On Symbol versus Bit Interleaving for Block Coded Binary Markov Channels

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Abstract

We examine bit and symbol interleaving strategies for linear non-binary block codes (under bounded distance decoding) over the family of binary additive noise finite-state Markov channel (FSMC) models with memory. We derive a simple analytical sufficient condition under which perfect (i.e., with infinite interleaving depth) symbol interleaving outperforms perfect bit interleaving in terms of the probability of codeword error (PCE). It is shown that the well-known Gilbert-Elliott channel (GEC) with positive noise correlation coefficient as well as the recently introduced Markovian queue-based channel (QBC) of memory M satisfy this condition. This result has been widely illustrated numerically (without proof) in the literature, particularly for the GEC. We also provide examples of binary FSMC models for which the reverse result holds, i.e., perfect bit interleaving outperforming perfect symbol interleaving. Finally, a numerical PCE study of imperfect symbol-interleaved non-binary codes over the QBC indicates that there is a linear relationship between the optimal interleaving depth and a function of a single parameter of the QBC.

Index Terms

Additive noise channels, binary finite-state Markov channels, Gilbert-Elliott channel, Markovian queue-based channel, Reed-Solomon and non-binary block codes, symbol and bit interleaving.

I. Introduction

An important class of non-binary error correcting codes used widely in data transmission and storage systems is the family of Reed-Solomon (RS) codes [1]. A commonly used strategy to employ an RS code to correct errors generated by a channel with (statistical) memory is to incorporate block interleaving into the communication system. It is also known that binary modulated time-correlated flat fading channels

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used in conjunction with hard-decision demodulation can be represented by stationary binary (modulo-2) additive noise channels with memory (e.g., see [2], [3]). When non-binary codewords are sent over such channels, two interleaving strategies are worth considering [4], [5]: (i) interleaving the code symbols; (ii) interleaving the code (or channel) bits which, under perfect or infinite interleaving depth, reduces the channel to the memoryless binary symmetric channel (BSC) [6].

In prior works, the performance of non-interleaved RS codes over correlated fading channels is analyzed in [7]–[9] using a two-step procedure. First, a binary finite state Markov channel (FSMC) model is introduced for the generation of the bit or symbol error process, and then a formula for the probability of codeword error (PCE) under bounded distance decoding is derived for the proposed model. In [7], the channel is modeled via the Gilbert-Elliott channel (GEC) [6] whose parameters are calculated using a simple threshold model. In [8], level crossing statistics are applied to characterize the fading arrival process and the fading durations, and the PCE is expressed in terms of the probability distribution of the fading durations. In [9], the bit error process resulting from the hard-decision demodulation of binary frequency-shift keying modulated signals over correlated Rician fading channels is modeled via a Fritchman channel [10]. Imperfect (finite-length) symbol interleaving is also considered in [8], [9]. In a recent work [11], the performance of symbol-interleaved RS codes over fading channels modeled via the GEC is examined in the context of CDMA2000 Broadcast and Multicast Services.

A numerical study of the superiority of symbol-interleaved (over bit-interleaved) RS codes is given in [4] for the case of slow fading channels. This result motivated the authors of [8] to consider only symbol interleaving in their investigations. An analytical expression for the PCE of RS codes over binary FSMC models under imperfect bit- and symbol-interleaving is derived in [5] for two decoding strategies (bounded distance decoding and error-forecasting decoding). The study conducted in [5] to compare the performance of these two interleaving strategies for the GEC corroborates the superiority of symbol-interleaving found in previous numerical studies. However, since there is no known analytical proof in the literature for this result, it is natural to investigate whether perfect symbol interleaving *always* outperforms perfect bit-interleaving for a given class of binary FSMC models or if there exist conditions on the channel parameters under which bit-interleaving provides better PCE performance.

In this work, we analytically investigate the merits of perfect symbol and bit interleaving for linear non-binary block codes under bounded-distance decoding over the class of binary FSMCs (with additive stationary hidden Markovian noise). This class of FSMC models includes the GEC (which has been widely shown to be a good model for flat fading channels [2], [3]) and the recently introduced queue-based channel

(QBC) [12]. The QBC, which features an Mth-order additive Markov noise process generated via a finite queue, has only four parameters (like the GEC), while allowing its memory order to be arbitrarily large. It also offers (unlike the GEC) closed form expressions for the block transition probability, capacity and autocorrelation function [12]. Furthermore, it has been shown that the QBC can accurately approximate the GEC [12] as well as (uncoded and RS coded) hard-decision demodulated Rician flat fading channels [13], [14].

Imperfect (i.e., with finite interleaving depth) interleaving is an important issue in practice. In particular, for non-binary block codes over the GEC it was found in [5] that perfect interleaving can be realized when the interleaving depth is a multiple of the channel's average burst length (e.g., the typical interleaving depth needed to achieve perfect symbol and bit interleaving is double and four times the average burst length of the GEC, respectively [5]). Another motivation for this work is to verify if a similar result also holds for the QBC. We provide PCE numerical results when imperfect symbol interleaved RS codes are sent over the QBC and investigate the choice of the optimal interleaving depth in terms of the parameters of this channel.

