The q-Foulkes Conjecture

François Bergeron, UQAM

Herstmonceux, July 2016

Historical Background
and
Mathematical Background

PLETHYSM

DUDLEY ERNEST LITTLEWOOD
(1903 – 1979)

POLYNOMIAL CONCOMITANTS AND INVARIANT MATRICES

J. LONDON MATH. SOC., 1936

A wider problem here suggests itself. Any invariant matrix of an invariant matrix must be an invariant matrix of the original matrix, and thus expressible as the direct sum of irreducible invariant matrices.

Thus
$$[A^{[\lambda]}]^{[\mu]} = \Sigma\, k_{\lambda\mu\nu} A^{[\nu]}.$$

Hence we may define a new type of multiplication of S-functions
$$\{\lambda\} \otimes \{\mu\} = \Sigma\, k_{\lambda\mu\nu} \{\nu\}.$$

$$\{\lambda\} \otimes \{\mu\} \qquad\qquad s_\mu[s_\lambda]$$

PLETHYSM

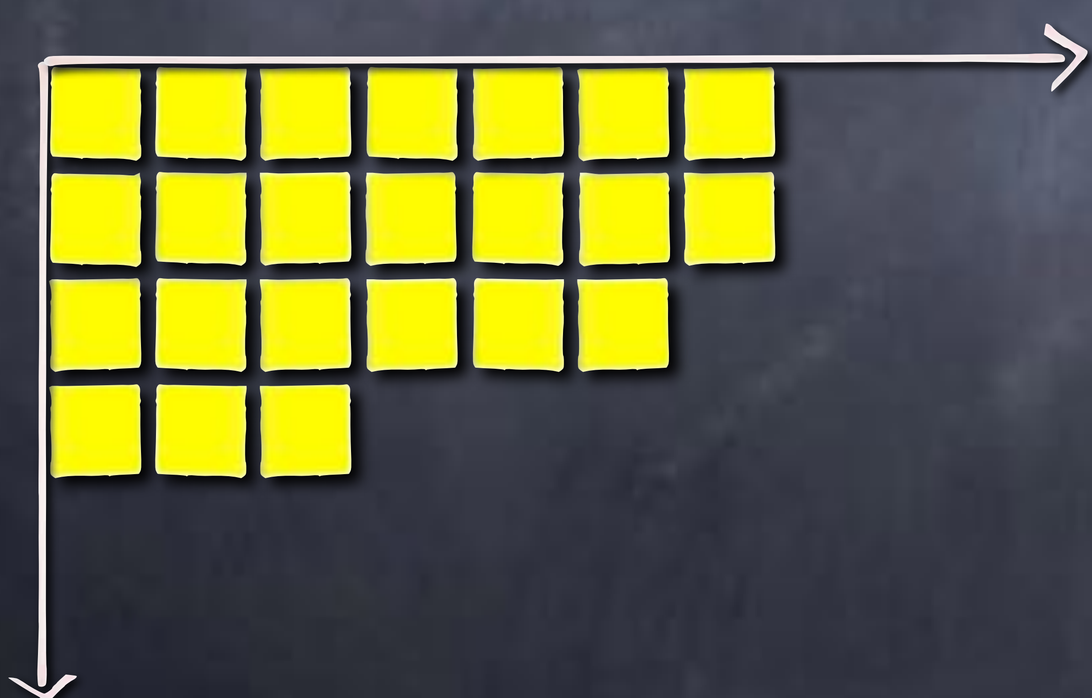
$\{\lambda\}\otimes\{\mu\}$

$s_\mu[s_\lambda]$

PARTITIONS

$\mu = \mu_1\mu_2\cdots\mu_k$

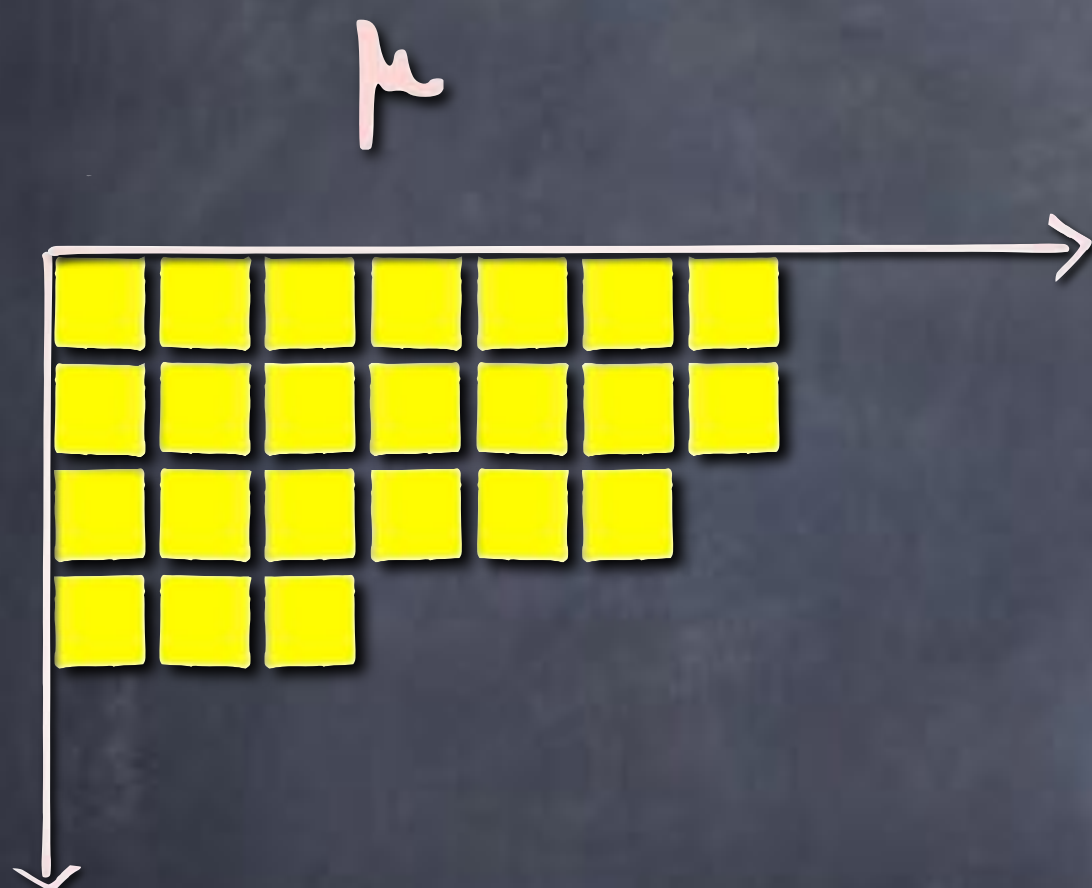
$\ell(\mu) = k$

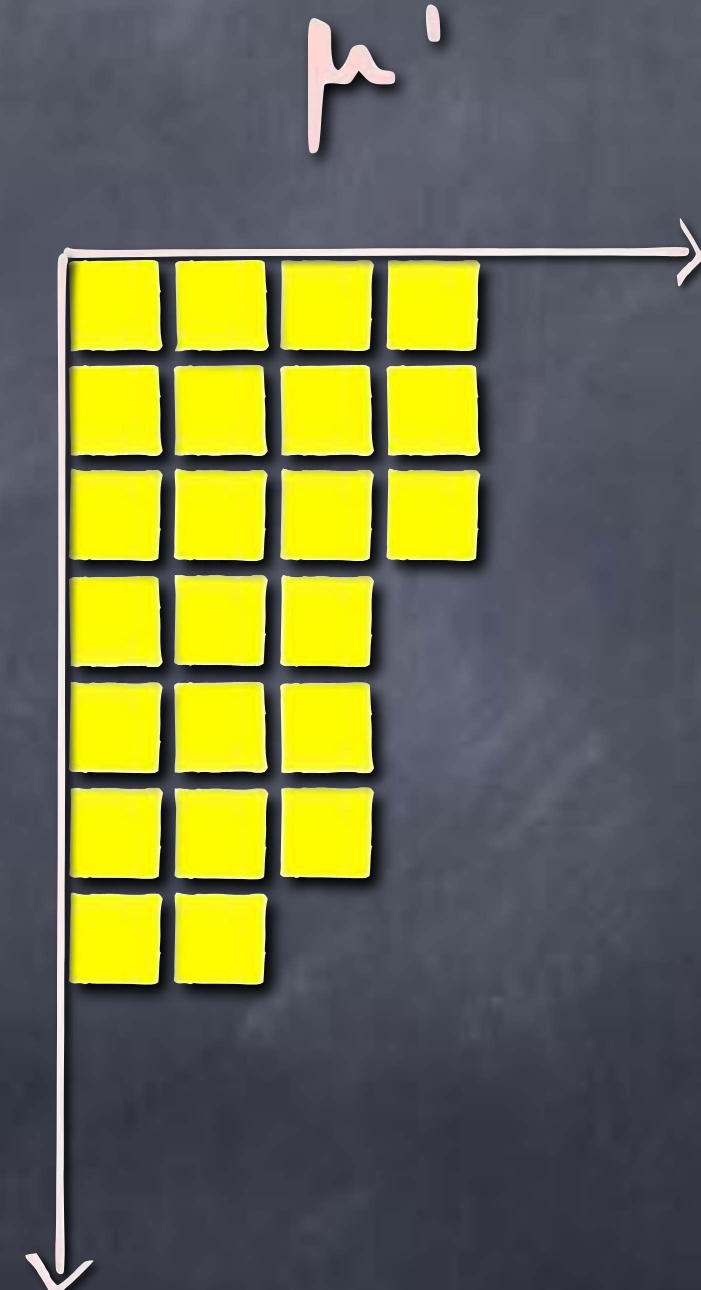


$\lambda = \lambda_1\lambda_2\cdots\lambda_j$

$\ell(\lambda) = j$

Partitions





RECALL: (SEE MACDONALD'S BOOK)

Λ RING OF SYMMETRIC FUNCTIONS

$$X = x_1, x_2, x_3, \cdots$$

$$\Lambda = \bigoplus_{d \geq 0} \Lambda_d$$

BASIS OF Λ_d INDEXED BY PARTITIONS OF d

μ PARTITION OF d $\qquad \mu = \mu_1 \mu_2 \cdots \mu_k \vdash d$

