1, r_2, s_1, s_2)$ DM-PTT-TWC is generally unknown. Below, we show that the capacity region can be analytically determined when the marginal channels exhibit the following symmetry property: for $j, j' = 1, 2$ with $j \neq j'$, $\boldsymbol{Q}_{j,x_{j'}}$'s are weakly-symmetric[5] for all $x_{j'} \in \mathcal{X}_{j'}$ and $C_{j,x_{j'}} = C_{j,1}$ for all $x_{j'} \neq 0$.

Recall that $\mathbf{1}\{\cdot\}$ denotes the indicator function. Letting $P^{\mathrm{U}_0}_{\mathcal{X}_j}$ denote the probability distribution that assigns zero probability mass to $X_j = 0$ and is uniform over the set $\mathcal{X}_j\backslash\{0\}$, $j = 1, 2$, we define six rate pairs and their associated input distributions for the generalized PTT-TWC with the above symmetry property as follows:

- $\boldsymbol{R}^*_1 \triangleq (0, 0)$, $P_{X_1,X_2}(x_1, x_2) = \mathbf{1}\{x_1 = 0\} \cdot \mathbf{1}\{x_2 = 0\}$;
- $\boldsymbol{R}^*_2 \triangleq (C_{1,1}, C_{2,1})$, $P_{X_1,X_2} = P^{\mathrm{U}_0}_{\mathcal{X}_1}P^{\mathrm{U}_0}_{\mathcal{X}_2}$;
- $\boldsymbol{R}^*_3 \triangleq (C_{1,0}, 0)$, $P_{X_1,X_2}(x_1, x_2) = P^{\mathrm{U}_0}_{\mathcal{X}_1}(x_1) \cdot \mathbf{1}\{x_2 = 0\}$;

[5] A channel is said to be weakly-symmetric if its transition matrix has identical column sums and its rows are permutations of each other [74, Section 7.2]; for such a channel, the mutual information is maximized by the uniform input distribution. We note that for more general symmetric transition matrices for which mutual information is maximized by the uniform input distribution (e.g. quasi-symmetric channels [78]), Theorem (2.16) does not necessarily hold.

- $\boldsymbol{R}_4^* \triangleq (0, C_{2,0}),\ P_{X_1,X_2}(x_1,x_2) = \mathbf{1}\{x_1 = 0\} \cdot P_{\mathcal{X}_2}^{\mathrm{U}_0}(x_2)$;
- $\boldsymbol{R}_5^* \triangleq (C_{1,1}, 0),\ P_{X_1,X_2}(x_1,x_2) = P_{\mathcal{X}_1}^{\mathrm{U}_0}(x_1) \cdot \mathbf{1}\{x_2 = 1\}$;
- $\boldsymbol{R}_6^* \triangleq (0, C_{2,1}),\ P_{X_1,X_2}(x_1,x_2) = \mathbf{1}\{x_1 = 1\} \cdot P_{\mathcal{X}_2}^{\mathrm{U}_0}(x_2)$.

Note that the $\boldsymbol{R}_l^*$'s are all attained via independent inputs.

**Theorem 2.16.** *For an $(r_1, r_2, s_1, s_2)$ DM-PTT-TWC that satisfies the above channel symmetry property, Shannon's inner bound is tight and the capacity region can be determined by taking the convex hull of $\boldsymbol{R}_1^*$, $\boldsymbol{R}_2^*$, $\max(\boldsymbol{R}_3^*, \boldsymbol{R}_5^*)$, and $\max(\boldsymbol{R}_4^*, \boldsymbol{R}_6^*)$.*[6]

The idea behind the proof of Theorem 2.16 is to show that any rate pair in Shannon's outer bound region $\mathcal{C}_{\mathrm{O}}$ can be upper-bounded component-wise by another rate pair that is a convex combination of the $\boldsymbol{R}_l^*$'s. More specifically, depending on the value of $C_{j,x_{j'}}$'s, we can use the four rate pairs: $\boldsymbol{R}_1^*$, $\boldsymbol{R}_2^*$, $\max(\boldsymbol{R}_3^*, \boldsymbol{R}_5^*)$, and $\max(\boldsymbol{R}_4^*, \boldsymbol{R}_6^*)$, to upper-bound any rate pair in $\mathcal{C}_{\mathrm{O}}$ and hence determine the capacity region. Here, we only prove the case where $\boldsymbol{R}_3^* = \max(\boldsymbol{R}_3^*, \boldsymbol{R}_5^*)$ and $\boldsymbol{R}_4^* = \max(\boldsymbol{R}_4^*, \boldsymbol{R}_6^*)$. The same argument can be used to prove other cases, and hence the details are omitted.

*Proof of Theorem 2.16:* Given any $P_{X_1,X_2}$, we bound the associated rate pair $(I(X_1;Y_2|X_2), I(X_2;Y_1|X_1))$ as follows:

$$\begin{aligned} I(X_1;Y_2|X_2) &= \sum_{x_2=0}^{r_2-1} P_{X_2}(x_2) \cdot I(X_1;Y_2|X_2 = x_2) && (2.31) \\ &\leq \sum_{x_2=0}^{r_2-1} P_{X_2}(x_2) \cdot \left[(1 - P_{X_1|X_2}(0|x_2)) \cdot C_{1,x_2}\right] && (2.32) \\ &= (P_{X_2}(0) - P_{X_1,X_2}(0,0)) \cdot C_{1,0} \end{aligned}$$

[6] We set $\max(\boldsymbol{A}, \boldsymbol{B}) = \boldsymbol{B}$ iff $\boldsymbol{A}$ is upper-bounded component-wise by $\boldsymbol{B}$.

$$
\begin{aligned}
&\quad + \sum_{x_2 \neq 0} (P_{X_2}(x_2) - P_{X_1,X_2}(0,x_2)) \cdot C_{1,x_2} \\
&= (P_{X_2}(0) - P_{X_1,X_2}(0,0)) \cdot C_{1,0} \\
&\quad + \sum_{x_2 \neq 0} (P_{X_2}(x_2) - P_{X_1,X_2}(0,x_2)) \cdot C_{1,1} \\
&\quad + \underbrace{(P_{X_1}(0) - P_{X_1,X_2}(0,0)) \cdot 0 + P_{X_1,X_2}(0,0) \cdot 0}_{=0}, \qquad (2.33)
\end{aligned}
$$

where (2.32) follows from Lemma A.2.2 in the Appendix and (2.33) holds since $C_{1,x_2} = C_{1,1}$ for all $x_2 \neq 0$. Similarly, we have

$$
\begin{aligned}
I(X_2;Y_1|X_1) &= \sum_{x_1=0}^{r_1-1} P_{X_1}(x_1) \cdot I(X_2;Y_1|X_1 = x_1) \\
&\leq \sum_{x_1=0}^{r_1-1} P_{X_1}(x_1) \cdot \left[(1 - P_{X_2|X_1}(0|x_1)) \cdot C_{2,x_1}\right] \\
&= (P_{X_1}(0) - P_{X_1,X_2}(0,0)) \cdot C_{2,0} \\
&\quad + \sum_{x_1 \neq 0} (P_{X_1}(x_1) - P_{X_1,X_2}(x_1,0)) \cdot C_{2,x_1} \\
&= (P_{X_1}(0) - P_{X_1,X_2}(0,0)) \cdot C_{2,0} \\
&\quad + \sum_{x_1 \neq 0} (P_{X_1}(x_1) - P_{X_1,X_2}(x_1,0)) \cdot C_{2,1} \\
&\quad + \underbrace{(P_{X_2}(0) - P_{X_1,X_2}(0,0)) \cdot 0 + P_{X_1,X_2}(0,0) \cdot 0}_{=0}. \qquad (2.34)
\end{aligned}
$$

Note that (2.33) and (2.34) and the fact that $\sum_{x_2 \neq 0}(P_{X_2}(x_2) - P_{X_1,X_2}(0,x_2)) = \sum_{x_1 \neq 0}(P_{X_1}(x_1) - P_{X_1,X_2}(x_1,0))$ imply that the pair $(I(X_1;Y_2|X_2), I(X_2;Y_1|X_1))$ is upper-bounded component-wise by

$$
P_{X_1,X_2}(0,0) \cdot \boldsymbol{R}_1^* + \left[\sum_{x_1 \neq 0} P_{X_1}(x_1) - P_{X_1,X_2}(x_1,0)\right] \cdot \boldsymbol{R}_2^* + \\
\left[P_{X_2}(0) - P_{X_1,X_2}(0,0)\right] \cdot \boldsymbol{R}_3^* + \left[P_{X_1}(0) - P_{X_1,X_2}(0,0)\right] \cdot \boldsymbol{R}_4^*.
$$

Since the coefficients of the above four rate pairs sum to one, any rate pair in $\mathcal{C}_\mathrm{O}$ is outer bounded by some convex combination of $\boldsymbol{R}_1^*$, $\boldsymbol{R}_2^*$, $\boldsymbol{R}_3^*$, and $\boldsymbol{R}_4^*$. Since the four rate pairs are achievable via independent inputs, we conclude that Shannon's inner bound is tight. ■

Clearly, the capacity region of Shannon's PTT-TWC can be easily determined via Theorem 2.16 without using the time-sharing argument [3] or the Lagrange multiplier method [32].

Moreover, we note that (2.31) can be interpreted as the average amount of information sent over a set of state-dependent one-way channels $\{P_{Y_2|X_1,X_2}(\cdot\,|\,\cdot, x_2) :\ x_2 \in \mathcal{X}_2\}$. Thus, terminal-2's input distribution $P_{X_2}$ not only carries its own message but also determines how often each one-way channel can be used for terminal 1. The same interpretation also applies to (2.34). Clearly, the best channel input distribution for one terminal may not create the most favorable one-way channel usage for the other terminal, necessitating a rate trade-off between the two terminals' transmissions.

Quantifying the trade-off is often the most involved part of determining the capacity region of general DM-TWCs. The prior approach to tackle the problem is to exploit (when they exist) channel symmetry or invariance properties so that for any $P_{X_1,X_2} = P_{X_2}P_{X_1|X_2}$, one can always find a $\tilde{P}_{X_1}$ such that $\mathcal{R}(P_{X_1,X_2}, P_{Y_1,Y_2|X_1,X_2}) \subseteq \mathcal{R}(\tilde{P}_{X_1}P_{X_2}, P_{Y_1,Y_2|X_1,X_2})$ [1, 3, 41]. However, this approach fails here since such $\tilde{P}_{X_1}$ may not exist for each $P_{X_1,X_2}$. This observation can be illustrated via Shannon's PTT-TWC as one can see that no single independent input distribution can achieve the rate pair $(R_1, R_2) = (\alpha, 1-\alpha)$, where $0 < \alpha < 1$. It is thus of interest to exploit other symmetry property as the one stated in Theorem 2.16 that allows us to show $\mathcal{C}_{\mathrm{O}} \subseteq \mathcal{C}_{\mathrm{I}}$ directly.

Table 2.2: Marginal transition matrices of a generalized PTT-TWC, where $0 \le a, b, c, d \le \frac{2}{3}$.

(a) $P_{Y_2|X_1,X_2}$

| $(X_1, X_2)$ | 0 | 1 | 2 |
|---|---|---|---|
| $(0,0)$ | $\frac{1}{3}$ | $\frac{1}{3}$ | $\frac{1}{3}$ |
| $(1,0)$ | $\frac{2}{3} - a$ | $a$ | $\frac{1}{3}$ |
| $(2,0)$ | $a$ | $\frac{2}{3} - a$ | $\frac{1}{3}$ |
| $(0,1)$ | $\frac{1}{3}$ | $\frac{1}{3}$ | $\frac{1}{3}$ |
| $(1,1)$ | $\frac{2}{3} - b$ | $b$ | $\frac{1}{3}$ |
| $(2,1)$ | $b$ | $\frac{2}{3} - b$ | $\frac{1}{3}$ |
| $(0,2)$ | $\frac{1}{3}$ | $\frac{1}{3}$ | $\frac{1}{3}$ |
| $(1,2)$ | $\frac{2}{3} - b$ | $b$ | $\frac{1}{3}$ |
| $(2,2)$ | $b$ | $\frac{2}{3} - b$ | $\frac{1}{3}$ |

(b) $P_{Y_1|X_1,X_2}$

| $(X_1, X_2)$ | 0 | 1 | 2 |
|---|---|---|---|
| $(0,0)$ | $\frac{1}{3}$ | $\frac{1}{3}$ | $\frac{1}{3}$ |
| $(0,1)$ | $\frac{2}{3} - c$ | $c$ | $\frac{1}{3}$ |
| $(0,2)$ | $c$ | $\frac{2}{3} - c$ | $\frac{1}{3}$ |
| $(1,0)$ | $\frac{1}{3}$ | $\frac{1}{3}$ | $\frac{1}{3}$ |
| $(1,1)$ | $\frac{2}{3} - d$ | $d$ | $\frac{1}{3}$ |
| $(1,2)$ | $d$ | $\frac{2}{3} - d$ | $\frac{1}{3}$ |
| $(2,0)$ | $\frac{1}{3}$ | $\frac{1}{3}$ | $\frac{1}{3}$ |
| $(2,1)$ | $\frac{2}{3} - d$ | $d$ | $\frac{1}{3}$ |
| $(2,2)$ | $d$ | $\frac{2}{3} - d$ | $\frac{1}{3}$ |

### 2.4.3 Case Study

In fact, the capacity result in Theorem 2.16 suggests a way to use different state-dependent one-way channels to optimize bi-directional transmission rates. In what follows, we illustrate all possible shapes of the capacity region via examples and discuss the optimal transmission strategy behind each result. Letting $(r_1, r_2, s_1, s_2) = (3, 3, 3, 3)$, we consider the generalized PTT-TWC with the parameterized marginal transition matrices as shown in Table 2.2 and the following settings:

Setting 1: $(a, b, c, d) = (0, 0.15, 0, 0.15) \Rightarrow$

$$C_{j,0} = 0.6667 > C_{j,x_k} = 0.1539$$

for $j, k = 1, 2$ with $j \neq k$ and all $x_k \neq 0$;

Setting 2: $(a, b, c, d) = (0, 0.05, 0, 0.01) \Rightarrow$

$$C_{1,0} = 0.6667 > C_{1,x_2} = 0.4105$$

$$C_{2,0} = 0.6667 > C_{2,x_1} = 0.5918$$

for all $x_1 \neq 0$ and $x_2 \neq 0$;

Setting 3: $(a, b, c, d) = (0.1, 0, 0, 0.01) \Rightarrow$

$$C_{1,0} = 0.2601 < C_{1,x_2} = 0.6667$$

$$C_{2,0} = 0.6667 > C_{2,x_1} = 0.5918$$

for all $x_1 \neq 0$ and $x_2 \neq 0$;

Setting 4: $(a, b, c, d) = (0.1, 0, 0.2, 0.05) \Rightarrow$

$$C_{1,0} = 0.2601 < C_{1,x_2} = 0.6667$$

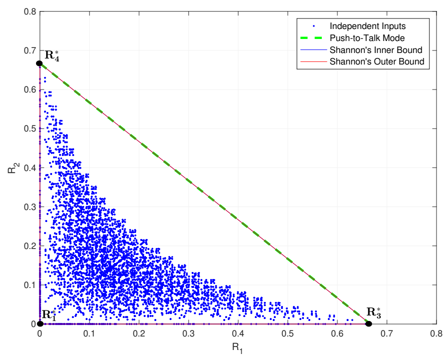
$$C_{2,0} = 0.0791 < C_{2,x_1} = 0.4105$$

for all $x_1 \neq 0$ and $x_2 \neq 0$.

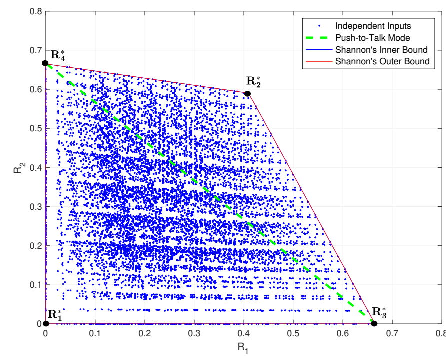
Note that, unlike for Shannon's original PTT-TWC, reliable full-duplex transmission is possible in the above settings since $C_{j,x_{j'}} > 0$ for all $j, j' = 1, 2$ and $x_{j'} \in \mathcal{X}_{j'}$.

In Figures 2.4(a)-2.4(d) (corresponding to Settings 1–4, respectively), the blue dots[7] are the achievable rate pairs via independent inputs of the form: $P_{X_1,X_2} = P_{X_1}P_{X_2}$; Shannon's inner bound region $\mathcal{C}_\text{I}$ is then given by taking the convex hull of those rate pairs. Shannon's outer bound $\mathcal{C}_\text{O}$ is obtained using a similar method, but the convex hull operation is not needed. We also depict the achievable rate region using the half-duplex transmission mode (via input symbol '0'). In all settings, we have that $\mathcal{C}_\text{I} = \mathcal{C}_\text{O}$ as expected.

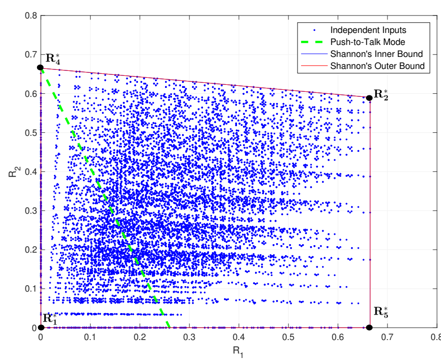
[7]In our computations, we discretized the standard 2-dimensional simplex to generate the input distributions for each terminal. The mutual information $I(X_j; Y_{j'}|X_{j'})$ is then evaluated under the product of the discretized input distributions. A similar approach is used to obtain rate pairs in Shannon's outer bound region.



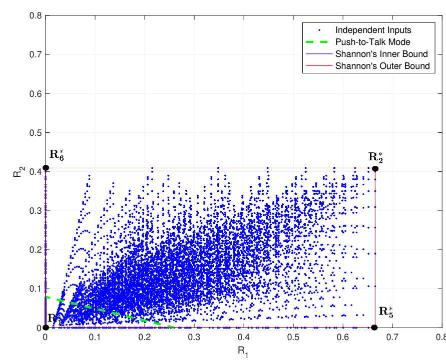

(a) Setting 1 (unused rate pairs: $\boldsymbol{R}_2^* = (0.1539, 0.1539)$, $\boldsymbol{R}_5^* = (0.1539, 0)$, and $\boldsymbol{R}_6^* = (0, 0.1539)$)




(b) Setting 2 (unused rate pairs: $\boldsymbol{R}_5^* = (0.4105, 0)$ and $\boldsymbol{R}_6^* = (0, 0.5918)$)




(c) Setting 3 (unused rate pairs: $\boldsymbol{R}_3^* = (0.2601, 0)$ and $\boldsymbol{R}_6^* = (0, 0.5918)$)




(d) Setting 4 (unused rate pairs: $\boldsymbol{R}_3^* = (0.2601, 0)$ and $\boldsymbol{R}_4^* = (0, 0.0791)$)

Figure 2.4: The capacity region of the generalized DM-PTT-TWCs in Table II. Except for Setting 1, the capacity region is determined by four rate pairs.

In Figure 2.4(a), we first observe that the half-duplex transmission can attain the entire capacity region. Indeed, although full-duplex transmission is reliable, the large difference between $C_{j,0}$ and $C_{j,x_{j'}}$ (for $x_{j'} \neq 0$) limits the rates achievable via two-way simultaneous transmission and hence the half-duplex transmission is still optimal (in the sense of achieving capacity). Nevertheless, the benefit of full-duplex transmission can be made significant by increasing the value of $C_{j,x_{j'}}$ for $x_{j'} \neq 0$. In Figure 2.4(b), we illustrate a situation where two-way simultaneous transmission achieves better rate pairs than using the half-duplex transmission.

Moreover, when the $C_{j,x_{j'}}$'s ($x_{j'} \neq 0$) are much larger than $C_{j,0}$, using $[P_{Y_2|X_1,X_2}(\cdot\,|\,\cdot\,,0)]$ and $[P_{Y_1|X_1,X_2}(\cdot\,|0,\,\cdot)]$ for information transmission becomes inefficient since they contribute very little to the overall transmission rates in (2.31) and (2.34). In this case, one should expect to abandon the (relatively) inefficient channels and use only the efficient ones. This is illustrated in Figs. 2.4(c) and 2.4(d). In an extreme case, such as Setting 4, the upper-right corner point of the capacity region is given by $\boldsymbol{R}_2^* = (0.6667, 0.4105) = (C_{1,1}, C_{2,1})$, implying that both terminals shut down the state-dependent one-way channels $[P_{Y_2|X_1,X_2}(\cdot\,|\,\cdot\,,0)]$ and $[P_{Y_1|X_1,X_2}(\cdot\,|0,\,\cdot)]$ and only use the remaining channels for information exchange.

## 2.5 A Non-trivial Outer Bound and Approximation Capacity Results

For DM-TWCs that are not symmetric in the sense of Theorems 2.6 or 2.8, calculating outer bounds to assess the capacity region is almost inevitable. The discussion in Section 2.3.3 highlighted the high computational demands for obtaining $\mathcal{C}_{\mathrm{O}}$. When using refined outer bounds such as in [29], the problem becomes even more complex due to the use of auxiliary random variables. In this section, we derive an easy-to-compute but non-trivial outer bound. Also, we give examples where our bound (together with $\mathcal{C}_{\mathrm{I}}$) results in a good estimate of $\mathcal{C}$. We note that our goal here is not to improve on any outer bound results; instead, we show that the computation of $\mathcal{C}_{\mathrm{I}}$ can in itself produce a useful outer bound.

### 2.5.1 A Relaxation of Channel Symmetry Properties

Our result is inspired by the proof of Theorem 2.6. To derive our simple outer bound, we relax the symmetry conditions of Theorem 2.6 as follows. Without loss of generality, the definitions are given for a specific direction of transmission.

**Definition 2.5.** *Given the set of one-way channels* $\{P_{Y_2|X_1,X_2}(\cdot|\cdot,x_2) : x_2 \in \mathcal{X}_2\}$, *let*

$$\alpha^* = \min_{\tilde{P}_{X_1}\in\mathcal{P}(\mathcal{X}_1)} \max_{x_2\in\mathcal{X}_2} \left|\mathcal{I}(\tilde{P}_{X_1}, P_{Y_2|X_1,X_2=x_2}) - \max_{P_{X_1}\in\mathcal{P}(\mathcal{X}_1)} \mathcal{I}(P_{X_1}, P_{Y_2|X_1,X_2=x_2})\right|. \quad (2.35)$$

*Such a collection of channels is said to have an* $\alpha^*$*-close common optimal input distribution.*

**Remark 2.8.** When $|\mathcal{X}_1| = 2$, we have that $\alpha^* \leq 0.011$ [99] for *any* finite collection of memoryless one-way channels under the uniform input: $\tilde{P}_{X_1}(0) = \tilde{P}_{X_1}(1) = 1/2$.

**Definition 2.6.** *Given the set of one-way channels* $\{P_{Y_1|X_1,X_2}(\cdot|x_1,\cdot) : x_1 \in \mathcal{X}_1\}$, *let*

$$\beta^* = \max_{P_{X_2}\in\mathcal{P}(\mathcal{X}_2)} \max_{\substack{x_1,x_1'\in\mathcal{X}_1\\ x_1\neq x_1'}} \left|\mathcal{I}(P_{X_2}, P_{Y_1|X_1=x_1,X_2}) - \mathcal{I}(P_{X_2}, P_{Y_2|X_1=x_1',X_2})\right|. \quad (2.36)$$

*Such a collection of channels is said to be* $\beta^*$*-invariant in the input-output mutual information.*

Based on Definitions 2.5-2.6, we obtain the following lemma, which will be used to obtain our capacity outer bound result.

**Lemma 2.1.** *For any DM-TWC and any achievable rate pair* $(R_1, R_2)$, *there exists a rate pair* $(R_1', R_2')$ *in* $\mathcal{C}_{\mathrm{I}}$ *such that* $R_1 \leq R_1' + \alpha^*$ *and* $R_2 \leq R_2' + 2\beta^*$.

*Proof:* Given any $P^{(1)}_{X_1,X_2} = P^{(1)}_{X_2}P^{(1)}_{X_1|X_2}$, let $P^{(2)}_{X_1,X_2} \triangleq P^{\alpha^*}_{X_1}P^{(1)}_{X_2}$, where $P^{\alpha^*}_{X_1}$ denotes the $\alpha^*$-close common optimal input distribution; i.e., $P^{\alpha^*}_{X_1}$ attains the $\alpha^*$ value in (2.35).

First, we have that

$$
\begin{aligned}
I^{(1)}(X_1;Y_2|X_2) &= \sum_{x_2} P_{X_2}^{(1)}(x_2)\cdot \mathcal{I}\big(P_{X_1|X_2=x_2}^{(1)}, P_{Y_2|X_1,X_2=x_2}\big)\\
&\le \sum_{x_2} P_{X_2}^{(1)}(x_2)\cdot \max_{P_{X_1|X_2=x_2}} \mathcal{I}\big(P_{X_1|X_2=x_2}, P_{Y_2|X_1,X_2=x_2}\big)\\
&\le \sum_{x_2} P_{X_2}^{(1)}(x_2)\cdot \Big(\mathcal{I}\big(P_{X_1}^{\alpha^*}, P_{Y_2|X_1,X_2=x_2}\big)+\alpha^*\Big)\\
&= \sum_{x_2} P_{X_2}^{(1)}(x_2)\cdot \mathcal{I}\big(P_{X_1}^{\alpha^*}, P_{Y_2|X_1,X_2=x_2}\big)+\alpha^*\\
&= I^{(2)}(X_1;Y_2|X_2)+\alpha^*,
\end{aligned}
$$

where the second inequality follows from Definition 2.5. Moreover,

$$
\begin{aligned}
I^{(1)}(X_2;Y_1|X_1) &= \sum_{x_1} P_{X_1}^{(1)}(x_1)\cdot \mathcal{I}\big(P_{X_2|X_1=x_1}^{(1)}, P_{Y_1|X_1=x_1,X_2}\big)\\
&\le \sum_{x_1} P_{X_1}^{(1)}(x_1)\cdot \left(\mathcal{I}\left(P_{X_2|X_1=x_1}^{(1)}, P_{Y_1|X_1=x_1',X_2}\right)+\beta^*\right)\\
&\le \mathcal{I}\big(P_{X_2}^{(1)}, P_{Y_1|X_1=x_1',X_2}\big)+\beta^*\\
&= \left(\sum_{x_1} P_{X_1}^{\alpha^*}(x_1)\right)\cdot \mathcal{I}\big(P_{X_2}^{(1)}, P_{Y_1|X_1=x_1',X_2}\big)+\beta^*\\
&\le \sum_{x_1} P_{X_1}^{\alpha^*}(x_1)\cdot \Big(\mathcal{I}\big(P_{X_2}^{(1)}, P_{Y_1|X_1=x_1,X_2}\big)+\beta^*\Big)+\beta^*\\
&= I^{(2)}(X_2;Y_1|X_1)+2\beta^*,
\end{aligned}
$$

where the first and the last inequalities are due to Definition 2.6 while the second inequality holds since $\mathcal{I}(\,\cdot\,,\,\cdot\,)$ is concave in the first argument. The claim is proved by noting that $(R_1', R_2') = (I^{(2)}(X_1;Y_2|X_2), I^{(2)}(X_2;Y_1|X_1)) \in \mathcal{C}_{\mathrm{I}}$. ■

**Theorem 2.17.** *For any DM-TWC,*

$$
\mathcal{C} \subseteq \overline{\mathrm{co}}\left(\bigcup_{(R_1',R_2')\in\mathcal{C}_{\mathrm{I}}} \{(R_1',R_2')\}\cup\{(R_1'+\alpha^*, R_2'+2\beta^*)\}\right)\cap \tilde{\mathcal{C}}_{\mathrm{O}},
$$

*where* $\tilde{\mathcal{C}}_{\mathrm{O}} \triangleq \{(R_1, R_2) : 0 \le R_1 \le I_1^*, 0 \le R_2 \le I_2^*\}$ *and*

$$I_j^* \triangleq \max_{x_{j'} \in \mathcal{X}_{j'}} \max_{P_{X_j} \in \mathcal{P}(\mathcal{X}_j)} I(X_j; Y_{j'} | X_{j'} = x_{j'}),$$

*for* $j \neq j'$*.*

The proof of Theorem 2.17 is omitted since it is straightforward given Lemma 2.1. We emphasize that $\alpha^*$ merely depends on the marginal input distributions on $\mathcal{X}_1$ (i.e., $P_{X_1}$ and $\tilde{P}_{X_1}$) and the marginal channel distribution $P_{Y_2|X_1,X_2}$. More importantly, the mutual information quantities in (2.35) are already found when computing $\mathcal{C}_{\mathrm{I}}$. One can thus efficiently obtain $\alpha^*$ within the framework of Shannon's inner bound computation; the same holds for $\beta^*$. As a result, forming this outer bound only requires subtraction and comparison operations.

Moreover, our outer bound coincides with $\mathcal{C}_{\mathrm{O}}$ when $\alpha^* = \beta^* = 0$, which is exactly equal to $\mathcal{C}_{\mathrm{I}}$ as can be deduced from the proof of Lemma 2.1 (and hence recovers the result in Theorem 2.6). For other cases, the values $\alpha^*$ and $2\beta^*$ roughly indicate how much our outer bound deviates from $\mathcal{C}_{\mathrm{I}}$ in the $R_1$ and $R_2$ axis, respectively. Using this fact, we can establish approximation capacity results for small deviations.

**Definition 2.7.** *For* $\epsilon \ge 0$*, the* $\epsilon$*-neighborhood* $\mathcal{C}_{\mathrm{I},\epsilon}$ *of Shannon's inner bound is defined as* $\mathcal{C}_{\mathrm{I},\epsilon} \triangleq \{(R_1, R_2) \in [0, I_1^*] \times [0, I_2^*] : \max\left(\frac{|R_1 - R_1'|}{I_1^*}, \frac{|R_2 - R_2'|}{I_2^*}\right) \le \epsilon$ *for some* $(R_1', R_2') \in \mathcal{C}_{\mathrm{I}}\}$*. If* $\mathcal{C}_{\mathrm{O}} \subseteq \mathcal{C}_{\mathrm{I},\epsilon}$*, then* $\mathcal{C}_{\mathrm{I},\epsilon}$ *is called an* $\epsilon$*-approximated capacity region.*

Combining this definition with Lemma 2.1, our outer bound is an $\epsilon$-approximated capacity region with $\epsilon = \max\left(\frac{\alpha^*}{I_1^*}, \frac{2\beta^*}{I_2^*}\right)$. Thus, a smaller value of $\epsilon$ gives an approximation $\mathcal{C}_{\mathrm{I},\epsilon} \approx \mathcal{C}$ with higher accuracy.

Table 2.3: Marginal channel transition matrices of Example 2.8

(a) $P_{Y_1|X_1,X_2}$

| $P_{Y_1\|X_1,X_2}$ | 0 | 1 | 2 |
|---|---|---|---|
| (0, 0) | 0.8 | 0.1 | 0.1 |
| (0, 1) | 0.1 | 0.8 | 0.1 |
| (0, 2) | 0.1 | 0.1 | 0.8 |
| (1, 0) | 0.8 | 0.1 | 0.1 |
| (1, 1) | 0.1 | 0.8 | 0.1 |
| (1, 2) | 0.1 | 0.1 | 0.8 |

(b) $P_{Y_2|X_1,X_2}$

| $P_{Y_2\|X_1,X_2}$ | 0 | 1 |
|---|---|---|
| (0, 0) | 0.7 | 0.3 |
| (1, 0) | 0.1 | 0.9 |
| (0, 1) | 1 | 0 |
| (1, 1) | 0.25 | 0.75 |
| (0, 2) | 0.5 | 0.5 |
| (1, 2) | 0 | 1 |

### 2.5.2 Examples

To end this section, we illustrate $\mathcal{C}_{\mathrm{I},\epsilon}$ via two examples. Example 2.8 also illustrates Remark 2.8; Example 2.9 shows a general interplay between $\mathcal{C}_{\mathrm{I},\epsilon}$ and the underlying channel parameters.

**Example 2.8.** Consider a DM-TWC with marginal channels given in Table 2.3. The channel $P_{Y_1|X_1,X_2}$ consists of two ternary sub-channels that satisfy condition (ii) of Theorem 2.6. The channel $P_{Y_2|X_1,X_2}$ is chosen to violate condition (i) of Theorem 2.6, which includes one Z-type, one inverse Z-type, and one pure asymmetric binary sub-channel; these sub-channels favor different input distributions. Based on the numerical computation for $\mathcal{C}_\mathrm{I}$, we obtain that $I_1^* = 0.5582$, $I_2^* = 0.6603$, $\alpha^* = 0.0102$, $\beta^* = 0$, and an outer bound $\mathcal{C}_{\mathrm{I},\epsilon}$ with $\epsilon = 0.0183$. As shown in Fig. 2.5, the region $\mathcal{C}_\mathrm{I}$ and $\mathcal{C}_{\mathrm{I},\epsilon}$ are quite close to each other, thus providing a good estimation to $\mathcal{C}$. Also, for any fixed $R_2$, the rate loss of $R_1$ is less than 0.011 (bits per channel use) when terminal 1 always uses the uniform inputs.

**Example 2.9.** Consider the DM-TWC with the marginal channels given in Table 2.4. The marginal channel $P_{Y_1|X_1,X_2}$ does not satisfy condition (ii) of Theorem 2.6, and its sub-channels are given by perturbing a BSC with crossover probability 0.04. To

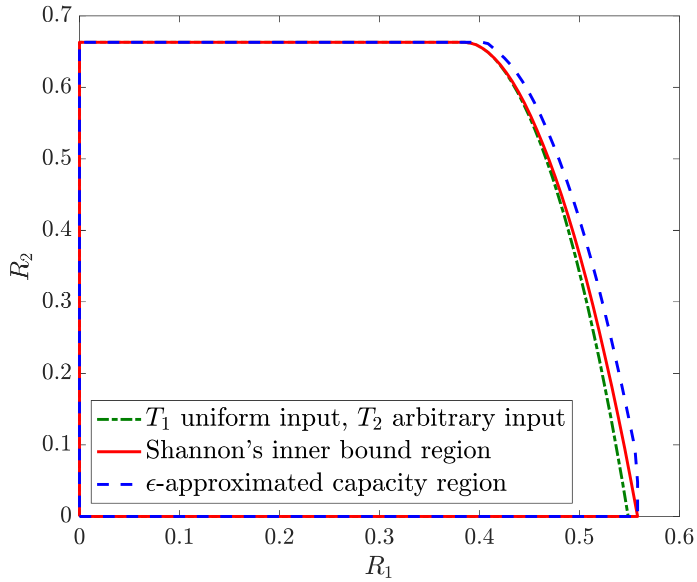



Figure 2.5: Numerical results for Example 2.8, where $\epsilon = 0.0183$.

Table 2.4: Marginal channel transition matrices of Example 2.9

(a) $P_{Y_1|X_1,X_2}$

| $P_{Y_1\|X_1,X_2}$ | 0 | 1 |
|---|---|---|
| $(0,0)$ | 0.96 | 0.04 |
| $(0,1)$ | 0.04 | 0.96 |
| $(1,0)$ | 0.961 | 0.039 |
| $(1,1)$ | 0.041 | 0.959 |
| $(2,0)$ | 0.96 | 0.04 |
| $(2,1)$ | 0.041 | 0.959 |

(b) $P_{Y_2|X_1,X_2}$

| $P_{Y_2\|X_1,X_2}$ | 0 | 1 | 2 |
|---|---|---|---|
| $(0,0)$ | 1 | 0 | 0 |
| $(1,0)$ | 0 | 1 | 0 |
| $(2,0)$ | 0.5 | 0.5 | 0 |
| $(0,1)$ | 0 | 0.1 | 0.9 |
| $(1,1)$ | 0.2 | $\gamma$ | $0.8-\gamma$ |
| $(2,1)$ | 0.2 | $0.8-\gamma$ | $\gamma$ |

Table 2.5: The values of $\alpha^*$ and $\epsilon$ under different settings of $\gamma$ in Example 2.9

| $\gamma$ | 0.1 | 0.15 | 0.2 | 0.25 | 0.3 | 0.35 | 0.375 | 0.4 |
|---|---|---|---|---|---|---|---|---|
| $\alpha^*$ | 0.1911 | 0.1808 | 0.1641 | 0.1398 | 0.1063 | 0.0608 | 0.0013 | 0.001 |
| $\epsilon$ | 0.1911 | 0.1808 | 0.1641 | 0.1398 | 0.1063 | 0.0608 | 0.0325 | 0.0066 |

demonstrate how $\mathcal{C}_{\mathrm{I},\epsilon}$ generally approximates $\mathcal{C}$, we also consider non-standard sub-channels for $P_{Y_2|X_1,X_2}$ parameterized by $\gamma \in [0, 0.8]$. Note that when $\gamma$ increases from 0.1 to 0.4, the sub-channel with transition matrix $[P_{Y_2|X_1,X_2}(\cdot\,|\,\cdot, 1)]$ becomes less

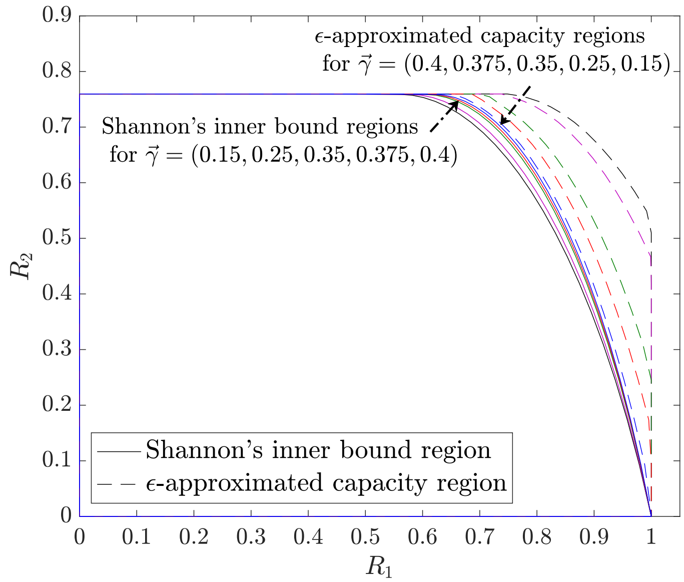



Figure 2.6: Numerical results for Example 2.9. The five inner regions are the $\mathcal{C}_{\mathrm{I}}$'s corresponding to the channel parameters $\gamma = 0.15$, 0.25, 0.35, 0.375, and 0.4, respectively, while the five outer regions are the corresponding $\mathcal{C}_{\mathrm{I},\epsilon}$'s (in reverse order).

noisy and the overall marginal channel $P_{Y_2|X_1,X_2}$ tends to be symmetric in the sense of condition (a). For this setup, we have that $\beta = 0.0025$, $I_1^* = 1$, and $I_2^* = 0.7577$. In Table 2.5, we list the values of $\alpha^*$ and $\epsilon$ for different values of $\gamma \in [0, 0.4]$. We also depict $\mathcal{C}_{\mathrm{I}}$ and $\mathcal{C}_{\mathrm{I},\epsilon}$ for selected values of $\gamma$ in Fig. 2.6. It is observed that when $\epsilon < 0.05$, our simple outer bound and $\mathcal{C}_{\mathrm{I}}$ determine the capacity region $\mathcal{C}$ with large accuracy. In other cases, our outer bound is still non-trivial.

To end this chapter, we present a DM-TWC whose capacity region can be exactly determined but the channel does not satisfy any of the tightness conditions in this thesis. The channel has the marginal transition matrices as shown in Table 2.6, and its capacity region is identical to the triangular capacity region of Shannon's PTT channels. This simple example, though artificial, reveals certain limitations of the

Table 2.6: The marginal transition matrices of the DM-TWC whose capacity is know but does not satisfy any tightness conditions.

(a) $P_{Y_2|X_1,X_2}$

| $(X_1, X_2)$ | 0 | 1 |
|---|---|---|
| $(0,0)$ | $\frac{1}{2}$ | $\frac{1}{2}$ |
| $(1,0)$ | $\frac{1}{2}$ | $\frac{1}{2}$ |
| $(2,0)$ | 1 | 0 |
| $(3,0)$ | 0 | 1 |
| $(0,1)$ | $\frac{1}{2}$ | $\frac{1}{2}$ |
| $(1,1)$ | $\frac{1}{2}$ | $\frac{1}{2}$ |
| $(2,1)$ | $\frac{1}{2}$ | $\frac{1}{2}$ |
| $(3,1)$ | $\frac{1}{2}$ | $\frac{1}{2}$ |
| $(0,2)$ | $\frac{1}{2}$ | $\frac{1}{2}$ |
| $(1,2)$ | $\frac{1}{2}$ | $\frac{1}{2}$ |
| $(2,2)$ | $\frac{1}{2}$ | $\frac{1}{2}$ |
| $(3,2)$ | $\frac{1}{2}$ | $\frac{1}{2}$ |
| $(0,3)$ | $\frac{1}{2}$ | $\frac{1}{2}$ |
| $(1,3)$ | $\frac{1}{2}$ | $\frac{1}{2}$ |
| $(2,3)$ | $\frac{1}{2}$ | $\frac{1}{2}$ |
| $(3,3)$ | $\frac{1}{2}$ | $\frac{1}{2}$ |

(b) $P_{Y_1|X_1,X_2}$

| $(X_1, X_2)$ | 0 | 1 |
|---|---|---|
| $(0,0)$ | $\frac{1}{2}$ | $\frac{1}{2}$ |
| $(0,1)$ | $\frac{1}{2}$ | $\frac{1}{2}$ |
| $(0,2)$ | 1 | 0 |
| $(0,3)$ | 0 | 1 |
| $(1,0)$ | $\frac{1}{2}$ | $\frac{1}{2}$ |
| $(1,1)$ | $\frac{1}{2}$ | $\frac{1}{2}$ |
| $(1,2)$ | $\frac{1}{2}$ | $\frac{1}{2}$ |
| $(1,3)$ | $\frac{1}{2}$ | $\frac{1}{2}$ |
| $(2,0)$ | $\frac{1}{2}$ | $\frac{1}{2}$ |
| $(2,1)$ | $\frac{1}{2}$ | $\frac{1}{2}$ |
| $(2,2)$ | $\frac{1}{2}$ | $\frac{1}{2}$ |
| $(2,3)$ | $\frac{1}{2}$ | $\frac{1}{2}$ |
| $(3,0)$ | $\frac{1}{2}$ | $\frac{1}{2}$ |
| $(3,1)$ | $\frac{1}{2}$ | $\frac{1}{2}$ |
| $(3,2)$ | $\frac{1}{2}$ | $\frac{1}{2}$ |
| $(3,3)$ | $\frac{1}{2}$ | $\frac{1}{2}$ |

current results and hence further studies are needed.

We also remark that for common-output TWCs, i.e., where both terminals receive the same channel outputs at each time instant, one can view the channels as a combination of two one-way channels with feedback. Although the common output is not exactly the same as the perfect or noisy feedback studied in one-way systems such as [100–102], one can possibly obtain capacity bounds for one system from the other system by identifying their relationship. Moreover, by viewing a DM-TWC as two multiple access channels with asymmetric side-information, the result in [103] could be useful to develop new capacity inner and outer bounds for TWCs. We leave these two directions for future research.

# Chapter 3

# Two-Terminal Two-Way Channels with Memory

Two-terminal TWCs with memory are similar to their memoryless counterparts, except that now their associated noise process can be correlated in time. There are many ways to model TWCs with memory, but a very general model is difficult to analyze. Shannon's original work considered a two-way channel with a recoverable state property [3, Section 16], where a limiting expression of the capacity region was given. As another attempt to tackle TWCs with memory, we restrict the focus to a simple model (Section 3.1) whose the channel inputs and outputs and noises are related by some deterministic functions. For such a channel model, we first derive inner and outer capacity bounds using standard techniques in Sections 3.2 and 3.3, respectively. Tightness conditions for the inner bound are also established in Section 3.4 for two special cases of this channel model, showing that their capacity can be achieved without the use of adaptive coding. A summary of these results is given in Fig. 3.1.