The contributions of this correspondence are summarized as follows. In Proposition 1, we establish a simple explicit condition (in terms of the FSMC noise statistics) under which perfect symbol interleaving results in a lower PCE compared to perfect bit interleaving for any linear non-binary block code used over the FSMC with bounded-distance decoding.[1] We analytically show that both the GEC with positive noise correlation coefficient (i.e., with persistent memory [6]) and the QBC satisfy this condition; see Propositions 2 and 3. Interestingly, we note an opposite behavior for the simplified Gilbert channel (SGC) [15] (i.e., the first-order Markov noise channel) when its noise correlation coefficient is negative[2] (this channel is a special instance of the GEC with oscillatory memory [6]); in this case, we show that perfect bit-interleaved non-binary codes outperform perfect symbol-interleaved ones. We also provide other examples of FSMC models (Fritchman channels with negative noise correlation coefficient) where bit interleaving can be better than symbol interleaving. Hence for some classes of channels (such as the GEC), the choice of the best interleaving strategy is directly related to the sign of the noise correlation coefficient. Finally, we conduct a numerical study to analyze the sensitivity of the QBC and interleaving

[1]It is worth pointing out that the result in Proposition 1 does not require that the noise process be hidden Markovian (we only need that the noise be stationary). We however restrict it be hidden Markovian since FSMC models are widely used to model fading channels.

[2]Note that the case of negative noise correlation coefficient can reflect situations involving very fast correlated Rician fading (cf. Remark 1). Even if such fast fading situations may occur rarely in practice, the fact that negative noise correlation leads to bit interleaving outperforming symbol interleaving (i.e., the reverse result of Propositions 1-3) is at least of conceptual interest.

parameters with respect to the PCE. We found that, similarly to the GEC, there is a linear relationship between the optimal interleaving depth and a function of a single parameter of the QBC.

II. System Description

We consider a coded communication system where non-binary transmitted symbols, assuming values from the Galois field GF(2^b), $b \geq 2$, are mapped one-to-one to a binary b-tuple and are transmitted across a binary FSMC model. The kth received binary symbol Y_k is described by $Y_k = X_k \oplus Z_k$, $k = 1, 2, \cdots$, where $\oplus$ denotes addition modulo-2, $X_k \in \{0, 1\}$ is the kth transmitted symbol and $Z_k \in \{0, 1\}$ is the kth channel noise symbol. We assume that the noise process $\{Z_k\}_{k=1}^{\infty}$ is a stationary hidden Markov source and is independent from the transmitted process $\{X_k\}_{k=1}^{\infty}$. Two channel models considered in this letter (one with an Mth-order Markovian noise and one with a hidden Markovian noise), which belong to the class of binary FSMC models, are next briefly described.

A. Queue-Based Channel

The queue-based channel (QBC) uses a simple approach to model an Mth-order Markov noise process via a finite queue [12]. At the kth time, the channel generates a noise output Z_k that depends on four parameters: the size of the queue, M, the channel bit error rate (BER), $p = \Pr(Z_k = 1)$, and correlation parameters ε and α, where $0 \leq \varepsilon < 1$, $\alpha \geq 0$. First, one of two parcels (an urn and a queue of size M) are selected with probability distribution $\{\varepsilon, 1 - \varepsilon\}$. If the urn is selected, the model generates an error $(Z_k = 1)$ with probability p. If the queue is selected, a binary noise symbol is selected with a probability distribution that depends on M and on the parameter α (α determines the bias for operating on the last cell of the queue of length M and is equal to 1 when $M = 1$ [12]). The channel state process $\{S_k\}_{k=-\infty}^{\infty}$, where $S_k \triangleq (Z_k, Z_{k-1}, \cdots, Z_{k-M+1})$ is a homogeneous first-order Markov process with an alphabet of size 2^M with $2^M \times 2^M$ transition probability matrix $\mathbf{P} = [p_{ij}]$ given by [12, Eq.(4)] and state stationary distribution column vector $\mathbf{\Pi} = [\pi_i]$ given by [12, Eq.(5)]. The QBC allows simple closed-form expressions for several statistics. In particular, the channel noise block probability $\Pr(Z_1 = z_1, Z_2 = z_2, \cdots, Z_n = z_n) = \Pr(Z^n = z^n)$ is expressed as [12]

- For blocklength $n \leq M$,

$$\Pr(Z^n = z^n) = \frac{\prod_{j=0}^{n-d_1^n-1}\left[j\frac{\varepsilon}{M-1+\alpha} + (1-\varepsilon)(1-p)\right]\prod_{j=0}^{d_1^n-1}\left[j\frac{\varepsilon}{M-1+\alpha} + (1-\varepsilon)p\right]}{\prod_{j=M-n}^{M-1}\left[1-(\alpha+j)\frac{\varepsilon}{M-1+\alpha}\right]} \quad (1)$$

  where $d_a^b = z_b + z_{b-1} + \cdots + z_a$ ($d_a^b = 0$ if $a > b$), and $\prod_{j=0}^{a}(\cdot) \triangleq 1$ if $a < 0$.