$$d = \mu_1 + \mu_2 + \cdots + \mu_\ell \qquad \mu_i \geq \mu_{i+1}$$

- MONOMIAL $m_\mu(x) = \sum_{\substack{j_i \\ \text{DISTINCTS}}} x_{j_1}^{\mu_1} x_{j_2}^{\mu_2} \cdots x_{j_k}^{\mu_k}$

- ELEMENTARY $e_n(x) := m_{\underbrace{11\cdots1}_{n}}(x)$

- COMPLETE HOMOGENOUS $h_n(x) = \sum_{|\mu|=n} m_\mu(x)$

- SCHUR $s_\mu(x) = \det\left(h_{\mu_i+j-i}(x)\right)_{1\le i,j\le k}$

- POWER SUM $p_d(x) := x_1^d + x_2^d + x_3^d + \cdots$

$$h_n = s_{(n)} \qquad f = \sum_{\lambda \vdash n} a_\lambda \prod_{i=1}^{k} p_{\lambda_i}$$

THE ω INVOLUTION

LINEAR MULTIPLICATIVE

SELF-ADJOINT OPERATOR

$$\omega(p_k) = (-1)^{k-1} p_k$$

$$\omega\, s_\mu = s_{\mu'}$$

POLYNOMIAL FUNCTORS

$$F : \mathcal{V}\mathrm{ECT}_{\mathbb{K}} \longrightarrow \mathcal{V}\mathrm{ECT}_{\mathbb{K}}$$

$$GL(\mathcal{V}) \xrightarrow{F(-)} GL(F(\mathcal{V}))$$

$S^\lambda(\mathcal{V})$ IRREDUCIBLE REPRESENTATION OF $GL(\mathcal{V})$

CHARACTER

$\mathcal{V} = \mathbb{K}\{v_1, v_2, \ldots, v_n\}$

$$\left.\begin{array}{l} T_X : \mathcal{V} \longrightarrow \mathcal{V} \\ \quad T_X(v_i) := x_i v_i \end{array}\right\} \text{ABSTRACT DIAGONAL MATRIX}$$

$X = x_1, x_2, \ldots, x_n$

$$\boxed{F(X) := \mathrm{TRACE}(F(T_X))}$$

THIS IS A SYMMETRIC FUNCTION OF THE x_i's.

