The formal Poincaré Lemma

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30/11/2005
Last updated: 28/02/2006

Abstract

The "δ-sequence" is defined and given an interpretation in terms of the exterior derivative of symmetric polynomial functions. The exactness of the sequence is proved. This leads to the formal Poincaré Lemma, which is a statement of the usual Poincaré Lemma in the setting where the rôle of functions is played by formal power series.

Introduction. The formal Poincaré Lemma is what comes out when one tries a power series approach to proving the usual Poincaré Lemma (which, it will be recalled, says that a closed form is locally exact). The key to the constructions we give is the realisation that the usual exterior derivative can be represented algebraically in terms of power series. We accomplish this by first looking at the formal exterior derivative for homogeneous power series (i.e., those where the coefficients are homogeneous polynomials). For this "fixed-order" exterior derivative we produce a sequence which we show is exact. This is the analogue of the Poincaré Lemma for this restricted class of differential forms. We then look at the case where coefficients are general power series; this leads to the formal Poincaré Lemma.

The rth δ-sequence. Although the δ-sequence as introduced by Spencer [1962] arises in terms of partial differential equations modelled as fibred submanifolds of jet bundles, the δ-sequence itself is purely an algebraic construction, (although with a differential interpretation, as we shall see). We thus consider an n-dimensional vector space V. We shall be interpreting V as a vector space *and* as a manifold. In the former case, elements of V will be denoted by v, and in the latter case by x. By $S_k(\mathsf{V}^*)$ we denote the set of symmetric $(0,k)$-tensors on V and by $\bigwedge_k(\mathsf{V}^*)$ the set of skew-symmetric $(0,k)$-tensors.

The following lemma concerning symmetric tensors will come up repeatedly. Let us denote by

$$P_k(\mathsf{V}) = \{f\colon \mathsf{V}\to\mathbb{R} \mid\ f(v) = A(v,\dots,v) \text{ for some } A\in T^0_k(\mathsf{V})\}$$

the symmetric homogeneous polynomial functions of degree k.

1 Lemma: (Characterisation of symmetric tensors) *Let V be a $\mathbb{R}$-vector space and let $f\in P_k(\mathsf{V})$. Then there exists a unique $A_f\in S_k(\mathsf{V}^*)$ such that $A_f(v,\dots,v) = f(v)$ for each $v\in\mathsf{V}$.*

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Proof: Let $A \in T^0_k(\mathsf{V})$ and denote by S_k the permutation group on k symbols. If $f(v) = A(v,\dots,v)$, $v \in \mathsf{V}$, define

$$A_f(v_1,\dots,v_k) = \frac{1}{k!}\sum_{\sigma\in S_k} A(v_{\sigma(1)},\dots,v_{\sigma(k)}).$$

It is evident that A_f is symmetric and that $f(v) = A_f(v,\dots,v)$, $v \in \mathsf{V}$. This gives the existence part of the result.

Now we prove uniqueness. For $k \in \mathbb{Z}_{>0}$ and $l \in \{1,\dots,k\}$, let

$$S(k,l) = \{\{j_1,\dots,j_l\} \subset \{1,\dots,k\} \mid r \neq s \implies j_r \neq j_s\}.$$

With this notation, the following result is useful.

1 Sublemma: *For $A \in T^0_k(\mathsf{V})$,*

$$\sum_{\sigma\in S_k} A(v_{\sigma(1)},\dots,v_{\sigma(k)}) = \sum_{l=1}^{k}\sum_{\substack{\{j_1,\dots,j_l\}\\ \in S(k,l)}} (-1)^{k-l} A(v_{j_1}+\dots+v_{j_l},\dots,v_{j_1}+\dots+v_{j_l}).$$

Proof: We prove this by examining the terms in the sum. For $l \in \{1,\dots,k\}$ and for $\{j_1,\dots,j_l\} \in S(k,l)$, if we expand the expression

$$A(v_{j_1}+\dots+v_{j_l},\dots,v_{j_1}+\dots+v_{j_l})$$

using multilinearity of A, we obtain the sum of all terms of the form $A(v_{r_1},\dots,v_{r_k})$ where $r_1,\dots,r_k \in \{j_1,\dots,j_l\}$. Thus this is a sum with l^k terms. Therefore, the right-hand side of the expression in the statement of the sublemma will itself be a linear combination of terms of the form $A(v_{r_1},\dots,v_{r_k})$ where $r_1,\dots,r_k \in \{1,\dots,k\}$. To prove the sublemma we shall show that the coefficient in the linear combination is 0 unless $r_1,\dots,r_k$ are distinct. When $r_1,\dots,r_k$ are distinct, we shall show that the coefficient in the linear combination is 1. This will prove the lemma since the terms on the right-hand side corresponding to the case when $r_1,\dots,r_k$ are distinct correspond exactly to the terms on the left-hand side of the expression in the sublemma.

