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Euler–Lehmer constants and a conjecture of Erdös

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ABSTRACT

The Euler–Lehmer constants $\gamma(a,q)$ are defined as the limits

$$\lim_{x\to\infty}\left(\sum_{\substack{n\leqslant x\\ n\equiv a \pmod q}}\frac{1}{n}-\frac{\log x}{q}\right).$$

We show that at most one number in the infinite list

$$\gamma(a,q), \quad 1\leqslant a<q,\ q\geqslant 2,$$

is an algebraic number. The methods used to prove this theorem can also be applied to study the following question of Erdös. If $f:\mathbb{Z}/q\mathbb{Z}\to\mathbb{Q}$ is such that $f(a)=\pm 1$ and $f(q)=0$, then Erdös conjectured that

$$\sum_{n=1}^{\infty}\frac{f(n)}{n}\neq 0.$$

If $q\equiv 3 \pmod 4$, we show that the Erdös conjecture is true.

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1. Introduction

Euler's constant γ, is defined as the limit:

$$\gamma := \lim_{x\to\infty}\left(\sum_{n\leqslant x}\frac{1}{n} - \log x\right) = 0.577215\ldots.$$

It is unknown at present, whether γ is transcendental or even irrational. There are numerous infinite series expressions for γ in the literature. From the plethora of such results, we give two examples:

$$\gamma = \sum_{n=2}^{\infty}(-1)^n\frac{\zeta(n)}{n},$$

where $\zeta(s)$ denotes the Riemann zeta function [5], and

$$\gamma = \sum_{n=1}^{\infty}(-1)^n\frac{[\log n]}{n},$$

where $[x]$ denotes the greatest integer less than or equal to x [17]. In 1975, Lehmer [8] defined generalized Euler constants as follows. Fix a natural number $q \geqslant 1$. For each a satisfying $0 \leqslant a < q$, the limit

$$\lim_{x\to\infty}\left(\sum_{\substack{n\leqslant x,\\ n\equiv a (\mathrm{mod}\ q)}}\frac{1}{n} - \frac{\log x}{q}\right),$$

exists and is denoted as $\gamma(a,q)$ by Lehmer. These constants satisfy several properties and we record some of them here. These are easily verified. However, if the reader wishes, the reader may consult [8] for details and proofs. We have

$$\gamma(0,q) = \frac{\gamma - \log q}{q}; \qquad \sum_{a=0}^{q-1}\gamma(a,q) = \gamma.$$

If $\gcd(a,q) = d$, then

$$q\gamma(a,q) = \frac{q}{d}\gamma(a/d, q/d) - \log d. \tag{1}$$

It follows easily that $\gamma(2,4) = \gamma/4$. The transcendence of each $\gamma(a,q)$ is unknown at present. In [10], we showed that for each $q > 1$, the finite list of $\varphi(q)+1$ numbers:

$$\gamma,\ \gamma(a,q), \quad 1 \leqslant a < q, \qquad \gcd(a,q) = 1 \tag{2}$$

contains at most one algebraic number. In this paper, we prove:

Theorem 1. *At most one number in the infinite list of numbers*

$$\gamma(a,q), \quad 1 \leqslant a < q,\ q \geqslant 2,$$

is an algebraic number. Further, if γ is algebraic, then only the number $\gamma(2,4) = \gamma/4$ from the above list is algebraic.

The method of proof has an interesting consequence which is of independent interest. In [10], the focus of interest was the digamma function $\psi(x)$ which is the logarithmic derivative of the classical gamma function $\Gamma(x)$ of Euler. Thus, an immediate consequence of the theorem is:

Corollary 2. *As x ranges over all rational numbers with $0 < x \leqslant 1$, at least one of $\Gamma(x)$, $\Gamma'(x)$ is transcendental, with at most one possible exceptional x.*

Presumably, the main assertion of the corollary is true for all rational x with $0 < x \leqslant 1$. We refer to [6] for more results in this direction.