- For blocklength $n \geq M+1$,

$$\Pr(Z^n = z^n) = L^{(M)} \prod_{i=M+1}^{n} \left[\left(d_{i-M+1}^{i-1} + \alpha z_{i-M} \right) \frac{\varepsilon}{M-1+\alpha} + (1-\varepsilon)p \right]^{z_i}$$
$$\left\{ \left[\left(M-1-d_{i-M+1}^{i-1} \right) + \alpha(1-z_{i-M}) \right] \frac{\varepsilon}{M-1+\alpha} + (1-\varepsilon)(1-p) \right\}^{1-z_i} \quad (2)$$

where

$$L^{(M)} = \frac{\prod_{j=0}^{M-1-d_1^M} \left[j \frac{\varepsilon}{M-1+\alpha} + (1-\varepsilon)(1-p) \right] \prod_{j=0}^{d_1^M-1} \left[j \frac{\varepsilon}{M-1+\alpha} + (1-\varepsilon)p \right]}{\prod_{j=0}^{M-1} \left[1 - (\alpha + j) \frac{\varepsilon}{M-1+\alpha} \right]}.$$

The noise correlation coefficient, Cor, for the QBC is a non-negative quantity given by

$$\text{Cor} = \frac{\mathbf{E}[Z_1\, Z_2] - \mathbf{E}[Z_1]\mathbf{E}[Z_2]}{\mathbf{E}[Z_1^2] - (\mathbf{E}[Z_1])^2} = \frac{\frac{\varepsilon}{M-1+\alpha}}{1 - (M-2+\alpha)\frac{\varepsilon}{M-1+\alpha}}$$

where $\mathbf{E}[\cdot]$ denotes expectation. When $\varepsilon = 0$ (Cor $= 0$), the resulting model reduces to the memoryless BSC with crossover probability p.

B. Gilbert-Elliott Channel

The GEC is driven by an underlying stationary ergodic two-state Markov chain composed of state 0, which produces errors with probability p_G, and state 1, where errors occur with probability p_B, where $p_G < p_B$. The transition probabilities of the Markov chain are $p_{01} = Q$ and $p_{10} = q$, where $0 < Q < 1$ and $0 < q < 1$. Mushkin and Bar-David [6] defined the "memory" of the GEC as $\mu = 1 - q - Q$. If $\mu > 0$ the channel has persistent memory, or if $\mu < 0$ the channel has oscillatory memory [6]. When $\mu = 0$ the model reduces to the memoryless BSC. We define two matrices $\mathbf{P}(0)$ and $\mathbf{P}(1)$, $\mathbf{P}(0) + \mathbf{P}(1) = \mathbf{P}$, where the (i,j)th entry of the matrix $\mathbf{P}(z)$, $z \in \{0,1\}$ is $\Pr(Z_k = z, S_k = j \mid S_{k-1} = i)$. The state stationary distribution vector is $\mathbf{\Pi} = [\pi_0, \pi_1]^T = [q/(q+Q), Q/(q+Q)]^T$ (where the superscript $[\cdot]^T$ indicates transposition), and the matrices $\mathbf{P}(0)$ and $\mathbf{P}(1)$ are given by

$$\mathbf{P}(0) = \begin{bmatrix} (1-Q)\,(1-p_G) & Q\,(1-p_B) \\ q\,(1-p_G) & (1-q)\,(1-p_B) \end{bmatrix} \quad \mathbf{P}(1) = \begin{bmatrix} (1-Q)\,p_G & Q\,p_B \\ q\,p_G & (1-q)\,p_B \end{bmatrix}.$$

The channel noise block probability can be expressed in matrix form as

$$\Pr(Z^n = z^n) = \mathbf{\Pi}^T \left(\prod_{k=1}^{n} \mathbf{P}(z_k) \right) \mathbf{1} \quad (3)$$

where $\mathbf{1}$ is a column vector of ones of length 2. For example, an expression for $p_0 \triangleq \Pr(Z_k = 0)$ is

$$p_0 = 1 - \text{BER} = \pi_0(1-p_G) + \pi_1(1-p_B). \quad (4)$$

The noise correlation coefficient for the GEC is expressed as

$$\text{Cor} = \frac{\mu(\text{BER} - p_G)(p_B - \text{BER})}{\text{BER}(1 - \text{BER})}. \tag{5}$$

The SGC [15] can be obtained from the GEC by setting $p_G = 0$ and $p_B = 1$. As a result, the SGC with BER p and noise correlation coefficient μ is a two-state first-order Markov noise channel with parameters $Q = (1 - \mu)p$ and $q = (1 - \mu)(1 - p)$. It directly follows from (4) and (5) that Cor and μ have identical signs.

III. Perfect Bit Interleaving vs Perfect Symbol Interleaving For Non-binary Codes

The objective of this section is to analytically compare the performance of non-binary codes under both perfect symbol interleaving and perfect bit interleaving when transmitted over the binary FSMC model described at the beginning of the previous section.

Let $\mathcal{C}$ be any non-binary linear block code over the Galois field GF(2^b) with length n and error correction capability t (e.g., a Reed-Solomon code). A transmitted symbol is received correctly if the stationary noise corrupting it is a sequence of zeros of length b, denoted as 0^b. Otherwise, the transmitted symbol is received incorrectly and a symbol error occurs. Let the probability that the channel produces the b-tuple all zeros be denoted by $F(b) = \Pr(Z^b = 0^b)$. Then the probability of correct decoding under bounded distance decoding, denoted P_c, for the perfect symbol-interleaved system is given by

$$P_c = \sum_{i=0}^{t} \binom{n}{i} (1 - F(b))^i (F(b))^{n-i}. \tag{6}$$

On the other hand, for the perfect bit-interleaved non-binary code, denote the probability of correct b transmissions by $G(b) \triangleq \Pr(Z = 0)^b$. Hence the probability of correct decoding for this interleaving scheme is given by (6) with replacing $F(b)$ by $G(b)$. The performance comparison carried out in this section is done in terms of P_c, or equivalently, in terms of PCE $= 1 - P_c$.