Let us fix $r_1,\dots,r_k \in \{1,\dots,k\}$ (not necessarily distinct) and examine how many terms of the form $A(v_{r_1},\dots,v_{r_k})$ appear in the sum on the right in the statement of the sublemma. This will depend on how many distinct elements of $\{1,\dots,k\}$ appear in the set $\{r_1,\dots,r_k\}$. Let us suppose that there are s distinct elements. For $l \geq s$, in the set $S(k,l)$ there will be $D(k,l,s)$ members which contain $\{r_1,\dots,r_k\}$ as a subset, where

$$D(k,l,s) = \frac{(k-s)!}{(l-s)!(k-l)!}.$$

To see this, note that to each member $\{j_1,\dots,j_l\} \in S(k,l)$ that contains $\{r_1,\dots,r_k\}$ as a subset, there corresponds a unique subset of $l-s$ elements of a set of $k-s$ elements (the complement to $\{r_1,\dots,r_k\}$ in $\{j_1,\dots,j_l\}$). There are $D(k,l,s)$ such subsets after we note that $D(k,l,s) = \binom{k-s}{l-s}$. This means that there will be $D(k,l,s)$ terms of the form $A(v_{r_1},\dots,v_{r_k})$ which appear in the sum

$$\sum_{\substack{\{j_1,\dots,j_l\}\\ \in S(k,l)}} (-1)^{k-l} A(v_{j_1}+\dots+v_{j_l},\dots,v_{j_1}+\dots+v_{j_l}).$$

Therefore, there will be $\sum_{l=s}^{k}(-1)^{k-l}D(k,l,s)$ terms of the form $A(v_{r_1},\dots,v_{r_k})$ in the right-hand side of the expression in the sublemma. We claim that

$$\sum_{l=s}^{k}(-1)^{k-l}D(k,l,s)=\begin{cases}1, & s=k,\\ 0, & s<k.\end{cases}$$

For $s=k$ the equality is checked directly. For $s<k$ we note that, for $x,y\in\mathbb{R}$ and for $k-s>0$, we have

$$(x+y)^{k-s}=\sum_{j=0}^{k-s}\binom{k-s}{j}x^jy^{k-s-j}=\sum_{l=s}^{k}D(k,l,s)x^{l-s}y^{k-l}.$$

Letting $x=1$ and $y=-1$ we obtain

$$\sum_{l=s}^{k}(-1)^{k-l}D(k,l,s)=0,$$

as desired. ▼

Now, if $A\in S_k(\mathsf{V}^*)$ satisfies $f(v)=A(v,\dots,v)$ for all $v\in\mathsf{V}$ we must have

$$\begin{aligned}A(v_1,\dots,v_k)&=\frac{1}{k!}\sum_{\sigma\in S_k}A(v_{\sigma(1)},\dots,v_{\sigma(k)})\\ &=\frac{1}{k!}\Big(\sum_{l=1}^{k}\sum_{\substack{\{j_1,\dots,j_l\}\\ \in S(k,l)}}(-1)^{k-l}f(v_{j_1}+\dots+v_{j_l})\Big)\\ &=\frac{1}{k!}\Big(\sum_{l=1}^{k}\sum_{\substack{\{j_1,\dots,j_l\}\\ \in S(k,l)}}(-1)^{k-l}A_f(v_{j_1}+\dots+v_{j_l},\dots,v_{j_1}+\dots+v_{j_l})\Big)\\ &=\frac{1}{k!}\sum_{\sigma\in S_k}A_f(v_{\sigma(1)},\dots,v_{\sigma(k)})\\ &=A_f(v_1,\dots,v_k),\end{aligned}$$

where we have used the symmetry of A in the first step and the symmetry of A_f in the last step. ■

The key point of the lemma is that a symmetric tensor is defined as soon as one knows its values on the diagonal in $\prod_{j=1}^{k}\mathsf{V}$. We shall use this idea frequently.

The following simple result is key to developing any intuition about the δ-sequence.