From these results, a natural question arises. Are the elements of the above set of Theorem 1 all distinct numbers? One can show that if any two elements above are equal, then γ is a Baker period, that is, an element of the $\overline{\mathbb{Q}}$-vector space spanned by 1 and logarithms of algebraic numbers. This is probably not the case though we have no proof of this at present. Indeed, Kontsevich and Zagier [7] have conjectured that γ is not even a period, much less a Baker period. In this context, we are able to show the following.

Theorem 3. *Let $q_1, q_2, \ldots$ be a sequence of mutually coprime numbers. The list of numbers consisting of γ and*

$$\gamma(a, q_i), \quad 1 \leqslant a \leqslant q_i, \qquad \gcd(a, q_i) = 1, \quad i \geqslant 1,$$

contains at most one pair of repetitions.

If we normalize our Euler constants by setting

$$\gamma^*(a, q) = q\gamma(a, q),$$

then we can show:

Theorem 4. *All of the numbers in the listing*

$$\gamma,\ \gamma^*(a, q), \quad 1 \leqslant a < q,\ q \geqslant 2, \quad (a, q) \neq (2, 4)$$

are distinct.

The method used to prove these theorems was nascent in two of our earlier papers [11] and [12]. In this paper, we bring it to the foreground and show that it has other applications. Most notably, we will apply it to prove the following theorems which have origins in a question of Chowla [3] and the work of Baker, Birch and Wirsing [2]. Given a function $f : \mathbb{Z}/q\mathbb{Z} \to \mathbb{C}$, Chowla introduced the Dirichlet series

$$L(s, f) = \sum_{n=1}^{\infty} \frac{f(n)}{n^s}.$$

One can show that this series admits an analytic continuation to the entire complex plane (see [10]) with a simple pole at $s = 1$ with residue

$$\frac{1}{q} \sum_{a=1}^{q} f(a).$$

Thus, $L(s, f)$ extends to an entire function if and only if the above sum is zero.

Theorem 5. *Let $f : \mathbb{Z}/q\mathbb{Z} \to \overline{\mathbb{Q}}$ be an algebraic-valued function not identically zero and ζ a primitive q-th root of unity. Suppose further that*

$$\sum_{a=1}^{q} f(a) = 0. \tag{3}$$

If

$$\frac{f(q)}{2q} + \frac{1}{q}\sum_{b=1}^{q-1} \frac{f(b)}{1-\zeta^b} \neq 0,$$

then, the number

$$\sum_{n=1}^{\infty} \frac{f(n)}{n},$$

is transcendental.

As a corollary, we deduce

Corollary 6. *If*

$$\sum_{b=1}^{q-1} f(b) \cot \frac{\pi b}{q} \neq 0,$$

then $L(1, f) \neq 0$.

Erdös conjectured (see [9]) that if $f : \mathbb{Z}/q\mathbb{Z} \to \mathbb{Q}$ with $f(a) = \pm 1$ and $f(q) = 0$, then

$$\sum_{n=1}^{\infty} \frac{f(n)}{n} \neq 0.$$

In 1973, Baker, Birch and Wirsing [2], using Baker's theory of linear forms in logarithms, proved a theorem which settles a conjecture of Chowla (see Lemma 12 below). We can apply their result to see that the conjecture of Erdös holds if q is a prime number. Their result is, however, not general enough to deal with the case q is composite. In 1982, Okada [13] showed that the conjecture of Erdös is true if $2\varphi(q) + 1 > q$. Thus, if q is a prime power or a product of two distinct primes, the conjecture is true. Subsequently, Saradha [14] extended this work. Tijdeman [16] showed that the conjecture is true if f is periodic and completely multiplicative (see Theorems 9 and 10 of [16]). The conjecture is also true if f is periodic and multiplicative with $|f(p^k)| < p - 1$ for every divisor p of q and every positive integer k (see Corollary 2 of [15]). In this paper, one of our goals is to prove the conjecture of Erdös under the assumption that $q \equiv 3 \pmod 4$.

