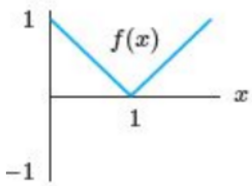
Unit #10 - Graphs of Antiderivatives, Substitution Integrals

Some problems and solutions selected or adapted from Hughes-Hallett Calculus.

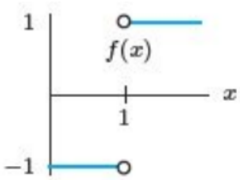
Graphs of Antiderivatives

In Questions 1 to 4, sketch two functions F such that $F' = f$. In one case, let $F(0) = 0$, and in the other let $F(0) = 1$.

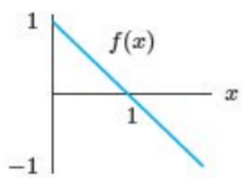
1.



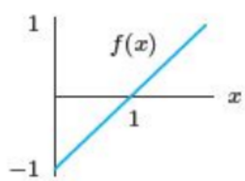
2.



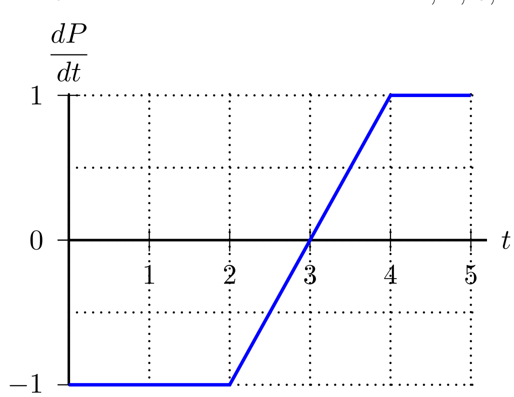
3.



4.



5. Using the graph below, and the fact that $P = 2$ when $t = 0$ to find values of P when $t = 1, 2, 3, 4$ and 5.

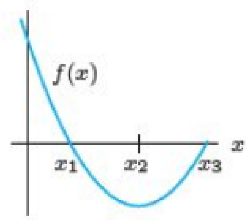


6. Given the values of the derivative $f'(x)$ in the table and that $f(0) = 100$, use the TRAP rule to estimate $f(x)$ for $x = 2, 4, 6$.

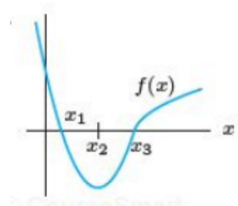
x	0	2	4	6
$f'(x)$	10	18	23	25

In Questions 7 to 10, sketch two functions F such that $F' = f$. In one case, let $F(0) = 0$, and in the other let $F(0) = 1$.
Mark the points x_1, x_2, and x_3 on the x-axis of your graph. Identify local maxima, minima and inflection points of $F(x)$.

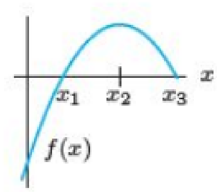
7.



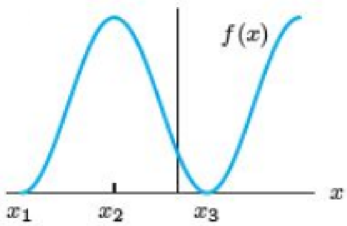
8.



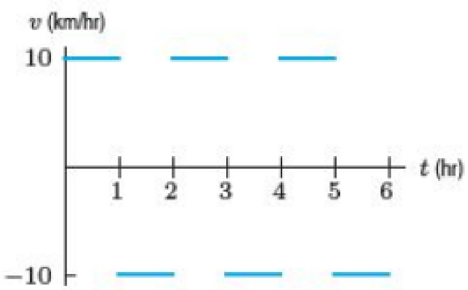
9.



10.



11. A particle moves back and forth along the x-axis. The graph below approximates the velocity of the particle as a function of time. Positive velocities represent movement to the right and negative velocities represent movement to the left. The particle starts at the point $x = 5$. Graph the distance of the particle from the origin, with distance measured in kilometers and time in hours.