2 PROPOSITION: (Characterisation of $\bigwedge_s(\mathsf{V}^*)\otimes S_r(\mathsf{V}^*)$) *Let $r,s\in\mathbb{Z}_{\geq 0}$. The map $\phi_{s,r}$ from $\bigwedge_s(\mathsf{V}^*)\otimes S_r(\mathsf{V}^*)$ to the set of differential s-forms on the manifold V satisfying*

$$\phi_{s,r}(\alpha\otimes A)(v_1,\dots,v_s)=A(x,\dots,x)\alpha(v_1,\dots,v_s),\qquad v_1,\dots,v_s\in\mathsf{T}_x\mathsf{V}\simeq\mathsf{V},$$

is a monomorphism of $\mathbb{R}$-vector spaces.

Proof: The map is clearly linear. To see that it is injective, suppose that $\phi_{s,r}(\alpha_1 \otimes A_1 + \cdots + \alpha_k \otimes A_k) = 0$. This means that the differential s-form

$$x \mapsto A_1(x, \dots, x)\alpha_1 + \cdots + A_k(x, \dots, x)\alpha_k$$

is zero. This means that

$$A_1(x, \dots, x)\alpha_1(v_1, \dots, v_s) + \cdots + A_k(x, \dots, x)\alpha_k(v_1, \dots, v_s) = 0$$

for every $x, v_1, \dots, v_s \in \mathsf{V}$. Now let $u_1, \dots, u_r, v_1, \dots, v_s \in \mathsf{V}$. Following the proof of Lemma 1, for each $j \in \{1, \dots, k\}$ we have

$$A_j(u_1, \dots, u_r) = \sum_{l=1}^{k} \sum_{\substack{\{j_1,\dots,j_l\}\\ \in S(k,l)}} (-1)^{k-l} A(u_{j_1} + \cdots + u_{j_l}, \dots, u_{j_1} + \cdots + u_{j_l}).$$

We may then write

$$\sum_{j=1}^{k} \alpha_j(v_1, \dots, v_j) A_j(u_1, \dots, u_r)$$
$$= \sum_{j=1}^{k} \sum_{\substack{\{j_1,\dots,j_l\}\\ \in S(k,l)}} (-1)^{k-l} \alpha_j(v_1, \dots, v_s) A_j(u_{j_1} + \cdots + u_{j_l}, u_{j_1} + \cdots + u_{j_l}).$$

The terms on the right vanish since $\alpha_1 \otimes A_1 + \cdots + \alpha_k \otimes A_k \in \ker(\phi_{s,r})$, and so we have $\alpha_1 \otimes A_1 + \cdots + \alpha_k \otimes A_k = 0$, giving injectivity of $\phi_{s,r}$. ■

This gives a handy identification of $\bigwedge_s(\mathsf{V}^*) \otimes S_r(\mathsf{V}^*)$ with a subspace of the vector space of differential s-forms on the manifold V. The next result shows that the set of all such differential forms is closed under exterior differentiation. We denote by $\boldsymbol{d}_s$ the exterior derivative on the manifold V restricted to differential s-forms.

3 PROPOSITION: (Invariance under exterior differentiation) *Let $r \in \mathbb{Z}_{>0}$ and $s \in \mathbb{Z}_{\geq 0}$. If $\alpha_1 \otimes A_1 + \cdots + \alpha_k \otimes A_k \in \bigwedge_s(\mathsf{V}^*) \otimes S_r(\mathsf{V}^*)$ then there exists $\beta_1 \otimes B_1 + \cdots + \beta_l \otimes B_l \in \bigwedge_{s+1}(\mathsf{V}^*) \otimes S_{r-1}(\mathsf{V}^*)$ such that*

$$\boldsymbol{d}_s \phi_{s,r}(\alpha_1 \otimes A_1 + \cdots + \alpha_k \otimes A_k) = \phi_{s+1,r-1}(\beta_1 \otimes B_1 + \cdots + \beta_l \otimes B_l).$$

Proof: Let us first consider the case when $s = 0$. In this case, elements of $\bigwedge_s(\mathsf{V}^*) \otimes S_r(\mathsf{V}^*)$ are to be thought of, under the map $\phi_{s,r}$, as polynomial functions on V that are symmetric and homogeneous of degree k. Explicitly, if $A \in S_r(\mathsf{V}^*)$ then we have the function $f_A \colon x \mapsto A(x, \dots, x)$. The exterior derivative satisfies

$$\langle \boldsymbol{d}_0 f_A(x); v\rangle = A(v, x, \dots, x).$$

Now define $A' \in \mathsf{V}^* \otimes S_{r-1}(\mathsf{V}^*)$ by asking that

$$A'(v, u, \dots, u) = rA(v, u, \dots, u), \qquad u, v \in \mathsf{V}.$$

Note that this uniquely defines A' by Lemma 1. Clearly we have $\phi_{1,r-1}(A') = \boldsymbol{d}\phi_{0,r}(A)$, so proving the result when $s = 0$.