Using these results, we can establish the following.

Theorem 7. *If $q \equiv 3 \pmod 4$, then the Erdös conjecture is true.*

In [10], we proved this theorem with the additional condition that f is an odd-valued odd function. Let us note that if q is even and

$$\sum_{n=1}^{\infty} \frac{f(n)}{n} = 0,$$

then (3) holds. Looking at this equation mod 2 gives a contradiction when q is even. Thus, the only case of the Erdös conjecture that is open is when $q \equiv 1 \pmod 4$.

2. Preliminary lemmas

We record in this section several results essentially established in [10] that will be needed in the proofs.

Lemma 8. *For $1 \leqslant a < q$, we have*

$$q\gamma(a,q) = \gamma - \sum_{b=1}^{q-1} e^{-2\pi iba/q} \log\bigl(1 - e^{2\pi ib/q}\bigr). \tag{4}$$

Proof. Let $\psi(x)$ denote the logarithmic derivative of the gamma function. By Theorem 7 of [8], we have

$$\gamma(a,q) = -\frac{1}{q}\bigl(\psi(a/q) + \log q\bigr).$$

By Lemma 21 of [10],

$$-\psi(a/q) - \gamma = \log q - \sum_{b=1}^{q-1} e^{-2\pi iba/q} \log\bigl(1 - e^{2\pi ib/q}\bigr). \tag{5}$$

Putting these formulas together gives us the stated result. (We take this opportunity to point out a typo on p. 312 of [10]. In the formula at the bottom of the page, the summation is from $b = 1$ to $b = q - 1$.) $\square$

We also record for future use the celebrated formula of Gauss [4] discovered by him in 1813: for $1 \leqslant a < q$,

$$\psi(a/q) + \gamma = -\log 2q - \frac{\pi}{2} \cot \frac{\pi a}{q} + 2 \sum_{0<j\leqslant q/2} \left(\cos \frac{2\pi aj}{q}\right) \log \sin \frac{\pi j}{q}. \tag{6}$$

In fact, this formula is easily obtained from (5) by equating the left hand side to the real part of the right hand side. Since the left hand side of (5) is real, we deduce that (5) and (6) are equivalent. Another simplified proof can be found in the paper of Lehmer [8]. (We alert the reader to a misprint in [8]. The term $\log(k/2)$ in the displayed formula after (20) should be $\log 2k$.) Thus, for $1 \leqslant a < q$, we have

$$q\gamma(a,q) = \gamma + \log 2 + \frac{\pi}{2} \cot \frac{\pi a}{q} - 2 \sum_{0<j\leqslant q/2} \left(\cos \frac{2\pi aj}{q}\right) \log \sin \frac{\pi j}{q}. \tag{7}$$

A pivotal role is played by the fundamental theorem of Baker concerning linear forms in logarithms. We record this as:

Lemma 9. *If* $\alpha_1, \dots, \alpha_n \in \overline{\mathbb{Q}} \setminus \{0\}$ *and* $\beta_1, \dots, \beta_n \in \overline{\mathbb{Q}}$*, then*

$$\beta_1 \log \alpha_1 + \dots + \beta_n \log \alpha_n$$

is either zero or transcendental. The latter case arises if $\log \alpha_1, \dots, \log \alpha_n$ *are linearly independent over* $\mathbb{Q}$ *and* $\beta_1, \dots, \beta_n$ *are not all zero.*

Proof. This is the content of Theorems 2.1 and 2.2 of [1]. Let us note that here and later, we interpret log as the principal value of the logarithm with the argument lying in the interval $(-\pi, \pi]$. $\square$

An important consequence of Lemma 9 will now be isolated as a separate lemma, as it will be the essential tool in many of our results below.

