Chaotic dynamics in deterministic dynamical systems

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Abstract

A survey is given of the rôle of chaotic dynamics in the field of nonlinear dynamics. Various phenomenon connected with chaotic dynamics are investigated, e.g., sensitivity to initial conditions, density of periodic orbits, transitivity/irreducibility of dynamics. The emphasis is on illustrating these phenomenon via representative examples. Means of detecting chaotic dynamics in dynamical systems are also discussed, including the use of Lyapunov exponents, topological entropy, and ergodic theory. In closing, a few broad areas where the current understanding of chaotic dynamics could be improved are discussed.

Keywords. Nonlinear dynamics, chaotic dynamics, Lyapunov exponents, topological entropy, ergodic theory

AMS Subject Classifications (2020). 34-00, 34A34, 34C28, 34D08, 37A05, 37B40

1. Introduction and historical remarks

Dynamical systems, or *dynamics*, or *nonlinear dynamics* are expansive and interconnected subjects with both broad theoretical foundations and wide ranges of application. We begin by fixing our objectives in the broad range of possible objectives. The intent is to outline the *qualitative* theory of dynamics—where one wishes to understand the character of dynamics—as opposed to the *quantitative* theory of dynamics—where one wishes to arrive at numerical answers to questions of interest, possibly by analytical methods or possibly by numerical simulation. We are particularly interested in the qualitative phenomenon colloquially known as *chaos*. Thus, the way to frame the objective of this survey is that it is an overview of the qualitative theory of dynamical systems, with a particular emphasis on chaotic dynamics.

1.1. A brief historical overview of ordinary differential equations and dynamics. Let us begin with some history concerning ordinary differential equations and dynamical systems. In the early days (late 1600's and early 1700's) of the subject—originating with the contributions of Leibniz, Newton, and the Bernoulli brothers—the emphasis was on understanding classes of ordinary differential equations and associated methods for solving these equations. Subsequent, still early (mid 1700's to early 1800's), contributions were made by Clairaut, D'Alembert, Euler, Lagrange, and Laplace. In the 1800's, the theory of differential equations was developed in various analytical directions by mathematicians such as Darboux, Dirichlet, Jacobi, and Picard, to name but a few of the prominent mathematicians involved. Particular attention deserves to be given to the contributions of Cauchy.

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Apart from his original contributions to the subject (e.g., series methods), his treatment of ordinary differential equations in his seminal *Cours d'analyse* in 1821 (reprinted as [Cauchy 1981]) presents a picture of the subject that has hardly changed up to the presently common means of teaching the subject to undergraduates. (Leaving aside what this says about how we teach undergraduates, it certainly gives proper credit to the contributions of Cauchy.)

Even for autonomous linear ordinary differential equations, however, obtaining closed-form solutions is generally hopeless, e.g., the roots of a polynomial of degree larger than four cannot be determined by radicals. Towards the end of the nineteenth century and the beginning of the twentieth century, attention began to be given to more qualitative aspects of dynamics, an early example of which is the *Poincaré–Bendixson criterion* for the existence of periodic orbits for planar ordinary differential equations.

Much of the early development in the theory of dynamics owes its context to the equations of *celestial mechanics*, bearing in mind that Newton's law of gravitational attraction itself led to some of the earliest of ordinary differential equations. And it is in the setting of celestial mechanics that the first developments of the qualitative theory of ordinary differential equations arose in the groundbreaking work of Poincaré on the *three-body problem*. Poincaré treated the three-body problem as a perturbation of the solvable two-body problem. During the course of his analysis, Poincaré observed that "[I]f one seeks to visualise the pattern formed by these two curves and their infinite number of intersections [...] [o]ne will be struck by the complexity of this figure, which I am not even attempting to draw" [Poincaré 1892-1899]. Here Poincaré is making reference to what we today call a *homoclinic tangle*, and it remains one of the most studied and best understood mechanisms leading to chaotic dynamics in ordinary differential equations.