For $s > 0$ we note that an arbitrary element in image$(\phi_{s,r})$ is a finite linear combination of elements of the form $x \mapsto f_A(x)\alpha$ for $A \in S_r(\mathsf{V}^*)$ and $\alpha \in \bigwedge_s(\mathsf{V}^*)$, thinking of α as a constant section of $\bigwedge_s(\mathsf{T}^*\mathsf{V})$. We can write this differential s-form as $f_A \wedge \alpha$. We then have

$$\boldsymbol{d}_s(f_A \wedge \alpha) = \boldsymbol{d}_0 f_A \wedge \alpha - f_A \wedge \boldsymbol{d}_s\alpha = \boldsymbol{d}_0 f_A \wedge \alpha,$$

using the fact that α is closed since it is constant. The definition of wedge product gives

$$(\boldsymbol{d}_0 f_A \wedge \alpha)(v_1, \dots, v_{s+1}) = \sum_{\sigma} \operatorname{sign}(\sigma)\boldsymbol{d}_0 f_A(v_{\sigma(1)})\alpha(v_{\sigma(2)}, \dots, v_{\sigma(s+1)}),$$

where the sum is over all permutations σ of $\{1, \dots, s+1\}$ which satisfy

$$\sigma(2) < \sigma(3) < \cdots < \sigma(s+1).$$

This amounts to

$$(\boldsymbol{d}_0 f_A \wedge \alpha)(v_1, \dots, v_{s+1}) = \sum_{j=1}^{s+1} (-1)^{j+1}\boldsymbol{d}_0 f_A(v_j)\alpha(v_1, \dots, \hat{v}_j, \dots, v_{s+1}),$$

where $\hat{\cdot}$ means the term is omitted from the argument. In the above expressions we have suppressed the base point $x \in \mathsf{V}$ at which all tensors are evaluated. Restoring this, and supposing that $v_1, \dots, v_{s+1} \in \mathsf{T}_x\mathsf{V}$, we have

$$\boldsymbol{d}_0 f_A(x) \wedge \alpha(x)(v_1, \dots, v_{s+1}) = \sum_{j=1}^{s+1} (-1)^{j+1} A'(v_j, x, \dots, x)\alpha(v_1, \dots, \hat{v}_j, \dots, v_{s+1}).$$

Now define $\omega \in \bigwedge_{s+1}(\mathsf{V}^*) \otimes S_{r-1}(\mathsf{V}^*)$ by asking that

$$\omega(v_1, \dots, v_{s+1}, u, \dots, u) = \sum_{j=1}^{s+1} (-1)^{j+1} A'(v_j, u, \dots, u)\alpha(v_1, \dots, \hat{v}_j, \dots, v_{s+1}).$$

By Lemma 1 this uniquely defines an element of $\bigwedge_{s+1}(\mathsf{V}^*) \otimes S_{r-1}(\mathsf{V}^*)$. Moreover, by construction we have $\phi_{s+1,r-1}(\omega) = \boldsymbol{d}\phi_{s,r}(\alpha \otimes A)$. The result now follows by $\mathbb{R}$-linearity of the exterior derivative and of $\phi_{s,r}$. ∎

This enables us to define $\delta_{s,r} \colon \bigwedge_s(\mathsf{V}^*) \otimes S_r(\mathsf{V}^*) \to \bigwedge_{s+1}(\mathsf{V}) \otimes S_{r-1}(\mathsf{V}^*)$ by asking that the diagram

$$\begin{array}{ccc} \bigwedge_s(\mathsf{V}^*) \otimes S_r(\mathsf{V}^*) & \xrightarrow{\delta_{s,r}} & \bigwedge_{s+1}(\mathsf{V}^*) \otimes S_{r-1}(\mathsf{V}^*) \\ \downarrow{\scriptstyle \phi_{s,r}} & & \downarrow{\scriptstyle \phi_{s+1,r-1}} \\ \Gamma^\infty(\bigwedge_s(\mathsf{TV})) & \xrightarrow[\boldsymbol{d}_s]{} & \Gamma^\infty(\bigwedge_{s+1}(\mathsf{TV})) \end{array}$$

commute. Explicitly, $\delta_{s,r}$ is defined by its satisfying

$$(\delta_{s,r}(\alpha \otimes A))(v_1, \dots, v_{s+1}, u_1, \dots, u_{r-1}) \\ = \sum_{j=1}^{s+1} (-1)^{j+1} r\alpha(v_1, \dots, \hat{v}_j, \dots, v_{s+1})A(v_j, u_1, \dots, u_{r-1}).$$

We adopt the usual convention of simply denoting $\delta_{s,r}$ by δ when this is convenient, supposing any possible ambiguity to be resolvable from context.