Lemma 10. *Let* $\alpha_1, \dots, \alpha_n$ *be positive algebraic numbers. If* $c_0, c_1, \dots, c_n$ *are algebraic numbers with* $c_0 \neq 0$*, then*

$$c_0 \pi + \sum_{j=1}^{n} c_j \log \alpha_j$$

is a transcendental number and hence non-zero.

Proof. We first choose a maximal set T of linearly independent numbers from the set

$$\log \alpha_j, \quad 1 \leqslant j \leqslant n.$$

Thus, for some set S,

$$T = \{\log \alpha_j \colon\ j \in S\}.$$

We first multiply our sum by i and rewrite it as

$$\Lambda := c_0 \pi i + \sum_{j \in S} d_j \log \alpha_j,$$

with d_j algebraic numbers. If the sum in question is zero, then by Baker's theorem we have that

$$\log(-1), \quad \log \alpha_j, \quad j \in S,$$

are linearly dependent over $\mathbb{Q}$. Thus, there are integers b_0, b_j with $j \in S$, not all zero, such that

$$b_0 \pi i = \sum_{j \in S} b_j \log \alpha_j.$$

But the right hand side is real. The left hand side is purely imaginary, unless $b_0 = 0$, in which case, we have

$$\sum_{j \in S} b_j \log \alpha_j = 0.$$

By the linear independence of the elements of T, we deduce $b_j = 0$ for all $j \in S$. Thus, $\Lambda \neq 0$ and by Baker's theorem, it is transcendental. This completes the proof. □

Our earlier work was partially motivated by a question of Chowla [3] and we record below some of the results that we will need later to prove our theorem. Let $f : \mathbb{Z}/q\mathbb{Z} \to \mathbb{C}$ be a complex valued function. One can show that

$$\sum_{n=1}^{\infty} \frac{f(n)}{n}$$

converges if and only if $\sum_{a=1}^{q} f(a) = 0$. The value of the series can be written in terms of generalized Euler constants.

Lemma 11. *If f is as above and*

$$\sum_{a=1}^{q} f(a) = 0,$$

then

$$\sum_{n=1}^{\infty} \frac{f(n)}{n} = \sum_{a=1}^{q} f(a)\gamma(a,q).$$

Proof. This is contained in [10] and can be derived from Theorem 16 of that paper, as indicated in Section 7 there. □

In the special case that f is rational-valued and q is a prime, we have by [2], the following result.

Lemma 12 *(Baker, Birch and Wirsing). If f is rational-valued on the residue classes* mod q *and not identically zero, then*

$$\sum_{n=1}^{\infty} \frac{f(n)}{n} \neq 0,$$

provided that $f(a) = 0$ whenever $1 < \gcd(a,q) < q$.

For the benefit of the reader, we recall the notion of S-units in an algebraic number field. If K is an algebraic number field and S is a finite set of rational primes, an algebraic integer $\alpha \in K$ is said to be an S-unit if every prime ideal occurring in the prime ideal decomposition of (α) lies above some prime of S.

3. Proof of Theorem 1

Suppose that the list contains at least two algebraic numbers. Assume first that one of these is $\gamma(2,4) = \gamma/4$ and the other one is $\gamma(a,q)$ for some q. If $\gcd(a,q) = 1$, then by (2) (see also Theorem 8 of [10]), we derive a contradiction. If $\gcd(a,q) \neq 1$, then writing $\gcd(a,q) = d$, a simple calculation shows that (see formula (1))

$$q\gamma(a,q) = q_1\gamma(a_1,q_1) - \log d,$$

where $da_1 = a$ and $dq_1 = q$. Using (7), we see that

$$q\gamma(a,q) - 4\gamma(2,4) = \log 2 - \log d + \frac{\pi}{2}\cot\frac{\pi a_1}{q_1} - 2\sum_{1\leqslant j\leqslant q_1/2}\cos\frac{2\pi a_1 j}{q_1}\log\sin\frac{\pi j}{q_1}.$$