Furthermore, motivated by Example 3.1 where adaptive coding can achieve higher bi-directional transmission rates, we generalize that example and propose adaptive coding schemes that achieve capacity for a class of additive-noise TWCs with memory

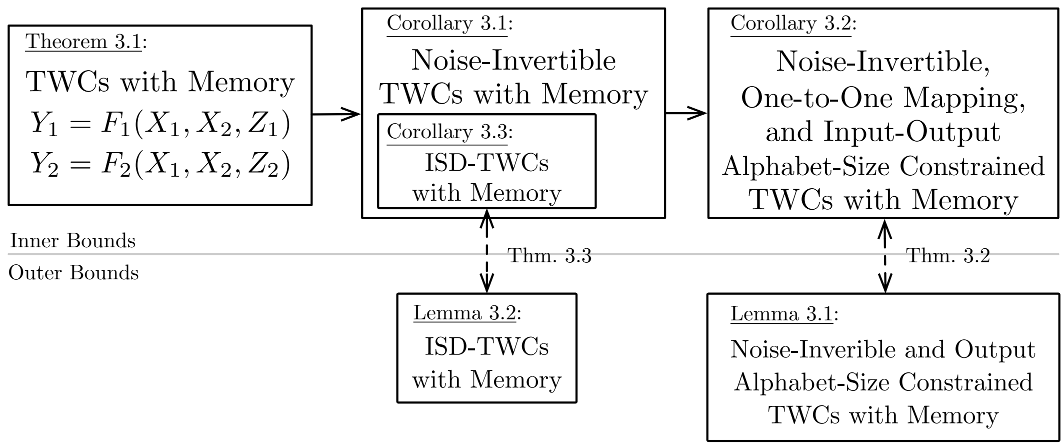



Figure 3.1: The relationships between the results for two-terminal TWCs with memory. Here, $A \overset{\text{Thm.C}}{\longleftrightarrow} B$ indicates that results $A$ and $B$ are combined in Theorem $C$ to determine the capacity region.

in Section 3.5. More specifically, given a finite impulse response noise process, we incorporate adaptive coding and superposition coding to cancel interference during transmissions.

## 3.1 System Model

Throughout this section, we consider the following two-terminal TWC with memory whose inputs and outputs are related via functions $F_1$ and $F_2$:

$$Y_{1,n} = F_1(X_{1,n}, X_{2,n}, Z_{1,n}), \tag{3.1a}$$

$$Y_{2,n} = F_2(X_{1,n}, X_{2,n}, Z_{2,n}), \tag{3.1b}$$

where $\{(Z_{1,n}, Z_{2,n})\}_{n=1}^{\infty}$ is a two-dimensional noise process which is independent of the terminals' messages $M_1$ and $M_2$. Note that this model is a special case of the general model introduced in Section 2.1.1; it is also a generalization of the discrete additive-noise TWC considered in [40].

## 3.2 Achievability Results

In this section, we assume that the individual noise processes $\{Z_{1,n}\}_{n=1}^{\infty}$ and $\{Z_{2,n}\}_{n=1}^{\infty}$ are stationary and ergodic. Note that we allow $\{Z_{1,n}\}_{n=1}^{\infty}$ and $\{Z_{2,n}\}_{n=1}^{\infty}$ to be dependent. For simplicity, we will omit to mention this assumption in our theorem statements throughout this section. We first state (without proof) an inner bound for arbitrary (time-invariant) functions $F_1$ and $F_2$. The bound can be proved via Shannon's standard random coding scheme (under non-adaptive independent inputs) for information stable one-way channels with memory [104], applied in each direction of the two-way transmission.

**Theorem 3.1** (**Inner Bound**). *For the channel described in* (3.1)*, a rate pair* $(R_1, R_2)$ *is achievable if*

$$R_1 \leq \lim_{N\to\infty} \frac{1}{N} I(X_1^N; Y_2^N | X_2^N),$$
$$R_2 \leq \lim_{N\to\infty} \frac{1}{N} I(X_2^N; Y_1^N | X_1^N),$$

*for some sequence of product input probability distributions* $\left\{P_{X_1^N} P_{X_2^N}\right\}_{N=1}^{\infty}$ *and the inputs* $X_j^N$ *are independent of* $\{(Z_{1,n}\}_{n=1}^{\infty}$ *and* $\{Z_{2,n})\}_{n=1}^{\infty}$, $j = 1, 2$.

We say that $F_j(X_1, X_2, Z_j)$ is invertible in $Z_j$ if $F_j(x_1, x_2, \cdot)$ is one-to-one and onto for any fixed $(x_1, x_2) \in \mathcal{X}_1 \times \mathcal{X}_2$. Under this invertibility condition, we obtain the following corollary.

**Corollary 3.1.** *Suppose that* $F_j$ *is invertible in* $Z_j$ *for* $j = 1, 2$. *A rate pair* $(R_1, R_2)$ *is achievable if*

$$R_1 \leq \lim_{N\to\infty} \frac{1}{N} H(Y_2^N | X_2^N) - \bar{H}(Z_2), \tag{3.2a}$$

$$R_2 \leq \lim_{N\to\infty} \frac{1}{N} H(Y_1^N|X_1^N) - \bar{H}(Z_1), \tag{3.2b}$$

*for some sequence of product input distributions* $\{P_{X_1^N} P_{X_2^N}\}_{N=1}^{\infty}$*, where* $\bar{H}(Z_j)$ *denotes the entropy rate of the noise process* $\{Z_{j,n}\}_{n=1}^{\infty}$ *and channel inputs are independent of* $\{(Z_{1,n}, Z_{2,n})\}_{n=1}^{\infty}$.

*Proof:* The proof follows from the fact that

$$\begin{aligned} I(X_1^N; Y_2^N|X_2^N) &= H(Y_2^N|X_2^N) - H(Y_2^N|X_1^N, X_2^N) \\ &= H(Y_2^N|X_2^N) - H(Z_2^N|X_1^N, X_2^N) \\ &= H(Y_2^N|X_2^N) - H(Z_2^N), \end{aligned}$$

where the second equality holds since $F_2$ is invertible in $Z_2$ and the last equality holds since the channel inputs are generated independently of the noise process $\{(Z_{2,n}, Z_{2,n})\}_{n=1}^{\infty}$. Applying a similar argument to $I(X_1^N; Y_2^N|X_2^N)$ completes the proof. ■

Let $F_j^{-1}$ denote the inverse of $F_j$ for fixed $(x_1, x_2) \in \mathcal{X}_1 \times \mathcal{X}_2$ so that $z_j = F_j^{-1}(x_1, x_2, y_j)$, $j = 1, 2$. If we further assume that $z_j = F_j^{-1}(x_1, x_2, y_j)$ is one-to-one in $x_{j'}$ for any fixed $(x_j, y_j) \in \mathcal{X}_j \times \mathcal{Y}_j$, where $j, j' = 1, 2$ with $j \neq j'$, and impose cardinality constraints on the alphabets, then the expressions in (3.2a) and (3.2b) can be simplified as follows.

**Corollary 3.2.** *Suppose that* $F_j$ *is invertible in* $Z_j$ *and* $F_j^{-1}$ *is one-to-one in* $X_{j'}$ *for* $j, j' = 1, 2$ *with* $j \neq j'$*. Also, assume that* $|\mathcal{X}_2| = |\mathcal{Y}_1| = |\mathcal{Z}_1| = q_1$ *and* $|\mathcal{X}_1| = |\mathcal{Y}_2| = |\mathcal{Z}_2| = q_2$ *for some integers* $q_1, q_2 \geq 2$*. Then, a rate pair* $(R_1, R_2)$ *is achievable if*

$$R_1 \leq \log q_2 - \bar{H}(Z_2),$$

$$R_2 \le \log q_1 - \bar{H}(Z_1).$$

*Proof:* The proof hinges on noting that $H(Y_j^N|X_j^N) \le N \log q_j$ and that the uniform input distribution $P_{X_1^N,X_2^N} = (P^{\mathrm{U}}_{\mathcal{X}_1} P^{\mathrm{U}}_{\mathcal{X}_2})^N$ achieves the upper bound. More specifically, we have to show that if $P_{X_1^n,X_2^n}$ is the uniform distribution, then $P_{Y_j^N|X_j^N}(y_j^N|x_j^N)$ is uniform on $\mathcal{Y}_j^N$ for any given $X_j^N = x_j^N$, and hence $H(Y_j^N|X_j^N = x_j^N) = N \log q_j$. By symmetry, we only provide the details for $H(Y_2^N|X_2^N)$. Suppose that $P_{X_1^N,X_2^N}$ is the uniform distribution on $\mathcal{X}_1^N \times \mathcal{X}_2^N$. Then, for any $x_2^N$ we have

$$\begin{aligned}
P_{Y_2^N|X_2^N}(y_2^N|x_2^N) &= \sum_{x_1^N} P_{Y_2^N|X_1^N,X_2^N}(y_2^N|x_1^N,x_2^N) P_{X_1^N|X_2^N}(x_1^N|x_2^N) \\
&= \left(\frac{1}{q_2}\right)^N \sum_{x_1^N} P_{Y_2^N|X_1^N,X_2^N}(F_2(x_1^N,x_2^N,z_2^N)|x_1^N,x_2^N) \\
&= \left(\frac{1}{q_2}\right)^N \sum_{x_1^N} P_{Z_2^N|X_1^N,X_2^N}(F_2^{-1}(x_1^N,x_2^N,y_2^N)|x_1^N,x_2^N) \\
&= \left(\frac{1}{q_2}\right)^N \sum_{z_2^N} P_{Z_2^N}(z_2^N) \qquad (3.3) \\
&= \left(\frac{1}{q_2}\right)^N,
\end{aligned}$$

where (3.3) holds since $(X_1^N, X_2^N)$ is independent of $Z_2^N$ and $F_2^{-1}(X_1, X_2, Y_2)$ is onto in $X_1$ due to the cardinality constraint. Clearly, $P_{Y_2^N|X_2^N=x_2^N}$ is the uniform distribution for any $x_2^N$, implying that $H(Y_2^N|X_2^N) = N \log q_2$. ■

Next we consider ISD-TWCs as in Example 2.3 and [41], but with the assumption that the noise process $\{(Z_{1,n}, Z_{2,n})\}_{n=1}^{\infty}$ can have memory. Note that any ISD-TWC with memory is a special case of the system model in (3.1) satisfying the injectivity condition in $Z_1$ and $Z_2$. Thus, Corollary 3.1 applies to ISD-TWCs with memory to obtain the following result.

**Corollary 3.3.** *For the ISD-TWC with memory, a rate pair* $(R_1, R_2)$ *is achievable if*

$$R_1 \leq \lim_{N\to\infty} \max_{P_{X_1^N}} \frac{1}{N} H(\tilde{h}_2(X_1^N, Z_2^N)) - \bar{H}(Z_2),$$
$$R_2 \leq \lim_{N\to\infty} \max_{P_{X_2^N}} \frac{1}{N} H(\tilde{h}_1(X_2^N, Z_1^N)) - \bar{H}(Z_1),$$

*where* $\bar{H}(Z_j)$ *denotes the entropy rate of the process* $\{Z_{j,n}\}_{n=1}^{\infty}$ *for* $j = 1, 2$.

Note that the two limits above exist since the output process $\{\tilde{h}_j(X_{j'}^N, Z_j^N)\}$ is stationary and ergodic when the input process $\{X_{j'}^N\}$'s is stationary and ergodic. We also remark that Corollary 3.2 applies to ISD-TWCs with memory under identical alphabet size constraints so that any rate pair in $\{(R_1, R_2) : R_1 \leq \log q_2 - \bar{H}(Z_2), R_2 \leq \log q_1 - \bar{H}(Z_1)\}$ is achievable for ISD-TWCs with memory.

### 3.3 Converse Results

In this section, we derive converses to Corollaries 3.2 and 3.3 under the assumption that the two-dimensional noise process $\{(Z_{1,n}, Z_{2,n})\}_{n=1}^{\infty}$ is stationary.

**Lemma 3.1 (Outer Bound for Noise-Invertible TWCs with Memory).** *Suppose that* $|\mathcal{Y}_j| = q_j$ *for some integer* $q_j \geq 2$. *If* $F_j$ *is invertible in* $Z_j$ *for* $j = 1, 2$, *any achievable rate pair* $(R_1, R_2)$ *must satisfy*

$$R_1 \leq \log q_2 - \lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} H(Z_{2,n}|Z_1^{n-1}, Z_2^{n-1}),$$
$$R_2 \leq \log q_1 - \lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} H(Z_{1,n}|Z_1^{n-1}, Z_2^{n-1}),$$

*provided that the above two limits exist for the given noise process* $\{(Z_{1,n}, Z_{2,n})\}_{n=1}^{\infty}$.

*Proof:* For an achievable rate pair $(R_1, R_2)$, we have

$$\begin{aligned}
NR_1 &= H(M_1|M_2) \\
&= I(M_1; Y_2^N|M_2) + H(M_1|Y_2^N, M_2) \\
&\le I(M_1; Y_2^N|M_2) + N\epsilon_N && (3.4) \\
&= \sum_{n=1}^{N} \Big[ H(Y_{2,n}|M_2, Y_2^{n-1}) - H(Y_{2,n}|M_1, M_2, Y_2^{n-1}) \Big] + N\epsilon_N && (3.5) \\
&\le \sum_{n=1}^{N} \Big[ \log q_2 - H(Y_{2,n}|M_1, M_2, Y_2^{n-1}) \Big] + N\epsilon_N && (3.6) \\
&\le \sum_{n=1}^{N} \Big[ \log q_2 - H(Y_{2,n}|M_1, M_2, Y_1^{n-1}, Y_2^{n-1}, X_{1,n}, X_{2,n}) \Big] + N\epsilon_N \\
&= \sum_{n=1}^{N} \Big[ \log q_2 - H(Z_{2,n}|M_1, M_2, Y_1^{n-1}, Y_2^{n-1}, X_1^n, X_2^n) \Big] + N\epsilon_N && (3.7) \\
&= \sum_{n=1}^{N} \Big[ \log q_2 - H(Z_{2,n}|M_1, M_2, Y_1^{n-1}, Y_2^{n-1}, X_1^n, X_2^n, Z_1^{n-1}, Z_2^{n-1}) \Big] + N\epsilon_N && (3.8) \\
&= \sum_{n=1}^{N} \Big[ \log q_2 - H(Z_{2,n}|Z_1^{n-1}, Z_2^{n-1}) \Big] + N\epsilon_N && (3.9) \\
&= N \log q_2 - \sum_{n=1}^{N} H(Z_{2,n}|Z_1^{n-1}, Z_2^{n-1}) + N\epsilon_N, && (3.10)
\end{aligned}$$

where (3.4) is due to Fano's inequality with $\epsilon_N \to 0$ as $N \to \infty$, (3.6) follows from $|\mathcal{Y}_2| = q_2$, (3.7) and (3.8) hold since $F_j$ is invertible in $Z_j$ given $(X_{1,n}, X_{2,n})$, and (3.9) holds since

$$\begin{aligned}
H(Z_{2,n}|Z_1^{n-1}, Z_2^{n-1}) &= H(Z_{2,n}|M_1, M_2, Z_1^{n-1}, Z_2^{n-1}) && (3.11) \\
&= H(Z_{2,n}|M_1, M_2, Z_1^{n-1}, Z_2^{n-1}, X_{1,1}, X_{2,1}) && (3.12) \\
&= H(Z_{2,n}|M_1, M_2, Z_1^{n-1}, Z_2^{n-1}, X_{1,1}, X_{2,1}, Y_{1,1}, Y_{2,1}) && (3.13) \\
&= H(Z_{2,n}|M_1, M_2, Z_1^{n-1}, Z_2^{n-1}, X_1^2, X_2^2, Y_{1,1}, Y_{2,1}) && (3.14)
\end{aligned}$$

$$= H(Z_{2,n}|M_1, M_2, Z_1^{n-1}, Z_2^{n-1}, X_1^n, X_2^n, Y_1^{n-1}, Y_2^{n-1}) \quad (3.15)$$

where (3.11) is due to the fact that $\{(Z_{1,n}, Z_{2,n})\}_{n=1}^{\infty}$ is independent of $(M_1, M_2)$, (3.12) and (3.14) hold since $X_{j,n} = f_{j,n}(M_j, Y_j^{n-1})$ for $j = 1, 2$, (3.13) follows from the identity $Y_{j,n} = F_j(X_{1,n}, X_{2,n}, Z_{j,n})$, and (3.15) is obtained by recursively using the same argument as in (3.12)-(3.14). Similarly, we have

$$NR_2 \leq N \log q_1 - \sum_{n=1}^{N} H(Z_{1,n}|Z_1^{n-1}, Z_2^{n-1}) + N\hat{\epsilon}_N. \quad (3.16)$$

The proof is completed by dividing both sides of (3.10) and (3.16) by $N$ and letting $N \to \infty$. ■

**Lemma 3.2** (**Outer Bound for ISD-TWCs with Memory**). *For the ISD-TWCs with stationary noise process* $\{(Z_{1,n}, Z_{2,n})\}_{n=1}^{\infty}$*, any achievable rate pair* $(R_1, R_2)$ *must satisfy*

$$R_1 \leq \liminf_{N\to\infty} \frac{1}{N}\left( \max_{P_{X_1^N}} H(\tilde{h}_2(X_1^N, Z_2^N)) - \sum_{n=1}^{N} H(Z_{2,n}|Z_1^{n-1}, Z_2^{n-1}) \right),$$
$$R_2 \leq \liminf_{N\to\infty} \frac{1}{N}\left( \max_{P_{X_2^N}} H(\tilde{h}_1(X_2^N, Z_1^N)) - \sum_{n=1}^{N} H(Z_{1,n}|Z_1^{n-1}, Z_2^{n-1}) \right).$$

*Proof:* The proof is similar to the proof of the previous lemma and hence the details are omitted. The main difference is that the first term in (3.5) is now upper bounded as follows

$$\sum_{n=1}^{N} H(Y_{2,n}|M_2, Y_2^{n-1}) = \sum_{n=1}^{N} H(h_2(X_{2,n}, T_{2,n})|M_2, Y_2^{n-1}, X_{2,n}, T_2^{n-1})$$
$$= \sum_{n=1}^{N} \sum_{x_2} P_{X_{2,n}}(x_2) \cdot H(h_2(X_{2,n}, T_{2,n})|M_2, Y_2^{n-1}, X_{2,n} = x_2, T_2^{n-1})$$

$$
\begin{aligned}
&= \sum_{n=1}^{N} \sum_{x_2} P_{X_{2,n}}(x_2) \cdot H(T_{2,n}|M_2, Y_2^{n-1}, X_{2,n} = x_{2,n}, T_2^{n-1}) \\
&= \sum_{n=1}^{N} H(T_{2,n}|M_2, Y_2^{n-1}, X_{2,n}, T_2^{n-1}) \\
&\le \sum_{n=1}^{N} H(T_{2,n}|T_2^{n-1}) \\
&= H(T_2^N) \\
&\le \max_{P_{X_1^N}} H(\tilde{h}_2(X_1^N, Z_2^N)),
\end{aligned}
$$

where the first equality holds since $X_{2,n}$ is a function of $M_2$ and $Y_2^{n-1}$ and $Y_2 = h_2(X_2, T_2)$ is injective in $T_2$ given $X_2$.

In the last step, we take limit infimum in $N$ (instead of taking limit) since the limit of $\frac{1}{N} \max_{P_{X_j^N}} H(\tilde{h}_{j'}(X_j^N, Z_{j'}^N))$ may not exist. Nevertheless, since $\{(Z_{1,n}, Z_{2,n})\}_{n=1}^{\infty}$ is stationary, the limit $\lim_{N\to\infty} \sum_{n=1}^{N} \frac{1}{N} H(Z_{j,n}|Z_j^{n-1}, Z_{j'}^{n-1})$ always exists for $j = 1, 2$. ■

## 3.4 Tightness Conditions for Random Coding Inner Bound

Based on the preceding inner and outer bounds, the capacity region for two classes of TWCs with memory (whose component noise processes are independent of each other) can be exactly determined as follows.

**Theorem 3.2.** *For a TWC with memory such that* $\{Z_{1,n}\}_{n=1}^{\infty}$ *and* $\{Z_{2,n}\}_{n=1}^{\infty}$ *are stationary, ergodic and mutually independent,* $F_j$ *is invertible in* $Z_j$ *and* $F_j^{-1}$ *is one-to-one in* $X_{j'}$ *for* $j, j' = 1, 2$ *with* $j \neq j'$*, and* $|\mathcal{X}_2| = |\mathcal{Y}_1| = |\mathcal{Z}_1| = q_1$ *and* $|\mathcal{X}_1| = |\mathcal{Y}_2| = |\mathcal{Z}_2| = q_2$ *for some integers* $q_1, q_2 \ge 2$*, the capacity region is given by*

$$
\mathcal{C} = \Big\{(R_1, R_2) : R_1 \le \log q_2 - \bar{H}(Z_2), R_2 \le \log q_1 - \bar{H}(Z_1)\Big\}. \tag{3.17}
$$

**Theorem 3.3.** *For a ISD-TWC with memory such that $\{Z_{1,n}\}_{n=1}^{\infty}$ and $\{Z_{2,n}\}_{n=1}^{\infty}$ are stationary, ergodic, and mutually independent, if stationary and ergodic inputs attain the outer bound, then the capacity region is given by*

$$\mathcal{C} = \Big\{(R_1, R_2) : R_1 \le \lim_{N\to\infty} \frac{1}{N} \max_{P_{X_1^N}} H(\tilde{h}_2(X_1^N, Z_2^N)) - \bar{H}(Z_2),$$
$$R_2 \le \lim_{N\to\infty} \frac{1}{N} \max_{P_{X_2^N}} H(\tilde{h}_1(X_2^N, Z_1^N)) - \bar{H}(Z_1)\Big\}. \tag{3.18}$$

**Remark 3.1.** Theorem 3.3 generalizes [41, Corollary 1] for memoryless ISD-TWCs. If one further consider the settings: $|\mathcal{X}_2| = |\mathcal{T}_1| = |\mathcal{Z}_1| = q_1$ and $|\mathcal{X}_1| = |\mathcal{T}_2| = |\mathcal{Z}_2| = q_2$ for some integers $q_1, q_2 \ge 2$, then $\lim_{N\to\infty} \frac{1}{N} \max_{P_{X_1^N}} H(\tilde{h}_2(X_1^N, Z_2^N)) = \log q_1$ and $\lim_{N\to\infty} \frac{1}{N} \max_{P_{X_2^N}} H(\tilde{h}_1(X_2^N, Z_1^N)) = \log q_2$.

The next example shows that if the noise processes $\{Z_{1,n}\}_{n=1}^{\infty}$ and $\{Z_{2,n}\}_{n=1}^{\infty}$ are *dependent*, then Shannon's random coding scheme is not optimal. In the next section, we will explore a class of TWCs with memory where adaptive coding is necessary to achieve capacity.

**Example 3.1** (**Adaptation is Useful**)**.** Let $q_1 = q_2 = 2$ and suppose that the channel is given by

$$Y_{1,n} = F_1(X_{1,n}, X_{2,n}, Z_{1,n}) = X_{1,n} \oplus_2 X_{2,n} \oplus_2 Z_{1,n},$$
$$Y_{2,n} = F_2(X_{1,n}, X_{2,n}, Z_{2,n}) = X_{1,n} \oplus_2 X_{2,n} \oplus_2 Z_{2,n},$$

where $\{Z_{1,n}\}_{n=1}^{\infty}$ is assumed to be memoryless with $Z_{1,n}$ uniformly distributed on the alphabet $\{0, 1\}$ for all $n$, and $\{Z_{2,n}\}_{n=1}^{\infty}$ is given by $Z_{2,1} = 0$ and $Z_{2,n} = Z_{1,n-1}$ for $n \ge 2$. Since the functions $F_1$ and $F_2$ are invertible in $Z_1$ and $Z_2$, the outer bound in

Lemma 3.1 indicates that

$$R_1 \leq \log 2 - \lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} H(Z_{2,n}|Z_1^{n-1}, Z_2^{n-1}) = 1 - 0 = 1,$$

$$R_2 \leq \log 2 - \lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} H(Z_{1,n}|Z_1^{n-1}, Z_2^{n-1}) = 1 - H(Z_{1,n}) = 0.$$

We claim that the rate pair $(R_1, R_2) = (1, 0)$ can be achieved by an adaptive coding scheme. Let $M_{1,n} \in \{0, 1\}$ denote the binary raw message to be sent from terminals 1 to 2 at time $n$. For each time $n \geq 1$, set the encoder output of terminal 1 as

$$X_{1,n} = M_{1,n} \oplus_2 X_{1,n-1} \oplus_2 Y_{1,n-1}$$

with initial conditions $X_{1,0} = X_{2,0} = Y_{1,0} = 0$, and set the encoder output of terminal 2 to be zero, i.e., $X_{2,n} = 0$ for all $n$. With this coding scheme, the received signal at terminal 2 is given by

$$\begin{aligned} Y_{2,n} &= X_{1,n} \oplus_2 X_{2,n} \oplus_2 Z_{2,n} \\ &= M_{1,n} \oplus_2 X_{1,n-1} \oplus_2 Y_{1,n-1} \oplus_2 Z_{2,n} \\ &= M_{1,n} \oplus_2 X_{1,n-1} \oplus_2 X_{1,n-1} \oplus_2 Z_{1,n-1} \oplus_2 Z_{2,n} = M_{1,n}, \end{aligned}$$

and thus the rate pair $(1, 0)$ is achievable. This achievability result together with the outer bound imply that the channel capacity is given by $\mathcal{C} = \{(R_1, R_2) : 0 \leq R_1 \leq 1, R_2 = 0\}$. However, the Shannon-type random coding scheme only provides $0 \leq R_1 \leq 1 - \bar{H}(Z_2) = 0$ and $0 \leq R_2 \leq 1 - \bar{H}(Z_2) = 0$, i.e., $(R_1, R_2) = (0, 0)$, by Corollary 3.2.[1]

[1] Corollary 3.2 is applicable here with a slight modification of Shannon's random coding scheme. Roughly speaking, information transmission occurs after the first time slot.

### 3.5 Adaptive Coding for a Class of TWCs with Memory

In this section, we consider the following additive-noise TWC model:

$$Y_{1,n} = F_1(X_{1,n}, X_{2,n}, Z_{1,n}) = X_{1,n} \oplus_q X_{2,n} \oplus_q Z_{1,n},$$
$$Y_{2,n} = F_2(X_{1,n}, X_{2,n}, Z_{1,n}) = X_{1,n} \oplus_q X_{2,n} \oplus_q Z_{2,n}.$$

Here, all system variables take value in the finite field $\mathbb{F}_q = (\{0, 1, \ldots, q\}, \oplus_q, \odot_q)$ and the two-dimensional noise process $\{(Z_{1,n}, Z_{2,n})\}_{n=1}^{\infty}$ is given by $Z_{1,n} = \tilde{Z}_n$ and[2]

$$Z_{2,n} = \sum_{m=0}^{M} \beta_m \tilde{Z}_{n-m}$$

where $\{\tilde{Z}_n\}_{n=1}^{\infty}$ is a stationary and ergodic noise process, $M$ denotes the finite memory order of the noise process $\{Z_{2,n}\}_{n=1}^{\infty}$ generated by $\{\tilde{Z}_n\}_{n=1}^{\infty}$ at each time instant, and the coefficients $\beta_m$'s are fixed and assumed to be known to both terminals. For simplicity, we assume that $\tilde{Z}_n = 0$ for $n \leq 0$.

For this channel model, we further consider two cases: (i) $\beta_0 = 0$ and (ii) $\beta_0 \neq 0$ to illustrate two different adaptive coding schemes. We will first apply Corollary 3.2 and Lemma 3.1 to obtain a random coding inner bound and a capacity outer bound for each case, and then give the details of how adaptive coding achieves channel capacity.

Case (i): In this case, the random coding inner bound in Corollary 3.2 yields the achievable rate region $\{(R_1, R_2) : 0 \leq R_1 \leq \log q - \bar{H}(Z_2), 0 \leq R_2 \leq \log q - \bar{H}(Z_1)\}$,[3] while Lemma 3.1 results in the outer bound region $\{(R_1, R_2) : 0 \leq R_1 \leq \log q, 0 \leq R_2 \leq \log q - \bar{H}(Z_1)\}$. The two regions generally do not coincide. However, the

[2]Note that the multiplication and summation are done over $\mathbb{F}_q$. To simplify our notation, we will omit $\odot_q$ when multiplication is considered.

[3]To apply Corollary 3.2 for this channel, one can modify Shannon's random coding scheme to send information after time $M$ without affecting the asymptotic transmission rate.

following adaptive coding scheme attains the outer bound region.

Our adaptive coding scheme consists of two phases of transmission. In the first phase, both terminals send pilot symbols to help terminal 1 collect information about the noise process $\tilde{Z}_n$ for future data transmission. In the second phase, terminal 1 then uses the information obtained in the first phase to combat the noise at terminal 2 and concurrently sends its own messages. Specifically, for $1 \leq n \leq M$, we simply set $X_{1,n} = X_{2,n} = 0$. For $M+1 \leq i \leq N$, terminal 1 employs the following rule to generate channel inputs:

$$\begin{aligned} X_{1,n} &= M_{1,n} \ominus_q \sum_{m=1}^{M} \beta_m (Y_{1,n-m} \ominus_q X_{1,n-m}) \\ &= M_{1,n} \ominus_q \sum_{m=1}^{M} \beta_m (X_{2,n-m} \oplus_q \tilde{Z}_{n-m}) \end{aligned}$$

where $M_{1,n} \in \mathbb{F}_q$ is the $n$th raw message of terminal 1 and $\ominus_q$ denotes the subtraction over $\mathbb{F}_q$, while terminal 2 applies a length standard random coding designed for the one-way channel $\tilde{Y}_{1,n} = X_{2,n} \oplus_q \tilde{Z}_{1,n} = X_{2,n} \oplus_q Z_{1,n}$ to transmit its message $M_2$.

Given the above transmission scheme, we now show how both receivers can recover the transmitted messages from their received signals. First, the received signal of terminal 1 is given by $Y_{1,n} = X_{1,n} \oplus_q X_{2,n} \oplus_q Z_{1,n}$ for $M+1 \leq n \leq N$. Since $X_{1,n}$ is known to terminal 1, $X_{1,n}$ can be removed perfectly from $Y_{1,n}$, which results in $\tilde{Y}_{1,n}$. Clearly, terminal 1 can reliably decode $M_2$ as long as $R_2 \leq \log q - \bar{H}(X_1)$. On the other hand, terminal 2 employs the following decoding function to recover $M_{1,n}$ for $M+1 \leq n \leq N$:

$$\hat{M}_{1,n} = Y_{2,n} \ominus_q X_{2,n} \oplus_q \sum_{m=1}^{M} \beta_m X_{2,n-m}$$

$$
\begin{aligned}
&= X_{1,n} \oplus_q Z_{2,n} \oplus_q \sum_{m=1}^{M} \beta_m X_{2,n-m} \\
&= M_{1,n} \ominus_q \sum_{m=1}^{M} \beta_m (X_{2,n-m} \oplus_q \tilde{Z}_{n-m}) \oplus_q Z_{2,n} \oplus_q \sum_{m=1}^{M} \beta_m X_{2,n-m} \\
&= M_{1,n}.
\end{aligned}
$$

The maximum information transmission rate $R_1$ is then given by $\frac{(N-M)}{N} \log q$, which tends to $\log q$ as $N \to \infty$. Combining the above then reveals that the outer bound region $\{(R_1, R_2) : 0 \le R_1 \le \log q, 0 \le R_2 \le \log q - \bar{H}(Z_1)\}$ is in fact the capacity region for this case.

Case (ii): In this case, the random coding inner bound in Corollary 3.2 yields the same achievable rate region as in Case (i), while Lemma 3.1 results in a different outer bound region $\{(R_1, R_2) : 0 \le R_1 \le \log q - \bar{H}(\tilde{Z}), 0 \le R_2 \le \log q - \bar{H}(Z_1)\}$. The two regions do not coincide when $\bar{H}(Z_2) \ne \bar{H}(\tilde{Z})$, but our adaptive coding scheme attains the outer bound region. Note that in general $\bar{H}(Z_2) \ge \bar{H}(\tilde{Z})$ since for $n \ge M$, we have that

$$
\begin{aligned}
H(Z_{2,n}|Z_{2,1}^{n-1}) &\ge H(Z_{2,n}|Z_{2,1}^{n-1}, \tilde{Z}_1^{n-1}) \\
&= H(\beta_0 \tilde{Z}_n|Z_{2,1}^{n-1}, \tilde{Z}_1^{n-1}) \\
&= H(\tilde{Z}_n|\tilde{Z}_1^{n-1}).
\end{aligned}
$$

Our adaptive coding scheme for this case still comprises two phases of transmission, where the first phase is identical to that in Case (i). For the second phase, terminal 1 first employs the standard random coding scheme with codeword length $N-M$ to generate intermediate inputs $\tilde{X}_{1,n}$ for the one-way channel $\tilde{Y}_{2,n} = \tilde{X}_{1,n} \oplus_q \tilde{Z}_n$, where $M+1 \le n \le N$. The message $M_1$ is conveyed in the $\tilde{X}_{1,n}$'s, and these $\tilde{X}_{1,n}$'s are next used to generate the channel inputs of the two-way channel via

$$
\begin{aligned}
X_{1,n} &= \beta_0 \tilde{X}_{1,n} \ominus_q \sum_{m=1}^{M} \beta_m (Y_{1,n-m} \ominus_q X_{1,n-m}) \\
&= \beta_0 \tilde{X}_{1,n} \ominus_q \sum_{m=1}^{M} \beta_m (X_{2,n-m} \oplus_q Z_{1,n-m}).
\end{aligned}
$$

Terminal 2 applies the standard random coding scheme with codeword length $N - M$ for the one-way channel $\tilde{Y}_{1,n} = X_{2,n} \oplus_q Z_{1,n}$ to transmit its message $M_2$.

To recover $M_2$ at terminal 1, the decoding scheme of Case (i) is employed and $M_2$ can be reliably decoded if $R_2 \leq \log q - \bar{H}(Z_1)$. To see how terminal 2 decodes $M_1$, we first look at the content of $Y_{2,n}$:

$$
\begin{aligned}
Y_{2,n} &= X_{1,n} \oplus_q X_{2,n} \oplus_q Z_{2,n} \\
&= \beta_0 \tilde{X}_{1,i} \ominus_q \sum_{m=1}^{M} \beta_m (X_{2,n-m} \oplus_q Z_{1,n-m}) \oplus_q X_{2,n} \oplus_q Z_{2,n} \\
&= \beta_0 \tilde{X}_{1,n} \ominus_q \sum_{m=1}^{M} \beta_m X_{2,n-m} \ominus_q \sum_{m=1}^{M} \beta_m \tilde{Z}_{n-m} \oplus_q X_{2,n} \oplus_q \sum_{m=1}^{M} \beta_m \tilde{Z}_{n-m} \oplus_q \beta_0 \tilde{Z}_n \\
&= \beta_0 (\tilde{X}_{1,n} \oplus_q \tilde{Z}_n) \ominus_q \sum_{m=1}^{M} \beta_m X_{2,n-m} \oplus_q X_{2,n}.
\end{aligned}
$$

Since terminal 2 knows $\beta_m$'s and $X_{2,n}$'s and $\beta_0 \neq 0$, terminal 2 can convert $Y_{2,n}$ into $\tilde{Y}_{2,n} = \tilde{X}_{1,n} \oplus_q \tilde{Z}_n$. Based on the channel outputs $\tilde{Y}_{2,n}$, $M + 1 \leq n \leq N$, the message $M_1$ can be reliably decoded if $R_1 \leq \log q - \bar{H}(\tilde{Z})$. Thus, our adaptive coding scheme achieves the outer bound region, and thus its associated coding rates determine the capacity region.

# Chapter 4

# Multiaccess-Broadcasting Memoryless Two-Way Channels

This section considers a three-terminal two-way communication scenario combining multiaccess and broadcasting. We first introduce the channel model and derive inner and outer bounds for the capacity region. Then, sufficient conditions for the two bounds to coincide are provided, along with illustrative examples.

## 4.1 Channel Model and Definitions

Two-way communication over a discrete additive-noise MA/DB TWC comprises three terminals as depicted in Fig. 4.1. Terminals 1 and 2 want to transmit messages $M_{13}$ and $M_{23}$, respectively, to terminal 3 through the TWC that acts as a MAC in the forward direction. Terminal 3 wishes to broadcast messages $M_{31}$ and $M_{32}$ to terminals 1 and 2, respectively, through the TWC that acts as a DBC in the reverse direction. The messages are assumed to be independent of each other and uniformly distributed over their alphabets. The joint distribution of all the variables for $N$

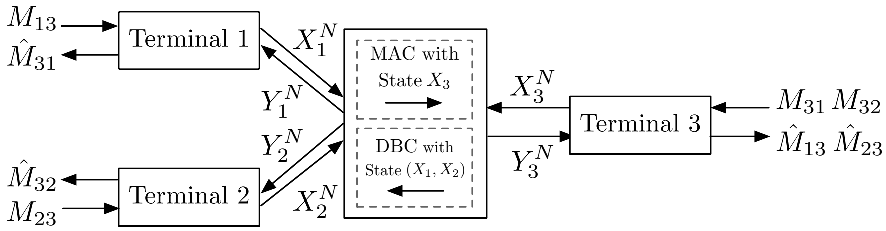



Figure 4.1: The information flow of MA/DB TWCs.

channel uses is given by

$$P_{M_{\{13,23,31,32\}},X^N_{\{1,2,3\}},Y^N_{\{1,2,3\}}} = P_{M_{13}}P_{M_{23}}P_{M_{31}}P_{M_{32}}\cdot\left(\prod_{n=1}^{N} P_{X_{1,n}|M_{13},Y_1^{n-1}}\right)$$
$$\cdot\left(\prod_{n=1}^{N} P_{X_{2,n}|M_{23},Y_2^{n-1}}\right)\cdot\left(\prod_{n=1}^{N} P_{X_{3,n}|M_{\{31,32\}},Y_3^{n-1}}\right)\cdot\left(\prod_{n=1}^{N} P_{Y_{1,n},Y_{2,n},Y_{3,n}|X^n_{\{1,2,3\}},Y^{n-1}_{\{1,2,3\}}}\right),$$

where $M_{\{13,23,31,32\}} \triangleq \{M_{13}, M_{23}, M_{31}, M_{32}\}$, $X^N_{\{1,2,3\}} \triangleq \{X_1^N, X_2^N, X_3^N\}$, and $Y^N_{\{1,2,3\}} \triangleq \{Y_1^N, Y_2^N, Y_3^N\}$. Thus, the $N$ transmissions can be described by the sequence of input-output conditional probabilities $\left\{P_{Y_{1,n},Y_{2,n},Y_{3,n}|X^n_{\{1,2,3\}},Y^{n-1}_{\{1,2,3\}}}\right\}_{n=1}^{N}$.

To simplify our analysis, we assume that the channel is memoryless in the sense that given current channel inputs, the current channel outputs are independent of past signals, i.e., $P_{Y_{1,n},Y_{2,n},Y_{3,n}|X^n_{\{1,2,3\}},Y^{n-1}_{\{1,2,3\}}} = P_{Y_{1,n},Y_{2,n},Y_{3,n}|X_{1,n},X_{2,n},X_{3,n}}$ for all $n$. Furthermore, the two directions of transmission are assumed to interact in a way such that $P_{Y_{1,n},Y_{2,n},Y_{3,n}|X_{1,n},X_{2,n},X_{3,n}} = P_{Y_{1,n},Y_{2,n}|X_{1,n},X_{2,n},X_{3,n}}P_{Y_{3,n}|X_{1,n},X_{2,n},X_{3,n}}$. Let all channel input and output alphabets other than $\mathcal{Y}_3$ equal $G_q = \{0, 1, \ldots, q-1\}$ for some $q \geq 2$. The MA/DB TWC is defined by the transition probability $P_{Y_3|X_1,X_2,X_3}$ in the MA direction and the transmission equations in the DB direction are given by

$$Y_{1,n} = X_{1,n} \oplus_q X_{3,n} \oplus_q Z_{1,n}, \tag{4.1a}$$

$$Y_{2,n} = X_{2,n} \oplus_q X_{3,n} \oplus_q Z_{1,n} \oplus_q Z_{2,n}, \tag{4.1b}$$

for $n = 1, 2, \cdots, N$, where $Z_{1,n}, Z_{2,n} \in G_q$ denote additive noise variables, the components of the memoryless and independent noise processes $\{Z_{1,n}\}_{n=1}^{\infty}$ and $\{Z_{2,n}\}_{n=1}^{\infty}$, respectively. We also assume that the channel noise processes are independent of all terminals' messages. Thus, the channel transition probability of this MA/DB TWC at time $n$ can be written as

$$\begin{aligned}
&P_{Y_{1,n},Y_{2,n},Y_{3,n}|X_1^n,X_2^n,X_3^n,Y_1^{n-1},Y_2^{n-1},Y_3^{n-1}}(y_{1,n}, y_{2,n}, y_{3,n}|x_1^n, x_2^n, x_3^n, y_1^{n-1}, y_2^{n-1}, y_3^{n-1}) \\
&\quad = P_{Y_{1,n},Y_{2,n},Y_{3,n}|X_{1,n},X_{2,n},X_{3,n}}(y_{1,n}, y_{2,n}, y_{3,n}|x_{1,n}, x_{2,n}, x_{3,n}) \\
&\quad = P_{Y_{3,n}|X_{1,n},X_{2,n},X_{3,n}}(y_{3,n}|x_{1,n}, x_{2,n}, x_{3,n}) P_{Y_{1,n}|X_{1,n},X_{2,n},X_{3,n},Y_{3,n}}(y_{1,n}|x_{1,n}, x_{2,n}, x_{3,n}, y_{3,n}) \\
&\qquad\qquad \cdot P_{Y_{2,n}|X_{1,n},X_{2,n},X_{3,n},Y_{1,n},Y_{3,n}}(y_{2,n}|x_{1,n}, x_{2,n}, x_{3,n}, y_{1,n}, y_{3,n}) \\
&\quad = P_{Y_3|X_1,X_2,X_3}(y_{3,n}|x_{1,n}, x_{2,n}, x_{3,n}) \\
&\qquad\qquad \cdot P_{Z_1}(y_{1,n} \ominus_q x_{1,n} \ominus_q x_{3,n}) P_{Z_2}(y_{2,n} \ominus_q x_{2,n} \ominus_q y_{1,n} \oplus_q x_{1,n}).
\end{aligned}$$

We remark that as our goal is to demonstrate a way to determine the capacity region for multi-terminal DM-TWCs, we consider a simple (additive) DBC model to simplify the derivation of tightness conditions. It is possible to consider general DBCs, but the obtained results will be more complex. We next define channel codes, achievable rates, and channel capacity for the MA/DB DM-TWC.