***Proposition** 1:* If $F(b) > G(b)$ for the binary FSMC model, then perfect symbol interleaving outperforms perfect bit interleaving for the transmission of $\mathcal{C}$ under bounded distance decoding.

Proof: [3] If x denotes the symbol error probability, then the PCE (under bounded distance decoding) can be expressed as a function of x as follows:

$$\text{PCE}(x) = \sum_{i=t+1}^{n} \binom{n}{i} x^i (1 - x)^{n-i} = I_x(t + 1, n - t)$$

[3]This proof, which is based on expressing the PCE (which is the binomial complementary cumulative distribution function) in terms of the regularized incomplete Beta function, is due to one of the anonymous reviewers (Reviewer 1). We herein include it in lieu of our original proof as it is simpler.

where $I_x(a,b)$ is the regularized incomplete Beta function given by

$$I_x(a,b) = \frac{1}{B(a,b)} \int_0^x y^{a-1} (1-y)^{b-1} \, dy = \sum_{i=a}^{a+b-1} \binom{a+b-1}{i} x^i (1-x)^{a+b-1-i}$$

for $a > 0$, $b > 0$ and $B(a,b) = \int_0^1 y^{a-1} (1-y)^{b-1} \, dy$ is the Beta function and is positive for all positive pairs (a,b). The function $I_x(a,b)$ is monotonically increasing with respect to x in the range $0 < x < 1$, since

$$\frac{dI_x(a,b)}{dx} = \frac{x^{a-1}(1-x)^{b-1}}{B(a,b)} > 0.$$

Therefore, the PCE under perfect symbol interleaving, PCE$(1-F(b))$, is smaller than the PCE under perfect bit interleaved, PCE$(1-G(b))$, whenever $F(b) > G(b)$. ■

In light of Proposition 1, we next show that perfect symbol interleaving is always better compared to perfect bit interleaving when the non-binary code is transmitted over either the QBC or the GEC with positive memory.

A. Queue-Based Channel

***Proposition** 2:* Under bounded distance decoding, perfect symbol interleaving outperforms perfect bit interleaving when non-binary codes over GF(2^b) are transmitted over the QBC, for $\varepsilon > 0$ and $p > 0$.

Proof: From Proposition 1, it is enough to show that $F(b) > G(b)$ for the QBC. For this channel, $G(b) = (1-p)^b$ and for $b \leq M$ we express $F(b)$ using (1) as

$$F(b) = \prod_{j=0}^{b-1} \frac{j\frac{\varepsilon}{M-1+\alpha} + (1-\varepsilon)(1-p)}{1-(\alpha+M-1-j)\frac{\varepsilon}{M-1+\alpha}}.$$

For each $j > 0$ we notice that for $p > 0$,

$$\frac{j\frac{\varepsilon}{M-1+\alpha} + (1-\varepsilon)(1-p)}{1-(\alpha+M-1-j)\frac{\varepsilon}{M-1+\alpha}} > (1-p).$$

Because $b > 1$ (for non-binary codes), we get

$$\prod_{j=0}^{b-1} \frac{j\frac{\varepsilon}{M-1+\alpha} + (1-\varepsilon)(1-p)}{1-(\alpha+M-1-j)\frac{\varepsilon}{M-1+\alpha}} > (1-p)^b$$

which implies that $F(b) > G(b)$. When $b > M$, $F(b)$ is expressed using (2) as

$$F(b) = \prod_{j=0}^{M-1} \frac{j\frac{\varepsilon}{M-1+\alpha} + (1-\varepsilon)(1-p)}{1-(\alpha+M-1-j)\frac{\varepsilon}{M-1+\alpha}} (\varepsilon + (1-\varepsilon)(1-p))^{b-M}.$$

We already remarked that $\frac{j\frac{\varepsilon}{M-1+\alpha}+(1-\varepsilon)(1-p)}{1-(\alpha+M-1-j)\frac{\varepsilon}{M-1+\alpha}} > (1-p)$ for $j > 0$. We also note that

$$\varepsilon + (1-\varepsilon)(1-p) = (1-p) + \varepsilon p \geq (1-p)$$

with equality if and only if either $p = 0$ or $\varepsilon = 0$. Therefore, we combine the above two inequalities to get that

$$\left(\prod_{j=0}^{M-1} \frac{j\frac{\varepsilon}{M-1+\alpha} + (1-\varepsilon)(1-p)}{1-(\alpha+M-1-j)\frac{\varepsilon}{M-1+\alpha}}(\varepsilon + (1-\varepsilon)(1-p))^{b-M}\right) > (1-p)^M(1-p)^{b-M} = (1-p)^b.$$

Therefore $F(b) > G(b)$ (the inequality is strict because we assume that both p and $\varepsilon \neq 0$). ■

