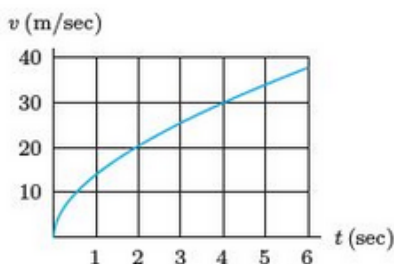# Unit #8 - The Integral

Some problems and solutions selected or adapted from Hughes-Hallett Calculus.

Distance And Velocity

1. The graph below shows the velocity, v, of an object (in meters/sec). Estimate the total distance the object traveled between $t = 0$ and $t = 6$.

2. The figure below shows the velocity of a particle, in cm/sec, along the t-axis for $-3 \leq t \leq 3$ (t in seconds).

   (a) Describe the motion in words. Is the particle changing direction or always moving in the same direction? Is the particle speeding up or slowing down?

   (b) Make over- and underestimates of the distance traveled for $-3 \leq t \leq 3$.
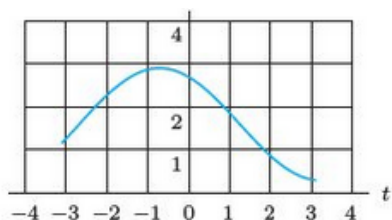
3. Consider the following table of values for $f(t)$.

t	15	17	19	21	23
$f(t)$	10	13	18	20	30

   (a) If we divide the time interval into $n = 4$ subintervals, what is Δt? What are t_0, t_1, t_2, t_3, t_4? What are $f(t_0), f(t_1), f(t_2), f(t_3), f(t_4)$?

   (b) Find the left and right sums using $n = 4$.

   (c) If we divide the time interval into $n = 2$ subintervals, what is Δt? What are t_0, t_1, t_2? What are $f(t_0), f(t_1), f(t_2)$?

   (d) Find the left and right sums using $n = 2$.

4. Consider the following table of values for $f(t)$.

t	0	4	8	12	16
$f(t)$	25	23	22	20	17

   (a) If we divide the time interval into $n = 4$ subintervals, what is Δt? What are t_0, t_1, t_2, t_3, t_4? What are $f(t_0), f(t_1), f(t_2), f(t_3), f(t_4)$?

   (b) Find the left and right sums using $n = 4$.

   (c) If we divide the time interval into $n = 2$ subintervals, what is Δt? What are t_0, t_1, t_2? What are $f(t_0), f(t_1), f(t_2)$?

   (d) Find the left and right sums using $n = 2$.

5. At time t, in seconds, your velocity, v, in meters/ second, is given by
$$v(t) = 1 + t^2 \text{ for } 0 \leq t \leq 6.$$
Use $\Delta t = 2$ to estimate the distance traveled during this time. Find the upper and lower estimates, and then average the two.

6. For time, t, in hours, $0 \leq t \leq 1$, a bug is crawling at a velocity, v, in meters/ hour given by
$$v = \frac{1}{1+t}.$$
Use $\Delta t = 0.2$ to estimate the distance that the bug crawls during this hour. Find an overestimate and an underestimate. Then average the two to get a new estimate.

For questions 7 to 10, the graph shows the velocity, in cm/sec, of a particle moving along the x-axis. Compute the particle's change in position, left (negative) or right (positive), between times $t = 0$ and $t = 5$ seconds.

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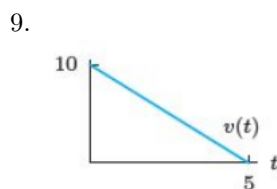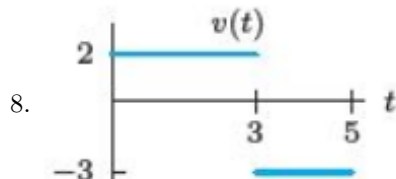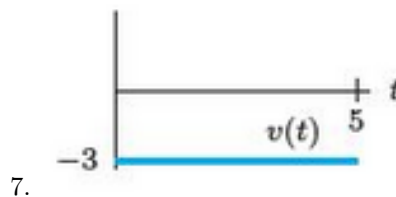
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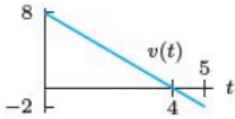
11. A car going 80 ft/s (about 90 km/h) brakes to a stop in five seconds. Assume the deceleration is constant.