IRREDUCIBLE CHARACTER

$$\boxed{S^\lambda(X) = \Delta_\lambda(X)} \longleftarrow \boxed{\text{SCHUR FUNCTIONS}}$$

SCHUR POSITIVITY

$$F = \bigoplus_{\lambda} a_\lambda S^\lambda \qquad a_\lambda \in \mathbb{N}$$

$$F(x) = \sum_{\lambda} a_\lambda s_\lambda(x)$$

ALSO GRADED VERSION

$$a_\lambda(q) \in \mathbb{N}[q]$$

INTEGER COEFFICIENT POLYNOMIAL

SCHUR-POSITIVITY IS RARE

AMONG
POSITIVE COEFFICIENT
HOMOGENEOUS DEGREE d
SYMMETRIC FUNCTIONS

$d = \quad 6$

PROPORTION OF SCHUR-POSITIVE $\quad 1/1027458432000$

FUNCTOR COMPOSITION
=
PLETHYSM OF CHARACTERS

$$\boxed{(F \circ G)(X) = F[G(X)]}$$

Rules of Plethysm

$$(f+g)[\quad] = f[\quad] + g[\quad]$$

$$(f\cdot g)[\quad] = f[\quad]\cdot g[\quad]$$

$$p_k[XY] = p_k[X]\,p_k[Y]$$

$$p_k[X+Y] = p_k[X] + p_k[Y]$$

$$p_k[x] = x^k \qquad p_k[cte] = cte$$

Plethysm

$$p_k[p_j] = p_{kj}$$

Hilbert scheme

Coinvariant spaces

DAHA

Chip firing

Sandpile model

Group harmonics

Parking functions

Bosons

Atomic states

Calogero-Sutherland

Quasisymmetric functions

Fermions

Torus knots

Diagonal invariants

Chemistry

Flag variety cohomology

Elliptic Hall-algebra

EXAMPLES

$$h_2[h_3] = s_6 + s_{42} \qquad s_\lambda = s_\lambda(X)$$

$$h_2[h_4] = s_8 + s_{44} + s_{62}$$

$$h_2[h_5] = s_{10} + s_{82} + s_{64}$$

$$h_3[h_2] = s_6 + s_{42} + s_{222}$$

$$h_4[h_2] = s_8 + s_{62} + s_{44} + s_{422} + s_{2222}$$

EXAMPLES

$$h_4[h_3] = s_{12} + s_{10,2} + s_{93}$$
$$+ s_{84} + s_{822} + s_{741} + s_{732}$$
$$+ s_{66} + s_{642} + s_{6222}$$
$$+ s_{5421} + s_{444}$$

$$\boxed{e_1^a[e_1^b] = e_1^{ab}}$$

Foulkes Conjecture

Herbert Owen Foulkes
(1907 – 1977)

CONCOMITANTS OF THE QUINTIC AND SEXTIC UP TO DEGREE FOUR IN THE COEFFICIENTS OF THE GROUND FORM

H. O. Foulkes*.

1. D. E. Littlewood (1) has shown that there is an exact correspondence between the S-functions appearing in the "new multiplication" $\{\lambda\}\otimes\{q\}$ and those concomitants, reducible or otherwise, of a ground form of type $\{\lambda\}$ which are of degree q in the coefficients of the ground form. He has also given several methods of computing such products and has obtained the number and types of concomitant for the cubic up to degree six in the coefficients, and for the quartic up to degree five in the coefficients†.

J. of London Math. Soc.
1950

A proof by S-functions of the general theorem underlying this assumption has not yet been obtained. The theorem is that for integers m, n, where $n > m$, the product $\{m\}\otimes\{n\}$ includes all terms of $\{n\}\otimes\{m\}$.