The ***r*th *δ*-sequence** is then given by

$$0 \longrightarrow S_r(\mathsf{V}^*) \xrightarrow{\delta} \mathsf{V}^* \otimes S_{r-1}(\mathsf{V}^*) \xrightarrow{\delta} \textstyle\bigwedge_2(\mathsf{V}^*) \otimes S_{r-2}(\mathsf{V}^*) \xrightarrow{\delta} \cdots \qquad (1)$$
$$\cdots \xrightarrow{\quad\delta\quad} \textstyle\bigwedge_n(\mathsf{V}^*) \otimes S_{r-n}(\mathsf{V}^*) \longrightarrow 0$$

The *r*th *δ*-sequence is exact. It is clear, since $\boldsymbol{d}_{s+1} \circ \boldsymbol{d}_s = 0$, that $\delta_{s+1,r-1} \circ \delta_{s,r} = 0$. Thus $\ker(\delta_{s+1,r-1}) \subset \operatorname{image}(\delta_{r,s})$. However, more is true; namely the δ-sequence is exact.

4 Proposition: (Exactness of the rth δ-sequence) *For each $r \in \mathbb{Z}_{>0}$, the sequence (1) is exact.*

Proof: For each $s, r \in \mathbb{Z}_{>0}$ we shall define a map $H_{s,r} \colon \bigwedge_{s+1}(\mathsf{V}^*) \otimes S_{r-1}(\mathsf{V}^*) \to \bigwedge_s(\mathsf{V}^*) \otimes S_r(\mathsf{V}^*)$ with the property that $H_{s+1,r-1} \circ \delta_{s+1,r-1} + \delta_{s,r} \circ H_{s,r}$ is the identity map on $\bigwedge_{s+1}(\mathsf{V}^*) \otimes S_{r-1}(\mathsf{V}^*)$. For $\beta \otimes B \in \bigwedge_{s+1}(\mathsf{V}^*) \otimes S_{r-1}(\mathsf{V}^*)$ define

$$H_{s,r}(\beta \otimes B)(v_1, \dots, v_s, u, \dots, u) = \frac{1}{r+s} B(u, \dots, u)\beta(u, v_1, \dots, v_s),$$

for $u, v_1, \dots, v_s \in \mathsf{V}$, noting that this uniquely defines $H_{s,r}(\beta \otimes B) \in \bigwedge_r(\mathsf{V}^*) \otimes S_s(\mathsf{V}^*)$ by Lemma 1. Using the definition of $\delta_{s,r}$ we compute

$$\begin{aligned} \delta_{s,r} \circ H_{s,r}(\beta \otimes B)&(v_1, \dots, v_{s+1}, u, \dots, u) \\ &= \frac{1}{r+s}\Big(\sum_{j=1}^{s+1}(-1)^{j+1}(r-1)B(v_j, u, \dots, u)\beta(u, v_1, \dots, \hat{v}_j, \dots, v_{s+1}) \\ &\qquad + \sum_{j=1}^{s+1}(-1)^{j+1}B(u, \dots, u)\beta(v_j, v_1, \dots, \hat{v}_j, \dots, v_{s+1})\Big), \end{aligned} \qquad (2)$$

for $u, v_1, \dots, v_{s+1} \in \mathsf{V}$. Using the definition of $\delta_{s+1,r-1}$ we have

$$\begin{aligned} \delta_{s+1,r-1}(\beta \otimes B)&(v_1, \dots, v_{s+2}, u, \dots, u) \\ &= \sum_{j=1}^{s+2}(-1)^{j+1}(r-1)B(v_j, u, \dots, u)\beta(v_1, \dots, \hat{v}_j, \dots, v_{s+2}) \end{aligned}$$

for $u, v_1, \dots, v_{s+2} \in \mathsf{V}$. Therefore, using the definition of $H_{s+1,r-1}$,

$$\begin{aligned} H_{s+1,r-1} \circ \delta_{s+1,r-1}(\beta \otimes B)&(v_1, \dots, v_{s+1}, u, \dots, u) \\ &= \frac{1}{r+s}\Big((r-1)B(u, \dots, u)\beta(v_1, \dots, v_{s+1}) \\ &\qquad + \sum_{j=1}^{s+1}(-1)^j(r-1)B(v_j, u, \dots, u)\beta(u, v_1, \dots, \hat{v}_j, \dots, v_{s+1})\Big) \end{aligned} \qquad (3)$$