   (a) Graph the velocity against time, t, for $0 \le t \le 5$ seconds.

   (b) Represent, as an area on the graph, the total distance traveled from the time the brakes are applied until the car comes to a stop.

   (c) Find this area and hence the distance traveled.

12. A 727 jet needs to attain a speed of 320 km/h to take off. If it can accelerate from 0 to 320 km/h in 30 seconds, how long must the runway be? (Assume constant acceleration.)

13. A student is speeding down Route 11 in his fancy red Porsche when his radar system warns him of an obstacle 400 feet ahead. He immediately applies the brakes, starts to slow down, and spots a skunk in the road directly ahead of him. The "black box" in the Porsche records the car's speed every two seconds, producing the following table. The speed decreases throughout the 10 seconds it takes to stop, although not necessarily at a uniform rate.

Time since brakes applied (sec)	0	2	4	6	8	10
Speed (ft/sec)	100	80	50	25	10	0

   (a) What is your best estimate of the total distance the student's car traveled before coming to rest?

   (b) Which one of the following statements can you justify from the information given?

      (i) The car stopped before getting to the skunk.
      (ii) The "black box" data is inconclusive. The skunk may or may not have been hit.
      (iii) The skunk was hit by the car.

14. Roger runs a marathon. His friend Jeff rides behind him on a bicycle and records his speed every 15 minutes. Roger starts out strong, but after an hour and a half he is so exhausted that he has to stop. Jeff's data follow:

Time since Start (min)	0	15	30	45	60	75	90
Speed (mph)	12	11	10	10	8	7	0

   (a) Assuming that Roger's speed is never increasing, give upper and lower estimates for the distance Roger ran during the **first half hour**.

   (b) Give upper and lower estimates for the distance Roger ran in total during the **entire hour and a half**.

   (c) How often would Jeff have needed to measure Roger's speed in order to find lower and upper estimates within 0.1 mile of the actual distance he ran?

15. The velocity of a particle moving along the x-axis is given by $f(t) = 6 - 2t$ cm/sec. Use a graph of $f(t)$ to find the exact change in position of the particle from time $t = 0$ to $t = 4$ seconds.

16. A baseball thrown directly upward at 96 ft/sec has velocity $v(t) = 96 - 32t$ ft/ sec at time t seconds.

   (a) Graph the velocity from $t = 0$ to $t = 6$.

   (b) When does the baseball reach the peak of its flight? How high does it go?

   (c) How high is the baseball at time $t = 5$?

17. Two cars start at the same time and travel in the same direction along a straight road. The graph below gives the velocity, v, of each car as a function of time, t. Which car:

   (a) Attains the larger maximum velocity?

   (b) Stops first?

   (c) Travels farther?
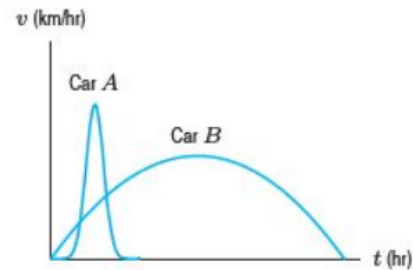
18. Two cars travel in the same direction along a straight road. The graph below shows the velocity, v, of each car at time t. Car B starts 2 hours after car A and car B reaches a maximum velocity of 50 km/ hr.
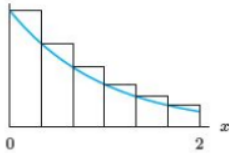
   (a) For approximately how long does each car travel?

   (b) Estimate car A's maximum velocity.