Two Formulations $a < b$

Symmetric Function Formulation

$h_b[h_a] - h_a[h_b]$ is Schur Positive

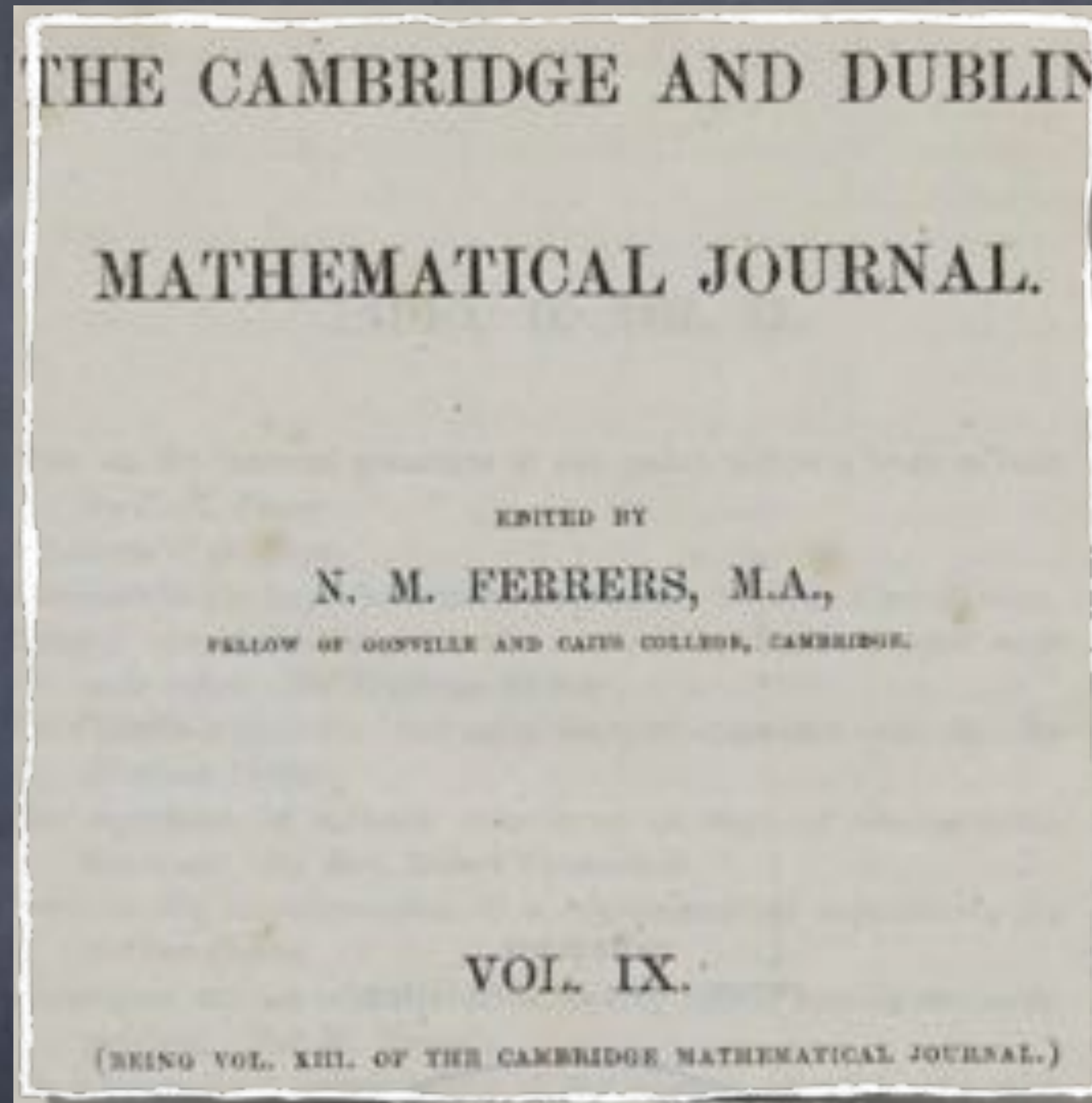
Functorial Formulation

$$S^a \circ S^b \overset{\exists}{\hookrightarrow} S^b \circ S^a$$

$$\twoheadleftarrow$$

$S^a(V)$ symmetric Power

CHARLES HERMITE
(1822–1901)



THE CAMBRIDGE AND DUBLIN

MATHEMATICAL JOURNAL.

EDITED BY

N. M. FERRERS, M.A.,

FELLOW OF GONVILLE AND CAIUS COLLEGE, CAMBRIDGE.

VOL. IX.

(BEING VOL. XIII. OF THE CAMBRIDGE MATHEMATICAL JOURNAL.)

$$h_b[h_a] = h_a[h_b] \qquad X = x_1, x_2$$

CHARLES HERMITE
(1822–1901)

SUR LA THEORIE DES FONCTIONS HOMOGENES À DEUX INDETERMINEES.

Par M. Hermite.

Mes premières recherches sur la théorie des formes à deux indéterminées, ont pour objet la démonstration de cette proposition arithmétique élémentaire, *que les formes à coefficients entiers et en nombre infini, qui ont les mêmes invariants, ne donnent qu'un nombre essentiellement limité de classes distinctes.*

Section I.—Loi de Réciprocité.

Elle est contenue dans le théoréme : *A tout covariant d'une forme de degré m, et qui par rapport aux coefficients de cette forme est du degré p, correspond un covariant du degré m par rapport aux coefficients, d'une forme du degré p.*

1854

$$h_b[h_a] = h_a[h_b] \qquad X = x_1, x_2$$

Roger Evans Howe

roc. Indian Acad. Sci. (Math. Sci.), Vol. 97, Nos 1–3, December 1987, pp. 85–109.
Printed in India.

(GL_n, GL_m)-duality and symmetric plethysm

ROGER HOWE*

The Institute for Advanced Studies, Hebrew University of Jerusalem, Givat Ram 91940, Jerusalem, Israel
Permanent Address: Department of Mathematics, Yale University, Box 2155 Yale Station, New Haven, CT 06520, USA

Abstract. In [7] the author has given an exposition of the theory of invariants of binary forms in terms of a particular version of Classical Invariant Theory. Reflection shows that many aspects of the development apply also to n-ary forms. The purpose of this paper is to make explicit this more general application. The plethysms $S^l(S^n(\mathbb{C}^n))$ are computed quite explicitly for $l = 2$, 3 and 4.

Keywords. (GL_n, GL_m)-duality; plethysm; symmetric plethysm; reciprocity laws; invariants.

$$\boxed{S^a \circ S^b \hookrightarrow S^b \circ S^a} \qquad \boxed{S^a \circ S^b \twoheadleftarrow S^b \circ S^a}$$

Natural candidate

Roger Evans Howe

(a) $$\alpha : \mathbb{S}(S^l(\mathbb{C}^n)) \to \sum_{f \geq 0} \mathbb{S}(\mathbb{C}^n \otimes \mathbb{C}^l)^{N^f}, \tag{25a}$$

which consists of maps between each pair of homogeneous components

(b) $$\alpha : S^p(S^l(\mathbb{C}^n)) \to S^l(S^p(\mathbb{C}^n)). \tag{25b}$$

When $n = 2$, it is not hard to see that the maps (25b) are all isomorphisms. This gives a very precise version of Hermite Reciprocity [7].