The work of Poincaré marked a turning point in the theory of dynamics, with the development of what we now know as *dynamical systems*. Birkhoff, in the 1920's and 1930's, continued work in some problems connected to the work of Poincaré, as well as making early developments in *ergodic theory*. In Russian, early developments in the same time period were made by a group around Andronov. An idea that emerged from this work is an attempt to *classify* the behaviour of dynamical systems. This is extremely difficult, even for planar ordinary differential equations, and an account of this can be found in [Andronov, Leontovich, Gordon, and Maĭer 1973].

After 1945, a great deal of the fundamental work in the theory of dynamical systems continued to be done in the USSR, with key contributions being made by Anosov, Arnol′d, Kolmogorov, and Sinai. From this work arose the first systematic examples of *strange attractors*, as outlined in [Devaney 2003, §2.5]. Another fundamental result from this period, especially as concerns the topic of this survey, is the *KAM Theorem* [Arnol′d 1963, Kolmogorov 1954, Moser 1962], which shows that the behaviour observed by Poincaré in his study of the three-body problem is typical of a perturbation of a class of *integrable systems*. In the 1950's and 1960's, the non-Soviet mathematical world joined the fray with the work of, for example, Moser, Smale, and Peixoto. During this period, there arose some serious efforts towards classification of dynamical systems, including a study of those attributes of dynamical systems that were *structurally stable* in the sense of being robust to perturbation. An important contribution here is an understanding of the primitive rôle of symbolic dynamics in some classes of dynamical systems via Markov partitions [Bowen 1970]. The mathematical subject of dynamical systems, initiated during this era, continues as an active area covering a broad array of subjects.

The theoretical study of dynamical systems, while offering deep insights into structures of interest in dynamics, is sometimes disconnected from the study of particular, sometimes physical, systems. Thus, in parallel with and in harmony with (subject to time delays in both directions) the development of the theoretical foundations, the 1960's and 1970's also saw an expansion of the subject of dynamical systems through applications. A famous example is the three-state differential equation model of Lorenz [1963] that arose in atmospheric modelling. This example exhibits, in a differential equation model, what appears to be a *strange attractor*. However, it was not until the careful numerical study of [Tucker 2002] that the Lorenz equations were rigorously shown to possess such an attractor.

Much of the discussion above has focussed on dynamics arising from ordinary differential equations, where the time variable is continuous. However, discrete-time dynamical systems arise in natural ways themselves, as well as being useful in the study of ordinary differential equation models, e.g., through Poincaré maps associated with systems with periodic time-dependence or with periodic orbits. An autonomous discrete-time dynamical system is simply an iteration of a mapping of the state space, as we shall point out in Example 2.2–2. Unlike the situation with ordinary differential equations, which define typical continuous-time dynamics, one can have elementary discrete-time dynamical systems that exhibit extremely complicated, e.g., chaotic behaviour. An early example of such a discrete-time dynamical system is the *logistic map*, introduced by the seemingly ubiquitous Lorenz [1964]. The logistic map is actually a one-parameter family of mappings, and the character of the dynamics as a function of the parameter is something that can be understood with some success, after much hard work. For example, the logistic map exhibits a sequence of *period doubling bifurcations*, and this phenomenon exhibits a *universal* behaviour revealed by Feigenbaum [1976]. Also, as this sequence of periodic doubling bifurcations takes place, a period three orbit appears. It is a theorem of Sharkovskii [1964] that, for a large class of interval mappings, the existence of a periodic three orbit implies the existence of orbits of all periods. As we shall see in Section 3, periodic orbits play an important rôle in characterising the presence of chaotic dynamics.

A comprehensive modern introduction to the mathematical foundations of the subject of dynamical systems is the book of Katok and Hasselblatt [1995]. A well written summary of the status of some aspects of applied dynamical systems as of the mid-1980's is [Guckenheimer and Holmes 1983].