for $u, v_1, \dots, v_{s+1} \in \mathsf{V}$. Combining (2) and (3) we arrive at

$$\begin{aligned}
(H_{s+1,r-1} \circ \delta_{s+1,r-1} + \delta_{s,r} \circ H_{s,r})(\beta \otimes B)(v_1, \dots, v_{s+1}, u, \dots, u) \\
&= \frac{1}{r+s}\Big((r-1)B(u,\dots,u)\beta(v_1,\dots,v_{s+1}) \\
&\quad + \sum_{j=1}^{s+1}(-1)^{j+1}B(u,\dots,u)\beta(v_j, v_1, \dots, \hat{v}_j, v_{s+1})\Big) \\
&= \frac{1}{r+s}\Big((r-1)B(u,\dots,u)\beta(v_1,\dots,v_{s+1}) \\
&\quad + (s+1)B(u,\dots,u)\beta(v_1,\dots,v_{s+1})\Big) \\
&= B(u,\dots,u)\beta(v_1,\dots,v_{s+1}) \\
&= \beta \otimes B(v_1,\dots,v_{s+1},u,\dots,u)
\end{aligned}$$

for $u, v_1, \dots, v_{s+1} \in \mathsf{V}$. By extending the above computations using linearity and by using Lemma 1, it follows that $H_{s+1,r-1} \circ \delta_{s+1,r-1} + \delta_{s,r} \circ H_{s,r}$ is the identity on $\bigwedge_{s+1}(\mathsf{V}^*) \otimes S_{r-1}(\mathsf{V}^*)$.

Now, if $\delta_{s+1,r-1}(\beta_1 \otimes B_1 + \dots + \beta_k \otimes B_k) = 0$ for $\beta_j \otimes B_j \in \bigwedge_{s+1}(\mathsf{V}^*) \otimes S_{r-1}(\mathsf{V}^*)$, $j \in \{1, \dots, k\}$, then we define $\alpha_j \otimes A_j \in \bigwedge_s(\mathsf{V}^*) \otimes S_r(\mathsf{V}^*)$ by $\alpha_j \otimes A_j = H_{s,r}(\beta_j \otimes B_j)$, $j \in \{1, \dots, k\}$. Then

$$\begin{aligned}
\delta_{s,r}\Big(\sum_{j=1}^k \alpha_j \otimes A_j\Big) &= \delta_{s,r} \circ H_{s,r}\Big(\sum_{j=1}^k \beta_j \otimes B_j\Big) \\
&= (\delta_{s,r} \circ H_{s,r} + H_{s+1,r-1} \circ \delta_{s+1,r-1})\Big(\sum_{j=1}^k \beta_j \otimes B_j\Big) = \sum_{j=1}^k \beta_j \otimes B_j,
\end{aligned}$$

showing that $\operatorname{image}(\delta_{s,r}) \subset \ker(\delta_{s+1,r-1})$ as desired. ■

5 REMARK: The proof above is an adaptation of one of the more common proofs of the usual Poincaré Lemma to the specific case needed, using the fact that $\delta_{s,r}$ is the restriction of $\boldsymbol{d}_s$ to $\operatorname{image}(\phi_{s,r})$. The proof we adapted may be found in, for example, [Warner 1983]. •

The formal Poincaré Lemma. Recall that the Poincaré Lemma says that, for a manifold M, if $\omega \in \Gamma^\infty(\bigwedge_{s+1}(\mathsf{TM}))$ satisfies $\boldsymbol{d}_{s+1}\omega = 0$, then for any $x_0 \in \mathsf{M}$ there exists a neighbourhood $\mathcal{U}$ of x_0 and $\theta \in \Gamma^\infty(\bigwedge_s(\mathsf{T}\mathcal{U}))$ such that $\omega|\mathcal{U} = \boldsymbol{d}_s\theta$. In the case when M and ω are analytic, one can imagine constructing θ by Taylor expanding ω, computing the exterior derivative in terms of the Taylor expansion, and then using this to determine the Taylor expansion of θ. If the Taylor expansion converges (and it does), then this can be used to define θ. Note that this construction involves only looking at things defined at x_0, and so is really an algebraic construction. The formal Poincaré Lemma simply eliminates all the annoyance of convergence dictated by analyticity, and works instead with formal power series. The exterior derivative is then just *defined* using the maps $\delta_{s,r}$ defined above, so eliminating any obligation to talk about limits in the definition of derivatives, etc.