For $n > 2$, the maps (25b) cannot always be isomorphisms. In a conversation with the author, A. Garsia remarked that numerical evidence suggests that there should exist a GL_n-module embedding of $S^p(S^l(\mathbb{C}^n))$ into $S^l(S^p(\mathbb{C}^n))$ when $l \geq p$. This conjecture was also made in [4]. Thus perhaps it is reasonable to expect that the maps (25b) should be injective if $p \leq l$, and surjective if $l \leq p$.

$$\boxed{S^a \circ S^b \hookrightarrow S^b \circ S^a} \qquad \boxed{S^a \circ S^b \twoheadleftarrow S^b \circ S^a}$$

~~Natural candidate~~

EXAMPLES

$$h_3[h_2] - h_2[h_3] = \Delta_{222}$$

$$h_4[h_2] - h_2[h_4] = \Delta_{422} + \Delta_{2222}$$

$$h_5[h_2] - h_2[h_5] = \Delta_{622} + \Delta_{442} + \Delta_{4222} + \Delta_{22222}$$

$$h_4[h_3] - h_3[h_4] = \Delta_{732} + \Delta_{6222} + \Delta_{5421}$$

MORE THAN 3 VARIABLES

$$\Delta_\lambda(x_1, x_2, \ldots, x_n) = 0$$

WHEN $n < \ell(\lambda)$

EXAMPLES

$$
\begin{aligned}
h_5[h_3] - h_3[h_5] = {} & s_{10,3,2} + s_{942} + s_{9222} \\
& + s_{843} + s_{8421} + s_{8322} \\
& + s_{762} + s_{7521} + s_{7431} \\
& + s_{7422} + s_{72222} + s_{6522} \\
& + s_{6441} + s_{64221} + s_{55311} \\
& + s_{5442}
\end{aligned}
$$

SOME ADVANCES AND GENERALIZATIONS

Robert M. Thrall

Source: *American Journal of Mathematics*, Vol. 64, No. 1 (1942), pp. 371-388
Published by: The Johns Hopkins University Press
Stable URL: http://www.jstor.org/stable/2371691

ON SYMMETRIZED KRONECKER POWERS AND THE STRUCTURE OF THE FREE LIE RING.*

By R. M. Thrall.

1. Introduction. This paper is divided into three chapters. In Chapter I foundations are laid for a general theory of representations of "power type" and their relationship with rings. Kronecker powers, symmetrized Kronecker powers, and the *Lie Representation* are introduced as transformations induced in certain modules of the free non-commutative ring, the free commutative ring, and the free Lie ring, respectively, by a class of ring automorphisms.

In Chapter II the starting point (§ 3) is a general discussion of a problem mentioned by Littlewood:[1] the analysis into irreducible invariant matrices of an invariant matrix of an invariant matrix. This is followed (§ 4) by more specific considerations in the case of the symmetrized Kronecker r-th power of a given invariant matrix. In § 5 formulas are obtained for the anaylsis of the symmetrized Kronecker r-th power of the symmetrized Kronecker m-th power for $r \leqq 3$, all m, and for $m \leqq 2$, all r. The chapter is concluded (§ 6) with a table giving the analysis of the symmetrized Kronecker r-th powers of the irreducible representations of the full linear group

Foulkes Conjecture thrue for $a = 2$

manuscripta math. 80, 347 - 371 (1993)

manuscripta mathematica

Stable properties of plethysm : on two conjectures of Foulkes

Michel BRION

Two conjectures made by H.O. Foulkes in 1950 can be stated as follows.
1) Denote by V a finite-dimensional complex vector space, and by $S_m V$ its m-th symmetric power. Then the $GL(V)$-module $S_n(S_m V)$ contains the $GL(V)$-module $S_m(S_n V)$ for $n > m$.
2) For any (decreasing) partition $\lambda = (\lambda_1, \lambda_2, \lambda_3, \ldots)$, denote by $S_\lambda V$ the associated simple, polynomial $GL(V)$-module. Then the multiplicity of $S_{(\lambda_1+mp, \lambda_2, \lambda_3, \ldots)} V$ in the $GL(V)$-module $S_n(S_{m+p} V)$ is an increasing function of p. We show that Foulkes' first conjecture holds for n large enough with respect to m (Corollary 1.3). Moreover, we state and prove two broad generalizations of Foulkes' second conjecture. They hold in the framework of representations of connected reductive groups, and they lead e.g. to a general analog of Hermite's reciprocity law (Corollary 1 in 3.3).

MICHEL BRION

THEOREM

FOULKES CONJECTURE THRUE FOR $a \ll b$

FUNCTORIAL FORMULATION

$$S^a \circ S^b \twoheadleftarrow S^b \circ S^a$$

$S^a(V)$ SYMMETRIC POWER

HOWE NATURAL CANDIDATE

Available online at www.sciencedirect.com
SCIENCE DIRECT
Journal of Algebra 277 (2004) 579–614
JOURNAL OF Algebra
www.elsevier.com/locate/jalgebra

Generalized Foulkes' Conjecture and tableaux construction

Rebecca Vessenes[1]
California Institute of Technology, Department of Mathematics, M.C. 253-37, Pasadena, CA 91125, USA

Received 25 June 2003
Available online 9 April 2004
Communicated by Jan Saxl

Journal of Pure and Applied Algebra 130 (1998) 85–98
JOURNAL OF PURE AND APPLIED ALGEBRA

On Foulkes' conjecture

William F. Doran IV *
Department of Mathematics, California Institute of Technology, Pasadena, CA 91125, USA