**Definition 4.1.** *An* $(N, R_{13}, R_{23}, R_{31}, R_{32})$ *channel code for the memoryless MA/DB TWC consists of four message sets* $\mathcal{M}_{13} = \{1, 2, \ldots, 2^{NR_{13}}\}$, $\mathcal{M}_{23} = \{1, 2, \ldots, 2^{NR_{23}}\}$, $\mathcal{M}_{31} = \{1, 2, \ldots, 2^{NR_{31}}\}$, $\mathcal{M}_{32} = \{1, 2, \ldots, 2^{NR_{32}}\}$, *three sequences of encoding functions:* $\boldsymbol{f}_1 = (f_{1,1}, f_{1,2}, \ldots, f_{1,N})$, $\boldsymbol{f}_2 = (f_{2,1}, f_{2,2}, \ldots, f_{2,N})$, $\boldsymbol{f}_3 = (f_{3,1}, f_{3,2}, \ldots, f_{3,N})$ *such that*

$$f_{1,1} : \mathcal{M}_{13} \to \mathcal{X}_1, \qquad f_{1,n} : \mathcal{M}_{13} \times \mathcal{Y}_1^{n-1} \to \mathcal{X}_1, \tag{4.2a}$$

$$f_{2,1} : \mathcal{M}_{23} \to \mathcal{X}_2, \qquad f_{2,n} : \mathcal{M}_{23} \times \mathcal{Y}_2^{n-1} \to \mathcal{X}_2, \tag{4.2b}$$

$$f_{3,1} : \mathcal{M}_{31} \times \mathcal{M}_{32} \to \mathcal{X}_3, \qquad f_{3,n} : \mathcal{M}_{31} \times \mathcal{M}_{32} \times \mathcal{Y}_3^{n-1} \to \mathcal{X}_3, \tag{4.2c}$$

*for $n = 2, 3, \ldots, N$, and three decoding functions $g_1$, $g_2$, and $g_3$, such that $\hat{M}_{31} = g_1(M_{13}, Y_1^N)$, $\hat{M}_{32} = g_2(M_{23}, Y_2^N)$, and $(\hat{M}_{13}, \hat{M}_{23}) = g_3(M_{31}, M_{32}, Y_3^N)$.*

When messages are encoded via the channel code, the average probability of decoding error is defined as

$$P_{\mathrm{e}}^{(N)}(\boldsymbol{f}_1, \boldsymbol{f}_2, \boldsymbol{f}_3, g_1, g_2, g_3) = \Pr\{\hat{M}_{13} \neq M_{13} \text{ or } \hat{M}_{23} \neq M_{23} \text{ or } \hat{M}_{31} \neq M_{31} \text{ or } \hat{M}_{32} \neq M_{32}\}.$$

**Definition 4.2.** *A rate quadruple $(R_{13}, R_{23}, R_{31}, R_{32})$ is said to be achievable for the memoryless MA/DB TWC if there exists a sequence of $(N, R_{13}, R_{23}, R_{31}, R_{32})$ codes with $\lim_{N\to\infty} P_e^{(N)} = 0$.*

**Definition 4.3.** *The capacity region $\mathcal{C}^{\text{MA-DBC}}$ of the memoryless MA/DB TWC is the closure of all achievable rate quadruples $(R_{13}, R_{23}, R_{31}, R_{32})$.*

## 4.2 Capacity Inner and Outer Bounds

Let $\mathcal{R}^{\text{MA-DBC}}(P_{X_1,X_2,X_3,V}, P_{Y_3|X_1,X_2,X_3}, P_{Z_1}, P_{Z_2})$ denote the set of rate quadruples $(R_{13}, R_{23}, R_{31}, R_{32})$ which satisfy the constraints

$$\begin{aligned} R_{13} &\leq I(X_1; Y_3|X_2, X_3), \\ R_{23} &\leq I(X_2; Y_3|X_1, X_3), \\ R_{13} + R_{23} &\leq I(X_1, X_2; Y_3|X_3), \\ R_{31} &\leq I(X_3; X_3 \oplus_q Z_1|V), \\ R_{32} &\leq I(V; X_3 \oplus_q Z_1 \oplus_q Z_2), \end{aligned}$$

where $V$ is an auxiliary random variable with alphabet $\mathcal{V}$ such that $|\mathcal{V}| \leq q+1$ and the mutual information terms are evaluated according to the joint probability distribution $P_{X_1,X_2,X_3,V,Y_3,Z_1,Z_2} = P_{X_1,X_2,X_3,V} P_{Y_3|X_1,X_2,X_3} P_{Z_1} P_{Z_2}$. We next establish a Shannon-type inner bound and an outer bound for the capacity of MA/DB TWCs in Theorems 4.1 and 4.2, respectively. Note that the achievable scheme in Theorem 4.1 is given by combining Shannon's standard (non-adaptive) coding schemes for the MAC [47, Theorem 4.2] and the DBC [47, Theorem 5.2], and hence the proof is omitted here.

**Theorem 4.1** (Inner Bound)**.** *For a memoryless MA/DB TWC with MA transition probability $P_{Y_3|X_1,X_2,X_3}$ and DB noise distributions $P_{Z_1}$ and $P_{Z_2}$, any rate quadruple $(R_{13}, R_{23}, R_{31}, R_{32}) \in \mathcal{C}_{\mathrm{I}}^{\text{MA-DBC}}(P_{Y_3|X_1,X_2,X_3}, P_{Z_1}, P_{Z_2})$ is achievable, where*

$$\mathcal{C}_{\mathrm{I}}^{\text{MA-DBC}}(P_{Y_3|X_1,X_2,X_3}, P_{Z_1}, P_{Z_2}) \triangleq \overline{\mathrm{co}}\left( \bigcup_{P_{X_1}, P_{X_2}, P_{V,X_3}} \mathcal{R}^{\text{MA-DBC}}(P_{X_1} P_{X_2} P_{V,X_3}, P_{Y_3|X_1,X_2,X_3}, P_{Z_1}, P_{Z_2}) \right).$$

**Theorem 4.2** (Outer Bound)**.** *For a memoryless MA/DB TWC with MA transition probability $P_{Y_3|X_1,X_2}$ and DB noise distributions $P_{Z_1}$ and $P_{Z_2}$, all achievable rate quadruples $(R_{13}, R_{23}, R_{31}, R_{32})$ belong to $\mathcal{C}_{\mathrm{O}}^{\text{MA-DBC}}(P_{Y_3|X_1,X_2,X_3}, P_{Z_1}, P_{Z_2})$, where*

$$\mathcal{C}_{\mathrm{O}}^{\text{MA-DBC}}(P_{Y_3|X_1,X_2,X_3}, P_{Z_1}, P_{Z_2}) \triangleq \overline{\mathrm{co}}\left( \bigcup_{P_{X_1,X_2,X_3,V}} \mathcal{R}^{\text{MA-DBC}}(P_{X_1,X_2,X_3,V}, P_{Y_3|X_1,X_2,X_3}, P_{Z_1}, P_{Z_2}) \right).$$

*Proof:* Given an achievable quadruple $(R_{13}, R_{23}, R_{31}, R_{32})$, we derive the necessary conditions for those rates by the standard converse method. For $R_{13}$, we have

$$\begin{aligned} &NR_{13} \\ &= H(M_{13}|M_{23}, M_{31}, M_{32}) \\ &= I(M_{13}; Y_3^N | M_{23}, M_{31}, M_{32}) + H(M_{13}|Y_3^N, M_{23}, M_{31}, M_{32}) \end{aligned}$$

$$
\begin{aligned}
&\leq I(M_{13}; Y_3^N | M_{23}, M_{31}, M_{32}) + N\epsilon_N && (4.3)\\
&\leq I(M_{13}; Y_2^N, Y_3^N | M_{23}, M_{31}, M_{32}) + N\epsilon_N\\
&= \sum_{n=1}^{N} I(M_{13}; Y_{2,n}, Y_{3,n} | Y_2^{n-1}, Y_3^{n-1}, M_{23}, M_{31}, M_{32}) + N\epsilon_N\\
&= \sum_{n=1}^{N} \Big( H(Y_{2,n}, Y_{3,n} | X_{2,n}, X_{3,n}, Y_2^{n-1}, Y_3^{n-1}, M_{23}, M_{31}, M_{32})\\
&\qquad - H(Y_{2,n}, Y_{3,n} | X_{2,n}, X_{3,n}, Y_2^{n-1}, Y_3^{n-1}, M_{23}, M_{31}, M_{32}, M_{13}) \Big) + N\epsilon_N && (4.4)\\
&\leq \sum_{n=1}^{N} \Big( H(Y_{2,n}, Y_{3,n} | X_{2,n}, X_{3,n}) - H(Y_{2,n}, Y_{3,n} | X_{1,n}, X_{2,n}, X_{3,n}) \Big) + N\epsilon_N && (4.5)\\
&= \sum_{n=1}^{n} I(X_{1,n}; Y_{2,n}, Y_{3,n} | X_{2,n}, X_{3,n}) + N\epsilon_N\\
&= \sum_{n=1}^{N} I(X_{1,n}; X_{2,n} \oplus_q X_{3,n} \oplus_q Z_{1,n} \oplus_q Z_{2,n}, Y_{3,n} | X_{2,n}, X_{3,n}) + N\epsilon_N\\
&= \sum_{n=1}^{N} I(X_{1,n}; Y_{3,n} | X_{2,n}, X_{3,n}) + I(X_{1,n}; Z_{1,n} \oplus_q Z_{2,n} | Y_{3,n}, X_{2,n}, X_{3,n}) + N\epsilon_N\\
&= \sum_{n=1}^{N} I(X_{1,n}; Y_{3,n} | X_{2,n}, X_{3,n}) + N\epsilon_N, && (4.6)
\end{aligned}
$$

where (4.3) follows from Fano's inequality with $\epsilon_N \to 0$ as $N \to \infty$, (4.4) holds since $X_{2,n} = f_{2,n}(M_{23}, Y_2^{n-1})$ and $X_{3,n} = f_{3,n}(M_{31}, M_{32}, Y_3^{n-1})$, (4.5) follows since the channel is memoryless, and (4.6) follows since $(Z_{1,n}, Z_{2,n})$ is independent of $(Y_{3,n}, X_{1,n}, X_{2,n}, X_{3,n})$. By symmetry, we also have

$$
NR_{23} \leq \sum_{n=1}^{N} I(X_{2,n}; Y_{3,n} | X_{1,n}, X_{3,n}) + N\epsilon_N. \qquad (4.7)
$$

For the sum rate $R_{13} + R_{23}$, we have

$$
\begin{aligned}
&N(R_{13} + R_{23})\\
&\quad = H(M_{13}, M_{23} | M_{31}, M_{32})
\end{aligned}
$$

$$
\begin{aligned}
&\le I(M_{13}, M_{23}; Y_3^N | M_{31}, M_{32}) + N\epsilon_N \\
&= \sum_{n=1}^{N} \Big( H(Y_{3,n} | X_{3,n}, Y_3^{n-1}, M_{31}, M_{32}) - H(Y_{3,n} | Y_3^{n-1}, M_{31}, M_{32}, M_{13}, M_{23}) \Big) + N\epsilon_N \\
&\le \sum_{n=1}^{N} \Big( H(Y_{3,n} | X_{3,n}) - H(Y_{3,n} | Y_3^{n-1}, M_{31}, M_{32}, M_{13}, M_{23}) \Big) + N\epsilon_N \\
&\le \sum_{n=1}^{N} \Big( H(Y_{3,n} | X_{3,n}) - H(Y_{3,n} | X_{1,n}, X_{2,n}, X_{3,n}) \Big) + N\epsilon_N \\
&= \sum_{n=1}^{N} I(X_{1,n}, X_{2,n}; Y_{3,n} | X_{3,n}) + N\epsilon_N,
\end{aligned}
$$

where $\epsilon_N \to 0$ as $N \to \infty$ by Fano's inequality. Therefore, for the rates in the MA direction, we have

$$
\begin{aligned}
R_{13} &\le \frac{1}{N} \sum_{n=1}^{N} I(X_{1,n}; Y_{3,n} | X_{2,n}, X_{3,n}) + \epsilon_N \le I(X_1; Y_3 | X_2, X_3) + \epsilon_N \\
R_{23} &\le \frac{1}{N} \sum_{n=1}^{N} I(X_{2,n}; Y_{3,n} | X_{1,n}, X_{3,n}) + \epsilon_N \le I(X_2; Y_3 | X_1, X_3) + \epsilon_N \\
R_{13} + R_{23} &\le \frac{1}{N} \sum_{i=1}^{N} I(X_{1,n}, X_{2,n}; Y_{3,n} | X_{3,n}) + \epsilon_N \le I(X_1, X_2; Y_3 | X_3) + \epsilon_N
\end{aligned}
$$

where the inequalities hold since the conditional mutual information $I(X_1; Y_3 | X_2, X_3)$, $I(X_2; Y_3 | X_1, X_3)$, and $I(X_1, X_2; Y_3 | X_3)$ are concave[1] in the joint input distribution $P_{X_1, X_2, X_3}$, where

$$
P_{X_1, X_2, X_3} = \frac{1}{N} \sum_{n=1}^{N} P_{X_{1,n}, X_{2,n}, X_{3,n}}.
$$

For the achievable rate $R_{32}$ in the DB direction, we have

$$
NR_{32}
$$

[1]This follows from the fact that $I(A; C|B)$ is concave in $P_{A,B}$ for fixed $P_{C|A,B}$ [3].

$$
\begin{aligned}
&= H(M_{32}|M_{23}) \\
&\leq I(M_{32}; Y_2^N|M_{23}) + N\epsilon_N \\
&= \sum_{n=1}^{N} I(M_{32}; Y_{2,n}|Y_2^{n-1}, M_{23}, X_2^n) + N\epsilon_N \\
&= \sum_{n=1}^{N} I(M_{32}; X_{3,n} \oplus_q Z_{1,n} \oplus_q Z_{2,n}|X_3^{n-1} \oplus_q Z_1^{n-1} \oplus_q Z_2^{n-1}, M_{23}, X_2^n) + N\epsilon_N \\
&= \sum_{n=1}^{N} I(M_{32}; X_{3,n} \oplus_q Z_{1,n} \oplus_q Z_{2,n}|X_3^{n-1} \oplus_q Z_1^{n-1} \oplus_q Z_2^{n-1}, M_{23}) + N\epsilon_N \qquad (4.8) \\
&\leq \sum_{n=1}^{N} I(M_{32}, X_3^{n-1} \oplus_q Z_1^{n-1} \oplus_q Z_2^{n-1}, M_{23}; X_{3,n} \oplus_q Z_{1,n} \oplus_q Z_{2,n}) + N\epsilon_N \qquad (4.9) \\
&\leq \sum_{n=1}^{N} I(M_{32}, M_{23}, M_{13}, X_3^{n-1} \oplus_q Z_1^{n-1} \oplus_q Z_2^{n-1}, X_3^{n-1} \oplus_q Z_1^{n-1}; \\
&\qquad\qquad X_{3,n} \oplus_q Z_{1,n} \oplus_q Z_{2,n}) + N\epsilon_N \\
&= \sum_{n=1}^{N} I(M_{\{32,23,13\}}, \tilde{Y}_1^{n-1}, \tilde{Y}_2^{n-1}; \tilde{Y}_{2,n}) + N\epsilon_N \qquad (4.10)
\end{aligned}
$$

where (4.8) holds since $X_2^n$ is a function of $(X_3^{n-1} \oplus_q Z_1^{n-1} \oplus_q Z_2^{n-1}, M_{23})$, (4.9) follows from the chain rule and the non-negativity of mutual information, and (4.10) is expressed in terms of $\tilde{Y}_{1,n} \triangleq X_{3,n} \oplus_q Z_{1,n}$, and $\tilde{Y}_{2,n} \triangleq X_{3,n} \oplus_q Z_{1,n} \oplus_q Z_{2,n} = \tilde{Y}_{1,n} \oplus_q Z_{2,n}$.

For $R_{31}$, we have

$$
\begin{aligned}
&NR_{31} \\
&= H(M_{31}|M_{\{32,23,13\}}) \\
&\leq I(M_{31}; Y_1^N, Y_2^N|M_{\{32,23,13\}}) + N\epsilon_N \\
&= \sum_{n=1}^{N} I(M_{31}; Y_{1,n}, Y_{2,n}|Y_1^{n-1}, Y_2^{n-1}, M_{\{32,23,13\}}) + N\epsilon_N \\
&\leq \sum_{n=1}^{N} I(M_{31}, X_{3,n}; Y_{1,n}, Y_{2,n}|Y_1^{n-1}, Y_2^{n-1}, M_{\{32,23,13\}}) + N\epsilon_N
\end{aligned}
$$

$$= \sum_{n=1}^{N} I(M_{31}, X_{3,n}; Y_{1,n}, Y_{2,n} | Y_1^{n-1}, Y_2^{n-1}, M_{\{32,23,13\}}, X_1^n, X_2^n) + N\epsilon_N \tag{4.11}$$

$$= \sum_{n=1}^{N} I(M_{31}, X_{3,n}; \tilde{Y}_{1,n}, \tilde{Y}_{2,n} | Y_1^{n-1}, Y_2^{n-1}, M_{\{32,23,13\}}, X_1^n, X_2^n) + N\epsilon_N$$

$$= \sum_{n=1}^{N} I(M_{31}, X_{3,n}; \tilde{Y}_{1,n}, \tilde{Y}_{2,n} | \tilde{Y}_1^{n-1}, \tilde{Y}_2^{n-1}, M_{\{32,13,23\}}) + N\epsilon_N \tag{4.12}$$

$$= \sum_{n=1}^{N} I(X_{3,n}; \tilde{Y}_{1,n}, \tilde{Y}_{2,n} | \tilde{Y}_1^{n-1}, \tilde{Y}_2^{n-1}, M_{\{32,13,23\}}) + \sum_{n=1}^{N} I(M_{31}; \tilde{Y}_{1,n}, \tilde{Y}_{2,n} | \tilde{Y}_1^{n-1}, \tilde{Y}_2^{n-1}, M_{\{32,13,23\}}, X_{3,n}) + N\epsilon_N$$

$$= \sum_{n=1}^{N} I(X_{3,n}; \tilde{Y}_{1,n}, \tilde{Y}_{2,n} | \tilde{Y}_1^{n-1}, \tilde{Y}_2^{n-1}, M_{\{32,13,23\}}) + N\epsilon_N \tag{4.13}$$

$$= \sum_{n=1}^{N} I(X_{3,n}; \tilde{Y}_{1,n} | \tilde{Y}_1^{n-1}, \tilde{Y}_2^{n-1}, M_{\{32,13,23\}}) + N\epsilon_N \tag{4.14}$$

where (4.11) holds since $X_{1,n} = f_{1,n}(M_{13}, Y_1^{n-1})$ and $X_{2,n} = f_{2,n}(M_{23}, Y_2^{n-1})$, (4.12) holds since $(Y_1^{n-1}, Y_2^{n-1}, X_1^n, X_2^n)$ can be generated knowing $(M_{13}, M_{23}, \tilde{Y}_1^{n-1}, \tilde{Y}_2^{n-1})$, (4.13) holds because $M_{31} -\!\circ\!- (\tilde{Y}_1^{n-1}, \tilde{Y}_2^{n-1}, M_{\{32,13,23\}}, X_{3,n}) -\!\circ\!- (\tilde{Y}_{1,n}, \tilde{Y}_{2,n})$ form a Markov chain, and (4.14) holds since $\tilde{Y}_{2,n} -\!\circ\!- (\tilde{Y}_{1,n}, \tilde{Y}_1^{n-1}, \tilde{Y}_2^{n-1}, M_{\{32,13,23\}}) -\!\circ\!- X_{3,n}$ form a Markov chain. Note that these Markov chain properties hold since $\{Z_{1,n}\}_{n=1}^N$ and $\{Z_{2,n}\}_{n=1}^N$ are independent memoryless processes and are independent of all terminals' messages.

Setting $V_n = (\tilde{Y}_1^{n-1}, \tilde{Y}_2^{n-1}, M_{\{32,13,23\}})$, we have the following Markov chain $V_n -\!\circ\!- X_{3,n} -\!\circ\!- (\tilde{Y}_{1,n}, \tilde{Y}_{2,n})$. From (4.10) and (4.14), we further obtain that $NR_{32} \leq \sum_{n=1}^N I(V_n; \tilde{Y}_{2,n}) + N\epsilon_N$ and $NR_{31} \leq \sum_{n=1}^N I(X_{3,n}; \tilde{Y}_{1,n} | V_n) + N\epsilon_N$. Let $Q$ be a time-sharing random variable that is uniform over $\{1, 2, \ldots, N\}$ and independent of all messages, inputs, and outputs. Setting $V = (Q, V_Q)$, $X_3 = X_{3,Q}$, $Z_1 = Z_{1,Q}$,

$Z_2 = Z_{2,Q}$ $\tilde{Y}_1 = X_3 \oplus_q Z_1 = \tilde{Y}_{1,Q}$, $\tilde{Y}_2 = X_3 \oplus_q Z_1 \oplus_q Z_2 = \tilde{Y}_{2,Q}$, we have

$$\begin{aligned} NR_{32} &\leq \sum_{n=1}^{N} I(V_n; \tilde{Y}_{2,n}) + N\epsilon_N \\ &= N \cdot I(V_Q; \tilde{Y}_{2,Q}|Q) + N\epsilon_N \\ &\leq N \cdot I(V; \tilde{Y}_2) + N\epsilon_N \\ &= N \cdot I(V; X_3 \oplus_q Z_1 \oplus_q Z_2) + N\epsilon_N, \end{aligned}$$

and

$$\begin{aligned} NR_{31} &\leq \sum_{n=1}^{N} I(X_{3,n}; \tilde{Y}_{1,n}|V_n) + N\epsilon_N \\ &= N \cdot I(X_3; \tilde{Y}_1|V) + N\epsilon_N \\ &= N \cdot I(X_3; X_3 \oplus_q Z_1|V) + N\epsilon_N \end{aligned}$$

for some $P_{Z_1,Z_2,X_3,V} = P_{X_3,V}P_{Z_1}P_{Z_2}$. Combining the obtained bounds for rates $R_{13}$ and $R_{23}$, the proof is completed by dividing $N$ on both sides of the bounds and letting $N \to \infty$. The bound on the alphabet size of $V$ can be established by the convex cover method [47]. ■

## 4.3 Tightness Conditions for Random Coding Inner Bound

The inner and outer bounds derived in the previous section are of the same form but have different restrictions on the joint distribution $P_{X_1,X_2,X_3,V}$, and hence they do not match. Here, we establish conditions under which the two bounds have matching input distributions, implying that they coincide and yield the capacity region.

**Theorem 4.3.** *The inner and outer capacity bounds in Theorems 4.1 and 4.2 coincide if for every conditional input distribution* $P^{(1)}_{X_1,X_2|X_3}$, *there exists a product input* *distribution* $P^{(2)}_{X_1,X_2|X_3} = \tilde{P}_{X_1}\tilde{P}_{X_2}$ *(which depends on* $P^{(1)}_{X_1,X_2|X_3}$*) such that*

$$I^{(1)}(X_1;Y_3|X_2,X_3=x_3) \le I^{(2)}(X_1;Y_3|X_2,X_3=x_3) \tag{4.15}$$

$$I^{(1)}(X_2;Y_3|X_1,X_3=x_3) \le I^{(2)}(X_2;Y_3|X_1,X_3=x_3) \tag{4.16}$$

$$I^{(1)}(X_1,X_2;Y_3|X_3=x_3) \le I^{(2)}(X_1,X_2;Y_3|X_3=x_3) \tag{4.17}$$

*hold for all* $x_3 \in \mathcal{X}_3$*. Under this condition, the capacity region is given by*

$$\mathcal{C}^{\text{MA-DBC}} = \overline{\text{co}}\left(\bigcup_{P_{X_1},P_{X_2},P_{V,X_3}} \mathcal{R}^{\text{MA-DBC}}\Big(P_{X_1}P_{X_2}P_{V,X_3}, P_{Y_3|X_1,X_2,X_3}, P_{Z_1}, P_{Z_2}\Big)\right).$$

*Proof:* Consider a MA-DB TWC governed by $P_{Y_3|X_1,X_2,X_3}$, $P_{Z_1}$, and $P_{Z_2}$. Recall that

$$\begin{aligned}\mathcal{R}^{\text{MA-DBC}}(P_{X_1,X_2,X_3,V}, P_{Y_3|X_1,X_2,X_3}, P_{Z_1}, P_{Z_2}) = \Big\{&(R_{13},R_{23},R_{31},R_{32}) :\\
R_{13} &\le I(X_1;Y_3|X_2,X_3), & (4.18)\\
R_{23} &\le I(X_2;Y_3|X_1,X_3), & (4.19)\\
R_{13}+R_{23} &\le I(X_1,X_2;Y_3|X_3), & (4.20)\\
R_{31} &\le I(X_3;X_3\oplus_q Z_1|V), & (4.21)\\
R_{32} &\le I(V;X_3\oplus_q Z_1\oplus_q Z_2)\Big\}. & (4.22)\end{aligned}$$

Since (4.18)-(4.20) do not depend on $V$ and (4.21) and (4.22) do not depend on $(X_1, X_2)$, we have

$$\begin{aligned}&\mathcal{R}^{\text{MA-DBC}}(P_{X_1,X_2,X_3,V}, P_{Y_3|X_1,X_2,X_3}, P_{Z_1}, P_{Z_2})\\
&\quad = \mathcal{R}^{\text{MA-DBC}}(P_{X_1,X_2|X_3}P_{V,X_3}, P_{Y_3|X_1,X_2,X_3}, P_{Z_1}, P_{Z_2}).\end{aligned} \tag{4.23}$$

To complete the proof, it suffices to show that for every $P_{X_1,X_2|X_3}$, the corresponding

$\tilde{P}_{X_1}\tilde{P}_{X_2}$ (which depends on $P_{X_1,X_2|X_3}$) given by our assumption, satisfies

$$\begin{aligned}&\mathcal{R}^{\text{MA-DBC}}(P_{X_1,X_2|X_3}P_{V,X_3}, P_{Y_3|X_1,X_2,X_3}, P_{Z_1}, P_{Z_2})\\&\subseteq \mathcal{R}^{\text{MA-DBC}}(\tilde{P}_{X_1}\tilde{P}_{X_2}P_{V,X_3}, P_{Y_3|X_1,X_2,X_3}, P_{Z_1}, P_{Z_2}),\end{aligned} \tag{4.24}$$

since then we clearly have

$$\mathcal{C}_{\text{O}}^{\text{MA-DBC}}(P_{Y_3|X_1,X_2,X_3}, P_{Z_1}, P_{Z_2}) \subseteq \mathcal{C}_{\text{I}}^{\text{MA-DBC}}(P_{Y_3|X_1,X_2,X_3}, P_{Z_1}, P_{Z_2}).$$

To show (4.24), consider two input distributions $P^{(1)}_{X_1,X_2,X_3,V} \triangleq P^{(1)}_{X_1,X_2|X_3}P^{(1)}_{V,X_3}$ and $P^{(2)}_{X_1,X_2,X_3,V} \triangleq \tilde{P}_{X_1}\tilde{P}_{X_2}P^{(1)}_{V,X_3}$, where $\tilde{P}_{X_1}\tilde{P}_{X_2}$ is given by the assumption. Then,

$$I^{(1)}(X_3; X_3 \oplus_q Z_1|V) = I^{(2)}(X_3; X_3 \oplus_q Z_1|V) \tag{4.25}$$

$$I^{(1)}(V; X_3 \oplus_q Z_1 \oplus_q Z_2) = I^{(2)}(V; X_3 \oplus_q Z_1 \oplus_q Z_2) \tag{4.26}$$

since $P^{(1)}_{X_1,X_2,X_3,V}$ and $P^{(2)}_{X_1,X_2,X_3,V}$ have the same marginal $P^{(1)}_{V,X_3}$. Furthermore,

$$\begin{aligned}I^{(1)}(X_1; Y_3|X_2, X_3) &= \sum_{x_3} P^{(1)}_{X_3}(x_3)\cdot I^{(1)}(X_1; Y_3|X_2, X_3 = x_3)\\&\leq \sum_{x_3} P^{(1)}_{X_3}(x_3)\cdot I^{(2)}(X_1; Y_3|X_2, X_3 = x_3)\\&= I^{(2)}(X_1; Y_3|X_2, X_3),\end{aligned}$$

where the inequality follows from (4.15) and the last equality holds since $P^{(1)}_{X_1,X_2,X_3,V}$ and $P^{(2)}_{X_1,X_2,X_3,V}$ have the same marginal $P^{(1)}_{X_3}$. Similarly, we obtain that

$$I^{(1)}(X_2; Y_3|X_1, X_3) \leq I^{(2)}(X_2; Y_3|X_1, X_3)$$

and

$$I^{(1)}(X_1, X_2; Y_3|X_3) \leq I^{(2)}(X_1, X_2; Y_3|X_3).$$

Consequently, (4.24) holds. ■

A special case of the above theorem is when $\tilde{P}_{X_1}\tilde{P}_{X_2}$ does not depend on $P_{X_1,X_2|X_3}$. This case may happen when $P_{Y_3|X_1,X_2,X_3}$ has a strong symmetry property.

**Corollary 4.1.** *The inner and outer capacity bounds in Theorems 4.1 and 4.2 coincide if there exists an input distributions* $P^{(2)}_{X_1,X_2} = P^*_{X_1}P^*_{X_2}$ *such that for all* $P^{(1)}_{X_1,X_2|X_3}$ *and* $x_3 \in \mathcal{X}_3$ *the inequalities given in* (4.15)-(4.17) *hold. In this case, the capacity region is given by*

$$\mathcal{C}^{\text{MA-DBC}} = \overline{\text{co}}\Bigg(\bigcup_{P_{V,X_3}} \mathcal{R}^{\text{MA-DBC}}\Big(P^*_{X_1}P^*_{X_2}P_{V,X_3}, P_{Y_3|X_1,X_2,X_3}, P_{Z_1}, P_{Z_2}\Big)\Bigg).$$

The next result is derived by treating the channel as a composition of state-dependent one-way channels.

**Theorem 4.4.** *The inner and outer capacity bounds in Theorems 4.1 and 4.2 coincide if the following conditions hold:*

*(i) There exists* $P^*_{X_1} \in \mathcal{P}(\mathcal{X}_1)$ *such that*

$$\operatorname*{arg\,max}_{P_{X_1|X_2=x_2,X_3=x_3}} I(X_1;Y_3|X_2=x_2,X_3=x_3) = P^*_{X_1}$$

*for all* $x_2 \in \mathcal{X}_2$ *and* $x_3 \in \mathcal{X}_3$*, and* $\mathcal{I}(P^*_{X_1}, P_{Y_3|X_1,X_2=x_2,X_3=x_3})$ *does not depend on* $x_2$ *for every fixed* $x_3$*;*

*(ii) For any* $P_{X_2} \in \mathcal{P}(\mathcal{X}_2)$*,* $\mathcal{I}(P_{X_2}, P_{Y_3|X_1=x_1,X_2,X_3=x_3})$ *does not depend on* $x_1 \in \mathcal{X}_1$ *and* $x_3 \in \mathcal{X}_3$*;*

*(iii) For any fixed* $P_{X_1,X_2}$*, we have that* $\mathcal{I}(P_{X_1,X_2}, P_{Y_3|X_1,X_2,X_3=x_3})$ *does not depend on*

$x_3 \in \mathcal{X}_3$*, and for each* $x_3 \in \mathcal{X}_3$ *we have that*

$$\mathcal{I}(P_{X_1,X_2}, P_{Y_3|X_1,X_2,X_3=x_3}) \leq \mathcal{I}(P^*_{X_1} P_{X_2}, P_{Y_3|X_1,X_2,X_3=x_3}),$$

*where* $P^*_{X_1}$ *is given by condition (i) and* $P_{X_2}$ *is marginalized from* $P_{X_1,X_2}$*.*

*Under this condition, the capacity region is given by*

$$\mathcal{C}^{\text{MA-DBC}} = \overline{\text{co}}\left(\bigcup_{P_{X_2},P_{V,X_3}} \mathcal{R}^{\text{MA-DBC}}\Big(P^*_{X_1} P_{X_2} P_{V,X_3}, P_{Y_3|X_1,X_2,X_3}, P_{Z_1}, P_{Z_2}\Big)\right).$$

*Proof:* Similar to the proof in Theorem 4.3, for any $P_{X_1,X_2|X_3} P_{V,X_3} = P_{X_2|X_3} P_{X_1|X_2,X_3} P_{V,X_3}$, it suffices to show that

$$\begin{aligned} &\mathcal{R}^{\text{MA-DBC}}(P_{X_1,X_2|X_3} P_{V,X_3}, P_{Y_3|X_1,X_2,X_3}, P_{Z_1}, P_{Z_2}) \\ &\quad \subseteq \mathcal{R}^{\text{MA-DBC}}(P^*_{X_1} P_{X_2|X_3} P_{V,X_3}, P_{Y_3|X_1,X_2,X_3}, P_{Z_1}, P_{Z_2}), \end{aligned} \tag{4.27}$$

where $P^*_{X_1}$ is given by conditions (i).

For any $P^{(1)}_{X_1,X_2,X_3,V} = P^{(1)}_{X_1,X_2|X_3} P^{(1)}_{V,X_3}$, let $P^{(2)}_{X_1,X_2,X_3,V} = P^*_{X_1} P^{(1)}_{X_2} P^{(1)}_{V,X_3}$, where $P^*_{X_1}$ is given by condition (i) and $P^{(1)}_{X_2}$ denotes the marginal distribution of $X_2$ derived from $P^{(1)}_{X_1,X_2,X_3,V}$. For the rate constraints in the DB direction, the same identities as in (4.25)-(4.26) can be obtained because $P^{(1)}_{X_1,X_2,X_3,V}$ and $P^{(2)}_{X_1,X_2,X_3,V}$ share a common marginal distribution given by $P^{(1)}_{V,X_3}$. For $R_{13}$ in the MA direction, we have

$$\begin{aligned} &I^{(1)}(X_1; Y_3|X_2, X_3) \\ &= \sum_{x_2,x_3} P^{(1)}_{X_2,X_3}(x_2, x_3) \cdot I^{(1)}(X_1; Y_3|X_2 = x_2, X_3 = x_3) \\ &= \sum_{x_2,x_3} P^{(1)}_{X_2,X_3}(x_2, x_3) \cdot \mathcal{I}\Big(P^{(1)}_{X_1|X_2=x_2,X_3=x_3}, P_{Y_3|X_1,X_2=x_2,X_3=x_3}\Big) \end{aligned}$$

$$
\begin{aligned}
&\leq \sum_{x_2,x_3} P^{(1)}_{X_2,X_3}(x_2,x_3) \cdot \left[ \max_{P_{X_1|X_2=x_2,X_3=x_3}} \mathcal{I}\Big(P_{X_1|X_2=x_2,X_3=x_3}, P_{Y_3|X_1,X_2=x_2,X_3=x_3}\Big)\right] \\
&= \sum_{x_2,x_3} P^{(1)}_{X_2,X_3}(x_2,x_3) \cdot \mathcal{I}\Big(P^*_{X_1}, P_{Y_3|X_1,X_2=x_2,X_3=x_3}\Big) \qquad (4.28) \\
&= \sum_{x_3} P^{(1)}_{X_3}(x_3) \sum_{x_2} P^{(1)}_{X_2|X_3}(x_2|x_3) \cdot \mathcal{I}\Big(P^*_{X_1}, P_{Y_3|X_1,X_2=x_2,X_3=x_3}\Big) \\
&= \sum_{x_3} P^{(1)}_{X_3}(x_3) \cdot \left(\sum_{x_2} P^{(1)}_{X_2|X_3}(x_2|x_3)\right) \cdot \mathcal{I}\Big(P^*_{X_1}, P_{Y_3|X_1,X_2=x_2',X_3=x_3}\Big) \qquad (4.29) \\
&= \sum_{x_2'} P^{(1)}_{X_2}(x_2') \sum_{x_3} P^{(1)}_{X_3}(x_3) \cdot \mathcal{I}\Big(P^*_{X_1}, P_{Y_3|X_1,X_2=x_2',X_3=x_3}\Big) \\
&= I^{(2)}(X_1;Y_3|X_2,X_3),
\end{aligned}
$$

where (4.28) and (4.29) directly follow from condition (i).

For $R_{23}$, we have

$$
\begin{aligned}
&I^{(1)}(X_2;Y_3|X_1,X_3) \\
&\quad= \sum_{x_1,x_3} P^{(1)}_{X_1,X_3}(x_1,x_3) \cdot I^{(1)}(X_2;Y_3|X_1=x_1,X_3=x_3) \\
&\quad= \sum_{x_1,x_3} P^{(1)}_{X_1,X_3}(x_1,x_3) \cdot \mathcal{I}\Big(P^{(1)}_{X_2|X_1=x_1,X_3=x_3}, P_{Y_3|X_1=x_1,X_2,X_3=x_3}\Big) \\
&\quad= \sum_{x_1,x_3} P^{(1)}_{X_1,X_3}(x_1,x_3) \cdot \mathcal{I}\Big(P^{(1)}_{X_2|X_1=x_1,X_3=x_3}, P_{Y_3|X_1=x_1',X_2,X_3=x_3'}\Big) \qquad (4.30) \\
&\quad\leq \mathcal{I}\left(\sum_{x_1,x_3} P^{(1)}_{X_1,X_3}(x_1,x_3) P^{(1)}_{X_2|X_1,X_3}(x_2|x_1,x_3), P_{Y_3|X_1=x_1',X_2,X_3=x_3'}\right) \qquad (4.31) \\
&\quad= \mathcal{I}\Big(P^{(1)}_{X_2}, P_{Y_3|X_1=x_1',X_2,X_3=x_3'}\Big) \\
&\quad= \sum_{x_1',x_3'} P^*_{X_1}(x_1') P^{(1)}_{X_3}(x_3') \cdot \mathcal{I}\Big(P^{(1)}_{X_2}, P_{Y_3|X_1=x_1',X_2,X_3=x_3'}\Big) \qquad (4.32) \\
&\quad= I^{(2)}(X_2;Y_3|X_2,X_3),
\end{aligned}
$$

where (4.30) and (4.32) follow from condition (ii) and (4.31) is due to convexity of

$\mathcal{I}(\cdot,\cdot)$ in its first argument.

Moreover, for the sum rate $R_{13}+R_{23}$, we have

$$
\begin{aligned}
&I^{(1)}(X_1,X_2;Y_3|X_3)\\
&=\sum_{x_3} P^{(1)}_{X_3}(x_3)\cdot I^{(1)}(X_1,X_2;Y_3|X_3=x_3)\\
&=\sum_{x_3} P^{(1)}_{X_3}(x_3)\cdot \mathcal{I}\left(P^{(1)}_{X_1,X_2|X_3=x_3},P_{Y_3|X_1,X_2,X_3=x_3}\right)\\
&=\sum_{x_3} P^{(1)}_{X_3}(x_3)\cdot \mathcal{I}\left(P^{(1)}_{X_1,X_2|X_3=x_3},P_{Y_3|X_1,X_2,X_3=x'_3}\right) && (4.33)\\
&\le \mathcal{I}\left(\sum_{x_3} P^{(1)}_{X_3}(x_3)P^{(1)}_{X_1,X_2|X_3}(x_1,x_2|x_3),P_{Y_3|X_1,X_2,X_3=x'_3}\right) && (4.34)\\
&=\mathcal{I}\left(P^{(1)}_{X_1,X_2},P_{Y_3|X_1,X_2,X_3=x'_3}\right)\\
&\le \mathcal{I}\left(P^{*}_{X_1}P^{(1)}_{X_2},P_{Y_3|X_1,X_2,X_3=x'_3}\right) && (4.35)\\
&=\sum_{x'_3} P^{(1)}_{X_3}(x'_3)\cdot \mathcal{I}\left(P^{*}_{X_1}P^{(1)}_{X_2},P_{Y_3|X_1,X_2,X_3=x'_3}\right)\\
&=I^{(2)}(X_1,X_2;Y_3|X_3),
\end{aligned}
$$

where (4.33) and (4.35) follow from condition (iii) and (4.34) is due to convexity of $\mathcal{I}(\,\cdot\,,\,\cdot\,)$ in its first argument. Therefore, (4.27) holds under conditions (i)-(iii). ■

Next, we derive our last sufficient condition by generalizing Shannon's condition (in Proposition 2.1) to the three-terminal setting. This new condition is easier to verify than the previous ones.

**Theorem 4.5.** *The inner and outer capacity bounds in Theorems 4.1 and 4.2 coincide if the following conditions hold:*

*(i) For any relabeling* $\tau^{\mathcal{X}_1}_{x'_1,x''_1}$ *on* $\mathcal{X}_1$*, there exists a permutation* $\pi^{\mathcal{Y}_3}[x'_1,x''_1]$ *on* $\mathcal{Y}_3$ *such*

*that for all $x_1$, $x_2$, $x_3$, and $y_3$, we have*

$$P_{Y_3|X_1,X_2,X_3}(y_3|x_1,x_2,x_3)= P_{Y_3|X_1,X_2,X_3}\big(\pi^{\mathcal{Y}_3}[x_1',x_1''](y_3)\big|\tau^{\mathcal{X}_1}_{x_1',x_1''}(x_1),x_2,x_3\big); \quad (4.36)$$

*(ii) For any relabeling $\tau^{\mathcal{X}_2}_{x_2',x_2''}$ on $\mathcal{X}_2$, there exists a permutation on $\pi^{\mathcal{Y}_3}[x_2',x_2'']$ on $\mathcal{Y}_3$ such that for all $x_1$, $x_2$, $x_3$, and $y_3$, we have*

$$P_{Y_3|X_1,X_2,X_3}(y_3|x_1,x_2,x_3)= P_{Y_3|X_1,X_2,X_3}\big(\pi^{\mathcal{Y}_3}[x_1',x_1''](y_3)\big|x_1,\tau^{\mathcal{X}_2}_{x_2',x_2''}(x_2),x_3\big). \quad (4.37)$$

*Under these conditions, the capacity region is given by*

$$\mathcal{C}^{\text{MA-DBC}}= \overline{\text{co}}\left(\bigcup_{P_{V,X_3}} \mathcal{R}^{\text{MA-DBC}}\Big(P^{\text{U}}_{\mathcal{X}_1}P^{\text{U}}_{\mathcal{X}_2}P_{V,X_3},P_{Y_3|X_1,X_2,X_3},P_{Z_1},P_{Z_2}\Big)\right), \quad (4.38)$$

*where $P^{\text{U}}_{\mathcal{X}_i}$ denotes uniform probability distribution on $\mathcal{X}_i$ for $i=1,2$.*

*Proof:* It suffices to show that

$$\begin{aligned}&\mathcal{R}^{\text{MA-DBC}}(P_{X_1,X_2|X_3}P_{V,X_3},P_{Y_3|X_1,X_2,X_3},P_{Z_1},P_{Z_2})\\ &\quad\subseteq \mathcal{R}^{\text{MA-DBC}}(P^{\text{U}}_{\mathcal{X}_1}P^{\text{U}}_{\mathcal{X}_2}P_{V,X_3},P_{Y_3|X_1,X_2,X_3},P_{Z_1},P_{Z_2}) \quad (4.39)\end{aligned}$$

for any $P_{X_1,X_2|X_3}P_{V,X_3}$. We first give a proof sketch. Analogous to Shannon's proof for point-to-point DM-TWCs (see Appendix A.1), we want to show that for any input distribution $P^{(1)}_{X_1,X_2,X_3,V}=P^{(1)}_{X_1,X_2|X_3}P^{(1)}_{V,X_3}$, if we set $P^{(2)}_{X_1,X_2,X_3,V}=P^{(2)}_{X_1,X_2|X_3}P^{(1)}_{V,X_3}$ and $P^{(3)}_{X_1,X_2,X_3,V}=P^{(3)}_{X_1,X_2|X_3}P^{(1)}_{V,X_3}$, where

$$P^{(2)}_{X_1,X_2|X_3}(\,\cdot\,,\cdot\,|\,\cdot\,) \triangleq P^{(1)}_{X_1,X_2|X_3}(\tau^{\mathcal{X}_1}_{x_1',x_1''}(\,\cdot\,),\,\cdot\,|\,\cdot\,), \quad (4.40)$$

$$P^{(3)}_{X_1,X_2|X_3}(\,\cdot\,,\cdot\,|\,\cdot\,) \triangleq \frac{1}{2}\Big(P^{(1)}_{X_1,X_2|X_3}(\,\cdot\,,\cdot\,|\,\cdot\,)+P^{(2)}_{X_1,X_2|X_3}(\,\cdot\,,\cdot\,|\,\cdot\,)\Big), \quad (4.41)$$

and $x_1', x_1'' \in \mathcal{X}_1$, then we have

$$\begin{aligned}
&\mathcal{R}^{\text{MA-DBC}}(P^{(1)}_{X_1,X_2|X_3}P^{(1)}_{V,X_3}, P_{Y_3|X_1,X_2,X_3}, P_{Z_1}, P_{Z_2}) \\
&\quad = \mathcal{R}^{\text{MA-DBC}}(P^{(2)}_{X_1,X_2|X_3}P^{(1)}_{V,X_3}, P_{Y_3|X_1,X_2,X_3}, P_{Z_1}, P_{Z_2}) && (4.42)\\
&\quad \subseteq \mathcal{R}^{\text{MA-DBC}}(P^{(3)}_{X_1,X_2|X_3}P^{(1)}_{V,X_3}, P_{Y_3|X_1,X_2,X_3}, P_{Z_1}, P_{Z_2}), && (4.43)
\end{aligned}$$

where the last inclusion is shown using (4.36) and extending Lemma A.1.2 to the MA/DBC setup. Then, we use an induction argument as in the proof of Lemma A.1.3 to obtain

$$\begin{aligned}
&\mathcal{R}^{\text{MA-DBC}}(P_{X_1,X_2|X_3}P_{V,X_3}, P_{Y_3|X_1,X_2,X_3}, P_{Z_1}, P_{Z_2}) \\
&\quad \subseteq \mathcal{R}^{\text{MA-DBC}}(P^{\text{U}}_{\mathcal{X}_1}P_{X_2|X_3}P_{V,X_3}, P_{Y_3|X_1,X_2,X_3}, P_{Z_1}, P_{Z_2}). && (4.44)
\end{aligned}$$

Next, we consider input distributions of the form $P^{(1)}_{X_1,X_2,X_3,V} = P^{\text{U}}_{\mathcal{X}_1}P^{(1)}_{X_2|X_3}P^{(1)}_{X_3,V}$ and set $P^{(2)}_{X_1,X_2,X_3,V} = P^{(2)}_{X_1,X_2|X_3}P^{(1)}_{V,X_3}$ and $P^{(3)}_{X_1,X_2,X_3,V}{=}P^{(3)}_{X_1,X_2|X_3}P^{(1)}_{V,X_3}$, where

$$\begin{aligned}
&P^{(2)}_{X_1,X_2|X_3}(\,\cdot\,,\cdot\,|\,\cdot\,) \triangleq P^{(1)}_{X_1,X_2|X_3}(\,\cdot\,,\tau^{\mathcal{X}_2}_{x_2',x_2''}(\,\cdot\,)|\,\cdot\,), \\
&P^{(3)}_{X_1,X_2|X_3}(\,\cdot\,,\cdot\,|\,\cdot\,) \triangleq \frac{1}{2}\left(P^{(1)}_{X_1,X_2|X_3}(\,\cdot\,,\cdot\,|\,\cdot\,) + P^{(2)}_{X_1,X_2|X_3}(\,\cdot\,,\cdot\,|\,\cdot\,)\right),
\end{aligned}$$

and $x_2', x_2'' \in \mathcal{X}_2$. It can be shown via (4.37) that (4.42)-(4.43) also hold, and thus applying an induction argument again yields

$$\begin{aligned}
&\mathcal{R}^{\text{MA-DBC}}(P^{\text{U}}_{\mathcal{X}_1}P_{X_2|X_3}P_{V,X_3}, P_{Y_3|X_1,X_2,X_3}, P_{Z_1}, P_{Z_2}) \\
&\quad \subseteq \mathcal{R}^{\text{MA-DBC}}(P^{\text{U}}_{\mathcal{X}_1}P^{\text{U}}_{\mathcal{X}_2}P_{V,X_3}, P_{Y_3|X_1,X_2,X_3}, P_{Z_1}, P_{Z_2}). && (4.45)
\end{aligned}$$

Combining (4.44) and (4.45) then proves our claim. Due to symmetry, we only prove (4.44). We begin with the following lemma whose proof is detailed in Appendix B.1.