   (c) Approximately how far does each car travel?
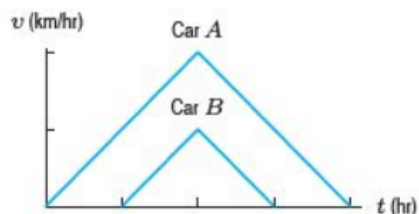
The Definite Integral

19. The figure below shows a Riemann sum approximation with n subdivisions to $\int_a^b f(x)dx$.

    (a) Is it a left- or right-hand approximation? Would the other one be larger or smaller?

    (b) What are a, b, n and Δx?

20. Using the figure below, draw rectangles representing each of the following Riemann sums for the function f on the interval $0 \le t \le 8$. Calculate the value of each sum.
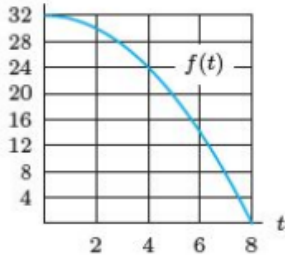

    (a) Left-hand sum with $\Delta t = 4$.

    (b) Right-hand sum with $\Delta t = 4$.

    (c) Left-hand sum with $\Delta t = 2$.

    (d) Right-hand sum with $\Delta t = 2$.

21. The graph of a function $f(t)$ is given in the figure below.
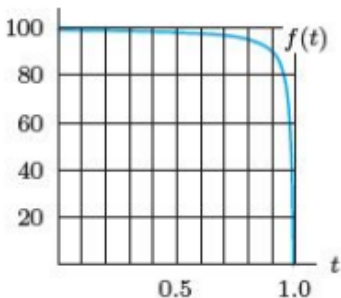
    Which of the following four numbers could be an estimate of $\int_0^1 f(t)\ dt$, accurate to two decimal places? Explain how you chose your answer.
    (a) -98.35 (b) 71.84 (c) 100.12 (d) 93.47

22. (a) What is the area between the graph of $f(x)$ shown below and the x-axis, between $x = 0$ and $x = 5$?

    (b) What is $\int_0^5 f(x)\ dx$?
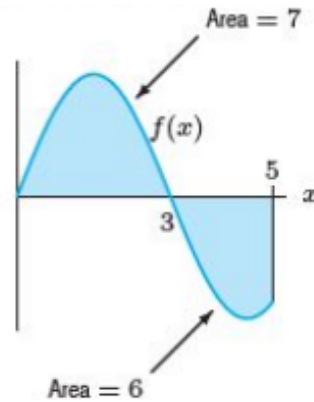
23. (a) On a sketch of $y = \ln(x)$, represent the left Riemann sum with $n = 2$ approximating $\int_1^2 \ln(x)\ dx$. Write out the terms in the sum, but do not evaluate it.

    (b) On another sketch, represent the *right* Riemann sum with $n = 2$ approximating $\int_1^2 \ln(x)\ dx$. Write out the terms in the sum, but do not evaluate it.

    (c) Which sum is an overestimate? Which sum is an underestimate?

24. Estimate $\int_1^2 x^2\ dx$ using left- and right-hand sums with four subdivisions, and then averaging them. How far from the true value of the integral could your final estimate be?

25. Without computing the sums, find the difference between the right- and left-hand Riemann sums if we use $n = 500$ subintervals to approximate $\int_{-1}^1 (2x^3 + 4)\ dx$.

26. Without computation, decide if $\int_0^{2\pi} e^{-x}\sin x\ dx$ is positive or negative. [Hint: Sketch $e^{-x}\sin(x)$].

27. (a) Graph $f(x) = \begin{cases} 1 - x & \text{if } 0 \le x \le 1 \\ x - 1 & \text{if } 1 < x \le 2 \end{cases}$

    (b) Find the *exact* value of $\int_0^2 f(x)\ dx$ (hint: sketch and see what shapes you get).

    (c) Calculate the 4-term left Riemann sum approximation to the definite integral. How does the approximation compare to the exact value?