Communicated by P.J. Freyd; received 11 October 1996

$a \leq c \leq b$ $\quad$ c divisor of ab

$$d := ab/c$$

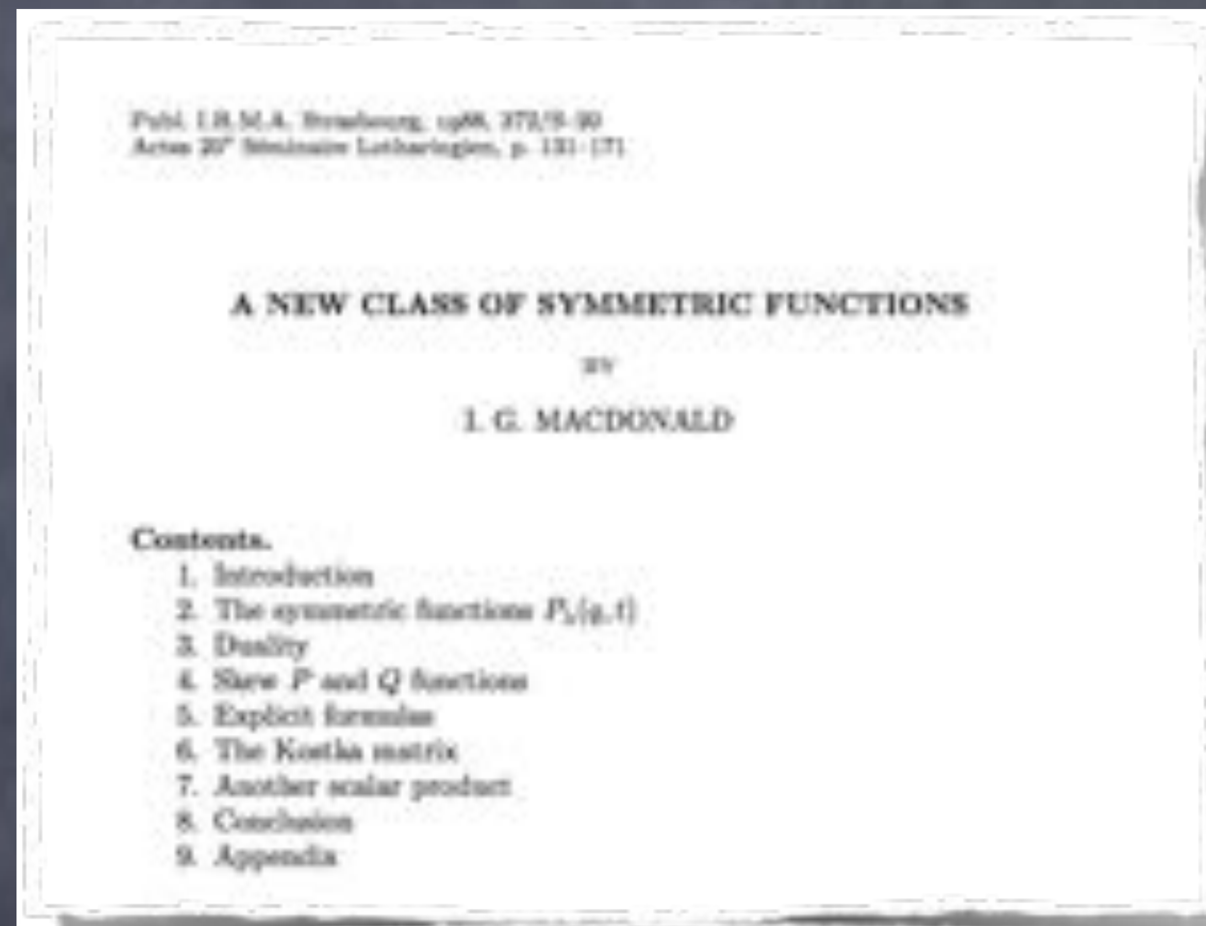
$$\boxed{h[h] - h_a[h_b] \text{ is Schur positive}}$$

$$\boxed{q\text{-ANALOG OF } \Delta_k}$$

Ian G. Macdonald

$$H_\mu(x;q,t)$$

Combinatorial Macdonald Polynomials



Publ. I.R.M.A. Strasbourg, 1988, 372/S-20
Actes 20e Séminaire Lotharingien, p. 131-171

A NEW CLASS OF SYMMETRIC FUNCTIONS

BY

I. G. MACDONALD

Contents.

1988

$$S_\mu(x;q) := q^{n(\mu')}\,\omega H_\mu(x;1/q,0)$$

Schur positive

Hall-Littlewood Polynomials

$$S_\mu(x;q) := q^{n(\mu')}\,\omega H_\mu(x;1/q,0)$$

ON CERTAIN SYMMETRIC FUNCTIONS

By D. E. LITTLEWOOD

[Received 10 December 1958. Revised 20 May 1960]

1. Introduction

THE derivation of the simple characters of symmetric groups from the easily obtained compound characters is usually performed by the use of alternants, by methods mainly due to Frobenius [(3) 67]. The existence of alternants to be used for this purpose seems to partake somewhat of the nature of a happy chance. Similar problems occur elsewhere for which

Proc. London Math. Soc. 1961

EXAMPLES

$$S_2 = \Delta_2 + q\,\Delta_{11}$$

$$S_{11} = \Delta_{11}$$

$$S_3 = \Delta_3 + (q+q^2)\,\Delta_{21} + q^3\Delta_{111}$$

$$S_{21} = \Delta_{21} + q\,\Delta_{111}$$

$$S_{111} = \Delta_{111}$$

$$S_4 = \Delta_4 + (q+q^2+q^3)\,\Delta_{31} + (q^2+q^4)\,\Delta_{22}$$
$$+ (q^3+q^4+q^5)\,\Delta_{211} + q^6\Delta_{1111}$$

$$\boxed{\begin{aligned} S_\mu(x;0) &= \Delta_\mu \\ S_\mu(x;1) &= e_{\mu'} \end{aligned}}$$

$$h_n = s_{(n)} \qquad H_n = S_{(n)}$$

$$\boxed{\begin{aligned} S_\mu(x;0) &= s_\mu \\ S_\mu(x;1) &= e_{\mu'} \end{aligned}} \qquad \boxed{\begin{aligned} H_n(x,0) &= h_n \\ H_n(x,1) &= e_1^n \end{aligned}}$$

$$H_2 = s_2 + q\, s_{11}$$

$$H_3 = s_3 + (q+q^2)\, s_{21} + q^3 s_{111}$$

$$H_n = S_{(n)} = h_n\left[\frac{X}{1-q}\right]\cdot\prod_{i=1}^{n}(1-q^i)$$

GRADED CHARACTER OF THE

- COINVARIANT RING OF $\mathfrak{S}_n$
- COHOMOLOGY RING OF THE FULL FLAG MANIFOLD
- MODULE OF $\mathfrak{S}_n$-HARMONIC POLYNOMIALS