First let us try to be a little careful about what we mean by a formal power series about $x_0 \in \mathsf{M}$. To make the ties with the above discussion clear, and to emphasise the point that this construction only involves objects defined at x_0, let us denote $\mathsf{V} = \mathsf{T}_{x_0}\mathsf{M}$.

6 Definition: (Formal power series) A ***formal power series*** at x_0 is an element of the direct product $\prod_{r\in\mathbb{Z}_{\geq 0}} S_r(\mathsf{V}^*)$ of $\mathbb{R}$-vector spaces. The set of formal power series at x_0 is denoted by $\mathbb{R}[[\mathsf{V}]]$. •

Let us be clear about what is meant by a formal power series at x_0. It is a map $P\colon \mathbb{Z}_{\geq 0} \to \cup_{r\in\mathbb{Z}_{\geq 0}} S_r(\mathsf{V}^*)$ which satisfies $P(r) \in S_r(\mathsf{V}^*)$. If one want to understand how a formal power series P relates to Taylor series, one can choose a coordinate chart $(\mathcal{U}, \phi)$ about x_0 for which $\phi(x_0) = \mathbf{0}$. Then one can attempt to define a function by

$$f_P(x) = \sum_{r\in\mathbb{Z}_{\geq 0}} P(r)(\phi(x), \dots, \phi(x)), \qquad x \in \mathcal{U}.$$

The matter of convergence of this series depends only on P, and so one can speak of the subset of $\mathbb{R}[[\mathsf{V}]]$ consisting of convergent power series; we denote this subset by $\hat{\mathbb{R}}[[\mathsf{V}]]$.

There is a natural $\mathbb{R}$-vector space structure defined on $\mathbb{R}[[\mathsf{V}]]$ by

$$(P_1 + P_2)(r) = P_1(r) + P_2(r), \quad (aP)(r) = a(P(r))$$

(this is just the vector space structure of the direct product). There is also a commutative ring structure on $\mathbb{R}[[\mathsf{V}]]$ as follows. Let $P_1 \in S_{r_1}(\mathsf{V}^*)$ and $P_2 \in S_{r_2}(\mathsf{V}^*)$ and note that these are elements of $\mathbb{R}[[\mathsf{V}]]$ since $S_r(\mathsf{V}^*)$ is a subspace of $\mathbb{R}[[\mathsf{V}]]$ for every $r \in \mathbb{Z}_{\geq 0}$. Define $P_1P_2 \in \mathbb{R}[[\mathsf{V}]]$ by

$$(P_1P_2)(\underbrace{u, \dots, u}_{\substack{r_1+r_2\\ \text{times}}}) = P_1(\underbrace{u, \dots, u}_{r_1 \text{ times}})P_2(\underbrace{u, \dots, u}_{r_2 \text{ times}}), \qquad u \in \mathsf{V}.$$

This defines $P_1P_2 \in S_{r_1+r_2}(\mathsf{V}^*)$ is the usual manner by noting that a symmetric tensor is uniquely defined by its values when all entries are equal, cf. Lemma 1. Indeed, fiddling with permutations gives $P_1P_2 \in S_{r_1+r_2}(\mathsf{V}^*)$ by the formula

$$(P_1P_2)(v_1, \dots, v_{r_1+r_2}) = \sum_{\sigma} P_1(v_{\sigma(1)}, \dots, v_{\sigma(r_1)})P_2(v_{\sigma(r_1+1)}, \dots, v_{\sigma(r_1+r_2)}),$$

where the sum is over all permutations satisfying

$$\sigma(1) < \sigma(2) < \dots < \sigma(r_1), \quad \sigma(r_1 + 1) < \sigma(r_1 + 2) < \dots < \sigma(r_1 + r_2).$$

This gives the product of elements of the subspaces $S_r(\mathsf{V}^*)$, $r \in \mathbb{Z}_{\geq 0}$, and this definition can be extended to all of $\mathbb{R}[[\mathsf{V}]]$ by linearity. If one restricts consideration to the set $\hat{\mathbb{R}}[[\mathsf{V}]]$ of convergent power series, the sum and product in $\mathbb{R}[[\mathsf{V}]]$ corresponds to the sum and product of analytic $\mathbb{R}$-valued functions. Put more precisely, the subring $\hat{\mathbb{R}}[[\mathsf{V}]]$ of $\mathbb{R}[[\mathsf{V}]]$ is isomorphic to the ring $\mathscr{C}^{\omega}_{x_0}(\mathsf{M})$ of germs of analytic functions at x_0. However, the ring structure in $\mathbb{R}[[\mathsf{V}]]$ makes sense, even for formal power series that do not converge.