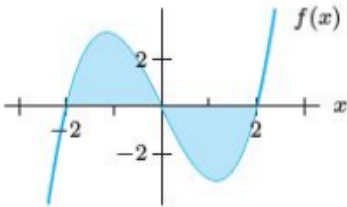28. Using the figure below, find the values of

    (a) $\int_{ac}^{b} f(x)\ dx$ (b) $\int_{bc}^{c} f(x)\ dx$

    (c) $\int_{a}^{c} f(x)\ dx$ (d) $\int_{a}^{c} |f(x)|\ dx$
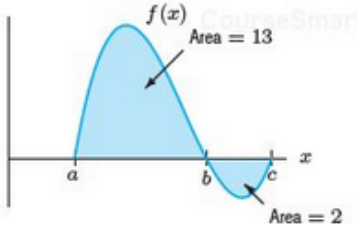
29. Given the figure below, and the statement that $\int_{-2}^{0} f(x)\ dx = 4$, estimate
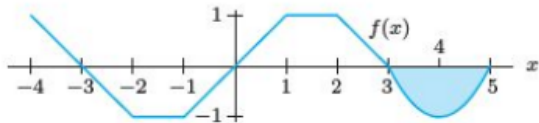
    (a) $\int_{0}^{2} f(x)\ dx$ (b) $\int_{-2}^{2} f(x)\ dx$

    (c) The total shaded area.

30. (a) Using the graph below, find $\int_{-3}^{0} f(x)\ dx$.

    (b) If the area of the shaded region is A, estimate
    $\int_{-3}^{4} f(x)\ dx$.

31. Calculate the following approximations to $\int_{0}^{6} x^2\ dx$.
    (a) LEFT(2); (b) RIGHT(2);
    (c) TRAP(2); (d) MID(2)

32. (a) Find LEFT(2) and RIGHT(2) for $\int_{0}^{4} (x^2 + 1)\ dx$.

    (b) Illustrate your answers to part (a) graphically. Is
    each approximation an underestimate or overestimate?

33. (a) Find MID(2) and TRAP(2) for $\int_{0}^{4} (x^2 + 1)\ dx$.

    (b) Illustrate your answers to part (a) graphically. Is
    each approximation an underestimate or overestimate?

34. Calculate the following approximations to $\int_{0}^{\pi} \sin(\theta)d\theta$.
    (a) LEFT(2); (b) RIGHT(2);
    (c) TRAP(2); (d) MID(2)

35. Using the table below, estimate the total distance traveled from time $t = 0$ to time $t = 6$ using LEFT, RIGHT, and TRAP.
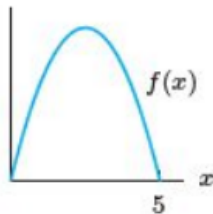
Time, t (s)	0	1	2	3	4	5	6
Velocity, v (m/s)	3	4	5	4	7	8	11

For the functions in Problems 36 − 39, pick which approximation- left, right, trapezoid, or midpoint- is guaranteed to give an *overestimate* for $\int_{0}^{5} f(x)\ dx$, and which is guaranteed to give an *underestimate*. (There may be more than one.)
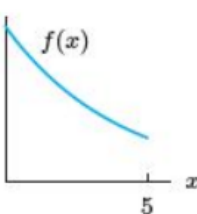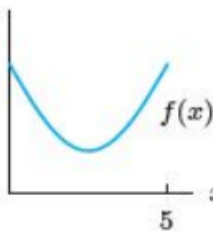
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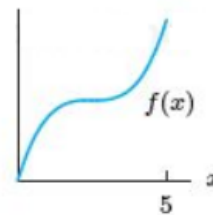
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40. (a) Find the exact value of $\int_{0}^{2\pi} \sin\theta\ d\theta$ without calculation (i.e. from a sketch).

(b) Explain, using pictures, why the MID(1) and MID(2) approximations to this integral give the exact value.

(c) Does MID(3) give the exact value of this integral? How about MID(n)? Explain.

41. The width, in feet, at various points along the fairway of a hole on a golf course is given in the figure below. If one pound of fertilizer covers 200 square feet, estimate the amount of fertilizer needed to fertilize the fairway.

Select the most accurate estimate approach from the methods covered in the class.