THE q-FOULKES CONJECTURE

FRANÇOIS BERGERON, UQAM

Conjecture 1 $a < b$

$$\boxed{\frac{H_b[H_a] - H_a[H_b]}{1-q} \text{ is Schur Positive}}$$

$$\Big\downarrow\ q=0$$

$$\boxed{h_b[h_a] - h_a[h_b] \text{ is Schur Positive}}$$

$$\mathcal{J}_{a,b}(x;q) := \frac{H_b[H_a] - H_a[H_b]}{1-q}$$

$H_b[H_a] - H_a[H_b]$ is divisible by $1-q$

$e_1^b[e_1^a] - e_1^a[e_1^b] = 0$ at $q=1$

$$e_1^b[e_1^a] = e_1^{ab}$$

$$\mathcal{F}_{a,b}(x;q) := \frac{H_b[H_a] - H_a[H_b]}{1-q}$$

$$\begin{aligned}\mathcal{F}_{23}(x;q) = {} & q^2(q+1)^2 s_{33} + q(q^2+1)(q+1)^2 s_{321} \\ & + q^2(q+1)^2 s_{3111} + (q+1)(q^2+1)\boxed{s_{222}} \\ & + q(q+1)(q^2+1)(q^2+q+1)\, s_{2211} \\ & + q^2(q+1)(2q^2+q+1)\, s_{21111} \\ & + q^3(q+1)(q^2+1)\, s_{111111}\end{aligned}$$

$$\mathcal{F}_{a,b}(x;0) = h_b[h_a] - h_a[h_b]$$

$$\mathcal{F}_{23}(x;0) = s_{222}$$

Expanding as a Polynomial in q

$$\mathcal{J}_{a,b} = (h_b[h_a] - h_a[h_b])$$
$$+ q\, h_1 (h_{b-1}[h_a]\cdot h_{a-1} - h_{a-1}[h_b]\cdot h_{b-1}) + \ldots$$

Leads to expect that

$$(h_{b-1}[h_a]\cdot h_{a-1} - h_{a-1}[h_b]\cdot h_{b-1})$$

is Schur-Positive

Theorem

$$\left(h_b[h_a] - h_a[h_b]\right)$$

Schur-Positive

$$\Downarrow$$

$$\left(h_{b-1}[h_a]\cdot h_{a-1} - h_{a-1}[h_b]\cdot h_{b-1}\right)$$

Schur-Positive

Conjecture 2

$a \le c \le b$ $\quad c$ divisor of ab $\quad d := ab/c$

$$\boxed{\frac{H_c[H_d] - H_a[H_b]}{1-q} \text{ is Schur positive}}$$

$$\Big\downarrow\ q = 0$$

$$\boxed{h_c[h_d] - h_a[h_b] \text{ is Schur positive}}$$

SUPPORTING EVIDENCE

- COMPUTER ALGEBRA CALCULATIONS

- $q = 1$, THE (a, b, c, d)-VERSION OF FOULKES CONJECTURE HOLDS.

$$\mathcal{F}_{a,b}(x;q) := \frac{H_b[H_a] - H_a[H_b]}{1-q}$$

$$\mathcal{F}_{23}(x,1) = 4\,s_{33} + 8\,s_{321} + 4\,s_{3111} + 4\,s_{222} + 12\,s_{2211} + 8\,s_{21111} + 4\,s_{111111}$$

$$= 4\,e_2^3$$

EXAMPLES

$$\boxed{\mathcal{F}_{a,b}(x,1) \in \mathbb{N}[e_1, e_2, h_2]}$$

$$a < b$$

$$\mathcal{F}_{23}(x,1) = 4\, e_2^3$$

$$\mathcal{F}_{24}(x,1) = 8\, e_2^4 + 16\, e_2^3 h_2$$

$$\mathcal{F}_{34}(x,1) = 24\, e_1^4 e_2^3 h_2$$

$$\mathcal{F}_{25}(x,1) = 16\, e_2^5 + 40\, e_2^4 h_2 + 40\, e_2^3 h_2^2$$

$$\mathcal{F}_{45}(x,1) = 16\, e_1^{10} e_2^5 + 80\, e_1^{10} e_2^3 h_2^2$$

$$\mathcal{F}_{56}(x,1) = 120\, e_1^{18} e_2^5 h_2 + 200\, e_1^{18} e_2^3 h_2^3$$

THEOREM

WE HAVE $\boxed{\mathcal{F}_{a,b}(x,1) \in \mathbb{N}[e_1, e_2, h_2]}$ $a < b$

EXPLICITLY GIVEN BY THE FORMULA:

$$\mathcal{F}_{a,b}(x,1) = \frac{e_1^{(a-2)b}}{2}\Big(ab(b-a)\, e_1^{2(b-1)} e_2 + \binom{a}{2}(P^b - Q^b) - \binom{b}{2}(P^a - Q^a)\Big)$$

WHERE

$$P := h_2 + e_2 \qquad Q := h_2 - e_2$$

THEOREM

$a \le c \le b$ $\quad c$ DIVISOR OF ab $\quad d := ab/c$

WE HAVE

$$\lim_{q \to 1} \frac{H_c[H_d] - H_a[H_b]}{1-q} = \frac{1}{2}\Big(n(b-d)\, e_1^{n-2} e_2 + \binom{a}{2} e_1^{n-2b} P^b - \binom{c}{2} e_1^{n-2d} Q^d \Big)$$