B. Gilbert-Elliott Channel

For the GEC model, $G(b) = p_0^b$, where p_0 is given by (4). We do not derive an explicit expression for $F(b)$. Alternatively, we define the generating series for $F(b)$ as

$$\mathcal{F}(z) \triangleq \sum_{b=0}^{\infty} F(b) z^b.$$

It follows from (3) that $F(b) = \mathbf{\Pi}^T \mathbf{P}^b(0)\mathbf{1}$. Then [16]

$$\mathcal{F}(z) = \mathbf{\Pi}^T (\mathbf{I} - \mathbf{P}(0)z)^{-1}\mathbf{1} \tag{7}$$

where $\mathbf{I}$ is the identity matrix. For the GEC, $\mathcal{F}(z)$ in (7) becomes

$$\mathcal{F}(z) = \frac{1 + a_1 z}{1 + b_1 z + b_2 z^2} \tag{8}$$

where

$$a_1 = -\mu\left[\pi_1(1-p_G) + \pi_0(1-p_B)\right], \quad b_1 = -[(1-\mu)p_0 + \mu(2-p_G-p_B)], \quad b_2 = \mu\,(1-p_G)(1-p_B).$$

The following recursion formula is derived directly from (8)

$$F(b) = -b_1 F(b-1) - b_2 F(b-2) \tag{9}$$

for $b \geq 2$, with initial conditions $F(0) = 1$ and $F(1) = p_0$. The condition stated in Proposition 1 holds for the GEC in light of the next lemma.

***Lemma** 1:* The following relation is satisfied for the GEC with $\mu > 0$

$$\frac{F(b)}{F(b-1)} > p_0, \quad \text{for} \quad b \geq 2. \tag{10}$$

Proof: The proof is by induction on b. For $b = 2$, the expressions for p_0 and $p_{00} \triangleq \Pr(Z_k = 0, Z_{k+1} = 0)$ calculated from (3) yield

$$\frac{F(2)}{F(1)} = \frac{p_{00}}{p_0} = -b_1 - \frac{b_2}{p_0} = p_0 + \mu\frac{\pi_0\pi_1(p_B - p_G)^2}{p_0} > p_0$$

since $\mu > 0$. Next assume that the statement (10) is true for a fixed $b \geq 2$. It follows from (9) that

$$F(b+1) = -b_1 F(b) - b_2 F(b-1)$$

or

$$\frac{F(b+1)}{F(b)} = -b_1 - b_2 \frac{F(b-1)}{F(b)}.$$

We conclude from the inductive hypothesis that $F(b-1)/F(b) < 1/p_0$, and since $b_2 > 0$ for $\mu > 0$, we obtain that

$$\frac{F(b+1)}{F(b)} > -b_1 - \frac{b_2}{p_0} = \frac{F(2)}{F(1)} > p_0.$$

■

By using (10) repeatedly for increasing values of b, we obtain a chain of inequalities of the form $F(b) > F(b-x)p_0^x$. In particular, when $x = b$, $F(b) > G(b)$. Thus, we have proved the following proposition.

***Proposition** 3:* Perfect symbol interleaved transmission of $\mathcal{C}$ performs better than the perfect bit interleaved one over the GEC with $\mu > 0$, assuming bounded distance decoding.

C. Channels with Negative Noise Correlation Coefficient

We next observe that for some classes of FSMC models with negative noise correlation coefficient, perfect bit interleaving can be better than perfect symbol interleaving.

Remark 1: The noise correlation coefficient of a communication fading system is generally (but not always) a positive quantity. To illustrate this, let us model (using the fitting method of [13]) via a GEC a discrete channel with binary frequency-shift keying modulation, Rician fading with Clarke's autocorrelation function, and hard quantized non-coherent demodulation [13]. For the case of Rayleigh fading, the correlation coefficient of this discrete channel is always non-negative. However, when the fading is Rician, there exists a range of fading parameters that yield a GEC with negative memory μ (or negative noise correlation coefficient). For example, for a discrete fading channel with signal-to-noise ratio (SNR) 13 dB, normalized Doppler frequency $f_D T = 0.6$ and Rician factor $K_R = 3$ dB, we obtain a fitting GEC with parameters $p_G = 0.0014$, $p_B = 0.06$, $q = 0.923$, $Q = 0.6175$. The resulting GEC BER is 0.025 and $\mu = -0.54$; also the capacity of the GEC and the (equivalent) BSC (under perfect bit interleaving) are 0.8323 and 0.8319 bits/channel use, respectively.

Remark 2: Note that in Proposition 1, if $F(b) < G(b)$, then we get the opposite result compared to the positive noise correlation case; i.e., perfect bit interleaving outperforms perfect symbol interleaving. For the simplified Gilbert channel, $F(b)$ and $G(b)$ are given by

$$F(b) = \left[(1-p)(\mu + (1-\mu)(1-p))^{b-1}\right] \quad \text{and} \quad G(b) = \left[(1-p)^b\right].$$

Note that if $\mu < 0$, then $F(b) < G(b)$.

Remark 3: Finally, note that we can construct examples of a simplified Fritchman channel [10] of negative noise correlation coefficient and with two good states and one bad state such that $F(2) < G(2)$ but $F(3) > G(3)$. [4] Thus, for this channel, neither perfect symbol interleaving nor perfect bit interleaving is always better, since this comparison depends on the code's field size 2^b.