Since the left hand side is algebraic, we conclude that the right side is algebraic. Noting that $\cos\pi x$ and $\sin\pi x$ are algebraic for rational values of x, we see by Lemma 10 that $\cot\pi a_1/q_1 = 0$. This means that $a_1 = 1$ and $q_1 = 2$ since $\gcd(a_1, q_1) = 1$. Thus,

$$q\gamma(a,q) - 4\gamma(2,4) = \log 2 - \log d.$$

This can be algebraic only if $d = 2$. Therefore, $a = 2$ and $q = 4$, contrary to our hypothesis. Hence, we may suppose that there are two algebraic numbers in the listing of the form $\gamma(A_1, Q_1)$ and $\gamma(A_2, Q_2)$ and these are different from $\gamma(2,4)$. Writing $d_1 = (A_1, Q_1)$ and $d_2 = (A_2, Q_2)$, we see that

$$Q_1\gamma(A_1, Q_1) - Q_2\gamma(A_2, Q_2) = q_1\gamma(a_1, q_1) - q_2\gamma(a_2, q_2) - \log d_1 + \log d_2, \tag{8}$$

is algebraic, where we have written $Q_1 = d_1 q_1$, $A_1 = d_1 a_1$ and $Q_2 = d_2 q_2$, $A_2 = d_2 a_2$. If

$$\cot\frac{\pi a_1}{q_1} \neq \cot\frac{\pi a_2}{q_2},$$

we again have a contradiction by Lemma 10. So, by the monotonicity of the cotangent function in $[0,1]$, we may suppose that $a_1/q_1 = a_2/q_2$. Since $(a_1, q_1) = (a_2, q_2) = 1$, this implies $a_1 = a_2$ and $q_1 = q_2$. Thus, $q_1\gamma(a_1, q_1) = q_2\gamma(a_2, q_2)$ and from (8), we deduce that

$$-\log d_1 + \log d_2$$

is algebraic. That is, $\log d_2/d_1$ is algebraic. By Lindemann's theorem, this means that $d_1 = d_2$. Hence, $A_1 = A_2$ and $Q_1 = Q_2$. This completes the proof.

4. Proof of Corollary 2

As remarked earlier, the method of proof of Theorem 1 allows us to deduce Corollary 2 asserting that apart from one possible rational number $0 < x \leqslant 1$, we have that at least one of $\Gamma(x)$ or $\Gamma'(x)$ is transcendental.

Suppose that there are two rationals x_1 and x_2 with $x_1 \neq x_2$, $0 < x_i \leqslant 1$, $i = 1, 2$ such that both $\Gamma(x_i)$, $\Gamma'(x_i)$ are algebraic for $i = 1, 2$. Let $x_i = a_i/q_i$, $i = 1, 2$. Since $\psi(x) = \Gamma'(x)/\Gamma(x)$, we get

$$\psi(a_i/q_i) \quad \text{is algebraic for } i = 1, 2. \tag{9}$$

Witout loss of generality we may assume that $\gcd(a_i, q_i) = 1$ as we deal with values of Γ, Γ' and ψ at a_i/q_i. We shall use the following result from [10] that at most one of the numbers

$$\gamma,\ \psi(a/q), \quad (a,q) = 1, \quad 1 \leqslant a \leqslant q,$$

is algebraic. Suppose $a_1/q_1 = 1$. Then $\psi(1) = -\gamma$ and $a_2/q_2 \neq 1$. Hence we use the above result with $q = q_2$, $a = a_2$ to get a contradiction to our assumption (9). Suppose now both a_1/q_1 and a_2/q_2 are unequal to 1. Since

$$\psi(a/q) + \log q = -q\gamma(a,q)$$

we find that

$$\log q_1 + q_1\gamma(a_1, q_1) \quad \text{and} \quad \log q_2 + q_2\gamma(a_2, q_2)$$

are both algebraic. Now we apply Lemma 10 to the difference of these two numbers and conclude as in the proof of Theorem 1 that

$$a_1 = a_2 \quad \text{and} \quad q_1 = q_2,$$

again a contradiction. This completes the proof.