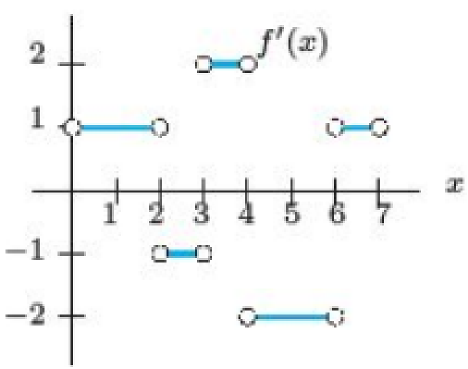


12. Assume f' is given by the graph shown below. Suppose f is continuous and that $f(3) = 0$.

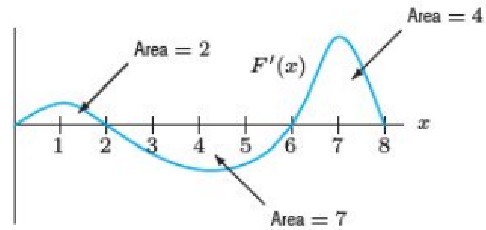
(a) Sketch a graph of f.

(b) Find $f(0)$ and $f(7)$.

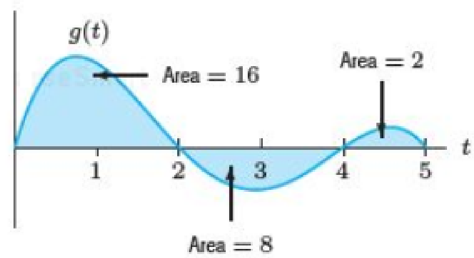
(c) Find $\int_0^7 f(x)\, dx$ in two different ways.



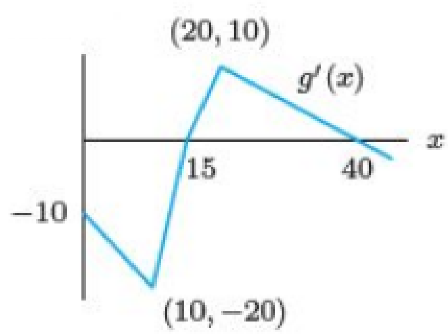
13. Use the graph of $F'(x)$ below and the fact that $F(2) = 3$ to sketch the graph of $F(x)$. Label the values of at least four points.



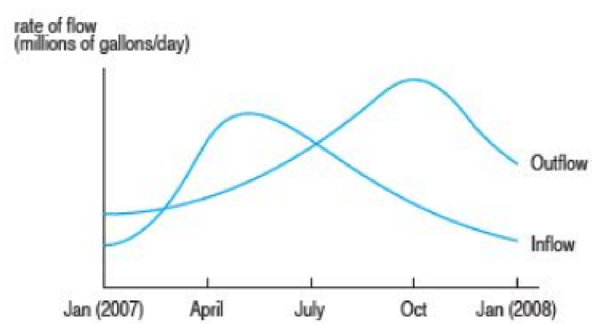
14. Using the graph of $g(t)$ below, sketch a graph of an antiderivative $G(t)$ of $g(t)$ satisfying $G(0) = 5$. Label each critical point of $G(t)$ with its coordinates.



15. Using the graph of g' shown below, and the fact that $g(0) = 50$, sketch the graph of $g(x)$. Give the coordinates of all critical points and inflection points of g.



16. The Quabbin Reservoir in the western part of Massachusetts provides most of Boston's water. The graph below represents the flow of water in and out of the Quabbin Reservoir throughout 2007.



(a) Sketch a graph of the quantity of water in the reservoir, as a function of time.

(b) When, in the course of 2007, was the quantity of water in the reservoir largest? Smallest? Mark and label these points on the graph you drew in part (a).

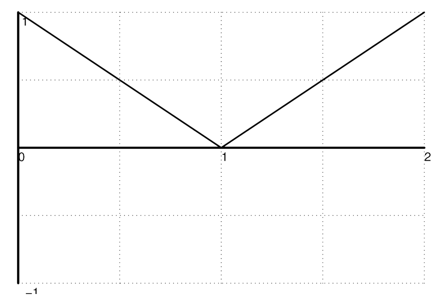
(c) When was the quantity of water increasing most rapidly? Decreasing most rapidly? Mark and label these times on both graphs.

(d) By July 2008 the quantity of water in the reservoir was about the same as in January 2007. Draw plausible graphs for the flow into and the flow out of the reservoir for the first half of 2008.

17. Consider the function

$$f(x) = \begin{cases} -x+1 & \text{for } 0 \leq x \leq 1 \\ x-1 & \text{for } 1 < x \leq 2\,. \end{cases}$$

The graph of this function is shown here:



(a) Find a continuous function F such that $F' = f$ and $F(1) = 1$. Hint: $F(x)$ will be a piecewise function.