Propositions 1-3 consider the PCE performance of non-binary codes under perfect interleaving. The next section provides a practical guideline to design the optimal interleaving depth (e.g. the typical interleaving depth needed to achieve perfect interleaving) for the QBC.

IV. Performance Evaluation under Imperfect Interleaving

In this section, we conduct a numerical PCE study of imperfect interleaved non-binary block codes (under bounded distance decoding) over the QBC. The performance is evaluated via the derivation of the probability of m errors in a block of length n, namely $P(m, n)$, yielding a PCE given by PCE $= \sum_{m=t+1}^{n} P(m, n)$. For our purposes, we numerically calculate $P(m, n)$ and the PCE using the method of [9]; however the recent analytical method of [5] can also be used. We consider an (n, k) RS code over GF(2^b) with codewords of length n and k information symbols. We assume block symbol interleaving with nb columns (codeword length in bits) and I_d (interleaving depth) rows. The b bits within each symbol are transmitted consecutively through the channel.

The superiority of imperfect symbol-interleaved to imperfect bit-interleaved non-binary codes over the GEC was observed in [5]. Similar results can be obtained for the QBC; see, for example, Fig. 1. This figure presents PCE versus I_d a bit and symbol interleaved shortened (73,57) RS code (with $b = 7$, $t = 8$ symbols) over the QBC with parameters $M = 2$, $\alpha = 1$, $p = 0.007$, and two values of Cor, Cor $= 0.75, 0.9$. We observe that imperfect symbol interleaving outperforms imperfect bit interleaving for all values of I_d. In particular, for sufficiently large I_d, these curves corroborate the result presented in Proposition 2. Motivated by these results we hereafter focus on symbol interleaving and our objective is to investigate the existence of a relationship between the optimal interleaving depth and the QBC parameters.

Fig. 2 presents PCE versus I_d for a symbol interleaved shortened (73,57) RS code (with $b = 7$, $t = 8$ symbols) over the QBC. The parameters of the QBC are $M = 1$ $(\alpha = 1)$, $p = 0.007$, and four values of Cor (or ε), Cor $= 0.5, 0.9, 0.95, 0.98$. For a given value of Cor, we observe that the PCE decreases as I_d increases until a threshold point at which it is no longer possible to improve the PCE. We denote this

[4] Using the notation of [10], consider for example a Fritchman channel with parameters $p_{11} = 0.11$, $p_{22} = 0.82$, $p_{31} = 0.42$ and $p_{32} = 0.3$; its noise correlation coefficient is -0.057. For this channel, we have $F(2) < G(2)$ and $F(3) > G(3)$.

value of I_d that renders the channel block memoryless (i.e., achieving perfect symbol-interleaving) by $I_d^\star$. The approximate values of $I_d^\star$ found from each curve of this figure are listed in Table I. We notice from this table a linear relationship between $I_d^\star$ and $1/(1-\varepsilon)$ which is expressed as $I_d^\star = \frac{1}{1-\varepsilon}$. We conduct in the following a similar analysis for a QBC with higher memory orders M.

Fig. 3 presents PCE versus I_d for a symbol interleaved shortened (73,57) RS code over the QBC with $M = 4$, $\varepsilon = 0.8$, $p = 0.007$, and two values of Cor, Cor $= 0.22$ $(\alpha = 11.2), 0.5$ $(\alpha = 1)$. The values of $I_d^\star$ are roughly the same for each curve, which allow us to conclude that, for a fixed ε, $I_d^\star$ is weakly dependent on the parameter α. A similar conclusion can be derived for the parameter p (curves not shown). We now fix $\alpha = 1$, $p = 0.007$, and plot in Fig. 4 the PCE versus I_d for a QBC with $M = 4$ and Cor $= 0.2$ $(\varepsilon = 0.5), 0.5$ $(\varepsilon = 0.8), 0.69$ $(\varepsilon = 0.9), 0.83$ $(\varepsilon = 0.95)$. A similar curve is presented in Fig. 5 for $M = 6$ and Cor $= 0.14$ $(\varepsilon = 0.5), 0.4$ $(\varepsilon = 0.8), 0.6$ $(\varepsilon = 0.9), 0.76$ $(\varepsilon = 0.95)$. The values of $I_d^\star$ achieved for each ε in these figures are shown in Table II, which can be expressed as $I_d^\star = \frac{\Gamma}{1-\varepsilon}$, where $\Gamma = 1.5$ for $M = 4$ and $\Gamma = 2.0$ for $M = 6$. Thus, for fixed $(\alpha,\ p,\ M)$, a linear relationship between $I_d^\star$ and $1/(1-\varepsilon)$ is valid for the QBC, where the proportional constant Γ increases with M. The same trend is observed for other values of code parameters (figures are not herein shown due to space limitations). This result provides the communication system designer with some insight for the practical interleaving design for the QBC. For example, in a recent work [14], QBC models at the packet level were developed for a non-interleaved RS coded communication system with time-correlated flat fading channel. For $f_D T = 0.0005$, SNR = 15 dB, and Rayleigh fading, an accurate QBC has parameters $M = 4$ and $\varepsilon = 0.8773$ (cf. Table I in [14]). The results of this section indicate that $I_d^\star = 12$ for this QBC.