**Lemma 4.1.** *For any* $P^{(1)}_{X_1,X_2,X_3,V} = P^{(1)}_{X_1,X_2|X_3} P^{(1)}_{V,X_3}$*, let* $P^{(2)}_{X_1,X_2,X_3,V} = P^{(2)}_{X_1,X_2|X_3} P^{(1)}_{V,X_3}$ *and* $P^{(3)}_{X_1,X_2,X_3,V} = P^{(3)}_{X_1,X_2|X_3} P^{(1)}_{V,X_3}$*, where* $P^{(2)}_{X_1,X_2|X_3}$ *and* $P^{(3)}_{X_1,X_2|X_3}$ *are given by* (4.40) *and* (4.41)*, respectively. Then,* (4.42)-(4.43) *hold.*

Without loss of generality, suppose that $\mathcal{X}_1 = \{1, 2, ..., \kappa\}$. For $1 \leq m \leq \kappa$, define $\Lambda_m$ as the set of all conditional probability distributions $P_{X_1,X_2|X_3}$ satisfying $P_{X_1,X_2|X_3}(1, x_2|x_3) = P_{X_1,X_2|X_3}(2, x_2|x_3) = \cdots = P_{X_1,X_2|X_3}(m, x_2|x_3)$ for any fixed $x_2 \in \mathcal{X}_2$ and $x_3 \in \mathcal{X}_3$. As in the proof of Lemma A.1.3, it can be shown by induction on $m$ that

$$\begin{aligned}&\mathcal{R}^{\text{MA-DBC}}(P_{X_1,X_2|X_3} P_{V,X_3}, P_{Y_3|X_1,X_2,X_3}, P_{Z_1}, P_{Z_2})\\ &\quad \subseteq \mathcal{R}^{\text{MA-DBC}}(\tilde{P}_{X_1,X_2|X_3} P_{V,X_3}, P_{Y_3|X_1,X_2,X_3}, P_{Z_1}, P_{Z_2})\end{aligned}$$

where $P_{X_1,X_2|X_3} \in \Lambda_m$ and $\tilde{P}_{X_1,X_2|X_3} \in \Lambda_{m+1}$ for $1 \leq m < \kappa$. Note that the base case $m = 1$ was proved in Lemma 4.1. Since $P_{X_1,X_2|X_3} \in \Lambda_\kappa$ can be expressed as $P_{X_1,X_2|X_3} = P^{\text{U}}_{\mathcal{X}_1} P_{X_2|X_3}$, (4.44) holds. To show (4.45), we consider input probability distributions of the form $P_{X_1,X_2,X_3,V} = P^{\text{U}}_{\mathcal{X}_1} P_{X_2|X_3} P_{X_3,V}$. By changing the roles of $X_1$ and $X_2$ in the above derivation, the rest of the proof is straightforward. ■

## 4.4 Examples

We next illustrate Theorems 4.3-4.5 via three examples.

**Example 4.1** (**Additive-Noise MA/DB DM-TWCs**)**.** Consider an additive-noise MA/DB DM-TWC in which the inputs and outputs of the DBC are described by (4.1a) and (4.1b) and the inputs and outputs of MAC are related via

$$Y_{3,n} = X_{1,n} \oplus_q X_{2,n} \oplus_q X_{3,n} \oplus_q Z_{3,n},$$

where $\{Z_{3,n}\}_{n=1}^{\infty}$ with $Z_{3,n} \in G_q$ is a discrete memoryless noise process which is independent of all messages of the terminals and the noise processes $\{Z_{1,n}\}_{n=1}^{\infty}$ and $\{Z_{2,n}\}_{n=1}^{\infty}$. For any $x_3 \in \mathcal{X}_3$, we have the following bounds:

$$I(X_1;Y_3|X_2,X_3=x_3) = H(Y_3|X_2,X_3=x_3) - H(Y_3|X_1,X_2,X_3=x_3) \leq \log_2 q - H_{\mathrm{b}}(Z_3),$$

$$I(X_2;Y_3|X_1,X_3=x_3) = H(Y_3|X_1,X_3=x_3) - H(Y_3|X_1,X_2,X_3=x_3) \leq \log_2 q - H_{\mathrm{b}}(Z_3),$$

$$I(X_1,X_2;Y_3|X_3=x_3) = H(Y_3|X_3=x_3) - H(Y_3|X_1,X_2,X_3=x_3) \leq \log_2 q - H_{\mathrm{b}}(Z_3),$$

where equalities hold when $P_{X_1,X_2} = P^{\mathrm{U}}_{\mathcal{X}_1} P^{\mathrm{U}}_{\mathcal{X}_2}$. Choosing $\tilde{P}_{X_1} = P^{\mathrm{U}}_{\mathcal{X}_1}$ and $\tilde{P}_{X_2} = P^{\mathrm{U}}_{\mathcal{X}_2}$, it is clear that (4.15)-(4.17) in Theorem 4.3 hold, and hence the capacity region is given by

$$\begin{aligned}
\mathcal{C}^{\text{MA-DBC}} &= \overline{\mathrm{co}}\left( \bigcup_{P_{V,X_3}} \mathcal{R}^{\text{MA-DBC}}\left(P^{\mathrm{U}}_{\mathcal{X}_1} P^{\mathrm{U}}_{\mathcal{X}_2} P_{U,X_3}, P_{Y_3|X_1,X_2,X_3}, P_{Z_1}, P_{Z_2}\right)\right) \\
&= \overline{\mathrm{co}}\left( \bigcup_{P_{V,X_3}} \Big\{(R_{13},R_{23},R_{31},R_{32}) : R_{13}+R_{23} \leq \log_2 q - H_{\mathrm{b}}(Z_3), \right. \\
&\qquad\qquad R_{31} \leq I(X_1; X_3 \oplus_2 Z_1|V), \\
&\qquad\qquad \left. R_{32} \leq I(X_2 \oplus Z_1 \oplus Z_2; V)\Big\}\right).
\end{aligned}$$

**Example 4.2.** Suppose that $\mathcal{X}_1 = \mathcal{X}_2 = \mathcal{X}_3 = \{0,1\}$, $\mathcal{Y}_1 = \mathcal{Y}_2 = \{0,1\}$, and $\mathcal{Y}_3 = \{0,1,2\}$. We consider a memoryless MA/DB TWC in which the DB direction is described by (4.1a) and (4.1b) and the channel transition matrix $[P_{Y_3|X_1,X_2,X_3}(\cdot\,|\,\cdot,\,\cdot,\,\cdot)]$

for the MA direction is given by

$$\begin{array}{c} \\ 000 \\ 100 \\ 010 \\ 110 \\ 001 \\ 101 \\ 011 \\ 111 \end{array}\begin{array}{c} \begin{array}{ccc} 0 & 1 & 2 \end{array} \\ \begin{pmatrix} 1-\varepsilon & 0 & \varepsilon \\ 1-\varepsilon & 0 & \varepsilon \\ 0 & 1-\varepsilon & \varepsilon \\ 0 & 1-\varepsilon & \varepsilon \\ 0 & \varepsilon & 1-\varepsilon \\ 0 & \varepsilon & 1-\varepsilon \\ 1-\varepsilon & \varepsilon & 0 \\ 1-\varepsilon & \varepsilon & 0 \end{pmatrix} \end{array}$$

where $0 \leq \varepsilon \leq 1$. Since each marginal channel governed by the transition matrix $[P_{Y_3|X_1,X_2,X_3}(\,\cdot\,|\,\cdot\,, x_2, x_3)]$ is quasi-symmetric, we immediately have that $P^*_{X_1} = P^{\mathrm{U}}_{\mathcal{X}_1}$. Also, since $[P_{Y_3|X_1,X_2,X_3}(\,\cdot\,|\,\cdot\,, x_2, x_3)]$, $x_2 \in \mathcal{X}_2$ and $x_3 \in \mathcal{X}_3$, are column permutations of each other, for any fixed $x_3 \in \mathcal{X}_3$, $\mathcal{I}(P^*_{X_1}, P_{Y_3|X_1,X_2=x_2,X_3=x_3})$ does not depend on $x_2 \in \mathcal{X}_2$. Thus, condition (i) of Theorem 4.4 holds. Moreover, condition (ii) holds since the matrices $[P_{Y_3|X_1,X_2,X_3}(\,\cdot\,|x_1,\,\cdot\,, x_3)]$, $x_1 \in \mathcal{X}_1$ and $x_3 \in \mathcal{X}_3$, are column permutations of each other.

Verifying condition (iii) involves several steps. We first observe that $\mathcal{I}(P_{X_1,X_2}, P_{Y_3|X_1,X_2,X_3=x_3})$ does not depend on $x_3 \in \mathcal{X}_3$ for any fixed $P_{X_1,X_2}$ since the matrices $[P_{Y_3|X_1,X_2,X_3}(\,\cdot\,|\,\cdot\,,\,\cdot\,, x_3)]$, $x_3 \in \mathcal{X}_3$, are column permutations of each other. Due to (4.33) and (4.34), it suffices to consider input distributions of this form: $P_{X_1,X_2,X_3,V} = P_{X_1,X_2} P_{X_3,V}$. Thus, given any $P^{(1)}_{X_1,X_2,X_3,V} = P^{(1)}_{X_1,X_2} P^{(1)}_{X_3,V}$, we define $P^{(2)}_{X_1,X_2,X_3,V}(x_1, x_2, x_3, v) = P^{(1)}_{X_1,X_2,X_3,V}(x_1 \oplus_2 1, x_2, x_3, v)$ for all $x_1, x_2, x_3, v$. Also, let

$$P^{(3)}_{X_1,X_2,X_3,V} = \frac{1}{2}\big(P^{(1)}_{X_1,X_2,X_3,V} + P^{(2)}_{X_1,X_2,X_3,V}\big)$$

so that we have $P^{(3)}_{X_1,X_2,X_3,V} = P^{(3)}_{X_1} P^{(1)}_{X_2} P^{(1)}_{X_3,V}$ with $P^{(3)}_{X_1} = P^{\mathrm{U}}_{\mathcal{X}_1} = P^{*}_{X_1}$. Now, since (4.36) holds in this example, one can directly obtain that $I^{(1)}(X_1, X_2; Y_3|X_3 = x_3) \leq I^{(3)}(X_1, X_2; Y_3|X_3 = x_3)$ from the proof of Lemma 4.1. As a result, this TWC satisfies all conditions of Theorem 4.4 and has capacity region given by

$$\mathcal{C}^{\text{MA-DBC}} = \overline{\text{co}}\left( \bigcup_{P_{X_2}, P_{V,X_3}} \mathcal{R}^{\text{MA-DBC}}\left( P^{\mathrm{U}}_{\mathcal{X}_1} P_{X_2} P_{V,X_3}, P_{Y_3|X_1,X_2,X_3}, P_{Z_1}, P_{Z_2} \right) \right).$$

**Example 4.3** (**Binary MA/DB TWCs with Erasures**)**.** Suppose that $\mathcal{X}_1 = \mathcal{X}_2 = \mathcal{X}_3 = \{0, 1\}$, $\mathcal{Y}_1 = \mathcal{Y}_2 = \{0, 1\}$, and $\mathcal{Y}_3 = \{0, 1, \mathbf{E}\}$, where $\mathbf{E}$ denotes erasure symbol. We consider a discrete memoryless MA/DB TWC in which the DBC direction is described by (4.1a) and (4.1b) and the MAC direction is described by

$$Y_{3,n} = (X_{1,n} \oplus_2 X_{2,n} \oplus_2 X_{3,n}) \cdot \mathbf{1}\{Z_{3,n} \neq \mathbf{E}\} + \mathbf{E} \cdot \mathbf{1}\{Z_{3,n} = \mathbf{E}\},$$

where $\{Z_{3,n}\}_{n=1}^{\infty}$ with $Z_{3,n} \in \{0, \mathbf{E}\}$ is a discrete memoryless noise process which is independent of all terminals' messages and the noise processes $\{Z_{1,n}\}_{n=1}^{\infty}$ and $\{Z_{2,n}\}_{n=1}^{\infty}$. Also, we assume that $\Pr\big(Z_{3,n} = \mathbf{E}\big) = \varepsilon$ for all $n$, thereby obtaining the channel transition matrix $[P_{Y_3|X_1,X_2,X_3}(\,\cdot\,|\,\cdot\,,\,\cdot\,,\,\cdot\,)]$ on the top of the next page. It can be directly verified that (4.36) and (4.37) in Theorem 4.5 hold. Hence, the inner and outer bounds coincide and the capacity region is given by

$$\begin{aligned}
\mathcal{C}^{\text{MA-DBC}} &= \overline{\text{co}}\left( \bigcup_{P_{V,X_3}} \mathcal{R}^{\text{MA-DBC}}\left( P^{\mathrm{U}}_{\mathcal{X}_1} P^{\mathrm{U}}_{\mathcal{X}_2} P_{V,X_3}, P_{Y_3|X_1,X_2,X_3}, P_{Z_1}, P_{Z_2} \right) \right) \\
&= \overline{\text{co}}\left( \bigcup_{P_{V,X_3}} \big\{ (R_{13}, R_{23}, R_{31}, R_{32}) : R_{13} + R_{23} \leq 1 - H_{\mathrm{b}}(\epsilon), \right. \\
&\qquad\qquad R_{31} \leq I(X_1; X_3 \oplus_2 Z_1 | V), \\
&\qquad\qquad \left. R_{32} \leq I(X_2 \oplus_2 Z_1 \oplus_2 Z_2; V) \big\} \right).
\end{aligned}$$

$$
[P_{Y_3|X_1,X_2,X_3}(\cdot\,|\,\cdot,\,\cdot,\,\cdot)] = \begin{array}{c} \\ 000 \\ 100 \\ 010 \\ 110 \\ 001 \\ 101 \\ 011 \\ 111 \end{array} \begin{array}{c} \begin{array}{ccc} 0 & 1 & \mathbf{E} \end{array} \\ \begin{pmatrix} 1-\varepsilon & 0 & \varepsilon \\ 0 & 1-\varepsilon & \varepsilon \\ 0 & 1-\varepsilon & \varepsilon \\ 1-\varepsilon & 0 & \varepsilon \\ 0 & 1-\varepsilon & \varepsilon \\ 1-\varepsilon & 0 & \varepsilon \\ 1-\varepsilon & 0 & \varepsilon \\ 0 & 1-\varepsilon & \varepsilon \end{pmatrix} \end{array}
$$

**Remark 4.1.** Examples 4.2 and 4.3 also satisfy Theorem 4.3 since the product distribution $\tilde{P}_{X_1}\tilde{P}_{X_2}$ required by Theorem 4.3 are explicitly given in these examples. Moreover, it is straightforward to show that Examples 4.2 and 4.3 do not satisfy the conditions of Theorems 4.5 and 4.4, respectively. In other words, Theorems 4.4 and 4.5 are neither equivalent nor special cases of each other.

# Chapter 5

# Two-Way Source-Channel Coding

## 5.1 Preliminaries

Recall the problem setup in Section 1.4, which is recapped in Fig. 5.1. Two terminals $T_1$ and $T_2$ exchange a block of correlated source messages $(S_1^K, S_2^K)$ of length-$K$ via $N$ uses of a noisy DM-TWC. Lossy reconstructions are allowed, and our objective is to seek direct and converse coding theorems for lossy source-channel transmissibility. Note that the noisy DM-TWC here is used without adopting any interactive communication protocol and the memoryless property of the channel implies that $P_{Y_{1,n},Y_{2,n}|X_1^n,X_2^n,Y_1^{n-1},Y_2^{n-1}} = P_{Y_{1,n},Y_{2,n}|X_{1,n},X_{2,n}} = P_{Y_1,Y_2|X_1,X_2}$ for all $n$.

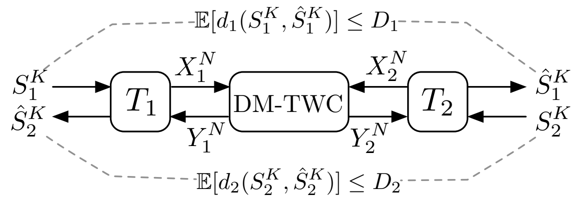



Figure 5.1: The block diagram for the lossy transmission of correlated source pair $(S_1^K, S_2^K)$ via $N$ uses of a noisy DM-TWC.

### 5.1.1 Definitions

In this section, we define joint source-channel codes and the achievable distortion region for source-channel communication over a DM-TWC. We also review various RD function expressions for point-to-point communication and channel coding results for DM-TWCs, which will be used in Section 5.3. For our problem setup, a joint source-channel code is defined as follows.

**Definition 5.1.** *An $(N, K)$ code for transmitting $(S_1^K, S_2^K)$ over a DM-TWC consists of two sequences of encoding functions $\boldsymbol{f}_1 \triangleq \{f_{1,n}\}_{n=1}^N$ and $\boldsymbol{f}_2 \triangleq \{f_{2,n}\}_{n=1}^N$ such that*

$$f_{1,1} : \mathcal{S}_1^K \to \mathcal{X}_1, \quad f_{1,n} : \mathcal{S}_1^K \times \mathcal{Y}_1^{n-1} \to \mathcal{X}_1$$
$$f_{2,1} : \mathcal{S}_2^K \to \mathcal{X}_2, \quad f_{2,n} : \mathcal{S}_2^K \times \mathcal{Y}_2^{n-1} \to \mathcal{X}_2$$

*for $n = 2, 3, \ldots, N$, and two decoding functions $g_1 : \mathcal{S}_1^K \times \mathcal{Y}_1^N \to \hat{\mathcal{S}}_2^K$ and $g_2 : \mathcal{S}_2^K \times \mathcal{Y}_2^N \to \hat{\mathcal{S}}_1^K$.*

The channel inputs at time $n = 1$ are only functions of the source messages, i.e., $X_{j,1} = f_{j,1}(S_j^K)$, but the subsequent channel inputs are generated by also adapting to the previous channel outputs via $X_{j,n} = f_{j,n}(S_j^K, Y_j^{n-1})$ for $n = 2, 3, \ldots, N$. Such encoding strategy is known as adaptive coding, in contrast to its non-adaptive counterpart where $X_{j,n} = f_{j,n}(S_j^k)$ for all $n$. We remark that our code definition also involves block-wise decoding; i.e., terminal $j$ reconstructs $S_{j'}^K$ via $\hat{S}_{j'}^K = g_j(S_j^K, Y_j^N)$ after receiving the entire $N$ channel outputs.

Moreover, the rate of the joint source-channel code is given by $K/N$ (source symbols/channel use), and the associated expected distortion is $D_j(K) \triangleq \mathbb{E}[d_j(S_j^K, \hat{S}_j^K)]$, where the expectation is taken with respect to the joint probability distribution

$$P_{S_1^K,S_2^K,X_1^N,X_2^N,Y_1^N,Y_2^N} = P_{S_1^K,S_2^K}\left(\prod_{n=1}^{N} P_{X_{1,n}|S_1^K,Y_1^{n-1}}\right) \cdot\left(\prod_{n=1}^{N} P_{X_{2,n}|S_2^K,Y_2^{n-1}}\right)\left(\prod_{n=1}^{N} P_{Y_{1,n},Y_{2,n}|X_{1,n},X_{2,n}}\right),$$

where $P_{Y_{1,n}Y_{2,n}|X_{1,n},X_{2,n}} = P_{Y_1,Y_2|X_1,X_2}$ for $n = 1, 2, \ldots, N$ (determined by the DM-TWC).

**Definition 5.2.** *A distortion pair* $(D_1, D_2)$ *is said to be achievable at rate* $R$ *if there exists a sequence of* $(N, K)$ *joint source-channel codes (where* $N$ *is a function of* $K$*) such that* $\lim_{K\to\infty} K/N = R$ *and* $\limsup_{K\to\infty} D_j(K) \le D_j$, $j = 1, 2$. *The achievable distortion region of a rate-*$R$ *two-way lossy transmission system is the convex closure of all achievable distortion pairs (at rate* $R$*).*

### 5.1.2 Rate-Distortion Functions

As a DM-TWC can be viewed as two state-dependent one-way channels, the following source coding related functions (each expressed in terms of a constrained minimization of a mutual information quantity) for one-way systems are also useful in the two-way channel setup.

- Standard RD function [47, Sec. 3.6]:

$$R^{(j)}(D_j) = \min_{P_{\hat{S}_j|S_j}:\mathbb{E}[d_j(S_j,\hat{S}_j)]\le D_j} I(S_j; \hat{S}_j). \tag{5.1}$$

- WZ RD function [48]: Letting $T_j \in \mathcal{T}_j$ with $|\mathcal{T}_j| \le |\mathcal{S}_j| + 1$ denote an auxiliary random variable that satisfies the Markov chain $T_j \multimap S_j \multimap S_{j'}$, we have

$$R_{\mathrm{WZ}}^{(j)}(D_j) = \min_{P_{T_j|S_j}} \min_{\substack{h:\mathcal{T}_j\times\mathcal{S}_{j'}\to\hat{\mathcal{S}}_j \\ \mathbb{E}[d_j(S_j,h(T_j,S_{j'}))]\le D_j}} I(S_j;T_j|S_{j'}). \tag{5.2}$$

- Conditional RD function [105]:

$$R_{S_j|S_{j'}}(D_j) = \min_{\substack{P_{\hat{S}_j|S_1,S_2} \\ \mathbb{E}[d_j(S_j,\hat{S}_j)]\le D_j}} I(S_j;\hat{S}_j|S_{j'}). \tag{5.3}$$

We remark that the source coding schemes that achieve the standard RD and WZ-RD functions can be the building blocks of an SSCC scheme for our overall system. For example, terminal $j$ can apply the WZ coding scheme to compress source $S_j^K$ given side-information $S_{j'}^K$. Although the coding scheme that achieves the conditional RD function cannot be applied in our problem setup (since there is no common side-information at the encoder and the decoder in general), the scheme is useful when $S_1$ and $S_2$ have a common part in the sense of Gács-Körner-Witsenhausen [47, Section 14.2.2]. We will use this result in Theorem 5.5 (see Section 5.4.2).

## 5.2 Forward JSCC Theorem Based on Adaptive Coding

This section establishes the most general achievability result in the paper. Without loss of generality, we only consider rate-one transmission, i.e., $N = K$; other rates can be obtained via suitable super-symbols.[1] First of all, we describe the key technical ingredients used in obtaining the main result in Theorem 5.1. Our approach is to construct an extended channel (from the original DM-TWC) and use a stationary

[1] To obtain a rate-$\frac{K_1}{N_1}$ result, we define a super source symbol (resp., a super channel input/output symbol) by combining $K_1$ source symbols (resp., $N_1$ channel input/output symbols).

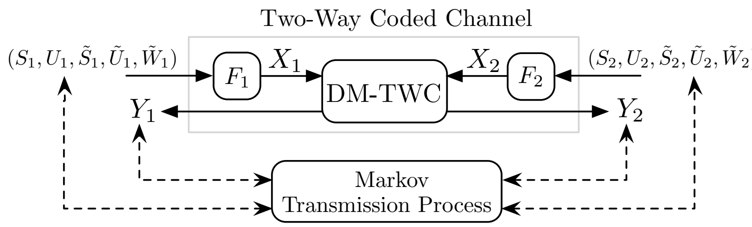



Figure 5.2: An illustration of the transmission over the two-way coded channel.

Markov chain to coordinate the terminals' transmissions.

### 5.2.1 Two-Way Coded Channel

Consider an auxiliary coded channel built on the original (physical) DM-TWC, as shown in the central box of Fig. 5.2. The coded channel has inputs $S_j, U_j, \tilde{S}_j, \tilde{U}_j$ and $\tilde{W}_j$ at terminal $j$. The input pairs $(S_j, U_j)$ and $(\tilde{S}_j, \tilde{U}_j)$ are used to carry the current and some prior source information, respectively, where $U_j$ (resp., $\tilde{U}_j$) denotes the coded version of $S_j$ (resp., $\tilde{S}_j$). The input $\tilde{W}_j$ carries some past channel inputs and outputs at terminal $j$. The new channel also involves two encoding functions $F_j : \mathcal{S}_j \times \mathcal{U}_j \times \tilde{\mathcal{S}}_j \times \tilde{\mathcal{U}}_j \times \tilde{\mathcal{W}}_j \to \mathcal{X}_j$, which transform the inputs of the coded channel into the inputs for the original DM-TWC. The outputs of the new channel are still $Y_1$ and $Y_2$. The joint input probability distribution of the coded channel is given by

$$P_{S_1,S_2,U_1,U_2,\tilde{S}_1,\tilde{S}_2,\tilde{U}_1,\tilde{U}_2,\tilde{W}_1,\tilde{W}_2} = P_{S_1,S_2} P_{U_1|S_1} P_{U_2|S_2} P_{\tilde{S}_1,\tilde{S}_2,\tilde{U}_1,\tilde{U}_2,\tilde{W}_1,\tilde{W}_2},$$

and the transition probability of the coded channel is given by

$$\begin{aligned} & P_{Y_1,Y_2|S_1,S_2,U_1,U_2,\tilde{S}_1,\tilde{S}_2,\tilde{U}_1,\tilde{U}_2,\tilde{W}_1,\tilde{W}_2}(y_1, y_2|s_1, s_2, u_1, u_2, \tilde{s}_1, \tilde{s}_2, \tilde{u}_1, \tilde{u}_2, \tilde{w}_1, \tilde{w}_2) \\ &= \sum_{x_1,x_2} \mathbf{1}\{x_1 = F_1(s_1, u_1, \tilde{s}_1, \tilde{u}_1, \tilde{w}_1)\} \\ &\qquad \mathbf{1}\{x_2 = F_2(s_2, u_2, \tilde{s}_2, \tilde{u}_2, \tilde{w}_2)\} P_{Y_1,Y_2|X_1,X_2}(y_1, y_2|x_1, x_2). \end{aligned} \tag{5.4}$$

### 5.2.2 Markov Chain for the Coded Channel

For the repeated use over time of the two-way coded channel, we next construct a discrete-time Markov chain for the overall system with state space:

$$\mathcal{S}_1 \times \mathcal{S}_2 \times \mathcal{U}_1 \times \mathcal{U}_2 \times \tilde{\mathcal{S}}_1 \times \tilde{\mathcal{S}}_2 \times \tilde{\mathcal{U}}_1 \times \tilde{\mathcal{U}}_2 \times \tilde{\mathcal{W}}_1 \times \tilde{\mathcal{W}}_2 \times \mathcal{X}_1 \times \mathcal{X}_2 \times \mathcal{Y}_1 \times \mathcal{Y}_2,$$

where $\tilde{\mathcal{S}}_j \triangleq \mathcal{S}_j$, $\tilde{\mathcal{U}}_j \triangleq \mathcal{U}_j$, and $\tilde{\mathcal{W}}_j \triangleq \mathcal{X}_j \times \mathcal{Y}_j$ for $j = 1, 2$. This Markov chain will be used to coordinate the transmissions of the two terminals as shown in Fig. 5.2. Let

$$Z^{(t)} \triangleq (S_1^{(t)}, S_2^{(t)}, U_1^{(t)}, U_2^{(t)}, \tilde{S}_1^{(t)}, \tilde{S}_2^{(t)}, \tilde{U}_1^{(t)}, \tilde{U}_2^{(t)}, \tilde{W}_1^{(t)}, \tilde{W}_2^{(t)}, X_1^{(t)}, X_2^{(t)}, Y_1^{(t)}, Y_2^{(t)})$$

denote the state of the Markov chain at time $t \in \mathbb{Z}_+$, where we set

$$\tilde{S}_j^{(t)} \triangleq S_j^{(t-1)}, \ \tilde{U}_j^{(t)} \triangleq U_j^{(t-1)}, \text{ and } \tilde{W}_j^{(t)} \triangleq (X_j^{(t-1)}, Y_j^{(t-1)}).$$

Given a parameter tuple $(P_{U_1|S_1}, P_{U_2|S_2}, P_{\tilde{S}_1,\tilde{S}_2,\tilde{U}_1,\tilde{U}_2,\tilde{W}_1,\tilde{W}_2}, F_1, F_2)$, we generate the quadruple $(S_1^{(t)}, S_2^{(t)}, U_1^{(t)}, U_2^{(t)})$ for all $t$ according to $P_{S_1,S_2,U_1,U_2} = P_{S_1,S_2}P_{U_1|S_1}P_{U_2|S_2}$ independently of $(\tilde{S}_1^{(t)}, \tilde{S}_2^{(t)}, \tilde{U}_1^{(t)}, \tilde{U}_2^{(t)}, \tilde{W}_1^{(t)}, \tilde{W}_2^{(t)})$. We also initialize the tuple $(\tilde{S}_1^{(1)}, \tilde{S}_2^{(1)}, \tilde{U}_1^{(1)}, \tilde{U}_2^{(1)}, \tilde{W}_1^{(1)}, \tilde{W}_2^{(1)})$ according to $P_{\tilde{S}_1,\tilde{S}_2,\tilde{U}_1,\tilde{U}_2,\tilde{W}_1,\tilde{W}_2}$, while the physical channel input at terminal $j$ is naturally produced as $X_j^{(t)} = F_j(S_j^{(t)}, U_j^{(t)}, \tilde{S}_j^{(t)}, \tilde{U}_j^{(t)}, \tilde{W}_j^{(t)})$, and the received channel output is $Y_j^{(t)}$. Based on this construction, the transition kernel of $\{Z^{(t)}\}$ is given by

$$\begin{aligned}
&P_{Z^{(t)}|Z^{(t-1)}}(s_1, s_2, u_1, u_2, \tilde{s}_1, \tilde{s}_2, \tilde{u}_1, \tilde{u}_2, \tilde{w}_1, \tilde{w}_2, x_1, x_2, y_1, y_2 \\
&\qquad\qquad\qquad |s'_1, s'_2, u'_1, u'_2, \tilde{s}'_1, \tilde{s}'_2, \tilde{u}'_1, \tilde{u}'_2, \tilde{w}'_1, \tilde{w}'_2, x'_1, x'_2, y'_1, y'_2) \\
&= P_{S_1,S_2}(s_1, s_2)P_{U_1|S_1}(u_1|s_1)P_{U_2|S_2}(u_2|s_2)\mathbf{1}\{\tilde{s}_1 = s'_1\}\mathbf{1}\{\tilde{s}_2 = s'_2\}\mathbf{1}\{\tilde{u}_1 = u'_1\}\mathbf{1}\{\tilde{u}_2 = u'_2\} \\
&\quad \cdot\mathbf{1}\{\tilde{w}_1 = (x'_1, y'_1)\}\mathbf{1}\{\tilde{w}_2 = (x'_2, y'_2)\}\mathbf{1}\{x_1 = F_1(s_1, u_1, \tilde{s}_1, \tilde{u}_1, \tilde{w}_1)\}
\end{aligned}$$

$$\cdot \mathbf{1}\{x_2 = F_2(s_2, u_2, \tilde{s}_2, \tilde{u}_2, \tilde{w}_2)\} P_{Y_1,Y_2|X_1,X_2}(y_1, y_2|x_1, x_2) \qquad (5.5)$$

for $t \geq 2$. It is easy to see that the process $\{Z^{(t)}\}$ is a first-order time-homogeneous Markov chain. However, whether or not the chain is stationary depends on the given parameters.

### 5.2.3 Stationary Distribution under Distortion Constraints

To obtain an achievability result with time-independent conditions, we only consider a stationary Markov chain for the coded channel. The following procedure can be used to find its parameters. Given $P_{S_1,S_2}$ and $P_{Y_1,Y_2|X_1,X_2}$, we first fix a choice of $P_{U_j|S_j}$ and $F_j$, $j = 1, 2$, and write the transition kernel (5.5) in matrix form as $Q_Z$. The matrix $Q_Z$ is stochastic, and since all alphabets are finite, an eigenvector of $Q_Z$ associated with the eigenvalue 1 exists and gives a stationary distribution $P_Z$ for $\{Z^{(t)}\}$, i.e., $P_Z = P_Z Q_Z$. Clearly, using the marginal distribution $P_{\tilde{S}_1,\tilde{S}_2,\tilde{U}_1,\tilde{U}_2,\tilde{W}_1,\tilde{W}_2}$ of $P_Z$ with the chosen $P_{U_j|S_j}$ and $F_j$, $j = 1, 2$, to initialize the Markov chain ensures stationarity. Note that for the stationary chain the two independent quadruples $(S_1^{(t)}, S_2^{(t)}, U_1^{(t)}, U_2^{(t)})$ and $(\tilde{S}_1^{(t)}, \tilde{S}_2^{(t)}, \tilde{U}_1^{(t)}, \tilde{U}_2^{(t)})$ have identical distributions for all $t$; thus $P_{S_1,S_2,U_1,U_2} = P_{\tilde{S}_1,\tilde{S}_2,\tilde{U}_1,\tilde{U}_2}$. Moreover, due to our construction of $\{Z^{(t)}\}$, we have the following necessary conditions for stationarity

$$P_{S_1,S_2} = P_{\tilde{S}_1,\tilde{S}_2}, \qquad (5.6)$$

$$P_{U_j|S_j} = P_{\tilde{U}_j|\tilde{S}_j}, \qquad (5.7)$$

for $j = 1, 2$. For source reconstruction, we next associate the parameters with decoding functions[2] $G_j : \tilde{\mathcal{U}}_{j'} \times \mathcal{S}_j \times \mathcal{U}_j \times \tilde{\mathcal{S}}_j \times \tilde{\mathcal{U}}_j \times \tilde{\mathcal{W}}_j \times \mathcal{Y}_j \to \hat{\tilde{\mathcal{S}}}_{j'}$, $j = 1, 2$. For simplicity, we call the tuple $(P_{U_1|S_1}, P_{U_2|S_2}, P_{\tilde{S}_1,\tilde{S}_2,\tilde{U}_1,\tilde{U}_2}, P_{\tilde{W}_1,\tilde{W}_2|\tilde{S}_1,\tilde{S}_2,\tilde{U}_1,\tilde{U}_2}, F_1, F_2, G_1, G_2)$ a *configuration*, which specifies a stationary distribution $P_Z$ given by

$$P_Z = \underbrace{P_{S_1,S_2} P_{U_1|S_1} P_{U_2|S_2}}_{=P_{S_1,S_2,U_1,U_2}} \underbrace{P_{\tilde{S}_1,\tilde{S}_2} P_{\tilde{U}_1|\tilde{S}_1} P_{\tilde{U}_2|\tilde{S}_2}}_{=P_{\tilde{S}_1,\tilde{S}_2,\tilde{U}_1,\tilde{U}_2}} P_{\tilde{W}_1,\tilde{W}_2|\tilde{S}_1,\tilde{S}_2,\tilde{U}_1,\tilde{U}_2} \cdot P_{X_1|S_1,U_1,\tilde{S}_1,\tilde{U}_1,\tilde{W}_1} P_{X_2|S_2,U_2,\tilde{S}_2,\tilde{U}_2,\tilde{W}_2} P_{Y_1,Y_2|X_1,X_2},$$

where $P_{S_1,S_2}$ and $P_{Y_1,Y_2|X_1,X_2}$ are fixed by the problem setup and $P_{X_j|S_j,U_j,\tilde{S}_j,\tilde{U}_j,\tilde{W}_j}$ is determined by $F_j$, $j = 1, 2$. We also let $\Pi_Z(D_1, D_2)$ denote the set of all configurations that induce a stationary chain and satisfy the distortion constraint $\mathbb{E}[d_j(\tilde{S}_j, \hat{\tilde{S}}_j)] \le D_j$ for $j = 1, 2$. Note that the set $\Pi_Z(D_1, D_2)$ might be empty for some $(D_1, D_2)$.

### 5.2.4 Main Result: JSCC Achievability

Based on the above setup, we establish the achievability result in Theorem 5.1 below. In Theorem 5.1, one can further convexify the achievable distortion region via a standard time-sharing argument [74].

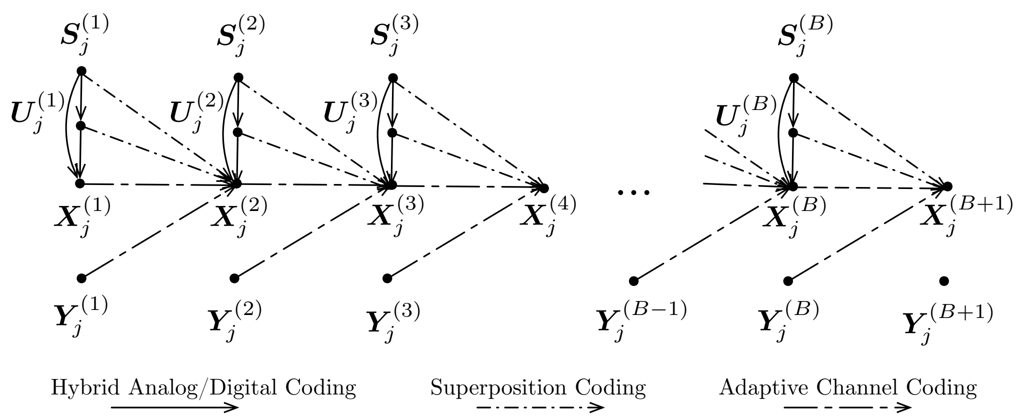
**Theorem 5.1** (Adaptive JSCC)**.** *A distortion pair $(D_1, D_2)$ is achievable for the rate-one lossy transmission of correlated sources over a DM-TWC if there exists a configuration in $\Pi_Z(D_1, D_2)$ such that*

$$I(\tilde{S}_1; \tilde{U}_1) < I(\tilde{U}_1; S_2, U_2, \tilde{S}_2, \tilde{U}_2, \tilde{W}_2, X_2, Y_2), \tag{5.8a}$$

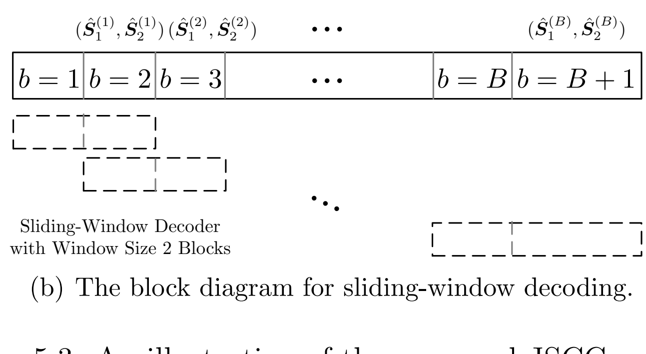
$$I(\tilde{S}_2; \tilde{U}_2) < I(\tilde{U}_2; S_1, U_1, \tilde{S}_1, \tilde{U}_1, \tilde{W}_1, X_1, Y_1). \tag{5.8b}$$

[2] As will be seen in the proof of Theorem 5.1, terminal $j$ reconstructs the prior source message $\tilde{S}_{j'}$ as $\hat{\tilde{S}}_{j'}$ after recovering $\tilde{U}_{j'}$; this reconstruction is done via $G_j$.

To facilitate the understanding of the conditions in (5.8), we sketch our coding method before giving a formal proof, which extends the hybrid analog/digital coding scheme of [89], used in conjunction with superposition block Markov encoding [28, 106] and a sliding-window decoder, as shown in Fig. 5.3. In our method, instead of exchanging a single block of source message pairs $(S_1^K, S_2^K)$ via $K$ channel uses, we exchange $B$ blocks of such source message pairs via $K(B+1)$ channel uses for some $B \in \mathbb{Z}_+$. The overall transmission rate is $\frac{B}{B+1}$, which approaches 1 as $B \to \infty$. The extra $K$ channel uses can be viewed as added redundancy for data protection.




(a) The encoding process of terminal $j$, where each node represents a block of variables and each node is a function of other nodes specified by the incoming edges.




(b) The block diagram for sliding-window decoding.

Figure 5.3: An illustration of the proposed JSCC method.

For $1 \le b \le B$, let $\boldsymbol{S}_j^{(b)} = (S_{j,1}^{(b)}, S_{j,2}^{(b)}, \ldots, S_{j,K}^{(b)})$ denote the $b$th source message block at terminal $j$; the same indexing convention applies to other variables. As shown in Fig. 5.3(a),[3] the encoding involves hybrid analog/digital coding, superposition coding, and adaptive channel coding. In the $b$th transmission block, terminal $j$ first encodes its source message $\boldsymbol{S}_j^{(b)}$ into the digital codeword $\boldsymbol{U}_j^{(b)}$. The codeword $\boldsymbol{U}_j^{(b)}$ not only plays the role of source compression but also data protection. Then, the current information $(\boldsymbol{S}_j^{(b)}, \boldsymbol{U}_j^{(b)})$ and the prior information $(\boldsymbol{S}_j^{(b-1)}, \boldsymbol{U}_j^{(b-1)})$ and $(\boldsymbol{X}_j^{(b-1)}, \boldsymbol{Y}_j^{(b-1)})$ are combined to generate the channel input $\boldsymbol{X}_j^{(b)}$.

To reconstruct source messages, we employ a sliding-window decoder as depicted in Fig. 5.3(b). The decoder is designed to operate on two consecutive transmission blocks, but each time it only decodes the earlier source block. For $2 \leq b \leq B+1$, suppose that the decoding window is now across the $(b-1)$st and the $b$th transmission blocks. Given that terminal $j$ has successfully recovered $\boldsymbol{U}_{j'}^{(b')}$ and reconstructed $\boldsymbol{S}_{j'}^{(b')}$ for all $b' < b-1$, the decoder uses all available information in the $(b-1)$st and the $b$th blocks to recover $\boldsymbol{U}_{j'}^{(b-1)}$ and reconstructs $\boldsymbol{S}_{j'}^{(b-1)}$ as $\hat{\boldsymbol{S}}_{j'}^{(b-1)}$ via $G_j$. Then, the decoder moves to the $b$th and the $(b+1)$st blocks to reconstruct $\boldsymbol{S}_{j'}^{(b)}$.