5. Proof of Theorem 4

As the proof proceeds along the same lines as the proof of Theorem 1, we will be brief. Using the notation of the earlier proof, suppose that $\gamma^*(A_1, Q_1) = \gamma^*(A_2, Q_2)$. Then,

$$q_1\gamma(a_1, q_1) - q_2\gamma(a_2, q_2) - \log d_1 + \log d_2 = 0.$$

By the formula (7) of Gauss and Lemma 10, we deduce that

$$\cot\frac{\pi a_1}{q_1} = \cot\frac{\pi a_2}{q_2}.$$

As before, we conclude that $a_1 = a_2$ and $q_1 = q_2$. Consequently, $d_1 = d_2$. This means that $(A_1, Q_1) = (A_2, Q_2)$.

6. Proof of Theorem 3

We first write each $q_i\gamma(a_i, q_i) - \gamma$ as a linear form in logarithms of algebraic numbers. Clearly, by (4), we may write this as

$$q_j\gamma(a_j, q_j) - \gamma = \sum_{h\in H}\beta_{jh}\log\bigl(1 - \zeta_{q_j}^h\bigr),$$

with H a set of positive integers such that

$$\bigl(1 - \zeta_{q_j}^h\bigr), \quad h \in H,$$

is a multiplicatively independent set and the β_{jh} are algebraic numbers. Now suppose the theorem is false and that $\gamma(a_1, q_1) = \gamma(a_2, q_2)$ and $\gamma(a_3, q_3) = \gamma(a_4, q_4)$. We may suppose that $q_1 \neq q_2$, for otherwise if $q_1 = q_2 = q$ (say), then $\gamma(a_1, q) = \gamma(a_2, q)$ implying $\gamma^*(a_1, q) = \gamma^*(a_2, q)$. By Theorem 4, this is not possible. (We could have also deduced this as follows. Choosing $f(a_1) = 1$ and $f(a_2) = -1$, with $f(a) = 0$ for the remaining residue classes mod q gives us a contradiction to the non-vanishing theorem of Lemma 12.) Similarly, we may suppose that $q_3 \neq q_4$. From the first equation $\gamma(a_1, q_1) = \gamma(a_2, q_2)$, we deduce that $(q_2 - q_1)\gamma$ is a Baker period. From the second equation $\gamma(a_3, q_3) = \gamma(a_4, q_4)$ we deduce that $(q_4 - q_3)\gamma$ is a Baker period. By eliminating γ we derive a vanishing linear form in logarithms. By Baker's theorem, these logarithms must be linearly dependent over $\mathbb{Q}$. First suppose that $\{q_1, q_2\}$ is disjoint from $\{q_3, q_4\}$. Exponentiating the linear form, we deduce that a product of S_1-units in $\mathbb{Q}(\zeta_{q_1}, \zeta_{q_2})$ with $S_1 = \{q_1, q_2\}$ is equal to a product of S_2-units in $\mathbb{Q}(\zeta_{q_3}, \zeta_{q_4})$ with $S_2 = \{q_3, q_4\}$. Since q_1, q_2, q_3, q_4 are mutually coprime by assumption, these fields are disjoint (that is, their intersection is $\mathbb{Q}$), and hence each of the products must be a rational number. On one hand, this rational number can only be divisible by prime divisors of q_1q_2. On the other hand, it can only be divisible by prime divisors of q_3q_4. Since $\gcd(q_1q_2, q_3q_4) = 1$, we deduce that the product must be ± 1.