(b) Use geometry to calculate the area under the graph of f and above the x−axis between $x = 0$ and $x = 2$ and show that it equals $F(2) - F(0)$.

(c) Use parts (a) and (b) to verify the Fundamental Theorem of Calculus for this example.

Substitution Integrals

To practice computing integrals using substitutions, do as many of the problems from this section as you feel you need. The problems trend from simple to the more complex.

Note: In the solutions to these problems, we always show the substitution used. On a test, if you can compute the antiderivative in your head, you do *not* need to go through all the steps shown here. They are included in these solutions as a learning and comprehension aid.

18. $\displaystyle\int te^{t^2}\,dt$

19. $\displaystyle\int e^{3x}\,dx$

20. $\displaystyle\int e^{-x}\,dx$

21. $\displaystyle\int 25e^{-0.2t}\,dt$

22. $\displaystyle\int t\cos(t^2)\,dt$

23. $\displaystyle\int \sin(2x)\,dx$

24. $\displaystyle\int \sin(3-t)\,dt$

25. $\displaystyle\int xe^{-x^2}\,dx$

26. $\displaystyle\int (r+1)^3\,dr$

27. $\displaystyle\int y(y^2+5)^8\,dy$

28. $\displaystyle\int t^2(t^3-3)^{10}\,dt$

29. $\displaystyle\int x^2(1+2x^3)^2\,dx$

30. $\displaystyle\int x(x^2+3)^2\,dx$

31. $\displaystyle\int x(x^2-4)^{7/2}\,dx$

32. $\displaystyle\int y^2(1+y)^2\,dy$

33. $\displaystyle\int (2t-7)^{73}\,dt$

34. $\displaystyle\int \frac{1}{y+5}\,dy$

35. $\displaystyle\int \frac{1}{\sqrt{4-x}}\,dx$

36. $\displaystyle\int (x^2+3)^2\,dx$

37. $\displaystyle\int x^2e^{x^3+1}\,dx$

38. $\displaystyle\int \sin(\theta)(\cos(\theta)+5)^7\,d\theta$

39. $\int \sqrt{\cos(3t)} \sin(3t)\ dt$

40. $\int \sin^6(\theta) \cos(\theta)\ d\theta$

41. $\int \sin^3(\alpha) \cos(\alpha) d\alpha$

42. $\int \sin^6(5\theta) \cos(5\theta)\ d\theta$

43. $\int \tan(2x)\ dx$

44. $\int \frac{(\ln z)^2}{z}\ dz$

45. $\int \frac{e^t + 1}{e^t + t}\ dt$

46. $\int \frac{y}{y^2 + 4}\ dy$

47. $\int \frac{\cos(\sqrt{x})}{\sqrt{x}}\ dx$

48. $\int \frac{e^{\sqrt{y}}}{\sqrt{y}}\ dy$

49. $\int \frac{1 + e^x}{\sqrt{x + e^x}}\ dx$

50. $\int \frac{e^x}{2 + e^x}\ dx$

51. $\int \frac{x + 1}{x^2 + 2x + 19}\ dx$

52. $\int \frac{t}{1 + 3t^2}\ dt$

53. $\int \frac{e^x - e^{-x}}{e^x + e^{-x}}\ dx$

54. $\int \frac{(t + 1)^2}{t^2}\ dt$

55. $\int \frac{x \cos(x^2)}{\sqrt{\sin(x^2)}}\ dx$

56. $\int_0^{\pi} \cos(x + \pi)\ dx$

57. $\int_0^{1/2} \cos(\pi x)\ dx$

58. $\int_0^{\pi/2} e^{-\cos(\theta)} \sin(\theta)\ d\theta$

59. $\int_1^2 2xe^{x^2}\ dx$

60. $\int_1^8 \frac{e^{\sqrt[3]{x}}}{\sqrt[3]{x^2}}\ dx$

61. $\int_{-1}^{e-2} \frac{1}{t + 2}\ dt$

62. $\int_1^4 \frac{\cos\sqrt{x}}{\sqrt{x}}\ dx$

63. $\int_0^2 \frac{x}{(1 + x^2)^2}\ dx$

64. If appropriate, evaluate the following integrals by substitution. If substitution is not appropriate, say so, and do not evaluate.