With the above sketch, the left-hand-side and the right-hand-side of (5.8) can be interpreted as source compression rates and as transmission rates for reliable communication, respectively. Moreover, the appearance of $(\tilde{S}_j, \tilde{U}_j)$ (rather than $(S_j, U_j)$) on the left-hand-side of (5.8) is due to the sliding-window decoder. The tuple $(S_j, U_j, \tilde{S}_j, \tilde{U}_j, \tilde{W}_j, X_j, Y_j)$ on the right-hand-side of (5.8) also illuminates the fact that the decoder at terminal $j$ uses all information within two blocks to decode

[3]To simplify the presentation of our encoding scheme, we write $\boldsymbol{S}_j^{(b-1)}, \boldsymbol{U}_j^{(b-1)}$, and $(\boldsymbol{X}_j^{(b-1)}, \boldsymbol{Y}_j^{(b-1)})$ in lieu of $\tilde{\boldsymbol{S}}_j^{(b)}$, $\tilde{\boldsymbol{U}}_j^{(b)}$, and $\tilde{\boldsymbol{W}}_j^{(b)}$, respectively, to refer to the prior information variables of the $b$th block, for $2 \leq b \leq B+1$. Later, when presenting our decoder, we also use $\hat{\boldsymbol{S}}_j^{(b-1)}$ (resp., $\hat{\boldsymbol{U}}_j^{(b-1)}$) rather than $\hat{\tilde{\boldsymbol{S}}}_j^{(b)}$ (resp., $\hat{\tilde{\boldsymbol{U}}}_j^{(b)}$) to denote the reconstruction of $\tilde{\boldsymbol{S}}_j^{(b)}$ (resp., $\tilde{\boldsymbol{U}}_j^{(b)}$).

$\tilde{U}_{j'}$. Now, we present the detailed proof of Theorem 5.1 as follows.

*Proof:* For the sake of brevity, the complete proof is presented using several auxiliary claims whose proofs are given in Appendix C.1. Let $\mathcal{T}_\epsilon^{(n)}$ denote the typical set of sequences with parameters $n \in \mathbb{Z}_+$ and $\epsilon > 0$ as defined in [47]; the domain of $\mathcal{T}_\epsilon^{(n)}$ will be clear from the context and hence omitted. Here, we set $n = N = K$ as we consider the rate-one transmission. For $j = 1, 2$ and $b = 1, 2, \cdots, B$, we define $2^{nR_j^{(b)}}$ as the size of terminal $j$'s codebook $\mathcal{C}_j^{(b)}$, which is used to encode the $b$th block $\boldsymbol{S}_j^{(b)}$ of source messages. For an event $\mathcal{E}$, we let $\overline{\mathcal{E}}$ denote its complement.

<u>*Codebook Generation*</u>: Given a configuration in $\Pi_Z(D_1, D_2)$, generate two length-$n$ sequences $(\tilde{\boldsymbol{S}}_1^{(1)}, \tilde{\boldsymbol{S}}_2^{(1)}, \tilde{\boldsymbol{U}}_1^{(1)}, \tilde{\boldsymbol{U}}_2^{(1)}, \tilde{\boldsymbol{W}}_1^{(1)}, \tilde{\boldsymbol{W}}_2^{(1)})$ and $(\boldsymbol{S}_1^{(B+1)}, \boldsymbol{S}_2^{(B+1)}, \boldsymbol{U}_1^{(B+1)}, \boldsymbol{U}_2^{(B+1)})$ to initialize and terminate the $(B+1)$-blocks encoding process with distributions

$$P_{\tilde{\boldsymbol{S}}_1^{(1)}, \tilde{\boldsymbol{S}}_2^{(1)}, \tilde{\boldsymbol{U}}_1^{(1)}, \tilde{\boldsymbol{U}}_2^{(1)}, \tilde{\boldsymbol{W}}_1^{(1)}, \tilde{\boldsymbol{W}}_2^{(1)}}(\tilde{\boldsymbol{s}}_1^{(1)}, \tilde{\boldsymbol{s}}_2^{(1)}, \tilde{\boldsymbol{u}}_1^{(1)}, \tilde{\boldsymbol{u}}_2^{(1)}, \tilde{\boldsymbol{w}}_1^{(1)}, \tilde{\boldsymbol{w}}_2^{(1)}) \\ = \prod_{i=1}^{n} P_{\tilde{S}_1, \tilde{S}_2, \tilde{U}_1, \tilde{U}_2, \tilde{W}_1, \tilde{W}_2}(\tilde{s}_{1,i}^{(1)}, \tilde{s}_{2,i}^{(1)}, \tilde{u}_{1,i}^{(1)}, \tilde{u}_{2,i}^{(1)}, \tilde{w}_{1,i}^{(1)}, \tilde{w}_{2,i}^{(1)}) \quad (5.9)$$

and

$$P_{\boldsymbol{S}_1^{(B+1)}, \boldsymbol{S}_2^{(B+1)}, \boldsymbol{U}_1^{(B+1)}, \boldsymbol{U}_2^{(B+1)}}(\boldsymbol{s}_1^{(B+1)}, \boldsymbol{s}_2^{(B+1)}, \boldsymbol{u}_1^{(B+1)}, \boldsymbol{u}_2^{(B+1)}) \\ = \prod_{i=1}^{n} P_{S_1, S_2, U_1, U_2}(s_{1,i}^{(B+1)}, s_{2,i}^{(B+1)}, u_{1,i}^{(B+1)}, u_{2,i}^{(B+1)}). \quad (5.10)$$

Moreover, generate codebooks $\mathcal{C}_j^{(b)} \triangleq \{\boldsymbol{U}_j^{(b)}(m_j^{(b)}) : m_j^{(b)} = 1, 2, \ldots, 2^{nR_j^{(b)}}\}$ for $b = 1, 2, \ldots, B$ and $j = 1, 2$, where $\boldsymbol{U}_j^{(b)}(m_j^{(b)})$ is a length-$n$ sequence distributed according to $P_{\boldsymbol{U}_j}(\boldsymbol{u}_j^{(b)}(m_j^{(b)})) = \prod_{i=1}^{n} P_{U_j}(u_{j,i}^{(b)}(m_j^{(b)}))$ and $\boldsymbol{U}_j^{(b)}(m_j^{(b)})$'s are independent of each other. The initialization and termination sequences and all codebooks are revealed to both terminals. We note that due to the construction of the Markov chain $\{Z^{(t)}\}$, the codebook $\mathcal{C}_j^{(b)}$ is also used for $\tilde{\boldsymbol{U}}_j^{(b+1)}$.

Encoding: Let $\epsilon_1 > \epsilon > 0$. For $b = 1, 2, \ldots, B$ and $j = 1, 2$, terminal $j$ finds $m_j^{(b)}$ such that $(\boldsymbol{S}_j^{(b)}, \boldsymbol{U}(m_j^{(b)})) \in \mathcal{T}_{\epsilon_1}^{(n)}$. If there is more than one such index, the encoder chooses one of them at random. If there is no such index, it chooses an index at random from $\{1, 2, \ldots, 2^{nR_j^{(b)}}\}$. The transmitter then sends $\boldsymbol{X}_j^{(b)}$, where

$$X_{j,i}^{(b)} = F_j(S_{j,i}^{(b)}, U_{j,i}^{(b)}(m_j^{(b)}), \tilde{S}_{j,i}^{(b)}, \tilde{U}_{j,i}^{(b)}, \tilde{W}_{j,i}^{(b)})$$

for $i = 1, 2, \ldots, n$, $\tilde{S}_{j,i}^{(b)} = S_{j,i}^{(b-1)}$, $\tilde{U}_{j,i}^{(b)} = U_{j,i}^{(b-1)}$, and $\tilde{W}_{j,i}^{(b)} = (X_{j,i}^{(b-1)}, Y_{j,i}^{(b-1)})$ for $b = 2, 3, \ldots, B$. For $b = B + 1$, $\boldsymbol{X}^{(B+1)}$ is generated in the same way using the termination sequence.

Decoding: For $b = 2, 3, \ldots, B + 1$ and $j, j' = 1, 2$ with $j \neq j'$, terminal $j$ finds an index $\hat{m}_{j'}^{(b-1)}$ such that

$$(\boldsymbol{S}_j^{(b)}, \boldsymbol{U}_j^{(b)}, \tilde{\boldsymbol{S}}_j^{(b)}, \tilde{\boldsymbol{U}}_j^{(b)}, \tilde{\boldsymbol{U}}_{j'}^{(b)}(\hat{m}_{j'}^{(b-1)}), \tilde{\boldsymbol{W}}_j^{(b)}, \boldsymbol{X}_j^{(b)}, \boldsymbol{Y}_j^{(b)}) \in \mathcal{T}_{\epsilon}^{(n)},$$

where $\tilde{\boldsymbol{U}}_{j'}^{(b)}(\hat{m}_{j'}^{(b-1)}) \in \mathcal{C}_{j'}^{(b-1)}$. If there is more than one choice, the decoder chooses one of them at random. If there is no such index, it chooses one at random from $\{1, 2, \ldots, 2^{nR_{j'}^{(b)}}\}$. The reconstruction for the source message $\boldsymbol{S}_{j'}^{(b-1)}$ is given by

$$\hat{S}_{j',i}^{(b-1)} = G_j(\tilde{U}_{j',i}^{(b)}(\hat{m}_{j'}^{(b-1)}), S_{j,i}^{(b)}, U_{j,i}^{(b)}, \tilde{S}_{j,i}^{(b)}, \tilde{U}_{j,i}^{(b)}, \tilde{W}_{j,i}^{(b)}, Y_{j,i}^{(b)})$$

for $i = 1, 2, \ldots, n$.

Performance Analysis: Let $M_j^{(b)}$ and $\hat{M}_j^{(b)}$ denote the random encoded and decoded indices for $\boldsymbol{S}_j^{(b)}$. We first define the following events for terminal 1.

$$\mathcal{E}_1^{(1)} \triangleq \{(\boldsymbol{S}_1^{(1)}, \boldsymbol{S}_2^{(1)}, \boldsymbol{U}_1^{(1)}(M_1^{(1)}), \boldsymbol{U}_2^{(1)}(M_2^{(1)}), \tilde{\boldsymbol{S}}_1^{(1)}, \tilde{\boldsymbol{S}}_2^{(1)},$$
$$\tilde{\boldsymbol{U}}_1^{(1)}, \tilde{\boldsymbol{U}}_2^{(1)}, \tilde{\boldsymbol{W}}_1^{(1)}, \tilde{\boldsymbol{W}}_2^{(1)}, \boldsymbol{X}_1^{(1)}, \boldsymbol{X}_2^{(1)}, \boldsymbol{Y}_1^{(1)}, \boldsymbol{Y}_2^{(1)}) \notin T_\epsilon^{(n)}\}. \tag{5.11a}$$

$$\mathcal{E}_1^{(B+1)} \triangleq \{(\boldsymbol{S}_1^{(B+1)}, \boldsymbol{S}_2^{(B+1)}, \boldsymbol{U}_1^{(B+1)}, \boldsymbol{U}_2^{(B+1)}, \tilde{\boldsymbol{S}}_1^{(B+1)}, \tilde{\boldsymbol{S}}_2^{(B+1)}, \tilde{\boldsymbol{U}}_1^{(B+1)}(\hat{M}_1^{(B)}), \tilde{\boldsymbol{U}}_2^{(B+1)}(M_2^{(B)}),$$
$$\tilde{\boldsymbol{W}}_1^{(B+1)}, \tilde{\boldsymbol{W}}_2^{(B+1)}, \boldsymbol{X}_1^{(B+1)}, \boldsymbol{X}_2^{(B+1)}, \boldsymbol{Y}_1^{(B+1)}, \boldsymbol{Y}_2^{(B+1)}) \notin T_\epsilon^{(n)}\}. \tag{5.11b}$$

$$\mathcal{E}_1^{(b)} \triangleq \{(\boldsymbol{S}_1^{(b)}, \boldsymbol{S}_2^{(b)}, \boldsymbol{U}_1^{(b)}(M_1^{(b)}), \boldsymbol{U}_2^{(b)}(M_2^{(b)}), \tilde{\boldsymbol{S}}_1^{(b)}, \tilde{\boldsymbol{S}}_2^{(b)}, \tilde{\boldsymbol{U}}_1^{(b)}(\hat{M}_1^{(b-1)}), \tilde{\boldsymbol{U}}_2^{(b)}(M_2^{(b-1)}),$$
$$\tilde{\boldsymbol{W}}_1^{(b)}, \tilde{\boldsymbol{W}}_2^{(b)}, \boldsymbol{X}_1^{(b)}, \boldsymbol{X}_2^{(b)}, \boldsymbol{Y}_1^{(b)}, \boldsymbol{Y}_2^{(b)}) \notin T_\epsilon^{(n)}\}, \tag{5.11c}$$

where $b = 2, 3, \ldots, B$.

We analogously define the events $\mathcal{E}_2^{(b)}$ for terminal 2 (not shown here) and consider the error event $\mathcal{E} = \cup_{b=1}^{B+1} \mathcal{E}_1^{(b)} \cup \mathcal{E}_2^{(b)}$. The expected distortion of terminal $j$'s source reconstruction (averaged with respect to all codebooks, source messages, channel inputs, and channel outputs) can be bounded by

$$\frac{1}{B}\sum_{b=1}^{B} \mathbb{E}[d_j(\boldsymbol{S}_j^{(b)}, \hat{\boldsymbol{S}}_j^{(b)})] \leq \Pr(\mathcal{E}) d_{j,\max} + \frac{1}{B}\sum_{b=1}^{B} \Pr\left(\overline{\mathcal{E}}\right) \mathbb{E}[d_j(\boldsymbol{S}_j^{(b)}, \hat{\boldsymbol{S}}_j^{(b)})|\overline{\mathcal{E}}] \tag{5.12}$$
$$\leq \Pr(\mathcal{E}) d_{j,\max} + \frac{1}{B}\sum_{b=1}^{B} (1+\epsilon) \mathbb{E}[d_j(S_j^{(b)}, \hat{S}_j^{(b)})] \tag{5.13}$$
$$= \Pr(\mathcal{E}) d_{j,\max} + (1+\epsilon) \mathbb{E}[d_j(S_j, \hat{S}_j)] \tag{5.14}$$
$$\leq \Pr(\mathcal{E}) d_{j,\max} + (1+\epsilon) D_j, \tag{5.15}$$

where (5.12) follows from $\mathbb{E}[d_j(\boldsymbol{S}_j^{(b)}, \hat{\boldsymbol{S}}_j^{(b)})|\mathcal{E}] \leq d_{j,\max}$ with $d_{j,\max} \triangleq \max_{s_j, \hat{s}_j} d_j(s_j, \hat{s}_j)$, (5.13) is due to the typical average lemma [47], (5.14) follows from the stationarity of the Markov chain, and the last inequality holds by assumption.

If we can further show that $\Pr\left(\mathcal{E}\right) \to 0$ and the joint source-channel coding rate goes to one as both $n$ and $B$ go to infinity, then the distortion pair $((1+\epsilon)D_1, (1+\epsilon)D_2)$

is achievable. Note that it suffices to show that $\Pr\left(\mathcal{E}_j^{(1)}\right) \to 0$ and $\Pr\left(\mathcal{E}_j^{(b)} \cap \overline{\mathcal{E}}_j^{(b-1)}\right) \to 0$ for all $j = 1, 2$ and $b = 2, 3, \ldots, B+1$ since by the identity

$$\bigcup_{b=1}^{B} \mathcal{E}_j^{(b)} = \mathcal{E}_j^{(1)} \cup \left( \bigcup_{b=2}^{B} \mathcal{E}_j^{(b)} \cap \overline{\mathcal{E}}_j^{(b-1)} \right),$$

we have

$$\Pr(\mathcal{E}) \leq \Pr\Big(\mathcal{E}_1^{(1)}\Big) + \Pr\Big(\mathcal{E}_2^{(1)}\Big) + \sum_{b=2}^{B+1} \left( \Pr\Big(\mathcal{E}_1^{(b)} \cap \overline{\mathcal{E}}_1^{(b-1)}\Big) + \Pr\Big(\mathcal{E}_2^{(b)} \cap \overline{\mathcal{E}}_2^{(b-1)}\Big) \right).$$

Due to symmetry, we only analyze $\Pr\left(\mathcal{E}_1^{(1)}\right)$ and $\Pr\left(\mathcal{E}_1^{(b)} \cap \overline{\mathcal{E}}_1^{(b-1)}\right)$. For $j = 1, 2$ and $b = 1, 2, \ldots, B+1$, we first define

$$\begin{aligned}
\mathcal{F}_j^{(b)} &= \{(\boldsymbol{S}_j^{(b)}, \boldsymbol{U}_j^{(b)}(m_j^{(b)})) \notin \mathcal{T}_{\epsilon_1}^{(n)} \text{ for all } m_j^{(b)}\}, \\
\mathcal{F}_3^{(b)} &= \{(\boldsymbol{S}_1^{(b)}, \boldsymbol{S}_2^{(b)}, \boldsymbol{U}_1^{(b)}(M_1^{(b)}), \boldsymbol{U}_2^{(b)}(M_2^{(b)}), \tilde{\boldsymbol{S}}_1^{(b)}, \tilde{\boldsymbol{S}}_2^{(b)}, \\
&\qquad \tilde{\boldsymbol{U}}_1^{(b)}(M_1^{(b-1)}), \tilde{\boldsymbol{U}}_2^{(b)}(M_2^{(b-1)}), \tilde{\boldsymbol{W}}_1^{(b)}, \tilde{\boldsymbol{W}}_2^{(b)} \boldsymbol{X}_1^{(b)}, \boldsymbol{X}_2^{(b)}, \boldsymbol{Y}_1^{(b)}, \boldsymbol{Y}_2^{(b)}) \notin \mathcal{T}_{\epsilon}^{(n)}\}, \\
\mathcal{F}_4^{(b)} &= \{\exists\, \hat{m}_1^{(b-1)} \neq M_1^{(b-1)} \text{ s.t. } (\boldsymbol{S}_2^{(b)}, \boldsymbol{U}_2^{(b)}(M_2^{(b)}), \tilde{\boldsymbol{S}}_2^{(b)}, \\
&\qquad \tilde{\boldsymbol{U}}_1^{(b)}(\hat{m}_1^{(b-1)}), \tilde{\boldsymbol{U}}_2^{(b)}(M_2^{(b-1)}), \tilde{\boldsymbol{W}}_2^{(b)}, \boldsymbol{X}_2^{(b)}, \boldsymbol{Y}_2^{(b)}) \in \mathcal{T}_{\epsilon}^{(n)}\},
\end{aligned}$$

with the exception that $\mathcal{F}_3^{(1)} \triangleq \mathcal{E}_1^{(1)}$ and $\mathcal{F}_3^{(B+1)} \triangleq \mathcal{E}_1^{(B+1)}$ due to the initialization and termination phases of the encoding process. We will use the following results to obtain (5.8a); detailed proofs of the claims are given in Appendix C.1.

*Claim 1*: For $b = 2, 3, \ldots, B+1$, the event $\overline{\mathcal{F}}_3^{(b)} \cap \overline{\mathcal{F}}_4^{(b)}$ implies that $\hat{M}_1^{(b-1)} = M_1^{(b-1)}$.

*Claim 2*: $\mathcal{E}_1^{(1)} \subseteq \mathcal{F}_1^{(1)} \cup \mathcal{F}_2^{(1)} \cup (\overline{\mathcal{F}}_1^{(1)} \cap \overline{\mathcal{F}}_2^{(1)} \cap \mathcal{E}_1^{(1)})$

*Claim 3*: The inclusion $\mathcal{E}_1^{(b)} \cap \overline{\mathcal{E}}_1^{(b-1)} \subseteq \mathcal{F}_1^{(b)} \cup \mathcal{F}_2^{(b)} \cup (\overline{\mathcal{F}}_1^{(1)} \cap \overline{\mathcal{F}}_2^{(1)} \cap \mathcal{F}_3^{(b)} \cap \overline{\mathcal{E}}_1^{(b-1)}) \cup \mathcal{F}_4^{(b)}$ holds for $b = 2, 3, \ldots, B$.

*Claim 4*: $\mathcal{E}_1^{(B+1)} \cap \overline{\mathcal{E}}_1^{(B)} \subseteq (\mathcal{F}_3^{(B+1)} \cap \overline{\mathcal{E}}_1^{(B)}) \cup \mathcal{F}_4^{(B+1)}$

*Claim 5*: If $R_j^{(1)} > I(S_j; U_j) + \delta_1(\epsilon_1)$, then $\lim_{n\to\infty} \Pr\left(\mathcal{E}_j^{(1)}\right) = 0$.

*Claim 6*: If $R_1^{(B)} < I(\tilde{U}_1; S_2, U_2, \tilde{S}_2, \tilde{U}_2, \tilde{W}_2, X_2, Y_2) - \delta(\epsilon)$, then $\lim_{n\to\infty} \Pr\left(\mathcal{E}_1^{(B+1)} \cap \overline{\mathcal{E}}_1^{(B)}\right) = 0$.

*Claim 7*: For $b = 2, 3, \ldots, B$, if $R_j^{(b)} > I(S_j; U_j) + \delta_1(\epsilon_1)$ and $R_1^{(b-1)} < I(\tilde{U}_1; S_2, U_2, \tilde{S}_2, \tilde{U}_2, \tilde{W}_2, X_2, Y_2) - \delta(\epsilon)$, then $\lim_{n\to\infty} \Pr\left(\mathcal{E}_1^{(b)} \cap \overline{\mathcal{E}}_1^{(b-1)}\right) = 0$.

The non-negative quantities $\delta_1(\epsilon_1)$ and $\delta(\epsilon)$ above arise from the standard typicality arguments and $\lim_{\epsilon_1\to 0} \delta_1(\epsilon_1) = 0$ and $\lim_{\epsilon\to 0} \delta(\epsilon) = 0$. Swapping the role of terminals 1 and 2, we obtain that $\lim_{n\to\infty} \Pr\left(\mathcal{E}_2^{(1)}\right) = 0$ and that $\lim_{n\to\infty} \Pr\left(\mathcal{E}_2^{(b)} \cap \overline{\mathcal{E}}_2^{(b-1)}\right) = 0$ for $b = 2, 3, \ldots, B+1$ provided that $R_j^{(b)} > I(S_j; U_j) + \delta_1(\epsilon_1)$ for $j = 1, 2$ and $b = 1, 2, \ldots, B$ and $R_2^{(b-1)} < I(\tilde{U}_2; S_1, U_1, \tilde{S}_1, \tilde{U}_1, \tilde{W}_1, X_1, Y_1) - \delta(\epsilon)$ for $b = 2, 3, \ldots, B+1$. Combining all conditions above then gives the two inequalities in (5.8). To complete the proof, we first increase $B$ so that the JSCC rate $B/(B+1)$ is close to one. Fixing this choice of $B$, we next make $n$ sufficiently large to ensure that all joint typicality requirements behind Claims 5-7 (and similar claims for terminal 2) are satisfied. As now we have $\lim_{n\to\infty} \Pr(\mathcal{E})=0$ (provided that all conditions hold) and $\epsilon$ is arbitrary, the distortion pair $(D_1, D_2)$ is achievable. ■

In the next section, we simplify the expressions in (5.8) by imposing some encoding constraints. Examples illustrating the main theorem will be given in Section 5.5.

## 5.3 Simplified Configurations and Special Cases

In this section, we consider two simplified forms of encoding to derive special cases from Theorem 5.1. Our objective is not only to obtain simpler achievability conditions but also to recover existing forward coding theorems for our problem setup.

By-products of the derivation are reduced-complexity coding schemes in those special cases. As we will see later in Section 5.4.2, the reduced-complexity schemes in the special cases are sometimes optimal in the sense that the associated achievable distortion region matches a certain outer bound; i.e., the scheme provides a complete JSCC theorem. In such a case, optimal performance can be achieved by a less complex coding scheme. To ease our presentation, we will not refer to the probability distributions $P_{S_1,S_2}$ and $P_{Y_1,Y_2|X_1,X_2}$ in the following result statements as they are fixed and given by the problem setup. Also, we continue to focus on the rate-one case.

### 5.3.1 A Non-Adaptive JSCC Scheme

Our first simplification disables the superposition and adaptive coding components, i.e., we let $X_j = F_j(S_j, U_j, \tilde{S}_j, \tilde{U}_j, \tilde{W}_j) \triangleq f_j(\tilde{S}_j, \tilde{U}_j)$ and $\hat{\tilde{S}}_{j'} = G_j(\tilde{U}_{j'}, S_j, U_j, \tilde{S}_j, \tilde{U}_j, \tilde{W}_j, Y_j) \triangleq g_j(\tilde{U}_{j'}, \tilde{S}_j, \tilde{U}_j, Y_j)$ for some $f_j$ and $g_j$, $j = 1, 2$. Set $P_{\tilde{S}_1,\tilde{S}_2} = P_{S_1,S_2}$, and set $P_{\tilde{U}_j|\tilde{S}_j} = P_{U_j|S_j}$ for a chosen $P_{U_j|S_j}$, $j = 1, 2$, so that (5.6) and (5.7) holds. We also set the pair $(\tilde{W}_1, \tilde{W}_2)$ to be *independent* of $(S_1, S_2, U_1, U_2, \tilde{S}_1, \tilde{S}_2, \tilde{U}_1, \tilde{U}_2, X_1, X_2, Y_1, Y_2)$ with joint probability distribution given by

$$P_{\tilde{W}_1,\tilde{W}_2}(\tilde{w}_1, \tilde{w}_2) = \sum_{a_1\in\mathcal{S}_1, a_2\in\mathcal{S}_2, b_1\in\mathcal{U}_1, b_2\in\mathcal{U}_2} P_{\tilde{S}_1,\tilde{S}_2}(a_1, a_2) P_{\tilde{U}_1|\tilde{S}_1}(b_1|a_1) P_{\tilde{U}_2|\tilde{S}_2}(b_2|a_2) \cdot \mathbf{1}\{\tilde{x}_1 = f_1(a_1, b_1)\}\mathbf{1}\{\tilde{x}_2 = f_2(a_2, b_2)\} P_{Y_1,Y_2|X_1,X_2}(\tilde{y}_1, \tilde{y}_2|\tilde{x}_1, \tilde{x}_2). \quad (5.16)$$

With the above setting, one can directly verify that

$$P_Z = P_{S_1,S_2} P_{U_1|S_1} P_{U_2|S_2} P_{\tilde{S}_1,\tilde{S}_2} P_{\tilde{U}_1|\tilde{S}_1} P_{\tilde{U}_2|\tilde{S}_2} P_{\tilde{W}_1,\tilde{W}_2} P_{X_1|\tilde{S}_1,\tilde{U}_1} P_{X_2|\tilde{S}_2,\tilde{U}_2} P_{Y_1,Y_2|X_1,X_2} \quad (5.17)$$

is a stationary distribution, i.e., $P_Z = Q_Z P_Z$. Given such $P_Z$, suppose that the chosen $g_j$ attains distortion level $D_j$, $j = 1, 2$, so that

$$(P_{U_1|S_1}, P_{U_2|S_2}, P_{\tilde{S}_1,\tilde{S}_2,\tilde{U}_1,\tilde{U}_2}, P_{\tilde{W}_1,\tilde{W}_2}, f_1, f_2, g_1, g_2) \in \Pi_Z(D_1, D_2).$$

For simplicity, we define the set $\Pi'_Z(D_1, D_2) \subset \Pi_Z(D_1, D_2)$ as the one that contains all such special configurations. Using $\Pi'_Z(D_1, D_2)$, Theorem 5.1 reduces to the following corollary.

**Corollary 5.1** (Non-Adaptive Hybrid Coding)**.** *A distortion pair* $(D_1, D_2)$ *is achievable for the rate-one lossy transmission of correlated sources over a DM-TWC if there exists a configuration in* $\Pi'_Z(D_1, D_2)$ *such that*

$$I(\tilde{S}_1; \tilde{U}_1|\tilde{S}_2, \tilde{U}_2) < I(\tilde{U}_1; Y_2|\tilde{S}_2, \tilde{U}_2), \tag{5.18a}$$

$$I(\tilde{S}_2; \tilde{U}_2|\tilde{S}_1, \tilde{U}_1) < I(\tilde{U}_2; Y_1|\tilde{S}_1, \tilde{U}_1). \tag{5.18b}$$

*Proof:* Since $\tilde{U}_{j'}$ is independent of $(S_j, U_j)$ and by definition $\tilde{W}_j$ is independent of $(\tilde{S}_{j'}, S_j, U_j, \tilde{S}_j, \tilde{U}_j, X_j, Y_j)$ for $j = 1, 2$, we can remove $(S_j, U_j, \tilde{W}_j)$ from (5.8) without changing the values on the right-hand-side of (5.8), e.g.,

$$I(\tilde{U}_1; S_2, U_2, \tilde{S}_2, \tilde{U}_2, \tilde{W}_2, X_2, Y_2) = I(\tilde{U}_1; \tilde{S}_2, \tilde{U}_2, X_2, Y_2) + \underbrace{I(\tilde{U}_1; S_2, U_2, \tilde{W}_2|\tilde{S}_2, \tilde{U}_2, X_2, Y_2)}_{=0}.$$

For (5.8a), we then have that

$$\begin{aligned} & I(\tilde{S}_1; \tilde{U}_1) < I(\tilde{U}_1; \tilde{S}_2, \tilde{U}_2, X_2, Y_2) \\ \Leftrightarrow \quad & H(\tilde{U}_1) - H(\tilde{U}_1|\tilde{S}_1) < I(\tilde{U}_1; \tilde{S}_2, \tilde{U}_2) + I(\tilde{U}_1; X_2, Y_2|\tilde{S}_2, \tilde{U}_2) \\ \Leftrightarrow \quad & H(\tilde{U}_1) - H(\tilde{U}_1|\tilde{S}_1, \tilde{S}_2, \tilde{U}_2) < H(\tilde{U}_1) - H(\tilde{U}_1|\tilde{S}_2, \tilde{U}_2) + I(\tilde{U}_1; X_2, Y_2|\tilde{S}_2, \tilde{U}_2) \end{aligned}$$

$$\Leftrightarrow \quad H(\tilde{U}_1|\tilde{S}_2, \tilde{U}_2) - H(\tilde{U}_1|\tilde{S}_1, \tilde{S}_2, \tilde{U}_2) < \underbrace{I(\tilde{U}_1; X_2|\tilde{S}_2, \tilde{U}_2)}_{=0} + \underbrace{I(\tilde{U}_1; Y_2|X_2, \tilde{S}_2, \tilde{U}_2)}_{=I(\tilde{U}_1; Y_2|\tilde{S}_2, \tilde{U}_2)} \quad (5.19)$$

$$\Leftrightarrow \quad I(\tilde{S}_1; \tilde{U}_1|\tilde{S}_2, \tilde{U}_2) < I(\tilde{U}_1; Y_2|\tilde{S}_2, \tilde{U}_2),$$

where the two equalities in (5.19) hold since $X_2 = f_2(\tilde{S}_2, \tilde{U}_2)$. By symmetry, one can analogously deduce (5.18b) from (5.8b). ∎

We remark that Corollary 5.1 further subsumes several special cases. In the following derivations, we will show that our chosen parameters form a configuration in $\Pi'_Z(D_1, D_2)$. As $P_{\tilde{W}_1,\tilde{W}_2}$ can be determined via (5.16) given other parameters, we will not specify $P_{\tilde{W}_1,\tilde{W}_2}$ for the sake of simplicity.

(i) **Uncoded transmission scheme**: Strictly speaking, the achievability result of an uncoded scheme cannot be deduced from Corollary 5.1 since the conditions in (5.18) have no impact on the scheme's performance. Nevertheless, we still can view it as a special case since every uncoded scheme can be converted into a configuration in our setup, which implies that our coding scheme (used to prove Theorem 5.1) can emulate uncoded transmission and attains the same distortion levels. Specifically, let $\mathcal{X}_j = \mathcal{S}_j$, $j = 1, 2$. Given encoding functions $\tilde{f}_j$ and decoding functions $\tilde{g}_j$ of an uncoded scheme such that $\mathbb{E}[d_j(\tilde{S}_j, \hat{\tilde{S}}_j)] \leq D_j$, we set $X_j = f_j(\tilde{U}_j, \tilde{S}_j) = \tilde{f}_j(\tilde{S}_j)$ and $\hat{\tilde{S}}_j = g_{j'}(\tilde{U}_j, \tilde{S}_{j'}, \tilde{U}_{j'}, Y_{j'}) = \tilde{g}_{j'}(\tilde{S}_{j'}, Y_{j'})$. Also, set $P_{\tilde{S}_1,\tilde{S}_1} = P_{S_1,S_2}$ and $U_j = \tilde{U}_j = \text{constant}$. This setting determines $P_{U_j|S_j}$ and $P_{\tilde{U}_j|\tilde{S}_j}$ uniquely and satisfies (5.6) and (5.7). We further obtain $P_{\tilde{W}_1,\tilde{W}_2}$ via (5.16). Clearly, the configuration $(P_{U_1|S_1}, P_{U_2|S_2}, P_{\tilde{S}_1,\tilde{S}_2,\tilde{U}_1,\tilde{U}_2}, P_{\tilde{W}_1,\tilde{W}_2}, \tilde{f}_1, \tilde{f}_2, \tilde{g}_1, \tilde{g}_2)$ belongs to $\Pi'_Z(D_1, D_2)$. Thus, one can establish the achievability result of uncoded transmission in our setup by giving appropriate functions $\tilde{f}_j$ and $\tilde{g}_j$.

(ii) **SSCC for the lossy transmission of independent sources**: To satisfy (5.6), we let $P_{S_1,S_2} = P_{\tilde{S}_1,\tilde{S}_2} = P_{S_1}P_{S_2}$. Define two independent random variables $V_1 \in \mathcal{X}_1$ and $V_2 \in \mathcal{X}_2$, whose joint probability distribution $P_{V_1}P_{V_2}$ achieves the rate pair $(I(V_1;Y_2|V_2), I(V_2;Y_1|V_1))$ in Shannon's capacity inner bound. For $j = 1, 2$, we let $\hat{S}_j$ denote the reconstruction variable in the standard RD function of $S_j$ in (5.1) and choose $P_{\hat{S}_j|S_j}$ that attains $R^{(j)}(D_j)$. Also, we define $(V_1', V_2') \in \mathcal{X}_1 \times \mathcal{X}_2$ with $P_{V_1'}P_{V_2'} = P_{V_1}P_{V_2}$ and define $\hat{S}_j' \in \hat{\mathcal{S}}_j$ as the reconstruction variable in the standard RD function of $\tilde{S}_j$ at distortion level $D_j$, i.e., we set $P_{\hat{S}_j'|\tilde{S}_j} = P_{\hat{S}_j|S_j}$. For $j = 1, 2$, let $U_j \triangleq (V_j, \hat{S}_j)$ and $\tilde{U}_j \triangleq (V_j', \hat{S}_j')$ and set $P_{U_j|S_j} = P_{V_j}P_{\hat{S}_j|S_j}$ and $P_{\tilde{U}_j|\tilde{S}_j} = P_{V_j'}P_{\hat{S}_j'|\tilde{S}_j}$. Clearly, the necessary condition in (5.7) is satisfied. Moreover, set

$$X_j = f_j(\tilde{U}_j, \tilde{S}_j) = f_j((V_j', \hat{S}_j'), \tilde{S}_j) = V_j'$$

and choose the decoding function $g_j$ as

$$\hat{\tilde{S}}_{j'} = g_j(\tilde{U}_{j'}, \tilde{U}_j, \tilde{S}_j, \tilde{Y}_j) = g_j((V_{j'}', \hat{S}_{j'}'), (V_j', \hat{S}_j'), \tilde{S}_j, \tilde{Y}_j) = \hat{S}_{j'}',$$

which yields $\mathbb{E}[d_j(\tilde{S}_j, \hat{\tilde{S}}_j)] \leq D_j$ for $j = 1, 2$. The above construction ensures that the tuple

$$(\underbrace{P_{V_1}P_{\hat{S}_1|S_1}}_{=P_{U_1|S_1}}, \underbrace{P_{V_2}P_{\hat{S}_2|S_2}}_{=P_{U_2|S_2}}, \underbrace{P_{\tilde{S}_1}P_{\tilde{S}_2}P_{V_1'}P_{\hat{S}_1'|\tilde{S}_1}P_{V_2'}P_{\hat{S}_2'|\tilde{S}_2}}_{=P_{\tilde{S}_1,\tilde{S}_2,\tilde{U}_1,\tilde{U}_2}}, P_{\tilde{W}_1,\tilde{W}_2}, f_1, f_2, g_1, g_2)$$

is a configuration in $\Pi_Z'(D_1, D_2)$. Next, using the fact that $S_1$ and $S_2$ are independent, one can simplify the sufficient conditions in (5.18) as follows (the details are given in Appendix C.2):

$$R^{(1)}(D_1) < I(X_1; Y_2|X_2)$$
$$R^{(2)}(D_2) < I(X_1; Y_1|X_2)$$

which is the achievability result for the SSCC scheme based on the standard lossy source coding and Shannon's random channel coding (without time-sharing).

(iii) **SSCC for the lossy transmission of correlated sources**: For $j = 1, 2$, we define pairs $(V_1, V_2) \in \mathcal{X}_1 \times \mathcal{X}_2$ and $(V_1', V_2') \in \mathcal{X}_1 \times \mathcal{X}_2$ in the same way as in the special case (ii); set the two pairs to have identical distributions, i.e., $P_{V_1}P_{V_2} = P_{V_1'}P_{V_2'}$. Letting $T_j \in \mathcal{T}_j$ denote the auxiliary random variable in the WZ RD function of $S_j$ in (5.2), we choose $P_{T_j|S_j}$ and the associated decoding function $h_{j'}(T_j, S_{j'})$ that achieves $R_{\text{WZ}}^{(j)}(D_j)$. Similarly, we use $T_j' \in \mathcal{T}_j$ in the WZ RD function of $\tilde{S}_j$ and set $P_{T_j'|\tilde{S}_j} = P_{T_j|S_j}$. Letting $U_j \triangleq (V_j, T_j)$ and $\tilde{U}_j \triangleq (V_j', T_j')$, we set $P_{U_j|S_j} = P_{V_j}P_{T_j|S_j}$ and $P_{\tilde{U}_j|\tilde{S}_j} = P_{V_j'}P_{T_j'|\tilde{S}_j}$. Also, set $P_{\tilde{S}_1,\tilde{S}_2} = P_{S_1,S_2}$. Thus, (5.6) and (5.7) are satisfied. Moreover, we set the encoding and decoding functions as

$$X_j = f_j(\tilde{U}_j, \tilde{S}_j) = f_j((V_j', T_j'), \tilde{S}_j) = V_j'$$

and

$$\hat{\tilde{S}}_j = g_{j'}(\tilde{U}_j, \tilde{U}_{j'}, \tilde{S}_{j'}, \tilde{Y}_{j'}) = g_{j'}((V_j', T_j'), (V_{j'}', T_{j'}'), \tilde{S}_{j'}, \tilde{Y}_{j'}) = h_{j'}(T_j', \tilde{S}_{j'}),$$

such that the decoder satisfies $\mathbb{E}[d_j(\tilde{S}_j, \hat{\tilde{S}}_j)] \leq D_j$ for $j = 1, 2$. With the above specifications, we next apply (5.16) to obtain $P_{\tilde{W}_1,\tilde{W}_2}$, yielding the following configuration in $\Pi'_Z(D_1, D_2)$:

$$(\underbrace{P_{V_1}P_{T_1|S_1}}_{=P_{U_1|S_1}}, \underbrace{P_{V_2}P_{T_2|S_2}}_{=P_{U_2|S_2}}, \underbrace{P_{\tilde{S}_1,\tilde{S}_2}P_{V_1'}P_{T_1'|\tilde{S}_1}P_{V_2'}P_{T_2'|\tilde{S}_2}}_{=P_{\tilde{S}_1,\tilde{S}_2,\tilde{U}_1,\tilde{U}_2}}, P_{\tilde{W}_1,\tilde{W}_1}, f_1, f_2, h_1, h_2).$$

Furthermore, using the Markov chain relationship: $T_1' \multimap \tilde{S}_1 \multimap \tilde{S}_2 \multimap T_2'$ and the memoryless property of the channel, one can easily deduce the following two inequalities from (5.18):

$$R^{(1)}_{\mathrm{WZ}}(D_1) < I(X_1; Y_2|X_2)$$
$$R^{(2)}_{\mathrm{WZ}}(D_2) < I(X_2; Y_1|X_1)$$

which is the achievability result for the SSCC scheme based on the WZ lossy source coding and Shannon's random channel coding (without time-sharing) [90]. As the derivation is very similar to the previous case (see Appendix C.2), we omit the details.

(iv) **Correlation-preserving coding scheme for (almost) lossless transmission of correlated sources [76]**: Suppose that $\mathcal{S}_j = \hat{\mathcal{S}}_j$ and consider the Hamming distortion measure [47, Sec. 3.6]. We first set $P_{\tilde{S}_1,\tilde{S}_2} = P_{S_1,S_2}$ to meet the necessary condition in (5.6). Recall the definitions of $(V_1, V_2)$ and $(V_1', V_2')$ in the special case (ii) with $P_{V_1}P_{V_2} = P_{V_1'}P_{V_2'}$, which achieve the same rate pair $(I(V_1; Y_2|V_2), I(V_2; Y_1|V_1))$ in Shannon's capacity inner bound. Moreover, we recall the variables $(\hat{S}_1, \hat{S}_2)$ and $(\hat{S}_1', \hat{S}_2')$ from the special case (ii), but here we choose $P_{\hat{S}_j|S_j}$ to achieve $R^{(j)}(0)$ in (5.1) and set $P_{\hat{S}_j'|\tilde{S}_j} = P_{\hat{S}_j|S_j}$ for $j = 1, 2$. Let $U_j \triangleq (V_j, \hat{S}_j)$ and $\tilde{U}_j \triangleq (V_j', \hat{S}_j')$, and set $P_{U_j|S_j} = P_{V_j}P_{\hat{S}_j|S_j}$ and $P_{\tilde{U}_j|\tilde{S}_j} = P_{V_j'}P_{\hat{S}_j'|\tilde{S}_j}$. The setting satisfies the condition in (5.7). We next consider the following encoding and decoding functions:

$$X_j = f_j(\tilde{U}_j, \tilde{S}_j) = f_j((V_j', \hat{S}_j'), \tilde{S}_j) = V_j'$$

and

$$\hat{\tilde{S}}_{j'} = g_j(\tilde{U}_{j'}, \tilde{U}_j, \tilde{S}_j, Y_j) = g_j((V_{j'}', \hat{S}_{j'}'), (V_j', \hat{S}_j'), \tilde{S}_j, \tilde{Y}_j) = \hat{S}_{j'}'.$$

Using (5.16) to obtain $P_{\tilde{W}_1,\tilde{W}_2}$, we ensure that the resulting configuration belongs to $\Pi_Z'(0,0)$. Furthermore, one can easily show that the sufficient conditions in (5.18) become

$$R^{(1)}(0) = H(\tilde{S}_1|\tilde{S}_2) < I(V_1'; Y_2|V_2', \tilde{S}_2) = I(X_1; Y_2|X_2, \tilde{S}_2)$$
$$R^{(2)}(0) = H(\tilde{S}_2|\tilde{S}_1) < I(V_2'; Y_1|V_1', \tilde{S}_1) = I(X_2; Y_1|X_1, \tilde{S}_1)$$

which recover the achievability conditions in [76, Cor. 8.1] (the rate-one case without coded time-sharing). Note that the block error rate for reconstructing the source messages is asymptotically vanishing here since the above conditions imply that $\lim_{K\to\infty} \Pr\left(\mathcal{E}\right) = 0$ (see the proof of Theorem 5.1 for the definition of the error event $\mathcal{E}$) and hence $\lim_{K\to\infty} \Pr\left((\tilde{S}_j^K, \hat{\tilde{S}}_j^K) \in \mathcal{T}_\epsilon^{(K)}\right) = 1$ for $j = 1, 2$, where $\mathcal{T}_\epsilon^{(K)}$ denotes the jointly typical set with parameters $K$ and $\epsilon$ as defined in [47]. This result implies that $\lim_{K\to\infty} \Pr\left(\{\tilde{S}_1^K \neq \hat{\tilde{S}}_1^K\} \cup \{\tilde{S}_2^K \neq \hat{\tilde{S}}_2^K\}\right) = 0$.