Now examine each of these products. By similar reasoning, we deduce that this product leads to an equality of two products, one being in $\mathbb{Q}(\zeta_{q_1})$ and the other being in $\mathbb{Q}(\zeta_{q_2})$. Again, since these two fields are disjoint, we deduce that each of the products must be ± 1. Finally, looking at the products and noting that each is a product of a multiplicatively independent set of numbers, we derive a contradiction. Now suppose that $\{q_1, q_2\}$ is not disjoint from $\{q_3, q_4\}$. Without loss of generality, we may assume that $q_2 = q_3$. Arguing as before, we deduce that a non-trivial multiplicative relation exists between certain S_3-units in $\mathbb{Q}(\zeta_{q_1}, \zeta_{q_4})$, with $S_3 = \{q_1, q_4\}$ and S-units in $\mathbb{Q}(\zeta_{q_2})$ with $S = \{q_2\}$. Since the field $\mathbb{Q}(\zeta_{q_1}, \zeta_{q_4})$ is disjoint from $\mathbb{Q}(\zeta_{q_2})$ (simply by ramification considerations), we deduce that the product must be a rational number, divisible on one hand by primes dividing q_1, q_4 and on the other hand, by primes dividing q_2. This forces the rational number to be ± 1. Again, by considering the product containing the S_3-units, and arguing as before, we deduce a contradiction. This completes the proof.

7. Proof of Theorem 5

By Theorem 19 of [10], we have

$$L(1, f) = \sum_{n=1}^{\infty} \frac{f(n)}{n} = -\sum_{a=1}^{q-1} \hat{f}(a) \log\bigl(1 - \zeta^a\bigr),$$

where $\zeta = e^{2\pi i/q}$. Changing a to $-a$ and writing

$$1 - \zeta^{-a} = \zeta^{-a/2}\bigl(\zeta^{a/2} - \zeta^{-a/2}\bigr),$$

we get

$$\log(1 - \zeta^{-a}) = \left(\frac{1}{2} - \frac{a}{q}\right)\pi i + \log\left(2 \sin \frac{\pi a}{q}\right).$$

Inserting this into the above expression shows that

$$L(1, f) = -\pi i\left(\sum_{a=1}^{q-1} \hat{f}(-a)\left(\frac{1}{2} - \frac{a}{q}\right)\right) - \sum_{a=1}^{q-1} \hat{f}(-a) \log\left(2 \sin \frac{\pi a}{q}\right).$$

If the coefficient of the term involving πi is non-zero, by Lemma 10, we are done. Hence, we are led to analyze the coefficient of πi. It is equal to

$$\frac{1}{q}\sum_{a=1}^{q-1}\left(\frac{1}{2} - \frac{a}{q}\right)\sum_{b=1}^{q} f(b)\zeta^{ba} = \frac{1}{q}\sum_{b=1}^{q} f(b)\sum_{a=1}^{q-1}\left(\frac{1}{2} - \frac{a}{q}\right)\zeta^{ba}. \tag{10}$$

When $b = q$, the inner sum is easily seen to be zero. Thus, the sum over b ranges from 1 to $q - 1$. Since

$$\sum_{a=1}^{q-1} \zeta^{ba} = -1,$$

we may re-write our coefficient as

$$\frac{1}{q}\sum_{b=1}^{q-1} f(b)\left(-\frac{1}{2} - \frac{1}{q}\sum_{a=0}^{q-1} a\zeta^{ba}\right).$$

The innermost sum can be evaluated as follows. Observe that

$$\sum_{a=0}^{q-1} x^a = \frac{x^q - 1}{x - 1}.$$

Differentiating both sides and putting $x = \zeta^b$ leads to

$$\sum_{a=0}^{q-1} a\zeta^{ba} = \frac{q}{\zeta^b - 1}.$$

Inserting this into (10), we get that the coefficient of πi is

$$\frac{f(q)}{2q} + \frac{1}{q}\sum_{b=1}^{q-1} \frac{f(b)}{1-\zeta^b}. \tag{11}$$

As noted earlier, when the expression in (11) does not vanish, $L(1, f)$ is transcendental. This completes the proof.