We close this brief historical overview of chaos in dynamics with a reflection on the popular science interest in *chaos theory* that arose in the 1980's and 1990's, and exemplified in the bestselling book of Gleick [1987]. The attractive theme of this popularisation of chaotic dynamical systems is that simple systems can exhibit complicated behaviour. Now, thirty years after this popular incarnation, the idea that seems to have permanently permeated the social conscience is that a butterfly might flap its wings on the banks of the Amazon River, resulting in a young woman drinking a somewhat subpar cup of tea on the Mongolian Steppe. This *sensitive dependence on initial condition* is indeed a part of chaotic dynamics, but we wish to give a far more substantial outline than this.

1.2. Prerequisites. Any moderately serious discussion of chaotic dynamics requires mathematics. Moreover, the more substantial is the discussion, the more mathematics will be required. It is neither possible nor advisable to try to make a survey such as this mathematically self-contained. Therefore, we present it with an assumption on the mathematical prerequisites of the reader that would be commensurate with them making real use of the the content of the survey. To this end, we offer the following guide on reading this article.

1. We assume that the reader has a basic understanding of elementary mathematical notation and set theoretic conventions. One place where we use nonstandard (but obvious) notation is our use of $\mathbb{R}_{\geq 0}$ and $\mathbb{R}_{>0}$ to denote the sets of nonnegative and positive real numbers, and our use of $\mathbb{Z}_{\geq 0}$ and $\mathbb{Z}_{>0}$ for similar sets of integers.
2. We assume that the reader is comfortable with basic ideas from analysis and topology. A good reference is [Willard 1970]. We shall make passing reference in Section 4.3 to topics in measure theory, for which a good reference is [Cohn 2013].
3. We assume that the reader is comfortable with ideas from differential geometry such as manifolds, submanifolds, mappings of various degrees of regularity, and vector fields and flows. A good reference is [Lee 2003].
4. We shall frequently use—and have already done so extensively above—terminology that we have not defined. The reader should look into the references to obtain definitions of concepts not given in the paper.

2. Dynamical system models

In this section we give specify the sorts of dynamical systems we shall work with in the paper, as well as give the examples of dynamical systems that we will use to illustrate various phenomenon.

2.1. Semiflows.

In order to make precise statements about dynamical systems, we should first give a proper definition of what we mean by a dynamical system. We wish to give a definition that is flexible enough to allow for all of the examples we consider, and this means allowing both continuous- and discrete-time models. In the first case, time is the set $\mathbb{R}_{\geq 0}$ of nonnegative real numbers. In the second case, time is the set $\mathbb{Z}_{\geq 0}$ of nonnegative integers. We will always use the symbol $\mathbb{T}$ to stand for either $\mathbb{R}_{\geq 0}$ or $\mathbb{Z}_{\geq 0}$.

2.1 Definition: (Semiflow) Let X be a set. A ***semiflow*** in X is a mapping $\Phi\colon \mathbb{T}\times X \to X$ satisfying

(i) $\Phi(0,x) = x$, $x \in X$, and

(ii) $\Phi(s+t,x) = \Phi(s,\Phi(t,x))$, $s,t \in \mathbb{T}$, $x \in X$.

We denote by $\Phi_t\colon X \to X$ the mapping defined by $\Phi_t(x) = \Phi(t,x)$. The set X is the ***state space*** and the mapping Φ is the ***dynamics***. If $\mathbb{T} = \mathbb{R}_{\geq 0}$, then Φ is a ***continuous-time*** semiflow and, if $\mathbb{T} = \mathbb{Z}_{\geq 0}$, then Φ is a ***discrete-time*** semiflow. •

One can also define the notion of a *flow*, where the time is allowed to be negative. For concreteness, we stick with semiflows.

If the set X is a topological space $\mathcal{S}$, then a semiflow Φ is a $\mathbf{C^0}$***-semiflow*** if Φ is a continuous map. If the set X is a smooth manifold M and if $r \in \mathbb{Z}_{>0} \cup \{\infty\}$, then a semiflow Φ is a $\mathbf{C^r}$***-semiflow*** if Φ is of class C^r.