In fact, since superposition coding is disabled in this simplified scheme, it is unnecessary to use the sliding window decoder. The decoding of each new source block can be done within the same transmission block. The block diagram of such coding system is depicted in Fig. 5.4 with the following system operations. The source messages $S_j^K$ are first mapped to a digital codeword $U_j^K(M_j)$ with index $M_j$. The

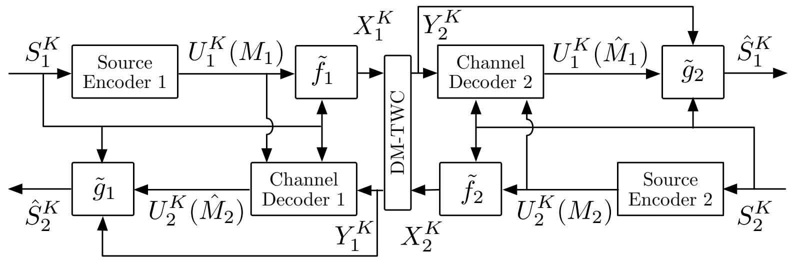



Figure 5.4: Rate-one non-adaptive hybrid coding scheme for the transmission of correlated sources over DM-TWCs.

channel inputs $X_j^K$ are then generated via the symbol-by-symbol map $\tilde{f}_j$, which combines the digital information $U_j^K(M_1)$ with the raw (or analog) information $S_j^K$. Upon receiving $Y_j^K$, terminal $j$ estimates the codeword index $M_{j'}$ based on all available information. Finally, the decoded codeword $U_{j'}(\hat{M}_{j'})$ and source message $S_j^K$ are passed together through the symbol-by-symbol map $\tilde{g}_j$ to produce $\hat{S}_{j'}^K$. The performance of this specific coding system is analyzed in [92]. The sufficient conditions in the achievability result are identical to those in (5.18) except that $(\tilde{S}_1, \tilde{S}_2, \tilde{U}_1, \tilde{U}_2)$ are replaced with $(S_1, S_2, U_1, U_2)$. We remark that one can also employ the unified coding results in [107] to obtain these conditions since the coded system in Fig. 5.4 involves block-wise operations without adaptation.

### 5.3.2 An SSCC Scheme with Adaptive Channel Coding

In the second simplification, we disable superposition coding for the raw source messages; i.e., we let $X_j = F_j(S_j, U_j, \tilde{S}_j, \tilde{U}_j, \tilde{W}_j) \triangleq f_j(U_j, \tilde{U}_j, \tilde{W}_j)$ and $\hat{\tilde{S}}_{j'} = G_j(\tilde{U}_{j'}, S_j, U_j, \tilde{S}_j, \tilde{U}_j, \tilde{W}_j, Y_j) \triangleq g_j(\tilde{U}_{j'}, \tilde{S}_j)$ for some $f_j$ and $g_j$, $j = 1, 2$. Set $P_{\tilde{S}_1,\tilde{S}_2} = P_{S_1,S_2}$ to satisfy (5.6). Let $V_j$, $\tilde{V}_j$, and $\tilde{W}_j$ be the auxiliary random variables used in Han's result [28] and let $\gamma_j : \mathcal{V}_j \times \tilde{\mathcal{V}}_j \times \tilde{\mathcal{W}}_j \to \mathcal{X}_j$ denote the encoding function of terminal $j$. Here, we choose $P_{V_1,V_2,\tilde{V}_1,\tilde{V}_2,\tilde{W}_1,\tilde{W}_2}$ and $\gamma_j$ that achieves the rate pair $(I(\tilde{V}_1; X_2, Y_2, \tilde{V}_2, \tilde{W}_2), I(\tilde{V}_2; X_1, Y_1, \tilde{V}_1, \tilde{W}_1))$ in Han's channel coding inner bound. Note that in Han's result, $P_{V_1,V_2,\tilde{V}_1,\tilde{V}_2,\tilde{W}_1,\tilde{W}_2} = P_{V_1}P_{V_2}P_{\tilde{V}_1}P_{\tilde{V}_2}P_{\tilde{W}_1,\tilde{W}_2|\tilde{V}_1,\tilde{V}_2}$ and $P_{\tilde{V}_j} = P_{V_j}$, $j = 1, 2$.

Moreover, recall in (5.2) the auxiliary random variable $T_j$ in the WZ-RD function for $S_j$, $j = 1, 2$; we choose $P_{T_j|S_j}$ and the associated decoding function $h_{j'}$ that attains $R^{(j)}_{\mathrm{WZ}}(D_j)$. We also define its counterpart $\tilde{T}_j$ for $\tilde{S}_j$ and set $P_{\tilde{T}_j|\tilde{S}_j} = P_{T_j|S_j}$ for $j = 1, 2$. Let $U_j \triangleq (V_j, T_j)$ and $\tilde{U}_j \triangleq (\tilde{V}_j, \tilde{T}_j)$ and set $P_{U_j|S_j} = P_{V_j}P_{T_j|S_j}$ and $P_{\tilde{U}_j|\tilde{S}_j} = P_{\tilde{V}_j}P_{\tilde{T}_j|\tilde{S}_j}$, which satisfy (5.7). Next, we consider the following encoding and decoding functions: $f_j(U_j, \tilde{U}_j, \tilde{W}_j) = \gamma_j(V_j, \tilde{V}_j, \tilde{W}_j)$ and $g_j(\tilde{U}_{j'}, \tilde{S}_j) = h_j(\tilde{T}_{j'}, \tilde{S}_j)$, which ensures that $\mathbb{E}[d_j(\tilde{S}_j, \hat{\tilde{S}}_j)] \le D_j$ for $j = 1, 2$. Under the above setting, the joint probability distribution of all involved random variables is then given by

$$P_Z = P_{S_1,S_2}\underbrace{P_{V_1}P_{T_1|S_1}}_{=P_{U_1|S_1}}\underbrace{P_{V_2}P_{T_2|S_2}}_{=P_{U_2|S_2}}P_{\tilde{S}_1,\tilde{S}_2}\underbrace{P_{\tilde{V}_1}P_{\tilde{T}_1|\tilde{S}_1}}_{=P_{\tilde{U}_1|\tilde{S}_1}}\underbrace{P_{\tilde{V}_2}P_{\tilde{T}_2|\tilde{S}_2}}_{=P_{\tilde{U}_2|\tilde{S}_2}}\underbrace{P_{\tilde{W}_1,\tilde{W}_2|\tilde{V}_1,\tilde{V}_2}}_{=P_{\tilde{W}_1,\tilde{W}_2|\tilde{S}_1,\tilde{S}_2,\tilde{U}_1,\tilde{U}_2}} \cdot P_{X_1|V_1,\tilde{V}_1,\tilde{W}_1}P_{X_2|V_2,\tilde{V}_2,\tilde{W}_2}P_{Y_1,Y_2|X_1,X_2}, \tag{5.23}$$

where $P_{\tilde{W}_1,\tilde{W}_2|\tilde{V}_1,\tilde{V}_2}$ is specified by Han's result [28] and $P_{X_j|V_j,\tilde{V}_j,\tilde{W}_j}$ is determined by $\gamma_j$, $j = 1, 2$. It can be shown (by definition) that $P_Z = P_Z Q_Z$, thus implying that

$$(P_{V_1}P_{T_1|S_1}, P_{V_2}P_{T_2|S_2}, P_{\tilde{S}_1,\tilde{S}_2}P_{\tilde{V}_1}P_{\tilde{T}_1|\tilde{S}_1}P_{\tilde{V}_2}P_{\tilde{T}_2|\tilde{S}_2}, P_{\tilde{W}_1,\tilde{W}_2|\tilde{V}_1,\tilde{V}_2}, \gamma_1, \gamma_2, h_1, h_2) \in \Pi_Z(D_1, D_2).$$

Letting $\Pi''_Z(D_1, D_2) \subseteq \Pi_Z(D_1, D_2)$ denote the set of all such special configurations, we obtain the following corollary from Theorem 5.1.

**Corollary 5.2** (SSCC with WZ Source Coding and Han's Adaptive Channel Coding)**.** *A distortion pair $(D_1, D_2)$ is achievable for the rate-one lossy transmission of correlated sources over a DM-TWC if there exists a configuration in $\Pi''_Z(D_1, D_2)$ such* *that*

$$R^{(1)}_{\mathrm{WZ}}(D_1) < I(\tilde{V}_1; X_2, Y_2, \tilde{V}_2, \tilde{W}_2), \tag{5.24a}$$

$$R^{(2)}_{\mathrm{WZ}}(D_2) < I(\tilde{V}_2; X_1, Y_1, \tilde{V}_1, \tilde{W}_1). \tag{5.24b}$$

*Proof:* For any configuration in $\Pi''_Z(D_1, D_2)$, the associated stationary distribution $P_Z$ can be factorized into the product form in (5.23). In addition to the independence between $(S_1, S_2, U_1, U_2)$ and $(\tilde{S}_1, \tilde{S}_2, \tilde{U}_1, \tilde{U}_2, \tilde{W}_1, \tilde{W}_2)$, the quadruple $(\tilde{S}_1, \tilde{S}_2, \tilde{T}_1, \tilde{T}_2)$ is independent of $(\tilde{V}_1, \tilde{V}_2)$. These facts imply the independence between $\tilde{V}_j$ and $(S_{j'}, V_{j'}, \tilde{S}_{j'}, \tilde{T}_{j'})$. Moreover, we have the following Markov chain relationships: $T_1 \multimap S_1 \multimap S_2 \multimap T_2$, $\tilde{T}_1 \multimap \tilde{S}_1 \multimap \tilde{S}_2 \multimap \tilde{T}_2$, and $\tilde{T}_j \multimap (\tilde{V}_j, S_{j'}, U_{j'}, \tilde{S}_{j'}, \tilde{T}_{j'}) \multimap (\tilde{V}_{j'}, \tilde{W}_{j'}, X_{j'}, Y_{j'})$, $j = 1, 2$. We now show that (5.8a) reduces to (5.24a):

$$\begin{aligned}
& I(\tilde{S}_1; \tilde{U}_1) < I(\tilde{U}_1; S_2, U_2, \tilde{S}_2, \tilde{U}_2, \tilde{W}_2, X_2, Y_2) \\
\Leftrightarrow\quad & I(\tilde{S}_1; \tilde{T}_1) + \underbrace{I(\tilde{S}_1; \tilde{V}_1 | \tilde{T}_1)}_{=0} < \underbrace{I(\tilde{U}_1; S_2, U_2)}_{=0} + I(\tilde{U}_1; \tilde{S}_2, \tilde{V}_2, \tilde{T}_2, \tilde{W}_2, X_2, Y_2 | S_2, U_2) \\
\Leftrightarrow\quad & I(\tilde{S}_1; \tilde{T}_1) - I(\tilde{U}_1; \tilde{S}_2, \tilde{T}_2 | S_2, U_2) < I(\tilde{U}_1; \tilde{V}_2, \tilde{W}_2, X_2, Y_2 | S_2, U_2, \tilde{S}_2, \tilde{T}_2) \\
\Leftrightarrow\quad & I(\tilde{S}_1; \tilde{T}_1) - I(\tilde{V}_1, \tilde{T}_1; \tilde{S}_2, \tilde{T}_2) < I(\tilde{V}_1, \tilde{T}_1; \tilde{V}_2, \tilde{W}_2, X_2, Y_2 | S_2, U_2, \tilde{S}_2, \tilde{T}_2) & (5.25) \\
\Leftrightarrow\quad & I(\tilde{S}_1; \tilde{T}_1 | \tilde{S}_2) < I(\tilde{V}_1; \tilde{V}_2, \tilde{W}_2, X_2, Y_2) & (5.26)
\end{aligned}$$

where (5.25) holds since $I(\tilde{U}_1; \tilde{S}_2, \tilde{T}_2 | S_2, U_2) = I(\tilde{U}_1; \tilde{S}_2, \tilde{T}_2)$ and $\tilde{U}_j = (\tilde{V}_j, \tilde{T}_j)$, and we have the equivalence in (5.26) since

$$\begin{aligned}
& I(\tilde{S}_1; \tilde{T}_1) - I(\tilde{V}_1, \tilde{T}_1; \tilde{S}_2, \tilde{T}_2) \\
& = I(\tilde{S}_1; \tilde{T}_1) - I(\tilde{T}_1; \tilde{S}_2, \tilde{T}_2) - \underbrace{I(\tilde{V}_1; \tilde{S}_2, \tilde{T}_2 | \tilde{T}_1)}_{=0}
\end{aligned}$$

$$= I(\tilde{S}_1;\tilde{T}_1) - I(\tilde{T}_1;\tilde{S}_2,\tilde{T}_2) - I(\tilde{S}_1;\tilde{T}_1|\tilde{S}_2) + I(\tilde{S}_1;\tilde{T}_1|\tilde{S}_2)$$

$$= H(\tilde{T}_1) - H(\tilde{T}_1|\tilde{S}_1) - H(\tilde{T}_1) + \underbrace{H(\tilde{T}_1|\tilde{S}_2,\tilde{T}_2)}_{=H(\tilde{T}_1|\tilde{S}_2)} - H(\tilde{T}_1|\tilde{S}_2) + \underbrace{H(\tilde{T}_1|\tilde{S}_1,\tilde{S}_2)}_{=H(\tilde{T}_1|\tilde{S}_1)} + I(\tilde{S}_1;\tilde{T}_1|\tilde{S}_2)$$

$$= I(\tilde{S}_1;\tilde{T}_1|\tilde{S}_2),$$

and

$$\begin{aligned}
&I(\tilde{V}_1,\tilde{T}_1;\tilde{V}_2,\tilde{W}_2,X_2,Y_2|S_2,U_2,\tilde{S}_2,\tilde{T}_2)\\
&\quad= I(\tilde{V}_1;\tilde{V}_2,\tilde{W}_2,X_2,Y_2|S_2,U_2,\tilde{S}_2,\tilde{T}_2) + \underbrace{I(\tilde{T}_1;\tilde{V}_2,\tilde{W}_2,X_2,Y_2|S_2,U_2,\tilde{S}_2,\tilde{T}_2,\tilde{V}_1)}_{=0}\\
&\quad= H(\tilde{V}_1|S_2,U_2,\tilde{S}_2,\tilde{T}_2) - H(\tilde{V}_1|S_2,U_2,\tilde{S}_2,\tilde{T}_2,\tilde{V}_2,\tilde{W}_2,X_2,Y_2)\\
&\quad= H(\tilde{V}_1) - H(\tilde{V}_1|\tilde{V}_2,\tilde{W}_2,X_2,Y_2) \qquad (5.27)\\
&\quad= I(\tilde{V}_1;\tilde{V}_2,\tilde{W}_2,X_2,Y_2),
\end{aligned}$$

where (5.27) holds since $\tilde{V}_1$ is independent of $(S_2,V_2,\tilde{S}_2,\tilde{T}_2)$ given $(\tilde{V}_2,\tilde{W}_2,X_2,Y_2)$. By symmetry, one can also deduce (5.24b) from (5.8b), thus completing the proof. ■

We note that by working with super-symbols, we obtain a rate-$K/N$ extension of Corollary 5.2.

**Corollary 5.3** (General Rate SSCC with WZ Source Coding and Han's Adaptive Channel Coding)**.** *A distortion pair $(D_1,D_2)$ is achievable for the rate-$K/N$ lossy transmission of correlated sources over a DM-TWC if*

$$K \cdot R^{(1)}_{\mathrm{WZ}}(D_1) < N \cdot I(\tilde{V}_1;X_2,Y_2,\tilde{V}_2,\tilde{W}_2), \qquad (5.28a)$$

$$K \cdot R^{(2)}_{\mathrm{WZ}}(D_2) < N \cdot I(\tilde{V}_2;X_1,Y_1,\tilde{V}_1,\tilde{W}_1), \qquad (5.28b)$$

*for some joint probability distribution $P_{\tilde{V}_1,\tilde{V}_2,\tilde{W}_1,\tilde{W}_2,X_1,X_2}$ as defined in [28, Section IV].*

As Han's channel coding result subsumes Shannon's result, the following corollary is immediate, which is perhaps the simplest SSCC result for our problem setup.

**Corollary 5.4** (General Rate SSCC with WZ Source Coding and Non-Adaptive Channel Coding)**.** *A distortion pair* $(D_1, D_2)$ *is achievable for the rate-*$K/N$ *lossy transmission of correlated sources over a DM-TWC if*

$$K \cdot R_{\mathrm{WZ}}^{(1)}(D_1) < N \cdot I(X_1; Y_2|X_2), \tag{5.29a}$$

$$K \cdot R_{\mathrm{WZ}}^{(2)}(D_2) < N \cdot I(X_2; Y_1|X_1), \tag{5.29b}$$

*for some* $P_{X_1} P_{X_2}$.

We remark that since our general JSCC scheme (in the proof of Theorem 5.1) does not consider time-sharing for the sake of simplicity, the channel coding rate pairs obtained by the convex closure operation in Han's and Shannon's inner bound (see Section 2.1.2) are excluded in Corollary 5.3 and Corollary 5.4, respectively. However, one can clearly incorporate time-sharing in our coding scheme and Theorem 5.1. After such convexification operation, one can include any achievable rate pair in Han's (resp., Shannon's) capacity inner bound region on the right-hand-side of (5.28) (resp., (5.29)). Furthermore, despite the fact that Corollary 5.3 strictly subsumes Corollary 5.4, the associated achievable distortion regions are identical when DM-TWCs are symmetric in the sense of Theorem 2.6, 2.7, or 2.8; i.e., when Shannon's inner bound is tight. In such situation, the simpler coding scheme of Corollary 5.4 is preferred.

## 5.4 Converse Results and Complete JSCC Theorems

The last two sections were devoted to the construction of achievable coding schemes. In this section, we derive two outer bounds to the achievable distortion region. Our objective is not only to identify unattainable distortion pairs but also to establish complete JSCC theorems.

### 5.4.1 Two Outer Bounds

Lemmas 5.1 and 5.2 provide two outer bounds. Lemma 5.2 is obtained via a genie-aided argument where the encoder at terminal $j$ can access the decoder side-information $S_{j'}^K$ at terminal $j'$. The proofs are standard and hence omitted. Details are given in [90] and [92], respectively.

**Lemma 5.1.** *If a rate-$K/N$ JSCC scheme achieves the distortion levels $D_1$ and $D_2$ for the lossy transmission of correlated sources over a DM-TWC, then*

$$K \cdot R^{(1)}(D_1) \leq K \cdot I(S_1; S_2) + N \cdot I(X_1; Y_2|X_2), \tag{5.30a}$$

$$K \cdot R^{(2)}(D_2) \leq K \cdot I(S_1; S_2) + N \cdot I(X_2; Y_1|X_1), \tag{5.30b}$$

*for some $P_{X_1,X_2}$.*

**Lemma 5.2** (Genie-Aided Outer Bound)**.** *If a rate-$K/N$ JSCC scheme achieves the distortion levels $D_1$ and $D_2$ for the lossy transmission of correlated sources over a DM-TWC, then we have*

$$K \cdot R_{S_1|S_2}(D_1) \leq N \cdot I(X_1; Y_2|X_2), \tag{5.31a}$$

$$K \cdot R_{S_2|S_1}(D_2) \leq N \cdot I(X_2; Y_1|X_1), \tag{5.31b}$$

*for some $P_{X_1,X_2}$.*

Lemmas 5.1 and 5.2 generally give different outer bounds; however, the regions are identical for independent sources $S_1$ and $S_2$ since in this case $I(S_1; S_2) = 0$ and

$R^{(j)}(D_j) = R_{S_j|S_{j'}}(D_j)$. The conditions in (5.30) and (5.31) are also equivalent for arbitrarily correlated sources for the specific distortion requirement $(D_1, D_2) = (0, 0)$ since $R_{S_j|S_{j'}}(0) = R^{(j)}(0) - I(S_1; S_2) = H(S_j|S_{j'})$.

### 5.4.2 Complete JSCC Theorems

Matching the achievability results in Section 5.3 with the converse results in Lemmas 5.1 and 5.2, we obtain three complete JSCC theorems (Theorems 5.2-5.4). We also establish a complete theorem (Theorem 5.5) for correlated source pairs that have common parts. In the results below, a "symmetric DM-TWC" is a DM-TWC that possesses the symmetry properties defined in Chapter 2. With these properties, Shannon's inner bound in (2.2) is tight and hence the capacity region is achieved via independent inputs. Moreover, taking the convex closure in (2.2) is not needed.

**Theorem 5.2** (Lossy Transmission of Indenpendent Sources)**.** *For the rate-$K/N$ lossy transmission of independent sources over a symmetric DM-TWC, a distortion pair $(D_1, D_2)$ is achievable if and only if*

$$K \cdot R^{(1)}(D_1) \leq N \cdot I(X_1; Y_2|X_2),$$
$$K \cdot R^{(2)}(D_2) \leq N \cdot I(X_2; Y_1|X_1),$$

*for some $P_{X_1}P_{X_2}$.*

*Proof:* This result is due to the special case (ii) of Corollary 5.1 and Lemma 5.1, together with the facts that $R^{(j)}_{\mathrm{WZ}}(D_j) = R^{(j)}(D_j)$ and $I(S_1; S_2) = 0$ for independent sources pair. ■

**Theorem 5.3** (Almost Lossless Transmission of Correlated Sources)**.** *For the rate-$K/N$ transmission of correlated sources over a symmetric DM-TWC, the almost lossless transmission is achievable if and only if*

$$K \cdot H(S_1|S_2) \leq N \cdot I(X_1; Y_2|X_2),$$

$$K \cdot H(S_2|S_1) \leq N \cdot I(X_2; Y_1|X_1),$$

*for some $P_{X_1}P_{X_2}$.*

*Proof:* In Lemma 5.1, we have that $K \cdot R^{(j)}(0) - K \cdot I(S_1; S_2) = K \cdot H(S_j|S_{j'})$. Combining this result with the special case (iv) of Corollary 5.1 then completes the proof. ■

**Theorem 5.4** (Lossy Transmission of Correlated Sources with Equal WZ and Conditional RD Functions)**.** *For the rate-$K/N$ lossy transmission of correlated sources whose WZ-RD functions equal to their conditional RD functions over a symmetric DM-TWC, a distortion pair $(D_1, D_2)$ is achievable if and only if*

$$K \cdot R_{S_1|S_2}(D_1) \leq N \cdot I(X_1; Y_2|X_2),$$

$$K \cdot R_{S_2|S_1}(D_2) \leq N \cdot I(X_2; Y_1|X_1),$$

*for some $P_{X_1}P_{X_2}$.*

*Proof:* The result follows from the special case (iii) of Corollary 5.1 and Lemma 5.2. ■

**Theorem 5.5** (Lossy Transmission of Correlated Sources with a Common Part)**.** *Assume that correlated sources $S_1$ and $S_2$ have a common part $S_0$ in the sense of Gács-Körner-Witsenhausen and the triplet $(S_0, S_1, S_2)$ forms a Markov chain $S_1 -\!\circ\!-$*

$S_0 \multimap S_2$. *For the rate-$K/N$ lossy transmission of such correlated sources over a symmetric DM-TWC, a distortion pair $(D_1, D_2)$ is achievable if and only if*

$$K \cdot R_{S_1|S_0}(D_1) \leq N \cdot I(X_1; Y_2|X_2), \tag{5.32a}$$

$$K \cdot R_{S_2|S_0}(D_2) \leq N \cdot I(X_2; Y_1|X_1), \tag{5.32b}$$

*for some $P_{X_1}P_{X_2}$.*

*Proof:* We construct a two-way coding scheme using two one-way SSCC schemes, one for each direction of the bi-directional transmission. Specifically, we employ the source coding scheme that achieves the distortion level $D_j$ of the conditional RD function $R^{(j)}_{S_j|S_0}(D_j)$ given in (5.3), $j = 1, 2$, followed by Shannon's one-way channel coding for data protection. The sufficient conditions for achieving the distortion pair $(D_1, D_2)$ as shown in (5.32) are thus immediate. Note that in this two-way coding scheme, we do not employ time-sharing and the channel inputs $X_1$ and $X_2$ are independent.

To derive outer bound, we let $S^{k_2}_{j,k_1} \triangleq (S_{j,k_1}, S_{j,k_1+1}, \ldots, S_{j,k_2})$ for $k_1 \leq k_2$. Given a rate-$K/N$ joint source-channel code that achieves the distortion pair $(D_1, D_2)$, we obtain (5.32a) by the following derivation:

$$K \cdot R_{S_1|S_0}(D_1) \leq K \cdot R_{S_1|S_0}\left(K^{-1}\sum_{k=1}^{K}\mathbb{E}\left[d_1(S_{1,k}, \hat{S}_{1,k})\right]\right) \tag{5.33}$$

$$\leq \sum_{k=1}^{K} R_{S_1|S_0}\left(\mathbb{E}[d_1(S_{1,k}, \hat{S}_{1,k})]\right) \tag{5.34}$$

$$\leq \sum_{k=1}^{K} I(S_{1,k}; \hat{S}_{1,k}|S_{0,k}) \tag{5.35}$$

$$\leq \sum_{k=1}^{K} I(S_{1,k}; S_2^K, Y_2^N|S_{0,k}) \tag{5.36}$$

$$
\begin{aligned}
&\leq \sum_{k=1}^{K} H(S_{1,k}|S_{0,k}) - H(S_{1,k}|S_0^k, S_2^K, Y_2^N) && (5.37)\\
&\leq \sum_{k=1}^{K} H(S_{1,k}|S_0^K, S_1^{k-1}, S_2^K) - H(S_{1,k}|S_0^K, S_1^{k-1}, S_2^K, Y_2^N) && (5.38)\\
&= \sum_{k=1}^{K} I(S_{1,k}; Y_2^N|S_0^K, S_1^{k-1}, S_2^K)\\
&= I(S_1^K; Y_2^N|S_0^K, S_2^K)\\
&= \sum_{n=1}^{N} I(S_1^K; Y_{2,n}|S_0^K, S_2^K, Y_2^{n-1})\\
&\leq \sum_{n=1}^{N} H(Y_{2,n}|X_{2,n}) - H(Y_{2,n}|S_0^K, S_1^K, S_2^K, Y_2^{n-1}, X_{1,n}, X_{2,n}) && (5.39)\\
&= \sum_{n=1}^{N} H(Y_{2,n}|X_{2,n}) - H(Y_{2,n}|X_{1,n}, X_{2,n}) && (5.40)\\
&= N \sum_{n=1}^{N} \frac{1}{N} \cdot I(X_{1,n}; Y_{2,n}|X_{2,n})\\
&\leq N \cdot I(X_1; Y_2|X_2), && (5.41)
\end{aligned}
$$

where (5.33) holds since $R_{S_1|S_0}(D_1)$ is non-increasing and the expected distortion of the code is not larger than $D_1$, (5.34) and (5.35) are respectively due to convexity and the definition of conditional RD function, (5.36) follows from the data-processing inequality, (5.37) holds since conditioning reduces entropy, (5.38) holds by the Markov chain relationships $S_{1,k} -\circ- S_{0,k} -\circ- (S_0^{k-1}, S_{0,k+1}^K, S_1^{k-1})$ and $S_1^K -\circ- S_0^K -\circ- S_2^K$ and since conditioning reduces entropy, (5.39) holds since $X_{2,n}$ is a function of $(Y_2^{n-1}, S_2^K)$ and since conditioning reduces entropy, (5.40) follows from the memoryless property of channel, and (5.41) holds with $P_{X_1,X_2} = N^{-1}\sum_{n=1}^{N} P_{X_{1,n},X_{2,n}}$ since $I(X_{1,n}; Y_{2,n}|X_{2,n})$ is concave in $P_{X_{1,n},X_{2,n}}$. By symmetry, a similar argument shows (5.32b).

Although the inputs $X_1$ and $X_2$ are arbitrarily correlated in the outer bound

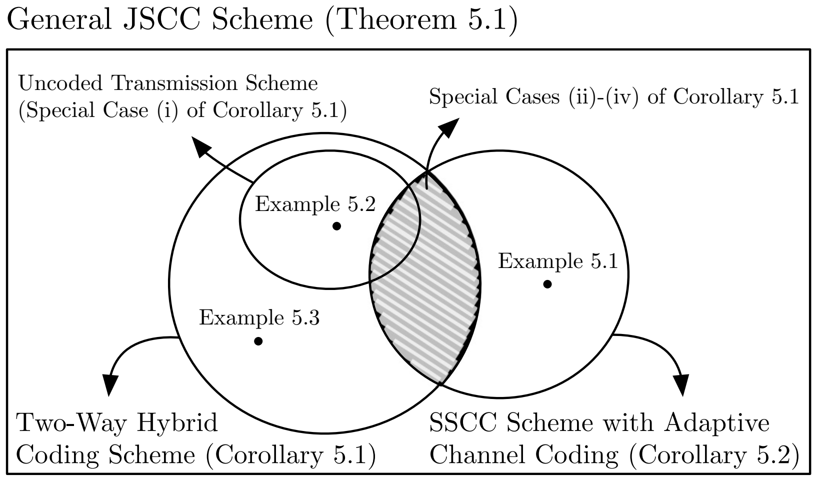



Figure 5.5: A general Venn diagram of the achievable distortion regions for the coding schemes presented in Sections 5.2 and 5.3, for a fixed source pair and channel. Moreover, Examples 5.1-5.3 in Section 5.5.1 show that certain inclusion relationships can be strict.

result, we can restrict to independent inputs without changing the outer bound region due to the channel symmetry property, i.e., the capacity region of the DM-TWC can be determined via independent channel inputs. Combining this fact with the achievability result then completes the proof. ■

## 5.5 Examples and Discussion

In this section, we illustrate our achievability results and discuss possible extensions. The Venn diagram in Fig. 5.5 summarizes the relationship of the achievable rate regions for the coding schemes in Sections 5.2 and 5.3. We begin with three examples showing that some inclusion relationships can be strict, followed by illustrative examples for Theorems 5.1, 5.4, and 5.5.

### 5.5.1 Examples

Examples 5.1 and 5.2 below show that Theorem 5.1 strictly generalizes Corollary 5.1 and Corollary 5.2, respectively. Example 5.3 not only illustrates a special use of the two-way hybrid coding scheme but also reveals that Corollary 5.1 strictly subsumes all of its special cases; see Section 5.3.1. Example 5.4 shows how a simple instance of our adaptive JSCC helps source transmission. At the end of this section, we provide two examples (Examples 5.5-5.6) for Theorem 5.4 and an example (Example 5.7) for Theorem 5.5. Note that except for the Gaussian case examined in Example 5.6, the Hamming distortion is considered in all examples. Let Ber($p$) denote a Bernoulli random variable with probability of success $p \in [0, 1]$. We will also need the following specialized converse result in Examples 5.1 and 5.4, whose proof is similar to Lemma 5.1.

**Proposition 5.1.** *Assume that the non-adaptive encoder $f_j : \mathcal{S}_j^K \to \mathcal{X}_j^K$ is used for $j = 1, 2$. If a distortion pair $(D_1, D_2)$ is achievable for the rate-one lossy transmission of independent sources over a DM-TWC, then*

$$R^{(1)}(D_1) \le I(X_1; Y_2|X_2, Q),$$
$$R^{(2)}(D_2) \le I(X_2; Y_1|X_1, Q),$$

*for some $P_Q P_{X_1|Q} P_{X_2|Q}$.*

Note that the pair $(I(X_1; Y_2|X_2, Q), I(X_2; Y_1|X_1, Q))$ under the distribution $P_Q P_{X_1|Q} P_{X_2|Q}$ in Proposition 5.1 is an alternative expression for the achievable rate pair in Shannon's inner bound (see (2.2)).

**Example 5.1** (**Transmitting Independent Binary Non-Uniform Sources over Dueck's DM-TWC [14]).** Consider the independent sources $S_1 = \text{Ber}(0.89)$ and $S_2 = \text{Ber}(0.89)$ so that $H(S_1) = H(S_2) \approx 0.5$. We recall Dueck's DM-TWC [14], where $\boldsymbol{X}_j = (X_{j,1}, X_{j,2})$,[4] $\mathbf{Y}_j = (X_{1,1} \cdot X_{2,1}, Z_j \oplus_2 X_{j',2}, Z_{j'})$, and $Z_1 = \text{Ber}(0.5)$ and $Z_2 = \text{Ber}(0.5)$ are independent channel noise variables that are independent of all channel inputs and sources. Han [28] showed that the channel coding rate pair $(R_{c,1}, R_{c,2}) = (0.5, 0.5)$ is not achievable via Shannon's random coding scheme but can be achieved via his adaptive channel coding scheme. Based on this fact and Proposition 5.1, we conclude that the hybrid coding scheme of Corollary 5.1 cannot achieve the distortion pair $(D_1, D_2) = (0, 0)$ (since it uses non-adaptive encoders and violates the necessary conditions in Proposition 5.1). By contrast, Corollary 5.2 shows that the distortion pair $(0, 0)$ is achievable via our general JSCC scheme as $R_{\text{WZ}}^{(j)}(0) = H(S_j) < R_{c,j}$ holds for $j = 1, 2$. Thus, Theorem 5.1 strictly subsumes Corollary 5.1.

**Example 5.2** (**Transmitting Correlated Binary Sources over Binary-Multiplying DM-TWCs [3]).** Consider the binary-multiplying TWC given by $Y_j = X_1 \cdot X_2$ for $j = 1, 2$. The capacity region of the channel is not known, but it is known that any symmetric achievable channel coding rate pair is component-wise upper bounded by $(0.646, 0.646)$ [31]. Suppose that we want to exchange binary correlated sources with joint probability distribution $P_{S_1,S_2}(0, 0) = 0$ and $P_{S_1,S_2}(s_1, s_2) = 1/3$ for $(s_1, s_2) \neq (0, 0)$. The WZ coding theorem indicates that the minimum source coding rate pair is $(H(S_1|S_2), H(S_2|S_1)) = (0.667, 0.667)$ to achieve the distortion

[4] As Dueck's DM-TWC has $\mathcal{X}_j = \{0, 1\}^2$ and $\mathcal{Y}_j = \{0, 1\}^3$, we here use $(X_{j,1}, X_{j,2}) \in \mathcal{X}_j$ to denote the two channel inputs of terminal $j$.

pair $(D_1, D_2) = (0, 0)$. Clearly, this pair is not achievable by *any* SSCC scheme, including the adaptive coding scheme of Corollary 5.2, because the source coding rate exceeds the largest possible transmission rate for reliable communication. However, the uncoded scheme: $X_j = S_j$ for $j = 1, 2$ can be easily shown to provide lossless transmission. As Corollary 5.2 and the uncoded scheme are special cases of our general JSCC method, Theorem 5.1 strictly subsumes Corollary 5.2.

**Example 5.3** (**Transmitting Correlated Binary Sources over a Mixed-Type DM-TWC).** Suppose that all alphabets are binary. Let the source messages $S_1$ and $S_2$ have the joint probability distribution $P_{S_1,S_2}(1, 0) = 0$ and $P_{S_1,S_2}(s_1, s_2) = 1/3$ for $(s_1, s_2) \neq (1, 0)$. Consider the DM-TWC described by $Y_1 = X_1 \oplus_2 X_2 \oplus_2 Z_1$ and $Y_2 = X_1 \cdot X_2$, where $Z_1 = \text{Ber}(0.05)$ that is independent of $S_j$'s and $X_j$'s. In other words, we have a (one-way) binary-multiplying channel in one direction and a binary additive channel with additive noise in another direction.

For this channel, none of the special cases of Corollary 5.1 can achieve the distortion pair $(D_1, D_2) = (0, 0)$. More specifically, the SSCC schemes in the special cases cannot attain the distortion pair since $H(S_1|S_2) < I(X_1; Y_2|X_2)$ and $H(S_2|S_1) < I(X_2; Y_1|X_1)$ cannot hold simultaneously. Moreover, using uncoded transmission in both directions yields the distortion pair $(D_1, D_2) = (0, 0.033)$. However, we can use the two-way hybrid coding scheme in Corollary 5.1 in the following way: use uncoded transmission from terminal 1 to 2 and use the concatenation of WZ source coding and Shannon's channel coding for the reverse direction. Then the distortion pair $(0, 0)$ is achievable. This example shows that Corollary 5.1 is a strictly generalization of its presented special cases.

**Example 5.4 (Transmitting Independent Binary Uniform Sources over Dueck's DM-TWC).** Consider the almost lossless transmission of the independent sources $S_1 = \text{Ber}(0.5)$ and $S_2 = \text{Ber}(0.5)$ through Dueck's DM-TWC (given in Example 1). Here, the binary noise variables $Z_1$ and $Z_2$ are assumed to be correlated with joint distribution given by $P_{Z_1,Z_2}(0,0) = 0$ and $P_{Z_1,Z_2}(z_1,z_2) = 1/3$ for $(z_1,z_2) \neq (0,0)$. For this channel, the optimal symmetric rate pair in Proposition 5.1 is obtained as $(I(X_1;Y_2|X_2), I(X_2;Y_1|X_1)) = (0.9503, 0.9503)$. Since the required source coding rate $R_{\text{WZ}}^{(j)}(0) = H(S_j) = 1$ (at terminal $j$) exceeds the outer bound in Proposition 5.1, the hybrid coding scheme in Corollary 5.1 cannot achieve the distortion pair $(D_1, D_2) = (0,0)$.

By contrast, the following use of our general JSCC scheme provides rate-one lossless transmission. Suppose that we exchange a length-$K$ of such source pair via $K+1$ channel uses. Clearly, the transmission rate approaches one as $K$ goes to infinity. For $j = 1,2$, we next set $(X_{j,1}^{(1)}, X_{j,2}^{(1)}) = (1, S_j^{(1)})$, $(X_{j,1}^{(K+1)}, X_{j,2}^{(K+1)}) = (Y_{j,3}^{(K)}, 1)$, and $(X_{j,1}^{(b)}, X_{j,2}^{(b)}) = (Y_{j,3}^{(n-1)}, S_j^{(b)})$ for $b = 2,3,\ldots,K$, where the superscripts represent time index. Via such adaptive encoding, terminal $j$ can exploit the correlation between $N_1$ and $N_2$ to perfectly decode $N_j^{(b-1)}$ from $Y_{j,1}^{(b)}$ and $Y_{j,3}^{(b-1)}$ and reconstruct $S_{j'}^{(b-1)}$ as $\hat{S}_{j'}^{(b-1)} = Z_j^{(b-1)} \oplus_2 Y_{j,2}^{(b-1)} = S_{j'}^{(b-1)}$ for all $2 \leq b \leq K+1$, thus achieving zero-error transmission. For $2 \leq b \leq K$, the above encoding and decoding procedure is depicted in Fig. 5.6. Note that whether or not the SSCC scheme in Corollary 5.2 achieves the same performance remains unclear.

**Example 5.5 (Transmitting Binary Correlated Sources with Z-channel Correlation over Binary Additive Noise DM-TWCs).** Suppose that all alphabets are binary. Given $0 \leq \epsilon_1, \epsilon_2 < 0.5$, the binary additive noise DM-TWC is

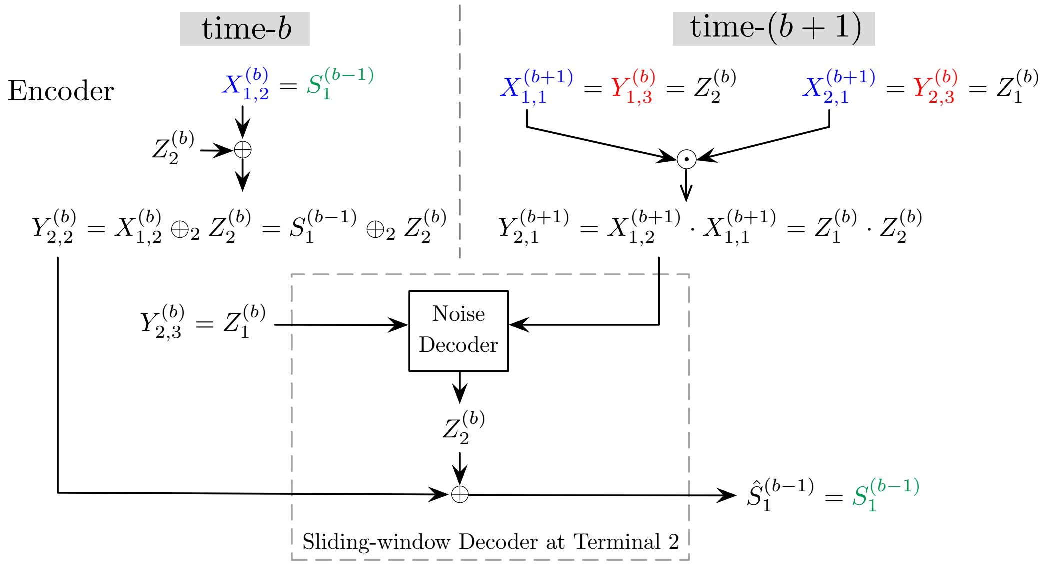



Figure 5.6: An illustration of adaptive encoding and sliding-window decoding in Example 4. At time-$b$, terminal 2 cannot perfectly decode $S_1^{(b-1)}$ from $Y_{2,2}^{(b)}$ due to the additive noise $Z_2^{(b)}$. However, at time-$(b+1)$, the adaptive channel inputs $X_{1,1}^{(b+1)}$ and $X_{2,1}^{(b+1)}$ enable a perfect decoding for $Z_2^{(b)}$ (based on $Y_{2,3}^{(b)}$, $Y_{2,1}^{(b+1)}$, and the noise correlation) at terminal 2, which can be used to eliminate the noise in $Y_{2,2}^{(b)}$ and achieve error-free transmission.

described by $Y_j = X_j \oplus_2 X_{j'} \oplus Z_j$, $j = 1, 2$, where the channel noise variables $Z_1 = \text{Ber}(\epsilon_1)$ and $Z_2 = \text{Ber}(\epsilon_2)$ are independent of each other, of the source messages, and of the channel inputs. The capacity region of the channel is given by [40]: $\{(R_{c_1}, R_{c_2}) : 0 \leq R_{c_1} \leq 1 - H_b(\epsilon_2), 0 \leq R_{c_2} \leq 1 - H_b(\epsilon_1)\}$. Consider the binary correlated source pair $(S_1, S_2)$ with Z-channel correlation [108]; i.e., the transition matrices $[P_{S_2|S_1}(\cdot|\cdot)]$ and $[P_{S_1|S_2}(\cdot|\cdot)]$ between the sources $S_1$ and $S_2$ can be interpreted as a Z-channel and a reverse Z-channel, respectively. Assume that the crossover probabilities of the Z-type channels are $\alpha_1$ and $\alpha_2$, respectively. Let $P_{S_1}(1) = q_1$ and $P_{S_2}(1) = q_2$, where $q_2$ is a function of $q_1$ and $\alpha_1$ (note that one may also write $q_1$ as a function of $q_2$ and $\alpha_2$). According to Theorem 5.4, the achievable distortion region for the rate-$K/N$ transmission consists of all pairs $(D_1, D_2)$ that satisfy the inequalities below:

$$K(1-q_1+q_1\alpha_1)\left[H_{\rm b}\left(\frac{q_1\alpha_1}{1-q_1+q_1\alpha_1}\right)-H_{\rm b}\left(\frac{D_1}{1-q_1+q_1\alpha_1}\right)\right]\le N(1-H_{\rm b}(\epsilon_2)),$$
$$K(1-q_2+q_2\alpha_2)\left[H_{\rm b}\left(\frac{q_2\alpha_2}{1-q_2+q_2\alpha_2}\right)-H_{\rm b}\left(\frac{D_2}{1-q_2+q_2\alpha_2}\right)\right]\le N(1-H_{\rm b}(\epsilon_1)).$$

**Example 5.6** (**Transmitting Correlated Gaussian Sources over DM-TWCs with Additive White Gaussian Noise (AWGN) DM-TWCs).** Consider the squared-error distortion measure. The AWGN DM-TWC is described by $Y_j = X_j + X_{j'} + Z_j,\ j, j' = 1, 2$, where $Z_1$ and $Z_2$ are independent zero mean Gaussian noises with variance $\sigma_1^2$ and $\sigma_2^2$, respectively, and are independent of the source messages and of the channel inputs. The average power of channel inputs $X_j$ is set as $P_j$ for $j = 1, 2$. Moreover, the correlated sources $S_1$ and $S_2$ are considered to be zero-mean unit-variance jointly Gaussian random variables with correlation coefficient $\rho$ for some $0 \le \rho \le 1$. For this setting, Theorem 5.4 (more specifically, Corollary 5.5) yields the achievable distortion region $\{(D_1, D_2) : D_j \ge (1-\rho^2)(1+\frac{P_j}{\sigma_{j'}})^{\frac{K}{N}}, j = 1, 2\}$, for the rate-$K/N$ transmission. The detailed derivation can be found in [90, Lemma 4].