   (a) $\int x \sin(x^2)\ dx$

   (b) $\int x^2 \sin(x)\ dx$

   (c) $\int \frac{x^2}{1 + x^2}\ dx$

   (d) $\int \frac{x}{(1 + x^2)^2}\ dx$

   (e) $\int x^3 e^{x^2}\ dx$

   (f) $\int \frac{\sin(x)}{2 + \cos(x)}\ dx$

65. Find the exact area under the graph of $f(x) = xe^{x^2}$ between $x = 0$ and $x = 2$.

66. Find the exact area under the graph of $f(x) = \frac{1}{x + 1}$ between $x = 0$ and $x = 2$.

67. Find $\int 4x(x^2 + 1)\ dx$ using two methods:

   (a) Do the multiplication first, and then antidifferentiate.

   (b) Use the substitution $w = x^2 + 1$.

   (c) Explain how the expressions from parts (a) and (b) are different. Are they both correct?

68. (a) Find $\int \sin\theta \cos\theta\ d\theta$

   (b) You probably solved part (a) by making the substitution $w = \sin\theta$ or $w = \cos\theta$. (If not, go back and do it that way.) Now find $\int \sin\theta \cos\theta\ d\theta$ by making the other substitution.

   (c) There is yet another way of finding this integral which involves the trigonometric identities:

$$\sin(2\theta) = 2 \sin\theta \cos\theta$$
$$\cos(2\theta) = \cos^2\theta - \sin^2\theta.$$

   Find $\int \sin\theta \cos\theta\ d\theta$ using one of these identities and then the substitution $w = 2\theta$.

   (d) You should now have three different expressions for the indefinite integral $\int \sin\theta \cos\theta\ d\theta$. Are they really different? Are they all correct? Explain.

Substitution Integrals - Applications

69. Let $f(t)$ be the rate of flow, in cubic meters per hour, of a flooding river at time t in hours. Give an integral for the total flow of the river:
    (a) Over the 3-day period, $0 \leq t \leq 72$ (since t is measured in hours).
    (b) In terms of time T in **days** over the same 3-day period.

70. Oil is leaking out of a ruptured tanker at the rate of $r(t) = 50e^{-0.02t}$ thousand liters per minute.
    (a) At what rate, in liters per minute, is oil leaking out at $t = 0$? At $t = 60$?
    (b) How many liters leak out during the first hour?

71. If we assume that wind resistance is proportional to velocity, then the downward velocity, v, of a body of mass m falling vertically is given by

$$v = \frac{mg}{k}(1 - e^{-kt/m})$$

    where g is the acceleration due to gravity and k is a constant. Find the height of the body, h, above the surface of the earth as a function of time. Assume the body starts at height h_0.

72. The rate at which water is flowing into a tank is $r(t)$ gallons/minute, with t in minutes.
    (a) Write an expression approximating the amount of water entering the tank during the interval from time t to time $t + \Delta t$, where Δt is small.
    (b) Write a Riemann sum approximating the total amount of water entering the tank between $t = 0$ and $t = 5$. Then write an exact expression for this amount.
    (c) By how much has the amount of water in the tank changed between $t = 0$ and $t = 5$ if $r(t) = 20e^{0.02t}$?
    (d) If $r(t)$ is as in part (c), and if the tank contains 3000 gallons initially, find a formula for $Q(t)$, the amount of water in the tank at time t.

73. After a spill of radioactive iodine, measurements at $t = 0$ showed the ambient radiation levels at the site of the spill to be four times the maximum acceptable limit. The level of radiation from an iodine source decreases according to the formula

$$R(t) = R_0 e^{-0.004t}$$

    where R is the radiation level (in millirems/ hour) at time t in hours and R_0 is the initial radiation level (at t =0).
    (a) How long will it take for the site to reach an acceptable level of radiation?
    (b) Engineers look up the safe limit of radiation and find it to be 0.6 millirems/hour. How much total radiation (in millirems) will have been emitted by the time found in part (a)?

74. David is learning about catalysts in his Chemistry course. He has read the definition:

    > Catalyst: A substance that helps a reaction to go faster without being used up in the reaction.

    In today's Chemistry lab exercise, he has to add a catalyst to a chemical mixture that produces carbon dioxide. When there is no catalyst, the carbon dioxide is produced at a rate of 8.37×10^{-9} moles per second. When C moles of the catalyst are present, the carbon dioxide is produced at a rate of $(6.15 \times 10^{-8})C + 8.37 \times 10^{-9}$ moles per second.

    The reaction begins at exactly 10:00 a.m. One minute later, at 10:01 sharp, David starts to add the catalyst at a constant rate of 0.5 moles per second.

    How much carbon dioxide is produced between 10:00 (sharp) and 10:05?