We should verify that this notion of semiflow captures the two principal sorts of dynamical systems in which we are interested.

2.2 Examples: (Classes of dynamical systems)

1. *(ODE models)* Let M be a smooth manifold and let $F\colon \mathsf{M} \to \mathsf{TM}$ be a vector field. Given $x \in \mathsf{M}$, let $t \mapsto \Phi^F(t,x)$ be the solution to the initial value problem

$$\dot{\xi}(t) = F \circ \xi(t), \quad \xi(0) = x.$$

   If F is locally Lipschitz, then this solution exists and is uniquely defined by the initial condition for times t in some open neighbourhood of 0 [Lee 2003, Theorems 17.17 and 17.18]. We shall assume that, in fact, the solution exists for all $t \in \mathbb{R}_{\geq 0}$, i.e., that F is a *forward complete* vector field. For example, if M is compact, then F is guaranteed to be forward complete [Lee 2003, Theorem 17.11]. In this case, we have a well-defined mapping $\Phi^F\colon \mathbb{R}_{\geq 0} \times \mathsf{M} \to \mathsf{M}$. With the Lipschitz assumption on F, Φ^F is continuous and also satisfies the requirements of Definition 2.1 to be a semiflow [Lee 2003, Theorem 17.19]. If, furthermore, F is of class C^r, $r \in \mathbb{Z}_{>0} \cup \{\infty\}$, then Φ^F is also of class C^r [Lee 2003, Theorem 17.19].

   That is to say, if F is locally Lipschitz and forward complete, then Φ^F is a continuous-time C^0-semiflow. If, additionally, F is of class C^r, then Φ^F is a continuous-time C^r-semiflow.

2. *(Iteration of mappings)* Let $\mathcal{S}$ be a topological space and let $\phi\colon \mathcal{S} \to \mathcal{S}$ be a continuous mapping. We define

$$\begin{aligned} \Phi^\phi\colon \mathbb{Z}_{\geq 0} \times \mathcal{S} &\to \mathcal{S} \\ (k,x) &\mapsto \underbrace{\phi \circ \cdots \circ \phi}_{k \text{ times}}(x), \end{aligned}$$

   with the understanding that $\Phi^\phi(0,x) = x$. It is easy to see that Φ^ϕ is a discrete-time C^0-semiflow, and is a particularly easy one to describe: one *iterates* the mapping ϕ.

   Of course, if the topological space is, additionally, a smooth manifold and if ϕ is of class C^r, $r \in \mathbb{Z}_{>0} \cup \{\infty\}$, then the above definition yields a discrete-time C^r-semiflow. •

2.2. A few dynamical systems concepts. While we attempt to avoid making this survey overloaded with dynamical systems chatter, it is convenient to have at hand a few general concepts so that we can sensibly talk in Section 3 about what constitutes chaotic dynamics.

The first notion is that of an orbit, which is a natural construction of interest as it tracks a specific initial condition.

2.3 Definition: (Orbit) Let X be a set and let $\Phi\colon \mathbb{T} \times X \to X$ be a semiflow. For $x \in X$, the ***orbit*** of x is $\mathrm{Orb}(x) = \{\Phi(t,x) \mid t \in \mathbb{T}\}$. •

Another general concept that will arise in our discussions is that of invariance.

2.4 Definition: (Invariance, restriction) Let X be a set and let $\Phi\colon \mathbb{T} \times X \to X$ be a semiflow.

(i) A subset $A \subseteq X$ is ***invariant*** for Φ if $\mathrm{Orb}(x) \subseteq A$ for $x \in A$.

(ii) If A is invariant for Φ, then the ***restriction*** of Φ to A is the semiflow

$$\begin{aligned} \Phi|A\colon \mathbb{T} \times A &\to A \\ (t,x) &\mapsto \Phi(x). \end{aligned}$$

•

Generally speaking, although one may not be able to understand the dynamics of a system in totality, sometimes one can find suitable invariant sets where the restriction of the dynamics can be understood.