**Example 5.7** (**Transmitting Quaternary Correlated Sources over Binary Additive Noise DM-TWCs).** Suppose that $\mathcal{S}_1 = \mathcal{S}_2 = \hat{\mathcal{S}}_1 = \hat{\mathcal{S}}_2 = \{A, B, C, D\}$ and $\mathcal{X}_1 = \mathcal{X}_2 = \mathcal{Y}_1 = \mathcal{Y}_2 = \{0, 1\}$. Consider the correlated source pair with joint probability distribution given by

$$P_{S_1,S_2}(s_1, s_2) = \begin{cases} \dfrac{1}{8} & \text{if } (s_1, s_2) \in \{A, B\} \times \{A, B\} \cup \{C, D\} \times \{C, D\}, \\ 0 & \text{otherwise.} \end{cases}$$

For such sources, we observe a binary common part $S_0$; $S_0 = 0$ and $S_0 = 1$ are corresponding to $S_1, S_2 \in \{A, B\}$ and $S_1, S_2 \in \{C, D\}$, respectively. Given this common part, we can decompose $S_j$ into $(S_0, S'_j)$, where $S'_j = \text{Ber}(0.5)$. It is easy to show that $S_j$ and $(S_0, S'_j)$ have a one-to-one correspondence and the Markov chain relationship $S'_1 \multimap S_0 \multimap S'_2$ holds. Moreover, the conditional RD function $P_{S'_j|S_0}(D_j)$ is given by $P_{S'_j|S_0}(D_j) = 1 - H_\text{b}(D_j)$ for $0 \le D_j \le 0.5$.

Due to the above decomposition, the terminals only need to exchange $(S'_1, S'_2)$. When transmitting the pair $(S'_1, S'_2)$ over the binary additive noise DM-TWCs (defined in Example 5.5) at rate-$K/N$, we can apply Theorem 5.5 to characterize the achievable distortion region of the overall system, which is the convex hull of all distortion pairs $(D_1, D_2)$ satisfying

$$K(1 - H_\text{b}(D_1)) \le N(1 - H_\text{b}(\epsilon_2)),$$
$$K(1 - H_\text{b}(D_2)) \le N(1 - H_\text{b}(\epsilon_1)).$$

### 5.5.2 Adaptive Coding with More Past Information

In our JSCC scheme (detailed in the proof of Theorem 5.1), we merely use the most recent channel inputs and outputs $(X_j^{(t-1)}, Y_j^{(t-1)})$ to generate the current channel input $X_j^{(t)}$. Although ideally one would use the entire past channel input and output history for adaptive coding, the accumulated information in this case causes the Markov chain not only to have a time-varying transition kernel but also to drastically expand the state space. The idea to jointly optimize the terminals' transmission via a stationary Markov chain becomes infeasible. In the following, we sketch two coding strategies to deal with this problem. Each of the strategies can be directly integrated into our JSCC scheme, but the encoding/decoding complexity will be higher and the sufficient conditions will be significantly more complicated than the current ones.

The first strategy is to generate $X_j^{(t)}$ as a function of the past $\mu$ channel inputs $(X_j^{(t-\mu)}, X_j^{(t-\mu+1)}, \ldots, X_j^{(t-1)})$ and outputs $(Y_j^{(t-\mu)}, Y_j^{(t-\mu+1)}, \ldots, Y_j^{(t-1)})$ for some $\mu > 1$, which is similar to the memory-$\mu$ channel coding for DM-TWCs [33, Section 4.4]. This strategy increases the encoding and decoding complexity, but the state space complexity of the Markov chain is constant.

The second strategy quantizes the past channel inputs and outputs at each terminal into a set with fixed size. The channel inputs can be then generated as a function of the quantized information in that set, rather than the entire past information. This strategy is similar to the Q-graph channel coding for single-output DM-TWCs [44], and it adds a minor encoding cost. However, as the quantized knowledge is not necessarily a sufficient statistic for optimal decoding, we still need to store all past information, which clearly increases system complexity.[5]

### 5.5.3 Adaptive Coding with Incremental Side-Information

Our adaptive coding mainly coordinates the terminals' transmission on the shared channel as we did not attempt to apply Kaspi's interactive source coding idea [46] to make the best use of the sequentially received signals. Here, we give an SSCC scheme that encompasses both ideas.

The exchange of correlated sources $S_1^K$ and $S_2^K$ is now accomplished in $L$ rounds for some $L \geq 1$, which comprises $N$ channel uses (note that $N$ is a function of $K$). Specifically, for $1 \leq l \leq L$, let $N_l$ denote the number of channel uses in the $l$th round of transmission, where $\sum_{l=1}^{L} N_l = N$. In each round, viewing the previously transmitted and decoded source codewords as side-information, each terminal applies binning for

[5] One can apply sliding-window decoding to limit the amount of past information at each receiver.

source coding, followed by Han's adaptive channel coding. Each terminal also decodes the other terminal's source codeword at the end of each transmission round. After $L$ rounds, each terminal reconstructs the other terminal's source messages from the side-information and its own source messages. Clearly, this simple SSCC scheme allows two-way simultaneous transmission and interactive source coding. We summarize the achievability result in Proposition 5.2 below (without proof). Here, $T_{j,l}$, $j = 1, 2$ and $l = 1, 2, \ldots, L$, are auxiliary random variables.

**Proposition 5.2.** *A distortion pair $(D_1, D_2)$ is achievable for the rate-$K/N$ lossy transmission of correlated sources over a DM-TWC if for all $1 \leq l \leq L$, we have that*

$$K \cdot I(S_1; T_{1,l} | S_2, T_1^{l-1}, T_2^{l-1}) < N_l \cdot I(\tilde{V}_{1,l}; X_{2,l}, Y_{2,l}, \tilde{V}_{2,l}, \tilde{W}_{2,l}),$$
$$K \cdot I(S_2; T_{2,l} | S_1, T_1^{l-1}, T_2^{l-1}) < N_l \cdot I(\tilde{V}_{2,l}; X_{1,l}, Y_{1,l}, \tilde{V}_{1,l}, \tilde{W}_{1,l}),$$

*for some joint probability distributions $P_{\tilde{V}_{1,l}, \tilde{V}_{2,l}, \tilde{W}_{1,l}, \tilde{W}_{2,l}, X_{1,l}, X_{2,l}}$ as defined in [28, Section IV] and*

$$P_{T_1^L, T_2^L | S_1, S_2} = \prod_{l=1}^{L} P_{T_{1,l} | S_1, T_1^{l-1}, T_2^{l-1}} P_{T_{2,l} | S_2, T_1^{l-1}, T_2^{l-1}}$$

*and two decoding functions $\hat{S}_{j'} = g_j(S_j, T_j^L, T_{j'}^L)$ such that $\mathbb{E}[d_j(S_j, \hat{S}_j)] \leq D_j$ for $j = 1, 2$.*

Note that the above proposition reduces to Corollary 5.3 when $L = 1$. In light of this, it is of interest to ask if there exists a general adaptive JSCC scheme that integrates both features and subsumes all of our presented achievability results. We leave this question for future research.

## 5.6 Case Study: Scalar Coding

In this section, we evaluate the performance of scalar coding defined below. Such coding scheme has low encoding/decoding latency and is very suitable for real-time communication.

**Definition 5.3** (**Scalar Coding**). *Set* $N = K$. *Scalar coding is a transmission scheme such that* $X_{j,n} = f_j(S_{j,n})$ *for some time-invariant function* $f_j : \mathcal{S}_j \to \mathcal{X}_j$ *and* $\hat{S}_{j',n} = g_j(S_{j,n}, Y_{j,n})$, *for* $j, j' = 1, 2$ *with* $j \neq j'$ *and* $n = 1, 2, \ldots, N$.

In Sections 5.6.1-5.6.2, we consider the following DM-TWC with $q$-ary modulo additive noise and Hamming distortion measure:

$$\begin{cases} Y_{1,n} = X_{1,n} \oplus_q X_{2,n} \oplus_q Z_{1,n} \\ Y_{2,n} = X_{1,n} \oplus_q X_{2,n} \oplus_q Z_{2,n}, \end{cases}$$

where $X_{1,n}$, $X_{2,n}$, $Z_{1,n}$, $Z_{2,n} \in G_q = \{1, 2, \ldots, q-1\}$, $\oplus_q$ is the modulo-$q$ addition, and $\{Z_{1,n}\}_{n=1}^{\infty}$ and $\{Z_{2,n}\}_{n=1}^{\infty}$ are memoryless noise processes which are independent of each other and of the correlated sources. For $j = 1, 2$ and for all $n$, we further assume that $\Pr\big(Z_{j,n} = 0\big) = 1 - \epsilon_j$ and $\Pr\big(Z_{j,n} = z_j\big) = \epsilon_j/(q-1)$ for $z_j = 1, 2, \ldots, q-1$, where $0 \leq \epsilon_j \leq (q-1)/q$.

In Section 5.6.3, we consider the following memoryless AWGN-TWCs and the mean-square-error distortion measure:

$$\begin{cases} Y_{1,n} = X_{1,n} + X_{2,n} + Z_{1,n}, \\ Y_{2,n} = X_{1,n} + X_{2,n} + Z_{2,n}, \end{cases}$$

where $\{Z_{1,n}\}_{n=1}^{\infty}$ and $\{Z_{2,n}\}_{n=1}^{\infty}$ are memoryless zero mean Gaussian noise processes with variance $\sigma_1^2$ and $\sigma_2^2$, respectively. Also, $\{Z_{1,n}\}_{n=1}^{\infty}$ and $\{Z_{2,n}\}_{n=1}^{\infty}$ are assumed to be independent of each other and of the sources. The $X_{j,n}$'s are additionally required to satisfy the power constraint $\mathbb{E}[\sum_{n=1}^{N} |X_{j,n}|^2] \leq NP_j$, where $P_j > 0$ is the average transmission power of terminal $j$.

### 5.6.1 Transmitting Doubly Binary Symmetric Sources over Binary Additive-Noise DM-TWCs

Consider a joint binary source whose marginal probability distributions are uniform such that the individual sources are respectively modeled as the input and output of a binary symmetric channel with crossover probability $\delta \in [0, 1/2]$. For this joint source, the correlation coefficient is $\rho = 1 - 2\delta$ and the associated RD function $R^{(j)}(D_j)$ under the Hamming distortion measure is given by [74]

$$R^{(j)}(D_j) = \begin{cases} 1 - H_{\mathrm{b}}(D_j), & 0 \leq D_j \leq 1/2, \\ 0, & D_j > 1/2. \end{cases} \tag{5.43}$$

By Lemma 5.1, we known that any rate-one source-channel coding scheme achieving distortion pair $(D_1, D_2)$ must satisfy

$$\begin{aligned} R^{(1)}(D_1) &\leq I(S_1; S_2) + I(X_1; Y_2|X_2) \\ &\leq 1 - H_{\mathrm{b}}(\delta) + (H(Y_2) - H(Y_2|X_1, X_2)) && (5.44) \\ &\leq 2 - H_{\mathrm{b}}(\delta) - H_{\mathrm{b}}(\epsilon_2) && (5.45) \end{aligned}$$

where (5.44) holds since $H(Y_2|X_2) \leq H(Y_2)$ and (5.45) follows that $H(Y_2) \leq 1$ and $H(Y_2|X_1, X_2) = H(Z_2|X_1, X_2) = H(Z_2) = H_{\mathrm{b}}(\epsilon_2)$. Similarly, we have $R^{(2)}(D_2) \leq 2 - H_{\mathrm{b}}(\delta) - H_{\mathrm{b}}(\epsilon_1)$. Using (5.45) and (5.43), lower bounds for the system distortions $D_1$ and $D_2$ can be found numerically for given $\delta$ and $\epsilon_j$'s.

Now, we consider the scalar coding scheme with $f_j(S_j) = S_j$ so that $X_{j,n} = S_{j,n}$

for $j = 1, 2$ and $n = 1, 2, \ldots, N$. For this encoder, it can be shown that the estimate $\hat{S}_{j,n} = Y_{j',n} \oplus_2 X_{j',n}$, $j \neq j'$, yields the optimum decoding performance, and the average distortions are given by $D_1 = \epsilon_2$ and $D_2 = \epsilon_1$. In Fig. 5.7, we plot the gap between the distortion lower bound and $\epsilon_2$ (for the direction from terminal 1 to 2). The numerical results show that scalar coding is sub-optimal. In particular, as the source correlation $\rho$ increases, the gap becomes larger. Also, when the quality of the channel deteriorates, the scalar coding scheme suffers a serious performance degradation. Nevertheless, when $S_1$ and $S_2$ are independent, i.e., $\rho = 0$, scalar coding becomes optimal (with the gap in Fig. 5.7 reducing to zero).

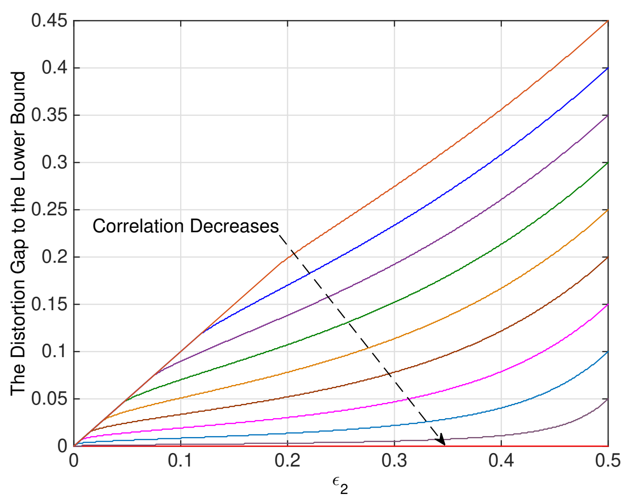



Figure 5.7: The performance loss of transmitting binary correlated sources via scalar coding. The curves from top to bottom correspond to $\rho$ ranging from 0.9 to 0 with a step size of 0.1.

### 5.6.2 Transmitting Non-Binary Independent Sources over Non-Binary Additive-Noise DM-TWCs

Suppose that $S_1$ and $S_2$ are independent and uniformly distributed $q$-ary sources, i.e., $\Pr(S_1 = s) = \Pr(S_2 = s) = 1/q$ for $s = 0, 1, \ldots, q-1$. For such sources, the RD function $R^{(j)}(D_j)$ under the Hamming distortion measure is given by [109]

$$R^{(j)}(D_j) = \begin{cases} \log_2 q - H_{\mathrm{b}}(D_j) - D_j \log_2(q-1), & 0 \le D_j \le \frac{q-1}{q}, \\ 0, & D_j > \frac{q-1}{q}, \end{cases} \tag{5.46}$$

for $j = 1, 2$. By Lemma 5.1 (with $r = 1$), one has

$$\begin{aligned} R^{(1)}(D_1) &\le I(S_1; S_2) + I(X_1; Y_2|X_2) \\ &\le \log_2 q - H(Z_2) \\ &= \log_2 q - H_{\mathrm{b}}(\epsilon_2) - \epsilon_2 \log_2(q-1) \end{aligned} \tag{5.47}$$

for the $q$-ary additive-noise DM-TWCs, where the last equation is obtained by evaluating $H(Z_2)$. From (5.46) and (5.47), we obtain that $D_1 \ge \epsilon_2$. Similarly, we have $D_2 \ge \epsilon_1$. On the other hand, one can easily show that the distortion achieved by the optimum decoder for scalar coding in this case is $D_1 = \epsilon_2$ and $D_2 = \epsilon_1$. Thus, scalar coding is optimal for this non-binary setting.

We remark that Theorem 5.2 has asserted that the SSCC scheme is optimal for this setting at an expense of long decoding delay. However, based on the above result, it is sufficient to employ the scalar coding to attain the optimal performance, thus significantly reducing the coding complexity and lowering the coding latency.

### 5.6.3 Transmitting Gaussian Sources over Memoryless AWGN-TWCs

The correlated sources $S_1$ and $S_2$ are herein considered to be jointly Gaussian with correlation coefficient $\rho$. Without loss of generality, $S_1$ and $S_2$ are assumed to have zero mean and unit variance. In this case, the RD function under the squared error distortion measure is given by [74]

$$R^{(j)}(D_j) = \begin{cases} \frac{1}{2}\log\frac{1}{D_j} & 0 < D_j \leq 1, \\ 0 & D_j > 1, \end{cases} \tag{5.48}$$

and $I(S_1; S_2) = -\frac{1}{2}\cdot\log\big(1-\rho^2\big)$, where $-1 \leq \rho \leq 1$.

We next derive a bound on the performance limit of rate-one transmission over the Gaussian TWCs. Let $\gamma_j \triangleq P_j/\sigma_{j'}^2$ be the signal-to-noise ratio (SNR) for $j, j' = 1, 2$ with $j \neq j'$. Combining (5.48) with Lemma (5.1), we obtain the lower bounds $D_1 \geq (1-\rho^2)/(1+\gamma_1)$ and $D_2 \geq (1-\rho^2)/(1+\gamma_2)$. In fact, as the WZ-RD and conditional RD functions are equal for the jointly Gaussian sources and adaptive coding cannot enlarge the channel capacity of the AWGN-TWC, we can apply the complete JSCC Theorem 5.5 to obtain the following corollary for general rate.

**Corollary 5.5.** *For the rate-$\frac{K}{N}$ lossy transmission of zero mean, unit variance, and correlation $\rho$ jointly Gaussian source $(S_1, S_2)$ over the memoryless AWGN-TWC with SNRs $\gamma_1$ and $\gamma_2$, $(D_1, D_2)$ is achievable if and only if*

$$\begin{cases} K\cdot R_{\mathrm{WZ}}^{(1)}(D_1) \leq \dfrac{N}{2}\log(1+\gamma_1), \\ K\cdot R_{\mathrm{WZ}}^{(2)}(D_2) \leq \dfrac{N}{2}\log(1+\gamma_2). \end{cases}$$

Now, consider the scalar coding from terminal $j$ to $j'$ with $f_j$ given by $X_{j,n} =$

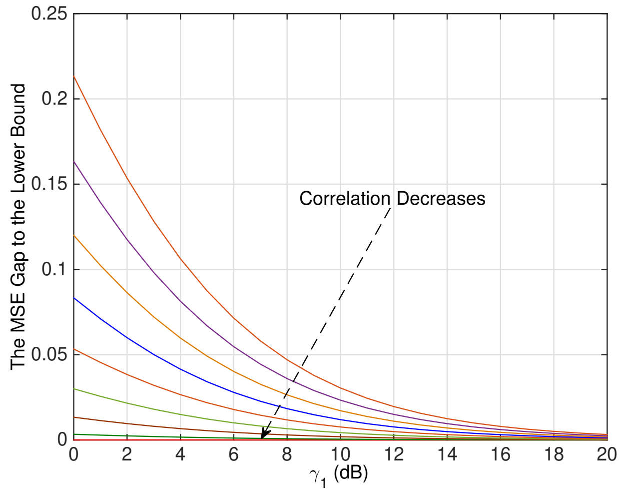



Figure 5.8: The performance loss of transmitting jointly Gaussian sources via scalar coding. The curves from top to bottom correspond to $\rho$ ranging from 0.9 to 0 with a step size of 0.1.

$f_j(S_{j,n}) = \alpha S_{j,n}$, where $\alpha = \sqrt{P_1}$ is set to satisfy the power constraint. At the receiver, we employ a minimum mean square error (MMSE) detector to yield the optimum estimate $\hat{S}_{j,n} = \sqrt{P_j}(Y_{j',n} - X_{j',n})/(P_j + \sigma_{j'}^2)$. From the numerical results shown in Fig. 5.8 (about the distortion gap from terminals 1 to 2), we observe a behavior similar to the discrete system of Fig. 5.7. In the extreme case of $\rho = 0$, i.e., when $S_1$ and $S_2$ are independent, scalar coding achieves the distortion lower bounds for both direction of transmission and is hence optimal.

# Chapter 6

# Conclusion and Future Work

We investigated capacity problems for TWCs and devised efficient JSCC schemes for sending correlated sources over two-terminal DM-TWCs. A large portion of this thesis was devoted to two-terminal DM-TWCs, laying the foundation to study transmission problems for other, more complex two-way networks. For the capacity problems, our approach relies on the viewpoint that a TWC can be decomposed into two interactive state-dependent one-way channels, which enables a more tractable analysis when the transmission of one terminal is not affected by the transmission of the other terminal. Taking this viewpoint, a rich set of conditions under which Shannon's random coding scheme is optimal was derived, together with a detailed examination of their relationships. The same approach was also used to obtain approximation results for two-terminal DM-TWCs and refine the current capacity results for TWCs with memory and multi-terminal DM-TWCs.

It is worth mentioning that our approach can be used to investigate whether or not Shannon-type random coding schemes (under independent and non-adaptive inputs) provide tight bounds for other classical communication scenarios such as MACs with feedback and one-way compound channels. In particular, our results can be used to identify compound channels where the availability of channel state information at the transmitter (in addition to the receiver) cannot improve capacity.

For two-terminal source transmission problems, we constructed an adaptive coding scheme to prove a forward JSCC theorem, which results in an achievable distortion region for two-way lossy simultaneous transmission. Our adaptive coding method demonstrates a way to coordinate the independent transmissions of the terminals; it also underscores the importance of preserving source correlation as illustrated via several examples. Moreover, our coding scheme subsumes several simple non-adaptive coding methods, providing a unified transmission framework that allows for diverse various system complexity and performance trade-offs. Although the general form of our scheme is complex, in many cases its SSCC instances suffice to achieve the optimal performance. Still, its potential use in practice needs a further study.

It is hoped that our results can serve as a basis for further study in two-way communication. To conclude this thesis, we outline some future research topics:

- Most of the available tightness conditions for Shannon's random coding inner bound only identify TWCs whose capacity region can be determined without the use of time-sharing. Those conditions are not applicable to determine capacity region for TWC where a time-sharing scheme is necessary to achieve capacity such as in the case of push-to-talk TWCs. It would be important to derive a more general tightness condition which takes both situations into account.

- In addition to our method to approximate the capacity region, one may use TWCs with symmetry properties to inner and outer bound the capacity region for asymmetric TWCs. The idea of degraded broadcast channels [88, 110] can be borrowed to define degradation among two TWCs. This research problem also needs a study on the notion of channel ordering for TWCs [111–114].

- For common-output DM-TWCs, one might develop a graph-based capacity outer bound using the techniques in [115] and find conditions under which the graph-based outer bound matches the graph-based inner bound [44]. Whether or not the concept of posterior matching [116] can be applied to design adaptive coding for such channels is also of interest.

- For source-channel communication with encoding/decoding latency constraint, one can study the optimality condition of scalar coding, as done in [117] for one-way systems. Causal coding [118] and zero-delay JSCC [119] are also interesting subjects to investigate in the context of TWCs. For practical considerations, one may refine the channel-optimized quantization design for DM-TWCs in [60] using a machine learning approach [120].

- The design of finite-length error correction codes for DM-TWCs is another challenging task in practice. Whether or not the start-of-the-art codes such as low-density parity-check codes [121] and polar codes [122] can perform well on DM-TWCs is not clear. How to incorporate the idea of adaptive coding into those channel codes also requires a further investigation.

- The two-way communication in this thesis only considers a single transmitter and receiver antenna pair. As multiple-input and multiple-output (MIMO) transmission systems [123] are now widely adopted in commercial standards, it will be useful to define a MIMO-TWC model and study its capacity from an information-theoretic perspective. We note that a possible way to define such a channel model is to consider finite-field matrix channels [124].

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# Appendix A

# Proofs of Supplementary Results for Chapter 2

## A.1 Proof of Proposition 2.1

The proof of Proposition 2.1 is based on the following Lemmas A.1.1 to A.1.3.

**Lemma A.1.1.** *If a DM-TWC satisfies the conditions in Proposition 2.1, then for any input distribution $P^{(1)}_{X_1,X_2}$, any pair of distinct symbols $x_1'$, $x_1'' \in \mathcal{X}_1$, and $P^{(2)}_{X_1,X_2}(x_1,x_2) \triangleq P^{(1)}_{X_1,X_2}(\tau^{\mathcal{X}_1}_{x_1',x_1''}(x_1),x_2)$, the following results hold:*

$$I^{(1)}(X_1;Y_2|X_2) = I^{(2)}(X_1;Y_2|X_2), \tag{A.1}$$

$$I^{(1)}(X_2;Y_1|X_1) = I^{(2)}(X_2;Y_1|X_1), \tag{A.2}$$

$$\mathcal{R}(P^{(1)}_{X_1,X_2}, P_{Y_1,Y_2|X_1,X_2}) = \mathcal{R}(P^{(2)}_{X_1,X_2}, P_{Y_1,Y_2|X_1,X_2}), \tag{A.3}$$

*where the superscript indicates the corresponding input distribution under evaluation.*

*Proof:* For any $P^{(1)}_{X_1,X_2}$ and $P^{(2)}_{X_1,X_2}(x_1,x_2) = P^{(1)}_{X_1,X_2}(\tau^{\mathcal{X}_1}_{x_1',x_1''}(x_1),x_2)$, we have

$$I^{(2)}(X_1;Y_2|X_2)$$
$$= \sum_{x_2} P^{(2)}_{X_2}(x_2) \cdot I^{(2)}(X_1;Y_2|X_2=x_2)$$

$$= \sum_{x_2} P^{(2)}_{X_2}(x_2) \sum_{x_1,y_2} P^{(2)}_{X_1|X_2}(x_1|x_2) P_{Y_2|X_1,X_2}(y_2|x_1,x_2) \log \frac{P_{Y_2|X_1,X_2}(y_2|x_1,x_2)}{P^{(2)}_{Y_2|X_2}(y_2|x_2)}$$

$$= \sum_{x_1,x_2,y_2} P^{(2)}_{X_1,X_2}(x_1,x_2) P_{Y_2|X_1,X_2}(y_2|x_1,x_2) \log \frac{P_{Y_2|X_1,X_2}(y_2|x_1,x_2)}{\sum_{\tilde{x}_1} P^{(2)}_{X_1|X_2}(\tilde{x}_1|x_2) P_{Y_2|X_1,X_2}(y_2|\tilde{x}_1,x_2)}$$

$$= \sum_{x_1,x_2,y_2} P^{(1)}_{X_1,X_2}(\tau^{\mathcal{X}_1}_{x_1',x_1''}(x_1),x_2) P_{Y_2|X_1,X_2}(\pi^{\mathcal{Y}_2}[x_1',x_1''](y_2)|\tau^{\mathcal{X}_1}_{x_1',x_1''}(x_1),x_2)$$

$$\cdot \log \frac{P_{Y_2|X_1,X_2}(\pi^{\mathcal{Y}_2}[x_1',x_1''](y_2)|\tau^{\mathcal{X}_1}_{x_1',x_1''}(x_1),x_2)}{\sum_{\tilde{x}_1} P^{(1)}_{X_1|X_2}(\tau^{\mathcal{X}_1}_{x_1',x_1''}(\tilde{x}_1)|x_2) P_{Y_2|X_1,X_2}(\pi^{\mathcal{Y}_2}[x_1',x_1''](y_2)|\tau^{\mathcal{X}_1}_{x_1',x_1''}(\tilde{x}_1),x_2)} \quad (A.4)$$

$$= \sum_{x_1,x_2,y_2} P^{(1)}_{X_1,X_2}(\tau^{\mathcal{X}_1}_{x_1',x_1''}(x_1),x_2) P_{Y_2|X_1,X_2}(\pi_2^{\mathcal{Y}_2}[x_1',x_1''](y_2)|\tau^{\mathcal{X}_1}_{x_1',x_1''}(x_1),x_2)$$

$$\cdot \log \frac{P_{Y_2|X_1,X_2}(\pi^{\mathcal{Y}_2}[x_1',x_1''](y_2)|\tau^{\mathcal{X}_1}_{x_1',x_1''}(x_1),x_2)}{\sum_{\tilde{x}_1} P^{(1)}_{X_1|X_2}(\tilde{x}_1|x_2) P_{Y_2|X_1,X_2}(\pi^{\mathcal{Y}_2}[x_1',x_1''](y_2)|\tilde{x}_1,x_2)} \quad (A.5)$$

$$= \sum_{x_1,x_2,y_2} P^{(1)}_{X_1,X_2}(\tau^{\mathcal{X}_1}_{x_1',x_1''}(x_1),x_2) P_{Y_2|X_1,X_2}(\pi_2^{\mathcal{Y}_2}[x_1',x_1''](y_2)|\tau^{\mathcal{X}_1}_{x_1',x_1''}(x_1),x_2)$$

$$\cdot \log \frac{P_{Y_2|X_1,X_2}(\pi^{\mathcal{Y}_2}[x_1',x_1''](y_2)|\tau^{\mathcal{X}_1}_{x_1',x_1''}(x_1),x_2)}{P^{(1)}_{Y_2|X_2}(\pi^{\mathcal{Y}_2}[x_1',x_1''](y_2)|x_2)}$$

$$= \sum_{x_1,x_2,\tilde{y}_2} P^{(1)}_{X_1,X_2}(\tau^{\mathcal{X}_1}_{x_1',x_1''}(x_1),x_2) P_{Y_2|X_1,X_2}(\tilde{y}_2|\tau^{\mathcal{X}_1}_{x_1',x_1''}(x_1),x_2) \log \frac{P_{Y_2|X_1,X_2}(\tilde{y}_2|\tau^{\mathcal{X}_1}_{x_1',x_1''}(x_1),x_2)}{P^{(1)}_{Y_2|X_2}(y_2|x_2)} \quad (A.6)$$

$$= \sum_{\tilde{x}_1,x_2,\tilde{y}_2} P^{(1)}_{X_1,X_2}(\tilde{x}_1,x_2) P_{Y_2|X_1,X_2}(\tilde{y}_2|\tilde{x}_1,x_2) \log \frac{P_{Y_2|X_1,X_2}(\tilde{y}_2|\tilde{x}_1,x_2)}{P^{(1)}_{Y_2|X_2}(\tilde{y}_2|x_2)} \quad (A.7)$$

$$= I^{(1)}(X_1;Y_2|X_2), \quad (A.8)$$

where (A.4) holds by the definition of $P^{(2)}_{X_1,X_2}(x_1,x_2)$ and the marginal

$P_{Y_2|X_1,X_2}(y_2|x_1,x_2) = P_{Y_2|X_1,X_2}(\pi^{\mathcal{Y}_2}[x_1',x_1''](y_2)|\tau^{\mathcal{X}_1}_{x_1',x_1''}(x_1),x_2)$ derived from the Shannon condition in (2.8), (A.5) and (A.7) hold since $\tau^{\mathcal{X}_1}_{x_1',x_1''}$ is a bijection, and (A.6) follows the bijection of $\pi^{\mathcal{Y}_2}[x_1',x_1'']$.

By a similar argument, we can verify that $I^{(1)}(X_2;Y_1|X_1){=}I^{(2)}(X_2;Y_1|X_1)$. The proof is then completed by noting that $\mathcal{R}(P^{(1)}_{X_1,X_2},P_{Y_1,Y_2|X_1,X_2}){=}\mathcal{R}(P^{(2)}_{X_1,X_2},P_{Y_1,Y_2|X_1,X_2})$, which follows from the definition of $\mathcal{R}$ in (2.1), (A.1), and (A.2). ■

**Lemma A.1.2.** *If a DM-TWC satisfies the condition in Proposition 2.1, then for any input distribution* $P^{(1)}_{X_1,X_2}$*, any pair of distinct symbols* $x_1'$, $x_1'' \in \mathcal{X}_1$*, and* $P^{(2)}_{X_1,X_2}(x_1,x_2) \triangleq P^{(1)}_{X_1,X_2}(\tau^{\mathcal{X}_1}_{x_1',x_1''}(x_1),x_2)$*, we have*

$$\mathcal{R}(P^{(1)}_{X_1,X_2},P_{Y_1,Y_2|X_1,X_2}) \subseteq \mathcal{R}(P^{(3)}_{X_1,X_2},P_{Y_1,Y_2|X_1,X_2}) \tag{A.9}$$

*where* $P^{(3)}_{X_1,X_2}(x_1,x_2) \triangleq \frac{1}{2}(P^{(1)}_{X_1,X_2}(x_1,x_2)+P^{(2)}_{X_1,X_2}(x_1,x_2))$.

*Proof:* The proof relies on the fact that both $I(X_1;Y_2|X_2)$ and $I(X_2;Y_1|X_1)$ are concave in input distribution $P_{X_1,X_2}$ [3]. For any given $P^{(1)}_{X_1,X_2}$, let $P^{(2)}_{X_1,X_2}(x_1,x_2) \triangleq P^{(1)}_{X_1,X_2}(\tau^{\mathcal{X}_1}_{x_1',x_1''}(x_1),x_2)$ and $P^{(3)}_{X_1,X_2}(x_1,x_2) = \frac{1}{2}(P^{(1)}_{X_1,X_2}(x_1,x_2)+P^{(2)}_{X_1,X_2}(x_1,x_2))$. The concavity then implies that

$$I^{(3)}(X_1;Y_2|X_2) \geq \frac{1}{2}I^{(1)}(X_1;Y_2|X_2)+\frac{1}{2}I^{(2)}(X_1;Y_2|X_2) \tag{A.10}$$

$$= I^{(1)}(X_1;Y_2|X_2), \tag{A.11}$$

and

$$I^{(3)}(X_2;Y_1|X_1) \geq \frac{1}{2}I^{(1)}(X_2;Y_1|X_1)+\frac{1}{2}I^{(2)}(X_2;Y_1|X_1) \tag{A.12}$$

$$= I^{(1)}(X_2;Y_1|X_1), \tag{A.13}$$

where (A.11) and (A.13) follow the result in Lemma A.1.1. The rest of the proof follows the definition of $\mathcal{R}$ in (2.1). ■

**Lemma A.1.3.** *If a DM-TWC satisfies the condition in Proposition 2.1, then for any given input distribution* $P_{X_1,X_2}$*, we have*

$$\mathcal{R}(P_{X_1,X_2}, P_{Y_1,Y_2|X_1,X_2}) \subseteq \mathcal{R}\Big(P^{\mathrm{U}}_{\mathcal{X}_1} P_{X_2}, P_{Y_1,Y_2|X_1,X_2}\Big), \tag{A.14}$$

*where* $P^{\mathrm{U}}_{\mathcal{X}_1}$ *is the uniform probability distribution on* $\mathcal{X}_1$ *and* $P_{X_2} = \sum_{x_1} P_{X_1,X_2}(x_1,x_2)$*.*

*Proof:* Without loss of generality, suppose that $\mathcal{X}_1 \triangleq \{1,2,\ldots,\kappa\}$. For any input distribution $P_{X_1,X_2}$ and some integer $1 \le l \le \kappa$, define

$$P^{\mathrm{U}_l}_{X_1,X_2}(x_1,x_2) = \begin{cases} \frac{1}{l}\sum_{i=1}^{l} P_{X_1,X_2}(i,x_2), & \text{if } 1 \le x_1 \le l, \\ P_{X_1,X_2}(x_1,x_2), & \text{if } l < x_1 \le \kappa. \end{cases} \tag{A.15}$$

Clearly, we have $P^{\mathrm{U}_1}_{X_1,X_2} = P_{X_1,X_2}$ and $P^{\mathrm{U}_\kappa}_{X_1,X_2} = P^{\mathrm{U}}_{\mathcal{X}_1} P_{X_2}$. We prove this lemma by using (finite) induction on $l$ showing that $\mathcal{R}(P_{X_1,X_2}, P_{Y_1,Y_2|X_1,X_2}) \subseteq \mathcal{R}(P^{\mathrm{U}_l}_{X_1,X_2}, P_{Y_1,Y_2|X_1,X_2})$ for $2 \le l \le \kappa$. For the base case, i.e., $l = 2$, we set $P^{(1)}_{X_1,X_2} = P_{X_1,X_2}$, $x_1' = 1$, and $x_1'' = 2$ in Lemma A.1.2, in which

$$\begin{aligned} P^{(3)}_{X_1,X_2}(x_1,x_2) &= \frac{1}{2}\Big(P^{(1)}_{X_1,X_2}(x_1,x_2) + P^{(2)}_{X_1,X_2}(x_1,x_2)\Big) \\ &= \frac{1}{2}\Big(P^{(1)}_{X_1,X_2}(x_1,x_2) + P^{(1)}_{X_1,X_2}(\tau^{\mathcal{X}_1}_{x_1',x_1''}(x_1),x_2)\Big) \\ &= \frac{1}{2}\Big(P_{X_1,X_2}(x_1,x_2) + P_{X_1,X_2}(\tau^{\mathcal{X}_1}_{1,2}(x_1),x_2)\Big) \\ &= \begin{cases} \frac{1}{2}\Big(P_{X_1,X_2}(1,x_2) + P_{X_1,X_2}(2,x_2)\Big), & \text{if } x_1 = 1 \text{ or } 2, \\ P_{X_1,X_2}(x_1,x_2), & \text{if } 2 < x_1 \le \kappa, \end{cases} \\ &= P^{\mathrm{U}_2}_{X_1,X_2}(x_1,x_2), \end{aligned}$$

thereby proving that $I^{\mathrm{U}_2}(X_1;Y_2|X_2) \geq I^{(1)}(X_1;Y_2|X_2)$ and $I^{\mathrm{U}_2}(X_2;Y_1|X_1) \geq I^{(1)}(X_2;Y_1|X_1)$, and hence $\mathcal{R}(P_{X_1,X_2}, P_{Y_1,Y_2|X_1,X_2}) \subseteq \mathcal{R}(P^{\mathrm{U}_2}_{X_1,X_2}, P_{Y_1,Y_2|X_1,X_2})$.

Now, assume that $\mathcal{R}(P_{X_1,X_2}, P_{Y_1,Y_2|X_1,X_2}) \subseteq \mathcal{R}(P^{\mathrm{U}_l}_{X_1,X_2}, P_{Y_1,Y_2|X_1,X_2})$ holds for $3 \leq l < m < \kappa$. We want to show that $\mathcal{R}(P_{X_1,X_2}, P_{Y_1,Y_2|X_1,X_2}) \subseteq \mathcal{R}(P^{\mathrm{U}_m}_{X_1,X_2}, P_{Y_1,Y_2|X_1,X_2})$. Observing that

$$P^{\mathrm{U}_m}_{X_1,X_2}(x_1,x_2) = \frac{1}{m}\left(P^{\mathrm{U}_1}_{X_1,X_2}(\tau^{\mathcal{X}_1}_{1,m}(x_1),x_2) + (m-1)P^{\mathrm{U}_{m-1}}_{X_1,X_2}(x_1,x_2)\right) \tag{A.16}$$

and by the concavity of $I(X_1;Y_2|X_2)$, we have

$$\begin{aligned} I^{\mathrm{U}_m}(X_1;Y_2|X_2) &\geq \frac{1}{m}\Big(I^{\mathrm{U}_1}(X_1;Y_2|X_2) + (m-1)I^{\mathrm{U}_{m-1}}(X_1;Y_2|X_2)\Big) \\ &\geq \frac{1}{m}\Big(I^{\mathrm{U}_1}(X_1;Y_2|X_2) + (m-1)I^{\mathrm{U}_1}(X_1;Y_2|X_2)\Big) \qquad \text{(A.17)} \\ &= I^{\mathrm{U}_1}(X_1;Y_2|X_2), \qquad \text{(A.18)} \end{aligned}$$

where (A.17) is due to the induction hypothesis. Similarly, we have

$$I^{\mathrm{U}_m}(X_2;Y_1|X_1) \geq I^{\mathrm{U}_1}(X_2;Y_1|X_1), \tag{A.19}$$

Combining the definition of $\mathcal{R}$ in (2.1), (A.18) and (A.19) then completes the proof.

■

We are ready to prove Proposition 2.1.

*Proof:* Note that

$$\begin{aligned} \mathcal{C}_{\mathrm{O}}(P_{Y_1,Y_2|X_1,X_2}) &= \bigcup_{P_{X_1,X_2}} \mathcal{R}(P_{X_1,X_2}, P_{Y_1,Y_2|X_1,X_2}) \\ &\subseteq \overline{\mathrm{co}}\left(\bigcup_{P_{X_2}} \mathcal{R}\Big(P^{\mathrm{U}}_{\mathcal{X}_1} P_{X_2}, P_{Y_1,Y_2|X_1,X_2}\Big)\right) \qquad \text{(A.20)} \end{aligned}$$

$$\subseteq \mathcal{C}_{\mathrm{I}}(P_{Y_1,Y_2|X_1,X_2}), \tag{A.21}$$

where (A.20) follows from Lemma A.1.3. Together with $\mathcal{C}_{\mathrm{I}}(P_{Y_1,Y_2|X_1,X_2}) \subseteq \mathcal{C}_{\mathrm{O}}(P_{Y_1,Y_2|X_1,X_2})$ then gives the conclusion:

$$\mathcal{C}(P_{Y_1,Y_2|X_1,X_2}) = \mathcal{C}_{\mathrm{I}}(P_{Y_1,Y_2|X_1,X_2}) = \mathcal{C}_{\mathrm{O}}(P_{Y_1,Y_2|X_1,X_2}) = \overline{\mathrm{co}}\left(\bigcup_{P_{X_2}} \mathcal{R}\left(P^{\mathrm{U}}_{\mathcal{X}_1} P_{X_2}, P_{Y_1,Y_2|X_1,X_2}\right)\right). \tag{A.22}$$

■

We remark that, based on the proof of Proposition 2.1, it is straightforward to prove Proposition 2.2, i.e., Shannon's full symmetry conditions, and hence the details are omitted.

## A.2 Auxiliary Results for the Proof of Theorem 2.16

The appendix establishes input-output mutual information results for one-way channels that are of the same type as the state-dependent one-way channels in the generalized PTT-TWC of Theorem 2.16. Let $\mathcal{X} = \{0, 1, \ldots, r-1\}$ and $\mathcal{Y} = \{0, 1, \ldots, s-1\}$ denote channel input and output alphabets, respectively, for some integers $r \geq 3$ and $s \geq 2$. Suppose that the set of probability vectors $\{[P_{Y|X}(\cdot|x_1)] : x_1 \in \mathcal{X}\backslash\{0\}\}$ specifies a weakly-symmetric channel and $P_{Y|X}(y|0) = 1/s$ for all $y \in \mathcal{Y}$. The input-output mutual information for a specific channel input symbol $x \in \mathcal{X}$ is defined as

$$I(X = x; Y) \triangleq \sum_{y \in \mathcal{Y}} P_{Y|X}(y|x) \cdot \log \frac{P_{Y|X}(y|x)}{P_Y(y)}.$$

The following results are needed in the proof of Theorem 2.16.