WHERE

$$P := h_2 + e_2 \qquad Q := h_2 - e_2$$

CONJECTURE 3

FOR α, β, γ AND δ PARTITIONS SUCH THAT

$$e_{\alpha'}[e_{\beta'}] = e_{\gamma'}[e_{\delta'}]$$

WE HAVE

$s_\gamma[s_\delta] - s_\alpha[s_\beta]$ SCHUR POSITIVE

if AND ONLY if

$\dfrac{S_\gamma[S_\delta] - S_\alpha[S_\beta]}{1-q}$ SCHUR POSITIVE

Conjecture 2

$\alpha = (a) \quad \beta = (b) \qquad \gamma = (c) \quad \delta = (d)$

$$e_1^a[e_1^b] = e_1^{ab} \qquad e_1^c[e_1^d] = e_1^{cd}$$

$$e_{\alpha'}[e_{\beta'}] = e_{\gamma'}[e_{\delta'}]$$

$\boxed{h_c[h_d] - h_a[h_b] \text{ is Schur Positive}}$

if and only if

$\boxed{\dfrac{H_c[H_d] - H_a[H_b]}{1-q} \text{ is Schur Positive}}$

$$\boxed{\frac{S_\gamma[S_\delta] - S_\alpha[S_\beta]}{1-q} \quad \text{Schur Positive}}$$

$$\langle [\gamma,\delta]:[\alpha,\beta]\rangle_q$$

$\langle [3, 22] : [2, 33]\rangle_q$

$\langle [2, 8] : [8, 2]\rangle_q$ $\langle [2, 8] : [4, 4]\rangle_q$ $\langle [2, 44] : [4, 22]\rangle_q$

$\langle [2, 9] : [3, 6]\rangle_q$, $\langle [2, 9] : [6, 3]\rangle_q$, $\langle [2, 9] : [9, 2]\rangle_q$,

$\langle [3, 6] : [6, 3]\rangle_q$, $\langle [2, 333] : [3, 222]\rangle_q$, $\langle [2, 63] : [3, 42]\rangle_q$

$\langle [2, 55] : [5, 22]\rangle_q$, $\langle [2, 10] : [4, 5]\rangle_q$, $\langle [2, 10] : [5, 4]\rangle_q$,

$\langle [2, 10] : [10, 2]\rangle_q$, $\langle [4, 5] : [5, 4]\rangle_q$

$\langle [2, 66] : [6, 22]\rangle_q$, $\langle [2, 66] : [3, 44]\rangle_q$, $\langle [2, 66] : [4, 33]\rangle_q$,

$\langle [3, 44] : [4, 33]\rangle_q$, $\langle [3, 55] : [5, 33]\rangle_q$, $\langle [2, 444] : [4, 222]\rangle_q$,

$\langle [2, 555] : [5, 222]\rangle_q$, $\langle [2, 3333] : [3, 2222]\rangle_q$, $\langle [2, 33333] : [3, 22222]\rangle_q$

$\langle [2, 63] : [3, 42]\rangle_q$, $\langle [2, 84] : [4, 42]\rangle_q$, $\langle [2, 93] : [3, 62]\rangle_q$,

$\langle [2, 96] : [3, 64]\rangle_q$, $\langle [2, 663] : [3, 442]\rangle_q$

$\langle [2, (10, 4)] : [4, 52]\rangle_q$, $\langle [2, (10, 5)] : [5, 42]\rangle_q$, $\langle [2, (12, 3)] : [3, 82]\rangle_q$

AMONG OTHERS

$\alpha = (a) \quad \beta = (\underbrace{b, b, \ldots, b}_{k \text{ copies}}) \qquad a < b$

$\gamma = (b) \quad \delta = (\underbrace{a, a, \ldots, a}_{k \text{ copies}})$

$$e_{\alpha'}[e_{\beta'}] = e_1^a[e_k^b]$$

$$e_{\gamma'}[e_{\delta'}] = e_k^{ab}$$

$$\boxed{\frac{S_\gamma[S_\delta] - S_\alpha[S_\beta]}{1-q} \quad \text{SCHUR POSITIVE}}$$

q-Stability

$\overline{\mu}$: Remove largest part from μ

$$\overline{\Delta_\mu} := \Delta_{\overline{\mu}}$$

$$\overline{43221} = 3221$$

CONJECTURE 4

$\overline{\mathcal{F}_{a,b+1}} - \overline{\mathcal{F}_{a,b}}$ IS SCHUR POSITIVE

AND COEFFICIENTS STABILIZE

THEOREM

MICHEL BRION

manuscripta math. 80, 347 - 371 (1993)

manuscripta mathematica

Stable properties of plethysm : on two conjectures of Foulkes

Michel BRION

Two conjectures made by H.O. Foulkes in 1950 can be stated as follows.
1) Denote by V a finite-dimensional complex vector space, and by $S_m V$ its m-th symmetric power. Then the $GL(V)$-module $S_n(S_m V)$ contains the $GL(V)$-module $S_m(S_n V)$ for $n > m$.
2) For any (decreasing) partition $\lambda = (\lambda_1, \lambda_2, \lambda_3, \ldots)$, denote by $S_\lambda V$ the associated simple, polynomial $GL(V)$-module. Then the multiplicity of $S_{(\lambda_1+mp, \lambda_2, \lambda_3, \ldots)} V$ in the $GL(V)$-module $S_n(S_{m+p} V)$ is an increasing function of p. We show that Foulkes' first conjecture holds for n large enough with respect to m (Corollary 1.3). Moreover, we state and prove two broad generalizations of Foulkes' second conjecture. They hold in the framework of representations of connected reductive groups, and they lead e.g. to a general analog of Hermite's reciprocity law (Corollary 1 in 3.3).

$g = 0$

Theorem $q=1$

$$\mathcal{F}_{a,b+1}(x,1) = e_1^a \mathcal{F}_{a,b}(x,1) + \Delta_{a,b}$$

where $\Delta_{a,b}$ is Schur positive.

This implies stability at $q=1$.

Fin