8. Proof of Corollary 6

To deduce the corollary, we observe that

$$\cot\frac{\pi a}{q} = i\frac{e^{\pi ia/q} + e^{-\pi ia/q}}{e^{\pi ia/q} - e^{-\pi ia/q}} = i + \frac{2i}{\zeta^a - 1},$$

by an easy computation. Thus,

$$\frac{1}{\zeta^a - 1} = \frac{1}{2i}\left(\cot\frac{\pi a}{q} - i\right).$$

Inserting this expression into our theorem yields the corollary.

9. Proof of Theorem 7

To prove Theorem 7, we use Theorem 5 and work modulo $2\mathcal{O}_K$ where $K = \mathbb{Q}(\zeta)$. Modulo 2, the second sum in Theorem 5 is congruent to

$$\sum_{b=1}^{q-1} \frac{1}{1-\zeta^b}.$$

We evaluate this sum. Notice that

$$1 + x + x^2 + \cdots + x^{q-1} = \prod_{b=1}^{q-1}\left(x - \zeta^b\right).$$

Logarithmically differentiating this expression and setting $x = 1$ gives

$$\sum_{b=1}^{q-1} \frac{1}{1-\zeta^b} = \frac{q-1}{2}.$$

If $q \equiv 3 \pmod 4$, then this sum is $1 \pmod{2\mathcal{O}_K}$. This proves the theorem.

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References

[1] A. Baker, Transcendental Number Theory, Cambridge University Press, 1975.
[2] A. Baker, B. Birch, E. Wirsing, On a problem of Chowla, J. Number Theory 5 (1973) 224–236.
[3] S. Chowla, The non-existence of non-trivial relations between the roots of a certain irreducible equation, J. Number Theory 2 (1970) 120–123.
[4] C.F. Gauss, Disquisitiones generales circa seriem infinitam etc., Comment. Gottingen 2 (1813) 86–101.
[5] X. Gourdon, P. Sebah, Constants and records of computation, http://numbers.computation.free.fr/Constants/Miscellaneous.Records.html.
[6] Kh. Hessami Pilehrood, T. Hessami Pilehrood, Infinite sums as linear combinations of polygamma functions, Acta Arith. 130 (2007) 231–254.
[7] M. Kontsevich, D. Zagier, Periods, in: Mathematics Unlimited, 2001 and Beyond, Springer, Berlin, 2001, pp. 771–808.
[8] D.H. Lehmer, Euler constants for arithmetical progressions, Acta Arith. 27 (1975) 125–142; see also selected papers of D.H. Lehmer, vol. 2, pp. 591–608.
[9] A.E. Livingston, The series $\sum_1^\infty f(n)/n$ for periodic f, Canad. Math. Bull. 8 (1965) 413–432.
[10] M. Ram Murty, N. Saradha, Transcendental values of the digamma function, J. Number Theory 125 (2007) 298–318.
[11] M. Ram Murty, N. Saradha, Special values of the polygamma function, Int. J. Number Theory 5 (2) (2009) 257–270.
[12] M. Ram Murty, N. Saradha, Transcendental values of the p-adic digamma function, Acta Arith. 133 (4) (2008) 349–362.
[13] T. Okada, On a certain infinite series for a periodic arithmetical function, Acta Arith. 40 (1982) 143–153.
[14] N. Saradha, Chowla's problem on the non-vanishing of certain infinite series and related questions, in: Number Theory and Discrete Geometry, in: Ramanujan Math. Soc. Lect. Notes, vol. 6, Ramanujan Math. Society, Mysore, 2008, pp. 171–179.
[15] N. Saradha, R. Tijdeman, On the transcendence of infinite sums of values of rational functions, J. London Math. Soc. 67 (3) (2003) 580–592.
[16] R. Tijdeman, Some applications of Diophantine approximation, in: M.A. Bennett, et al. (Eds.), Number Theory for the Millenium III, A.K. Peters, Natick, MA, 2002, pp. 261–284.
[17] G. Vacca, A new series for the Eulerian constant, Quart. J. Pure Appl. Math. 41 (1910) 363–368.