**Lemma A.2.1.** *The capacity of the channel with the above properties is given by*

$C^* = \max_{P_X} I(X;Y) = \log s - H([P_{Y|X}(\,\cdot\,|1)])$, *where* $H([P_{Y|X}(\,\cdot\,|1)])$ *denotes the entropy of the probability vector* $[P_{Y|X}(\,\cdot\,|1)]$. *The capacity-achieving input distribution is given by:*

$$P_X^*(x) = \begin{cases} 0 & \textit{if } x = 0, \\ \frac{1}{r-1} & \textit{otherwise.} \end{cases}$$

*Proof:* We apply the KKT condition for channel capacity [77, Theorem 4.5.1] to check the optimality of $P_X^*$. Under $P_X^*$, we first have that $I(X = x; Y) = \log s - H([P_{Y|X}(\,\cdot\,|1)])$ for $x \neq 0$ [74, Theorem 7.2.1] since $P_X^*$ is a uniform distribution when restricted to the input alphabet $\mathcal{X} \setminus \{0\}$ and the channel with the restricted inputs is weakly-symmetric. Moreover, for $x \neq 0$, we have

$$\begin{aligned} I(X=0;Y) &= \sum_{y=0}^{s-1} \frac{1}{s} \cdot \log \frac{1/s}{\sum_{x' \neq 0} P_{Y|X}(y|x')/(r-1)} \\ &= \log \frac{r-1}{s} - \sum_{y=0}^{s-1} \frac{1}{s} \cdot \log \left( \sum_{x' \neq 0} P_{Y|X}(y|x') \right) \\ &= \log \frac{r-1}{s} - \log \left( \sum_{x' \neq 0} P_{Y|X}(y'|x') \right) && \text{(A.23)} \\ &= -\log s + \log(r-1) \\ &\qquad - \underbrace{\left( \sum_{y'=0}^{s-1} P_{Y|X}(y'|x) \right)}_{=1} \cdot \log \left( \sum_{x' \neq 0} P_{Y|X}(y'|x') \right) \\ &\leq -H(Y|X=x) + \log(r-1) \\ &\qquad - \sum_{y'=0}^{s-1} P_{Y|X}(y'|x) \cdot \log \left( \sum_{x' \neq 0} P_{Y|X}(y'|x') \right) && \text{(A.24)} \\ &= \sum_{y'=0}^{s-1} P_{Y|X}(y'|x) \cdot \log \frac{P_{Y|X}(y'|x)}{\sum_{x' \neq 0} P_{Y|X}(y'|x')/(r-1)} \end{aligned}$$

$$= I(X = x; Y),$$

where $y' \in \mathcal{Y}$ is arbitrary in (A.23) since $\sum_{x' \neq 0} P_{Y|X}(y|x')$ does not depend on $y$ and (A.24) holds since $H(Y|X = 0) \leq \log s$. Combining the above results then gives that $I(X = 0; Y) \leq I(X = x; Y)$ for all $x \neq 0$, thus implying the optimality of $P_X^*$. Finally, we conclude that $C^* = \max_{P_X} I(X; Y) = I(X = x; Y)$ for any $x \neq 0$ by the KKT condition. ■

**Lemma A.2.2.** *For any* $0 \leq \alpha \leq 1$*, consider the following channel input distribution:*

$$P_X^{(1)}(x) = \begin{cases} \alpha & \text{if } x = 0, \\ \frac{1-\alpha}{r-1} & \text{otherwise}, \end{cases}$$

*Let* $P_X^{(2)}$ *denote any input distribution with* $P_X^{(2)}(0) = \alpha$*. Then, we have that* $I^{(2)}(X; Y) \leq I^{(1)}(X; Y) = (1 - \alpha)C^*$ *(here the superscript indicates which input distribution is used for evaluation).*

*Proof:* First, we have that $H^{(2)}(Y) \leq \log s = H^{(1)}(Y)$. Also, since $H(Y|X = x) = H(Y|X = 1)$ for all $x \neq 0$ due to the weakly-symmetric structure, one can easily conclude that $H^{(1)}(Y|X) = H^{(2)}(Y|X)$. The above results then imply that $I^{(2)}(X; Y) \leq I^{(1)}(X; Y)$. Moreover, a direct computation (with the result in Lemma A.2.1) yields that $I^{(1)}(X; Y) = (1 - \alpha)C^*$, thereby completing the proof. ■

# Appendix B

# Proofs of Supplementary Results for Chapter 4

## B.1 Proof of Lemma 4.1

*Proof:* We have

$$
\begin{aligned}
&I^{(2)}(X_1;Y_3|X_2,X_3=x_3)\\
&=\sum_{x_1,x_2,y_3} P^{(2)}_{X_1,X_2|X_3}(x_1,x_2|x_3)\cdot P_{Y_3|X_1,X_2,X_3}(y_3|x_1,x_2,x_3)\\
&\qquad\cdot\log\frac{P_{Y_3|X_1,X_2,X_3}(y_3|x_1,x_2,x_3)}{\sum_{\tilde{x}_1} P^{(2)}_{X_1|X_2,X_3}(\tilde{x}_1|x_2,x_3)\cdot P_{Y_3|X_1,X_2,X_3}(y_3|\tilde{x}_1,x_2,x_3)}\\
&=\sum_{x_1,x_2,y_3} P^{(1)}_{X_1,X_2|X_3}(\tau^{\mathcal{X}_1}_{x_1',x_1''}(x_1),x_2|x_3)P_{Y_3|X_1,X_2,X_3}(\pi^{\mathcal{Y}_3}[x_1',x_1''](y_3)|\tau^{\mathcal{X}_1}_{x_1',x_1''}(x_1),x_2,x_3)\\
&\quad\cdot\Bigg[\log P_{Y_3|X_1,X_2,X_3}(\pi^{\mathcal{Y}_3}[x_1',x_1''](y_3)|\tau^{\mathcal{X}_1}_{x_1',x_1''}(x_1),x_2,x_3)\\
&\quad-\log\left(\sum_{\tilde{x}_1} P^{(1)}_{X_1|X_2,X_3}(\tau^{\mathcal{X}_1}_{x_1',x_1''}(\tilde{x}_1)|x_2,x_3)P_{Y_3|X_1,X_2,X_3}(\pi^{\mathcal{Y}_3}[x_1',x_1''](y_3)|\tau^{\mathcal{X}_1}_{x_1',x_1''}(\tilde{x}_1),x_2,x_3)\right)\Bigg]
\end{aligned}
\tag{B.1}
$$

$$
\begin{aligned}
&=\sum_{x_1,x_2,y_3} P^{(1)}_{X_1,X_2|X_3}(x_1,x_2|x_3)\cdot P_{Y_3|X_1,X_2,X_3}(y_3|x_1,x_2,x_3)\\
&\qquad\cdot\log\frac{P_{Y_3|X_1,X_2,X_3}(y_3|x_1,x_2,x_3)}{\sum_{\tilde{x}_1} P^{(1)}_{X_1|X_2,X_3}(\tilde{x}_1|x_2,x_3)P_{Y_3|X_1,X_2,X_3}(y_3|\tilde{x}_1,x_2,x_3)}
\end{aligned}
\tag{B.2}
$$

$$= I^{(1)}(X_1; Y_2|X_2, X_3 = x_3),$$

where (B.1) follows from (4.36) and (4.40), (B.2) holds since $\pi^{\mathcal{Y}_3}[x_1', x_1'']$ and $\tau^{\mathcal{X}_1}_{x_1', x_1''}$ are bijections. By a similar argument, we have that $I^{(2)}(X_2; Y_3|X_1, X_3 = x_3) = I^{(1)}(X_2; Y_3|X_1, X_3 = x_3)$ and that $I^{(2)}(X_1, X_2; Y_3|X_3 = x_3) = I^{(1)}(X_1, X_2; Y_3|X_3 = x_3)$. Next, using the concavity of $I(X_1; Y_3|X_2, X_3 = x_3)$, $I(X_2; Y_3|X_1, X_3 = x_3)$, and $I(X_1, X_2; Y_3|X_3 = x_3)$ in $P_{X_1,X_2|X_3}(\cdot\,,\cdot\,|x_3)$[1] yields that

$$\begin{aligned}
I^{(3)}(X_1; Y_3|X_2, X_3 = x_3) &\geq \frac{1}{2}\big(I^{(1)}(X_1; Y_3|X_2, X_3 = x_3) + I^{(2)}(X_1; Y_3|X_2, X_3 = x_3)\big) \\
&= I^{(1)}(X_1; Y_3|X_2, X_3 = x_3), \\
I^{(3)}(X_2; Y_3|X_1, X_3 = x_3) &\geq \frac{1}{2}\big(I^{(1)}(X_2; Y_3|X_1, X_3 = x_3) + I^{(2)}(X_2; Y_3|X_1, X_3 = x_3)\big) \\
&= I^{(1)}(X_2; Y_3|X_1, X_3 = x_3), \\
I^{(3)}(X_1, X_2; Y_3|X_3 = x_3) &\geq \frac{1}{2}\big(I^{(1)}(X_1, X_2; Y_3|X_3 = x_3) + I^{(2)}(X_1, X_2; Y_3|X_3 = x_3)\big) \\
&= I^{(1)}(X_1, X_2; Y_3|X_3 = x_3),
\end{aligned}$$

and hence

$$\begin{aligned}
I^{(3)}(X_1; Y_3|X_2, X_3) &\geq I^{(1)}(X_1; Y_3|X_2, X_3), \\
I^{(3)}(X_2; Y_3|X_1, X_3) &\geq I^{(1)}(X_2; Y_3|X_1, X_3), \\
I^{(3)}(X_1, X_2; Y_3|X_3) &\geq I^{(1)}(X_1, X_2; Y_3|X_3),
\end{aligned}$$

since $P^{(1)}_{X_3} = P^{(3)}_{X_3}$. Together with the definition of $\mathcal{R}^{\text{MA-DBC}}$ given in Section 4.2, the inclusions in (4.42)-(4.43) are proved. ■

[1] $I(X_1; Y_3|X_2, X_3 = x_3)$ and $I(X_2; Y_3|X_1, X_3 = x_3)$ are concave function of $P_{X_1,X_2|X_3}(\cdot\,,\cdot\,|x_3)$ since $I(X_1; Y_2|X_2)$ and $I(X_2; Y_1|X_1)$ are both concave in the input distribution $P_{X_1,X_2}$ [3].

# Appendix C

# Proofs of Supplementary Results for Chapter 5

## C.1 Auxiliary Results for the Proof of Theorem 5.1

Here, we prove Claims 1-7 in the proof of Theorem 5.1. The notation $\mathcal{T}_\epsilon^{(n)}(\,\cdot\,|\,\cdot\,)$ is used to denote conditional typical sets.

**Claim 1**: For $b = 2, 3, \ldots, B+1$, the event $\overline{\mathcal{F}}_3^{(b)} \cap \overline{\mathcal{F}}_4^{(b)}$ implies that $\hat{M}_1^{(b-1)} = M_1^{(b-1)}$.

*Proof:* $\overline{\mathcal{F}}_3^{(b)}$ implies that

$$(\boldsymbol{S}_2^{(b)}, \boldsymbol{U}_2^{(b)}, \tilde{\boldsymbol{S}}_2^{(b)}, \tilde{\boldsymbol{U}}_1^{(b)}(M_1^{(b-1)}), \tilde{\boldsymbol{U}}_2^{(b)}(M_2^{(b-1)}), \tilde{\boldsymbol{W}}_2^{(b)}, \boldsymbol{X}_2^{(b)}, \boldsymbol{Y}_2^{(b)}) \in \mathcal{T}_\epsilon^{(n)}.$$

Thus, we have that $\hat{M}_1^{(b-1)} = M_1^{(b-1)}$ under $\overline{\mathcal{F}}_3^{(b)} \cap \overline{\mathcal{F}}_4^{(b)}$. ■

**Claim 2**: $\mathcal{E}_1^{(1)} \subseteq \mathcal{F}_1^{(1)} \cup \mathcal{F}_2^{(1)} \cup (\overline{\mathcal{F}}_1^{(1)} \cap \overline{\mathcal{F}}_2^{(1)} \cap \mathcal{E}_1^{(1)})$

*Proof:* This follows since the right-hand-side is equal to $\mathcal{E}_1^{(1)} \cup \mathcal{F}_1^{(1)} \cup \mathcal{F}_2^{(1)}$. ■

**Claim 3**: The inclusion $\mathcal{E}_1^{(b)} \cap \overline{\mathcal{E}}_1^{(b-1)} \subseteq \mathcal{F}_1^{(b)} \cup \mathcal{F}_2^{(b)} \cup (\overline{\mathcal{F}}_1^{(b)} \cap \overline{\mathcal{F}}_2^{(b)} \cap \mathcal{F}_3^{(b)} \cap \overline{\mathcal{E}}_1^{(b-1)}) \cup \mathcal{F}_4^{(b)}$ holds for $b = 2, 3, \ldots, B$.

*Proof:* Claim 1 implies that $\overline{\mathcal{F}}_3^{(b)} \cap \overline{\mathcal{F}}_4^{(b)} \subseteq \overline{\mathcal{E}}_1^{(b)}$ and hence $\mathcal{E}_1^{(b)} \subseteq \mathcal{F}_3^{(b)} \cup \mathcal{F}_4^{(b)}$. Together with the facts that

$$\mathcal{E}_1^{(b)} \cap \overline{\mathcal{E}}_1^{(b-1)} \subseteq (\mathcal{F}_3^{(b)} \cap \overline{\mathcal{E}}_1^{(b-1)}) \cup (\mathcal{F}_4^{(b)} \cap \overline{\mathcal{E}}_1^{(b-1)}) \subseteq (\mathcal{F}_3^{(b)} \cap \overline{\mathcal{E}}_1^{(b-1)}) \cup \mathcal{F}_4^{(b)}$$

and that

$$\begin{aligned}\mathcal{F}_3^{(b)} \cap \overline{\mathcal{E}}_1^{(b-1)} &= (\mathcal{F}_3^{(b)} \cap \overline{\mathcal{E}}_1^{(b-1)} \cap (\mathcal{F}_1^{(b)} \cup \mathcal{F}_2^{(b)})) \cup (\mathcal{F}_3^{(b)} \cap \overline{\mathcal{E}}_1^{(b-1)} \cap \overline{\mathcal{F}_1^{(b)} \cup \mathcal{F}_2^{(b)}}) \\ &\subseteq \mathcal{F}_1^{(b)} \cup \mathcal{F}_2^{(b)} \cup (\overline{\mathcal{F}_1^{(b)} \cup \mathcal{F}_2^{(b)}} \cap \mathcal{F}_3^{(b)} \cap \overline{\mathcal{E}}_1^{(b-1)}),\end{aligned}$$

we obtain the desired inclusion relationship. ■

**Claim 4**: $\mathcal{E}_1^{(B+1)} \cap \overline{\mathcal{E}}_1^{(B)} \subseteq (\mathcal{F}_3^{(B+1)} \cap \overline{\mathcal{E}}_1^{(B)}) \cup \mathcal{F}_4^{(B+1)}$

*Proof:* The result follows from the proof of Claim 3. ■

**Claim 5**: If $R_j^{(1)} > I(S_j; U_j) + \delta_1(\epsilon_1)$, then $\lim_{n\to\infty} \Pr\left(\mathcal{E}_j^{(1)}\right) = 0$.

*Proof:* Due to Claim 2, it suffices to show that $\lim_{n\to\infty} \Pr\left(\mathcal{F}_j^{(1)}\right) = 0$ for $j = 1, 2$ and $\lim_{n\to\infty} \Pr\left(\overline{\mathcal{F}_1^{(1)} \cup \mathcal{F}_2^{(1)}} \cap \mathcal{E}_1^{(1)}\right) = 0$ under the hypothesis. For $\Pr\left(\mathcal{F}_j^{(1)}\right)$, we define a non-typical set $\mathcal{A}_j = \{\boldsymbol{S}_j^{(1)} \notin \mathcal{T}_{\epsilon_0}^{(n)}\}$ for some $\epsilon_0 < \epsilon_1$, $j = 1, 2$. Then, $\mathcal{F}_j^{(1)} \subseteq \mathcal{A}_j \cup (\mathcal{F}_j^{(1)} \cap \overline{\mathcal{A}}_j)$. Clearly, $\lim_{n\to\infty} \Pr\left(\mathcal{A}_j\right) = 0$ due to the weak law of large numbers, and $\Pr\left(\mathcal{F}_j^{(1)} \cap \overline{\mathcal{A}}_j\right) \le \Pr\left(\mathcal{F}_j^{(1)} | \overline{\mathcal{A}}_j\right)$. For $\Pr\left(\mathcal{F}_j^{(1)} | \overline{\mathcal{A}}_j\right)$, we apply the covering lemma [47, Lemma 3.3] with the correspondences

$$X \leftrightarrow \emptyset, U \leftrightarrow S_j, \hat{X} \leftrightarrow U_j, R \leftrightarrow R_j^{(1)}, \epsilon' \leftrightarrow \epsilon_0, \text{ and } \epsilon \leftrightarrow \epsilon_1$$

to obtain that if $R_j^{(1)} > I(S_j; U_j) + \delta(\epsilon_1)$, then $\lim_{n\to\infty} \Pr\left(\mathcal{F}_j^{(1)} | \overline{\mathcal{A}}_j\right) = 0$. Thus, we obtain $\lim_{n\to\infty} \Pr\left(\mathcal{F}_j^{(1)}\right) = 0$ under the hypothesis for $j = 1, 2$.

The proof of $\lim_{n\to\infty} \Pr\left(\overline{\mathcal{F}_1^{(1)} \cup \mathcal{F}_2^{(1)}} \cap \mathcal{E}_j^{(1)}\right) = 0$ is more involved. For $\epsilon_2$ and $\epsilon_3$

such that $\epsilon_1 < \epsilon_2 < \epsilon_3$, let

$$\mathcal{B}_1 \triangleq \{(\boldsymbol{S}_1^{(1)}, \boldsymbol{S}_2^{(1)}, \boldsymbol{U}_1^{(1)}(M_1^{(1)})) \notin \mathcal{T}_{\epsilon_2}^{(n)}\}$$

and

$$\mathcal{B}_2 \triangleq \{(\boldsymbol{S}_1^{(1)}, \boldsymbol{S}_2^{(1)}, \boldsymbol{U}_1^{(1)}(M_1^{(1)}), \boldsymbol{U}_2^{(1)}(M_2^{(1)})) \notin \mathcal{T}_{\epsilon_3}^{(n)}\}.$$

We first show that conditional on the event $\overline{\mathcal{F}_1^{(1)} \cup \mathcal{F}_2^{(1)}}$, we have that $\lim_{n\to\infty} \Pr(\mathcal{B}_2) = 0$. We begin by considering the inclusion relationship:

$$\mathcal{B}_2 \subseteq \mathcal{F}_1^{(1)} \cup (\mathcal{B}_1 \cap \overline{\mathcal{F}}_1^{(1)}) \cup \mathcal{F}_2^{(1)} \cup (\mathcal{B}_2 \cap \overline{\mathcal{B}}_1 \cap \overline{\mathcal{F}}_2^{(1)}).$$

Using union bound, we have that

$$\begin{aligned}\Pr(\mathcal{B}_2) &\leq \Pr\Big(\mathcal{F}_1^{(1)}\Big) + \Pr\Big(\mathcal{F}_2^{(1)}\Big) + \Pr\Big(\mathcal{B}_1 \cap \overline{\mathcal{F}}_1^{(1)}\Big) + \Pr\Big(\mathcal{B}_2 \cap \overline{\mathcal{B}}_1 \cap \overline{\mathcal{F}}_2^{(1)}\Big) \\ &\leq \Pr\Big(\mathcal{F}_1^{(1)}\Big) + \Pr\Big(\mathcal{F}_2^{(1)}\Big) + \Pr\Big(\mathcal{B}_1 | \overline{\mathcal{F}}_1^{(1)}\Big) + \Pr\Big(\mathcal{B}_2 | \overline{\mathcal{B}}_1 \cap \overline{\mathcal{F}}_2^{(1)}\Big). \end{aligned} \tag{C.1}$$

Now, applying the conditional typicality lemma [47, Section 2.5] with the correspondences

$$X \leftrightarrow (S_1, U_1), Y \leftrightarrow S_2, \epsilon' \leftrightarrow \epsilon_1, \text{ and } \epsilon \leftrightarrow \epsilon_2,$$

we have that $\lim_{n\to\infty} \Pr\Big(\overline{\mathcal{B}}_1 | \overline{\mathcal{F}}_1^{(1)}\Big) = 1$. Similarly, applying the conditional typical lemma with the correspondences:

$$X \leftrightarrow (S_1, S_2, U_1), Y \leftrightarrow U_2, \epsilon' \leftrightarrow \epsilon_2, \text{ and } \epsilon \leftrightarrow \epsilon_3,$$

one further obtains that $\lim_{n\to\infty} \Pr\Big(\overline{\mathcal{B}}_2 | \overline{\mathcal{B}}_1 \cap \overline{\mathcal{F}}_2^{(1)}\Big) = 1$. Together with the first part of the proof and (C.1), we conclude that $\lim_{n\to\infty} \Pr(\mathcal{B}_2) = 0$.

We next use the inclusion $\overline{\mathcal{F}_1^{(1)} \cup \mathcal{F}_2^{(1)}} \cap \mathcal{E}_j^{(1)} \subseteq \mathcal{B}_2 \cup \mathcal{E}_j^{(1)} \subseteq \mathcal{B}_2 \cup (\mathcal{E}_j^{(1)} \cap \overline{\mathcal{B}}_2)$, which yields the inequality

$$\Pr\left(\overline{\mathcal{F}_1^{(1)} \cup \mathcal{F}_2^{(1)}} \cap \mathcal{E}_j^{(1)}\right) \leq \Pr(\mathcal{B}_2) + \Pr\left(\mathcal{E}_j^{(1)} | \overline{\mathcal{B}}_2\right).$$

For $\Pr\left(\mathcal{E}_j^{(1)} | \overline{\mathcal{B}}_2\right)$, since $(\tilde{\boldsymbol{S}}_1^{(1)}, \tilde{\boldsymbol{S}}_2^{(1)}, \tilde{\boldsymbol{U}}_1^{(1)}, \tilde{\boldsymbol{U}}_2^{(1)}, \tilde{\boldsymbol{W}}_1^{(1)}, \tilde{\boldsymbol{W}}_2^{(1)})$ is generated according to (5.9) (and is independent of $(\boldsymbol{S}_1^{(1)}, \boldsymbol{S}_2^{(1)}, \boldsymbol{U}_1^{(1)}, \boldsymbol{U}_2^{(1)})$) and the channel input $\boldsymbol{X}_1^{(1)}$ is generated component-wise, the conditional typicality lemma implies that

$$\lim_{n\to\infty} \Pr\left(\boldsymbol{S}_1^{(1)}, \boldsymbol{S}_2^{(1)}, \boldsymbol{U}_1^{(1)}, \boldsymbol{U}_2^{(1)}, \tilde{\boldsymbol{S}}_1^{(1)}, \tilde{\boldsymbol{S}}_2^{(1)}, \tilde{\boldsymbol{U}}_1^{(1)}, \tilde{\boldsymbol{U}}_2^{(1)}, \tilde{\boldsymbol{W}}_1^{(1)}, \tilde{\boldsymbol{W}}_2^{(1)}, \boldsymbol{X}_1^{(1)}, \boldsymbol{X}_2^{(1)}\right) \in \mathcal{T}_{\epsilon_4}^{(n)}) = 1$$

under $\overline{\mathcal{B}}_2$ for some $\epsilon_4 > \epsilon_3$. Applying the conditional typicality lemma again with the correspondences

$$X \leftrightarrow (S_1, S_2, U_1, U_2, \tilde{S}_1, \tilde{S}_2, \tilde{U}_1, \tilde{U}_2, \tilde{W}_1, \tilde{W}_2, \tilde{X}_1, \tilde{X}_2), Y \leftrightarrow (Y_1, Y_2), \epsilon' \leftrightarrow \epsilon_4, \text{ and } \epsilon \leftrightarrow \epsilon,$$

and using the memoryless property of the channel, we further have that $\lim_{n\to\infty} \Pr\left(\mathcal{E}_j^{(1)} | \overline{\mathcal{B}}_2\right) = 0$. Combining this with (C.1) implies

$$\lim_{n\to\infty} \Pr\left(\overline{\mathcal{F}_1^{(1)} \cup \mathcal{F}_2^{(1)}} \cap \mathcal{E}_j^{(1)}\right) = 0,$$

which completes the proof of the claim. ■

**Claim 6**: If $R_1^{(B)} < I(\tilde{U}_1; S_2, U_2, \tilde{S}_2, \tilde{U}_2, \tilde{W}_2, X_2, Y_2) - \delta(\epsilon)$, then $\lim_{n\to\infty} \Pr\left(\mathcal{E}_1^{(B+1)} \cap \overline{\mathcal{E}}_1^{(B)}\right) = 0$.

*Proof:* With the help of Claim 4, it suffices to show that $\lim_{n\to\infty} \Pr\left(\mathcal{F}_3^{(B+1)} \cap \overline{\mathcal{E}}_1^{(B)}\right) = 0$ and $\lim_{n\to\infty} \Pr\left(\mathcal{F}_4^{(B+1)}\right) = 0$ under the hypothesis. To obtain the first result, we follow the proof of Claim 5. Consider the inequality

$$\Pr\Big(\mathcal{F}_3^{(B+1)} \cap \overline{\mathcal{E}}_1^{(B)}\Big) \leq \Pr\Big(\mathcal{F}_3^{(B+1)} | \overline{\mathcal{E}}_1^{(B)}\Big).$$

Conditioning on $\overline{\mathcal{E}}_1^{(B)}$ clearly imposes a joint typicality constraint on the sequence $(\tilde{\boldsymbol{S}}_1^{(B+1)}, \tilde{\boldsymbol{S}}_2^{(B+1)}, \tilde{\boldsymbol{U}}_1^{(B+1)}, \tilde{\boldsymbol{U}}_2^{(B+1)}, \tilde{\boldsymbol{W}}_1^{(B+1)}, \tilde{\boldsymbol{W}}_2^{(B+1)})$ in the event $\mathcal{F}_3^{(B+1)}$. We also know that the sequence $(\boldsymbol{S}_1^{(B+1)}, \boldsymbol{S}_2^{(B+1)}, \boldsymbol{U}_1^{(B+1)}, \boldsymbol{U}_2^{(B+1)})$ in the event $\mathcal{F}_3^{(B+1)}$ will be jointly typical with high probability due to (5.10) and the weak law of large numbers. Using these observations, we apply the conditional typicality lemma twice, as in the last part of the proof of Claim 5, to conclude that $\lim_{n\to\infty} \Pr\Big(\mathcal{F}_3^{(B+1)} \cap \overline{\mathcal{E}}_1^{(B)}\Big) = 0$.

To analyze $\Pr\Big(\mathcal{F}_4^{(B+1)}\Big)$, we may assume that $(M_1^{(B)}, M_2^{(B)}) = (1,1) \triangleq \boldsymbol{M}_{1,1}^{(B)}$ by the symmetry of random codebook generation and the encoding procedure. Then, we have two Markov chain relationships for $m_1 \neq 1$:

$$\begin{aligned}\tilde{\boldsymbol{U}}_1^{(B+1)}(m_1) -\!\!\circ\!\!- (&\boldsymbol{S}_1^{(B+1)}, \boldsymbol{S}_2^{(B+1)}, \boldsymbol{U}_1^{(B+1)}, \boldsymbol{U}_2^{(B+1)}, \tilde{\boldsymbol{S}}_1^{(B+1)}, \tilde{\boldsymbol{S}}_2^{(B+1)}, \tilde{\boldsymbol{U}}_1^{(B+1)}(1),\\ &\tilde{\boldsymbol{U}}_2^{(B+1)}(1), \tilde{\boldsymbol{W}}_1^{(B+1)}, \tilde{\boldsymbol{W}}_2^{(B+1)}) -\!\!\circ\!\!- (\boldsymbol{X}_1^{(B+1)}, \boldsymbol{X}_2^{(B+1)}) -\!\!\circ\!\!- (\boldsymbol{Y}_1^{(B+1)}, \boldsymbol{Y}_2^{(B+1)})\end{aligned} \tag{C.2}$$

and

$$\begin{aligned}\tilde{\boldsymbol{U}}_1^{(B+1)}(m_1) -\!\!\circ\!\!- (\tilde{\boldsymbol{S}}_1^{(B+1)}, \tilde{\boldsymbol{U}}_1^{(B+1)}(1)) -\!\!\circ\!\!- (&\boldsymbol{S}_1^{(B+1)}, \boldsymbol{S}_2^{(B+1)}, \boldsymbol{U}_1^{(B+1)}, \boldsymbol{U}_2^{(B+1)},\\ &\tilde{\boldsymbol{S}}_2^{(B+1)}, \tilde{\boldsymbol{U}}_2^{(B+1)}(1), \tilde{\boldsymbol{W}}_1^{(B+1)}, \tilde{\boldsymbol{W}}_2^{(B+1)}, \boldsymbol{X}_1^{(B+1)}, \boldsymbol{X}_2^{(B+1)}).\end{aligned} \tag{C.3}$$

To simplify the derivation, we define

$$\boldsymbol{A}_1(\hat{m}_1^{(B)}) = (\boldsymbol{S}_2^{(B+1)}, \boldsymbol{U}_2^{(B+1)}, \tilde{\boldsymbol{S}}_2^{(B+1)}, \tilde{\boldsymbol{U}}_1^{(B+1)}(\hat{m}_1^{(B)}), \tilde{\boldsymbol{U}}_2^{(B+1)}(1), \tilde{\boldsymbol{W}}_2^{(B+1)}, \boldsymbol{X}_2^{(B+1)}, \boldsymbol{Y}_2^{(B+1)})$$

and let $\boldsymbol{a}_1 = (\boldsymbol{s}_2, \boldsymbol{u}_2, \tilde{\boldsymbol{s}}_2, \tilde{\boldsymbol{u}}_1, \tilde{\boldsymbol{u}}_2, \tilde{\boldsymbol{w}}_2, \boldsymbol{x}_2, \boldsymbol{y}_2)$ to denote a realization of $\boldsymbol{A}_1(\hat{m}_1^{(B)})$. When excluding the variable $\tilde{\boldsymbol{U}}_1^{(B+1)}(\hat{m}_1^{(B)})$ (resp., $(\tilde{\boldsymbol{U}}_1^{(B+1)}(\hat{m}_1^{(B)}), \boldsymbol{X}_2^{(B+1)}, \boldsymbol{Y}_2^{(B+1)})$) from

$\boldsymbol{A}_1(\hat{m}_1^{(B)})$, we let the remaining tuples denoted as $\boldsymbol{A}_2$ (resp., $\boldsymbol{A}_3$). Note that when $\boldsymbol{a}_1$ is given, $\boldsymbol{a}_2$ and $\boldsymbol{a}_3$ are determined as well. Moreover, we define

$$\boldsymbol{B} = (\boldsymbol{S}_1^{(B+1)}, \boldsymbol{U}_1^{(B+1)}, \tilde{\boldsymbol{S}}_1^{(B+1)}, \tilde{\boldsymbol{U}}_1^{(B+1)}(1), \tilde{\boldsymbol{W}}_1{}^{(B+1)})$$

and let $\boldsymbol{b} = (\boldsymbol{s}_1, \boldsymbol{u}_1', \tilde{\boldsymbol{s}}_1, \tilde{\boldsymbol{u}}_1', \boldsymbol{w}_1)$ to denote a realization of it. In the following, we find an upper bound for $\Pr\Big(\mathcal{F}_4^{(B+1)}\Big)$ using the fact that $\Pr\Big(\mathcal{F}_4^{(B+1)}\Big) = \Pr\Big(\mathcal{F}_4^{(B+1)}|\boldsymbol{M}_{1,1}^{(B)}\Big)$:

$$\begin{aligned}
&\Pr\Big(\mathcal{F}_4^{(B+1)}|\boldsymbol{M}_{1,1}^{(B)}\Big)\\
&\le \sum_{\hat{m}_1=2}^{2^{nR_1^{(B)}}} \sum_{\boldsymbol{a}_1\in\mathcal{T}_\epsilon^{(n)}} \Pr\Big(\boldsymbol{A}_1(\hat{m}_1) = \boldsymbol{a}_1|\boldsymbol{M}_{1,1}^{(B)}\Big) && \text{(C.4)}\\
&= \sum_{\hat{m}_1=2}^{2^{nR_1^{(B)}}} \sum_{\boldsymbol{a}_1\in\mathcal{T}_\epsilon^{(n)}} \sum_{\boldsymbol{b}} \Pr\Big(\boldsymbol{A}_1(\hat{m}_1) = \boldsymbol{a}_1, \boldsymbol{B} = \boldsymbol{b}|\boldsymbol{M}_{1,1}^{(B)}\Big) && \text{(C.5)}\\
&= \sum_{\hat{m}_1=2}^{2^{nR_1^{(B)}}} \sum_{\boldsymbol{a}_1\in\mathcal{T}_\epsilon^{(n)}} \sum_{\boldsymbol{b}} \Pr\Big(\boldsymbol{X}_2^{(B+1)} = \boldsymbol{x}_2, \boldsymbol{Y}_2^{(B+1)} = \boldsymbol{y}_2|\boldsymbol{M}_{1,1}^{(B)}\Big)\\
&\qquad \Pr\Big(\boldsymbol{A}_3 = \boldsymbol{a}_3, \boldsymbol{B} = \boldsymbol{b}|\boldsymbol{X}_2^{(B+1)} = \boldsymbol{x}_2, \boldsymbol{Y}_2^{(B+1)} = \boldsymbol{y}_2, \boldsymbol{M}_{1,1}^{(B)}\Big)\\
&\qquad\qquad \Pr\Big(\tilde{\boldsymbol{U}}_1^{(B+1)}(\hat{m}_1) = \tilde{\boldsymbol{u}}_1|\boldsymbol{A}_2 = \boldsymbol{a}_2, \boldsymbol{B} = \boldsymbol{b}, \boldsymbol{M}_{1,1}^{(B)}\Big) && \text{(C.6)}\\
&= \sum_{\hat{m}_1=2}^{2^{nR_1^{(B)}}} \sum_{\boldsymbol{a}_1\in\mathcal{T}_\epsilon^{(n)}} \sum_{\boldsymbol{b}} \Pr\Big(\boldsymbol{X}_2^{(B+1)} = \boldsymbol{x}_2, \boldsymbol{Y}_2^{(B+1)} = \boldsymbol{y}_2|\boldsymbol{M}_{1,1}^{(B)}\Big)\\
&\qquad \Pr\Big(\boldsymbol{A}_3 = \boldsymbol{a}_3, \boldsymbol{B} = \boldsymbol{b}|\boldsymbol{X}_2^{(B+1)} = \boldsymbol{x}_2, \boldsymbol{Y}_2^{(B+1)} = \boldsymbol{y}_2, \boldsymbol{M}_{1,1}^{(B)}\Big)\\
&\qquad\qquad \Pr\Big(\tilde{\boldsymbol{U}}_1^{(B+1)}(\hat{m}_1) = \tilde{\boldsymbol{u}}_1|\tilde{\boldsymbol{S}}_1^{(B+1)} = \tilde{\boldsymbol{s}}_1, \tilde{\boldsymbol{U}}_1^{(B+1)}(1) = \boldsymbol{u}_1', M_1^{(B)} = 1\Big) && \text{(C.7)}\\
&\le \sum_{\hat{m}_1=2}^{2^{nR_1^{(B)}}} \sum_{\boldsymbol{a}_1\in\mathcal{T}_\epsilon^{(n)}} \Pr\Big(\boldsymbol{X}_2^{(B+1)} = \boldsymbol{x}_2, \boldsymbol{Y}_2^{(B+1)} = \boldsymbol{y}_2|\boldsymbol{M}_{1,1}^{(B)}\Big) \cdot (1+\epsilon) \prod_{i=1}^{n} P_{\tilde{U}_1^{(B+1)}}(\tilde{u}_{1,i})\\
&\qquad \sum_{\boldsymbol{b}} \Pr\Big(\boldsymbol{A}_3 = \boldsymbol{a}_3, \boldsymbol{B} = \boldsymbol{b}|\boldsymbol{X}_2^{(B+1)} = \boldsymbol{x}_2, \boldsymbol{Y}_2^{(B+1)} = \boldsymbol{y}_2, \boldsymbol{M}_{1,1}^{(B)}\Big) && \text{(C.8)}
\end{aligned}$$

$$= (1+\epsilon) \sum_{\hat{m}_1=2}^{2^{nR_1^{(B)}}} \sum_{\boldsymbol{a}_2 \in \mathcal{T}_\epsilon^{(n)}} \Pr\Big(\boldsymbol{A}_2 = \boldsymbol{a}_2 | \boldsymbol{M}_{1,1}^{(B)}\Big) \prod_{i=1}^{n} P_{\tilde{U}_1^{(B+1)}}(\tilde{u}_{1,i})$$

$$\leq (1+\epsilon) \cdot 2^{nR_1^{(B)}} \sum_{\boldsymbol{a}_2 \in \mathcal{T}_\epsilon^{(n)}} \sum_{\tilde{\boldsymbol{u}}_1 \in \mathcal{T}_\epsilon^{(n)}(\tilde{U}_1|\boldsymbol{a}_2)} \Pr\Big(\boldsymbol{A}_2 = \boldsymbol{a}_2 | \boldsymbol{M}_{1,1}^{(B)}\Big) \prod_{i=1}^{n} P_{\tilde{U}_1}(\tilde{u}_{1,i}^{(B+1)})$$

$$\leq (1+\epsilon) \cdot 2^{nR_1^{(B)}} \sum_{\boldsymbol{a}_2 \in \mathcal{T}_\epsilon^{(n)}} |\mathcal{T}_\epsilon^{(n)}(\tilde{U}_1|\boldsymbol{a}_2)| \cdot \Pr\Big(\boldsymbol{A}_2 = \boldsymbol{a}_2 | \boldsymbol{M}_{1,1}^{(B)}\Big) \cdot 2^{-n(H(\tilde{U}_1)-\delta_1(\epsilon))} \tag{C.9}$$

$$\leq (1+\epsilon) \cdot 2^{nR_1^{(B)}} 2^{n(H(\tilde{U}_1|S_2,U_2,\tilde{S}_2,\tilde{U}_2,\tilde{W}_2,X_2,Y_2)+\delta_2(\epsilon))} \cdot 2^{-n(H(\tilde{U}_1)-\delta_1(\epsilon))} \tag{C.10}$$

$$\leq (1+\epsilon) \cdot 2^{n(R_1^{(B)} - I(\tilde{U}_1;S_2,U_2,\tilde{S}_2,\tilde{U}_2,\tilde{W}_2,X_2,Y_2)+\delta(\epsilon))} \tag{C.11}$$

where (C.4) is due to the union bound, (C.5) and (C.6) respectively follow from the law of total probability and the chain rule, (C.7) is due to the Markov chain relationships in (C.2) and (C.3), the inequality in (C.8) is obtained using [89, Lemma 1] with the correspondences

$$S \leftrightarrow \tilde{S}_1^{(B)}, U \leftrightarrow \tilde{U}_1^{(B)}, \epsilon' \leftrightarrow \epsilon_1, \text{ and } M \leftrightarrow M_1^{(B)},$$

(C.9)-(C.11) follow standard bounds for typical sets, and in the last equation we set $\delta(\epsilon) \triangleq \delta_1(\epsilon) + \delta_2(\epsilon)$.[1] Therefore, if

$$R_1^{(B)} < I(\tilde{U}_1; S_2, U_2, \tilde{S}_2, \tilde{U}_2, \tilde{W}_2, X_2, Y_2) - \delta(\epsilon)$$

holds, then $\lim_{n\to\infty} \Pr\Big(\mathcal{F}_4^{(B+1)}\Big) = 0$. By symmetry, one can easily obtain a similar condition for terminal 2. Combining the first part then completes the proof. ■

**Claim 7**: For $b = 2, 3, \ldots, B$, if $R_j^{(b)} > I(S_j; U_j) + \delta_1(\epsilon_1)$ and $R_1^{(b-1)} < I(\tilde{U}_1; S_2, U_2, \tilde{S}_2, \tilde{U}_2, \tilde{W}_2, X_2, Y_2) - \delta(\epsilon)$, then $\lim_{n\to\infty} \Pr\left(\mathcal{E}_1^{(b)} \cap \overline{\mathcal{E}}_1^{(b-1)}\right) = 0$.

*Proof:* We sketch the proof since the details follow similar lines of the proofs for

[1]Note that $\lim_{\epsilon\to 0} \delta_1(\epsilon) = 0$ and $\lim_{\epsilon\to 0} \delta_2(\epsilon) = 0$.

Claims 5 and 6. Using Claim 3, it suffices to show that under the hypothesis, we have that $\lim_{n\to\infty}\Pr\left(\mathcal{F}_j^{(b)}\right)=0$ for $j=1,2$, $\lim_{n\to\infty}\Pr\left(\overline{\mathcal{F}_1^{(b)}\cup\mathcal{F}_2^{(b)}}\cap\mathcal{F}_3^{(b)}\cap\overline{\mathcal{E}}_1^{(b-1)}\right)=0$, and $\lim_{n\to\infty}\Pr\left(\mathcal{F}_4^{(b)}\right)=0$. Note that the first two quantities can be easily proved using the argument in the first part of the proof for Claim 5, which imposes the condition $R_j^{(b)}>I(S_j;U_j)+\delta(\epsilon_1)$ for $j=1,2$.

To show $\lim_{n\to\infty}\Pr\left(\overline{\mathcal{F}_1^{(b)}\cup\mathcal{F}_2^{(b)}}\cap\mathcal{F}_3^{(b)}\cap\overline{\mathcal{E}}_1^{(b-1)}\right)=0$, we follow the proofs of Claim 5 and 6. Based on the proof of Claim 5, it is straightforward to obtain that $\lim_{n\to\infty}\Pr\left(\overline{\mathcal{F}_1^{(b)}\cup\mathcal{F}_2^{(b)}}\right)=1$ under the conditions $R_j^{(b)}>I(S_j;U_j)+\delta(\epsilon_1)$, $j=1,2$. Consider the inequality

$$\Pr\left(\overline{\mathcal{F}_1^{(b)}\cup\mathcal{F}_2^{(b)}}\cap\mathcal{F}_3^{(b)}\cap\overline{\mathcal{E}}_1^{(b-1)}\right)\le\Pr\left(\mathcal{F}_3^{(b)}|\overline{\mathcal{F}_1^{(b)}\cup\mathcal{F}_2^{(b)}}\cap\overline{\mathcal{E}}_1^{(b-1)}\right),$$

where the event $\overline{\mathcal{E}}_1^{(b-1)}$ implies that $(\tilde{\boldsymbol{S}}_1^{(b)},\tilde{\boldsymbol{S}}_2^{(b)},\tilde{\boldsymbol{U}}_1^{(b)},\tilde{\boldsymbol{U}}_2^{(b)},\tilde{\boldsymbol{W}}_1^{(b)},\tilde{\boldsymbol{W}}_2^{(b)})$ is a jointly typical sequence. Noting that the right-hand-side of the inequality is now at a position similar to $\Pr\left(\mathcal{F}_3^{(B+1)}|\overline{\mathcal{E}}_1^{(B)}\right)$ in the proof of Claim 6, we obtain the desired result by applying conditional typicality lemma twice as done before.

For the probability $\Pr\left(\mathcal{F}_4^{(b)}\right)$, we adopt the proof of Claim 6 with the correspondence $B+1\leftrightarrow b$, which imposes the sufficient condition $R_1^{(b-1)}<I(\tilde{U}_1;S_2,U_2,\tilde{S}_2,\tilde{U}_2,\tilde{W}_2,X_2,Y_2)-\delta(\epsilon)$ for $\lim_{n\to\infty}\Pr\left(\mathcal{F}_4^{(b)}\right)=0$. Combining the above results then completes the proof. ■

## C.2 Auxiliary Result for Special Case (ii) of Corollary 5.1

By symmetry, we only show that $I(\tilde{S}_1;\tilde{U}_1|\tilde{S}_2,\tilde{U}_2) < I(\tilde{U}_1;Y_2|\tilde{S}_2,\tilde{U}_2)$ reduces to $R^{(1)}(D_1) < I(X_1;Y_2|X_2)$. First, observe that

$$\begin{aligned}
I(\tilde{S}_1;\tilde{U}_1|\tilde{S}_2,\tilde{U}_2) &= I(\tilde{S}_1;V_1',\hat{S}_1'|\tilde{S}_2,V_2',\hat{S}_2') \\
&= \underbrace{I(\tilde{S}_1;V_1'|\tilde{S}_2,V_2',\hat{S}_2')}_{=0} + I(\tilde{S}_1;\hat{S}_1'|\tilde{S}_2,V_2',\hat{S}_2',V_1') \\
&= H(\hat{S}_1'|\tilde{S}_2,V_2',\hat{S}_2',V_1') - H(\hat{S}_1'|\tilde{S}_2,V_2',\hat{S}_2',V_1',\tilde{S}_1) \\
&= H(\hat{S}_1') - H(\hat{S}_1'|\tilde{S}_1) && \text{(C.12)} \\
&= I(\tilde{S}_1;\hat{S}_1') \\
&= R^{(1)}(D_1) && \text{(C.13)}
\end{aligned}$$

where (C.12) holds since $\tilde{S}_1$ and $\tilde{S}_2$ are independent and hence $\hat{S}_1'$ is independent of $(\tilde{S}_2,V_2',\hat{S}_2',V_1')$, and (C.13) follows since the joint probability distribution $P_{\tilde{S}_1,\hat{S}_1'} = P_{S_1,\hat{S}_1}$ achieves $R^{(1)}(D_1)$.

Moreover, we have that

$$\begin{aligned}
I(\tilde{U}_1;Y_2|\tilde{S}_2,\tilde{U}_2) &= I(V_1',\hat{S}_1';Y_2|\tilde{S}_2,V_2',\hat{S}_2') \\
&= I(V_1';Y_2|\tilde{S}_2,V_2',\hat{S}_2') + I(\hat{S}_1';Y_2|\tilde{S}_2,V_2',\hat{S}_2',V_1') \\
&= I(X_1;Y_2|\tilde{S}_2,X_2,\hat{S}_2') + I(\hat{S}_1';Y_2|\tilde{S}_2,X_2,\hat{S}_2',X_1) && \text{(C.14)} \\
&= H(Y_2|\tilde{S}_2,X_2,\hat{S}_2') - H(Y_2|\tilde{S}_2,X_2,\hat{S}_2',X_1) && \text{(C.15)} \\
&= H(Y_2|X_2) - H(Y_2|X_2,X_1) && \text{(C.16)} \\
&= I(X_1;Y_2|X_2)
\end{aligned}$$

where (C.14) follows since $X_j = V_j'$, (C.15) holds since given channel inputs $X_1$ and

$X_2$, the output $Y_2$ is independent of other variables, and (C.16) holds due to the Markov chain relationship $(\tilde{S}_2, \hat{S}_2') \multimap X_2 \multimap Y_2$.