V. Conclusions

In this work, we mathematically demonstrate that for a class of binary additive noise finite-state channels satisfying an explicit (sufficient) condition expressed in terms of the channel noise statistics, perfectly interleaving the channel at the (code) symbol level always outperforms perfectly interleaving it at the bit level when transmitting non-binary linear block codes over such channels. We show that the Gilbert-Elliott channel (GEC) with positive noise correlation and the recently introduced Markovian queue-based channel (QBC) are two finite-state channels for which the condition holds. Both of these channels have been previously shown to accurately model hard-decision demodulated time-correlated Rayleigh fading channels as well as slow fading Rician channels (e.g., see [3], [13]). Furthermore, we remark that there exist finite-state channels (such as the GEC and Fritchman channels with negative noise correlations which can model Rician channels with fast fading) for which a reverse result holds; i.e., for which bit-interleaving outperforms symbol-interleaving. Finally, we conduct a numerical study to evaluate the effects of finite-length (imperfect) symbol-interleaving on the performance of Reed-Solomon codes sent over the QBC. We observe that, as for the case of the GEC [5], there exists a simple linear relationship between the optimal interleaving depth and a function of a channel correlation parameter; such property provides useful interleaving design criteria when operating over the QBC and the underlying fading channels it represents.

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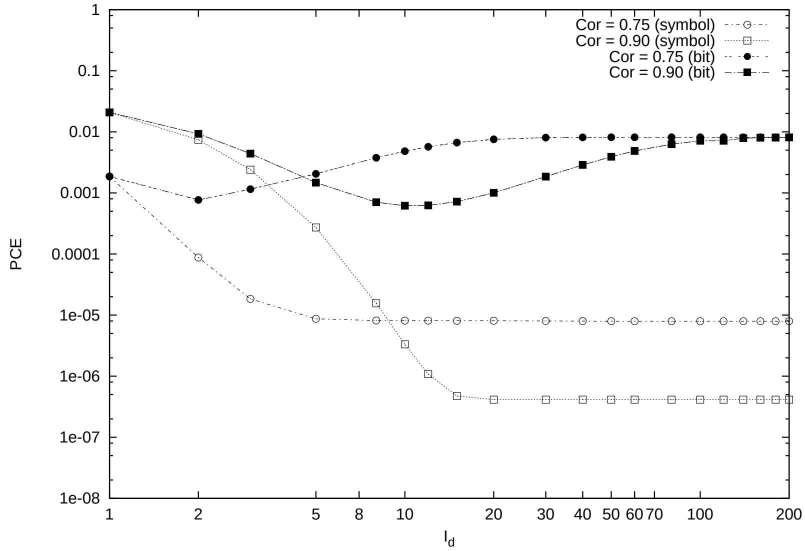

Fig. 1. PCE versus I_d for (73,57) RS, $b = 7, t = 8$, over the QBC with parameters $M = 2$, $\alpha = 1$, $p = 0.007$, Cor $= 0.75, 0.90$. Symbol and bit interleaving.

TABLE I

OPTIMAL INTERLEAVING DEPTH FOR THE QBC WITH PARAMETERS $M = 1$, $\alpha = 1$, $p = 0.007$.

ε	$I_d^\star$
0.5	2
0.9	10
0.95	20
0.98	50

TABLE II

OPTIMAL INTERLEAVING DEPTH FOR THE QBC WITH $M = 4$ AND $M = 6$ DERIVED FROM FIGS. 4 AND 5

ε	$I_d^\star$ $(M = 4)$	$I_d^\star$ $(M = 6)$
0.5	3	4
0.8	8	10
0.9	15	20
0.95	30	40

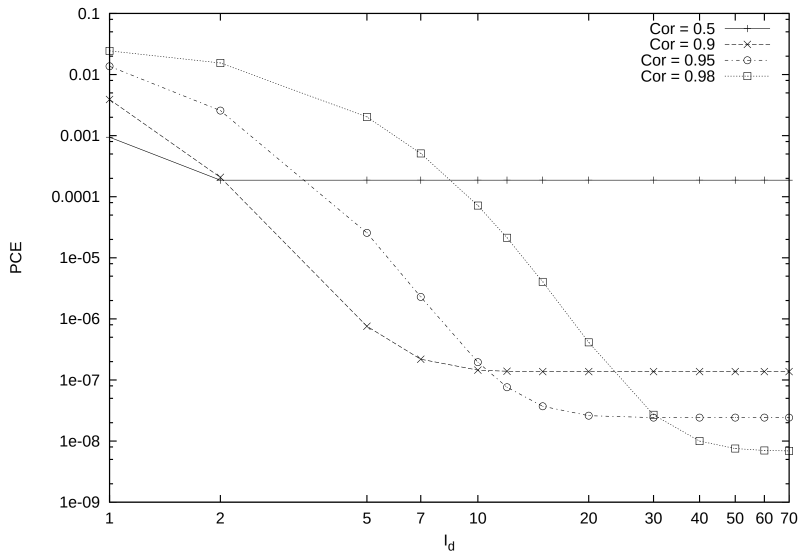

Fig. 2. PCE versus I_d for (73,57) RS, $b=7, t=8$, over the QBC with parameters $M=1$, $\alpha=1$, $p=0.007$. Cor $=0.5, 0.9, 0.95, 0.98$. Symbol interleaving.