Now note that $\bigwedge_s(\mathsf{V}^*) \otimes \mathbb{R}[[\mathsf{V}]]$ can be thought of as the vector space of those differential s-forms on V with coefficients in $\mathbb{R}[[\mathsf{V}]]$ in exactly the way that $\bigwedge_s(\mathsf{V}^*) \otimes S_r(\mathsf{V}^*)$ is, by Proposition 2, to be thought of as the set of differential s-forms with coefficients that are homogeneous polynomials functions. Indeed, if one wishes to be a little more precise about this, one notes that

$$\textstyle\bigwedge_s(\mathsf{V}^*) \otimes \mathbb{R}[[\mathsf{V}]] \simeq \displaystyle\prod_{r\in\mathbb{Z}_{\geq 0}} \textstyle\bigwedge_s(\mathsf{V}^*) \otimes S_r(\mathsf{V}^*), \tag{4}$$

and so $\bigwedge_s(\mathsf{V}^*) \otimes \mathbb{R}[[\mathsf{V}]]$ is isomorphic to the direct product of the subspaces image$(\phi_{s,r})$, $r \in \mathbb{Z}_{\geq 0}$, of differential forms on V. Of course, this only makes sense formally. But if one restricts to $\hat{\mathbb{R}}[[\mathsf{V}]]$, one recovers *bona fide* analytic differential s-forms. The $\mathbb{R}$-vector space $\bigwedge_s(\mathsf{V}^*) \otimes \mathbb{R}[[\mathsf{V}]]$ has the structure of a module over the ring $\mathbb{R}[[\mathsf{V}]]$ if one defines the scalar product to satisfy

$$P(\alpha \otimes Q) = \alpha \otimes (PQ).$$

The module $\bigwedge_s(\mathsf{V}^*) \otimes \hat{\mathbb{R}}[[\mathsf{V}]]$ over the ring $\hat{\mathbb{R}}[[\mathsf{V}]]$ is then isomorphic to the module of germs of analytic differential s-forms on M at x_0 which is thought of as a module over the ring $\mathscr{C}^{\omega}_{x_0}(\mathsf{M})$ (using the fact that this latter ring is isomorphic to $\hat{\mathbb{R}}[[\mathsf{V}]]$).

We now define the analogue of the exterior derivative for the module $\bigwedge_s(\mathsf{V}^*) \otimes \mathbb{R}[[\mathsf{V}]]$.

7 Definition: (Formal exterior derivative) For $s \in \mathbb{Z}_{\geq 0}$ define $\delta_s \colon \bigwedge_s(\mathsf{V}^*) \otimes \mathbb{R}[[\mathsf{V}]] \to \bigwedge_{s+1}(\mathsf{V}^*) \otimes \mathbb{R}[[\mathsf{V}]]$ to satisfy

$$(\delta_s(\alpha \otimes P))(r) = \delta_{s,r}(\alpha \otimes P(r))$$

(using the identification (4)). •

If one restricts to $\bigwedge_s(\mathsf{V}^*) \otimes \hat{\mathbb{R}}[[\mathsf{V}]]$, then δ_s is the representation of $\boldsymbol{d}_s$ under the isomorphism of $\bigwedge_s(\mathsf{V}^*) \otimes \hat{\mathbb{R}}[[\mathsf{V}]]$ with the germs of analytic differential s-forms at x_0. Again, when it is convenient we shall simply write δ for δ_s.

This leads to the sequence

$$\begin{aligned} 0 \longrightarrow \mathbb{R}[[\mathsf{V}]] \xrightarrow{\delta} \mathsf{V}^* \otimes \mathbb{R}[[\mathsf{V}]] \xrightarrow{\delta} \textstyle\bigwedge_2(\mathsf{V}^*) \otimes \mathbb{R}[[\mathsf{V}]] \xrightarrow{\quad\delta\quad} \cdots \\ \cdots \xrightarrow{\quad\delta\quad} \textstyle\bigwedge_n(\mathsf{V}^*) \otimes \mathbb{R}[[\mathsf{V}]] \longrightarrow 0 \end{aligned} \tag{5}$$

One can now recast Proposition 4 as the following result, which tells us that the Poincaré Lemma holds for formal power series, even when they do not converge.

8 Theorem: (Formal Poincaré Lemma) *The sequence (5) is exact.*

References

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