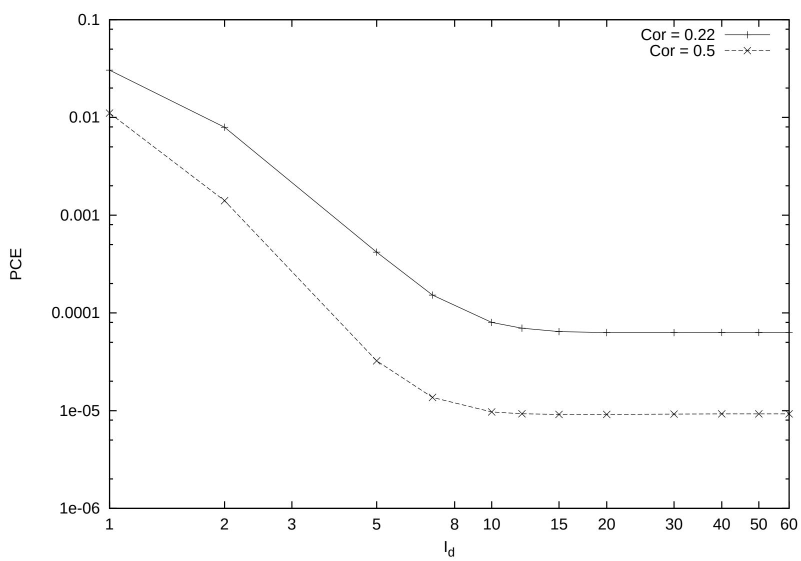

Fig. 3. PCE versus I_d for (73,57) RS, $b=7, t=8$, over the QBC with parameters $M=4$, $\varepsilon=0.8$, $p=0.007$, Cor $=0.22, 0.5$. Symbol interleaving.

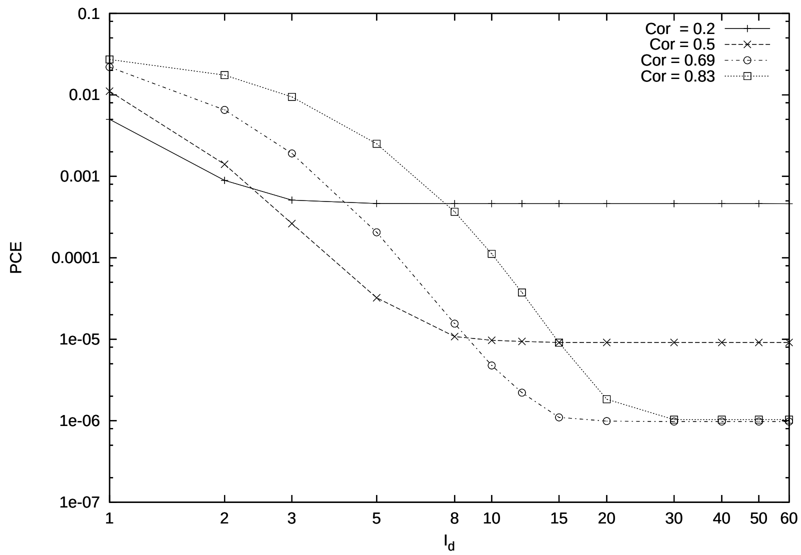

Fig. 4. PCE versus I_d for (73,57) RS, $b = 7, t = 8$, over the QBC with parameters $M = 4$, $\alpha = 1$, $p = 0.007$, Cor $= 0.2, 0.5, 0.7, 0.83$. Symbol interleaving.

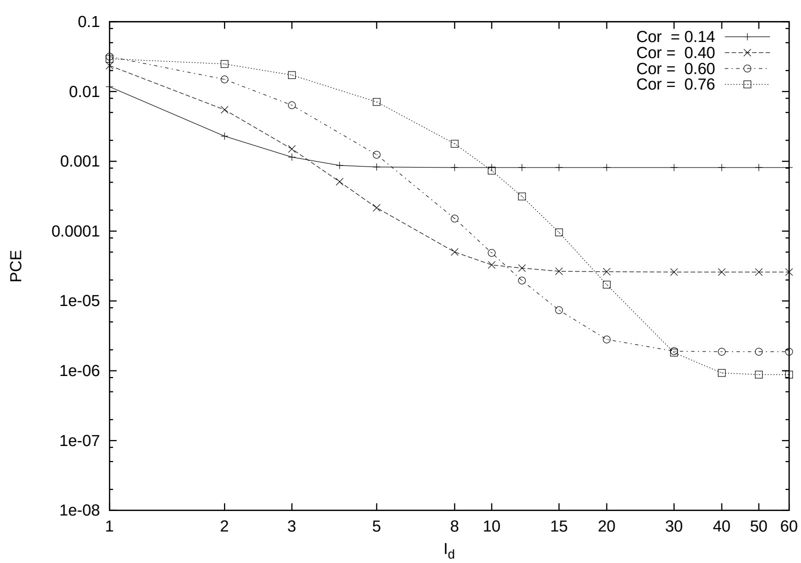

Fig. 5. PCE versus I_d for (73,57) RS, $b = 7, t = 8$, over the QBC with parameters $M = 6$, $\alpha = 1$, $p = 0.007$, Cor $= 0.14, 0.4, 0.6, 0.76$. Symbol interleaving.