2.3. Examples. In this section we introduce the examples that we use in the paper. A number of these examples are of the "academic" variety, because it is only with such simple examples that one can come to some decent understanding of their dynamics within the confines of a short survey. The "serious" physical examples that we consider cannot be analysed within the boundaries of a paper such as this, and so, for these examples, we shall only be able to make broad statements, referring to references for details. While most of the examples are used to illustrate chaotic dynamics, some are used only to show that the definition of chaos cannot be oversimplified.

Rotation of the circle. The example we begin with is indeed extremely simple. As state space, we take the unit circle

$$\mathbb{S}^1 = \{(x_1, x_2) \in \mathbb{R}^2 \mid \; x_1^2 + x_2^2 = 1\}.$$

As dynamics, we take the discrete-time semiflow defined as in Example 2.2–2 by counter-clockwise rotation by an angle $2\pi\alpha$ for $\alpha \in [0, 1)$:

$$\begin{aligned}\phi_\alpha \colon \mathbb{S}^1 &\to \mathbb{S}^1\\ (x_1, x_2) &\mapsto (x_1 \cos(2\pi\alpha) + x_2 \sin(2\pi\alpha), -x_1 \sin(2\pi\alpha) + x_2 \cos(2\pi\alpha)).\end{aligned}$$

One way to view the mapping is that it is the restriction to the unit circle of the linear rotation of the plane, noting that circles are invariant submanifolds for linear rotations.

It is interesting to consider the orbits of Φ^{ϕ_α}. Here there is an essential difference between the cases of α being rational or irrational. In Figure 1 we depict orbits in particular

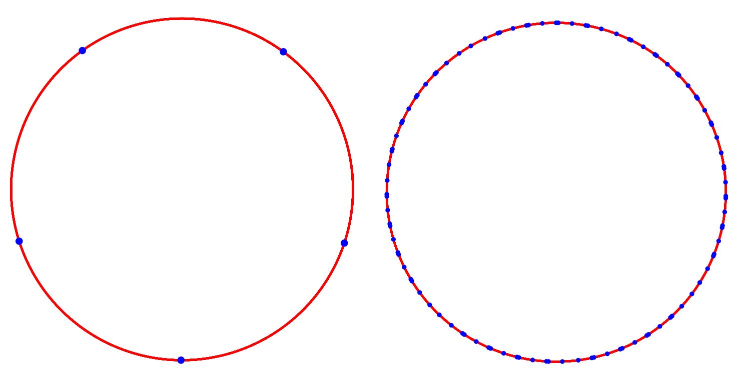

Figure 1. Orbits for rational (left) and irrational (right) rotations of $\mathbb{S}^1$

instances of each of these cases. The point is that the orbits are finite when α is rational and infinite (indeed, dense in $\mathbb{S}^1$) when α is irrational.

The dynamics of more general mappings of $\mathbb{S}^1$ are considered in [Hasselblatt and Katok 2003, §4.3].

The logistic map. The famous logistic map, apparently introduced by Lorenz [1964], is the mapping

$$\begin{aligned}\lambda_\mu\colon [0,1] &\to [0,1]\\ x &\mapsto \mu x(1-x),\end{aligned}$$

which is well-defined for $\mu \in [0,4]$ (meaning that, for these values of μ, it takes values in $[0,1]$). This defines, as in Example 2.2–2, a discrete-time semiflow with state space $[0,1]$.

There is a standard method for representing the orbits of interval maps such as the logistic map. One chooses an initial state, then one draws a vertical line from this initial point on the x-axis until it intersects the graph. Then one draws an horizontal line that intersects the diagonal line "$y = x$." Then one again draws a vertical line to intersect the graph. Then one carries on in this way. The result is sometimes called a ***cobweb plot***, and we show one such for the logistic map on the left in Figure 2. While a cobweb plot

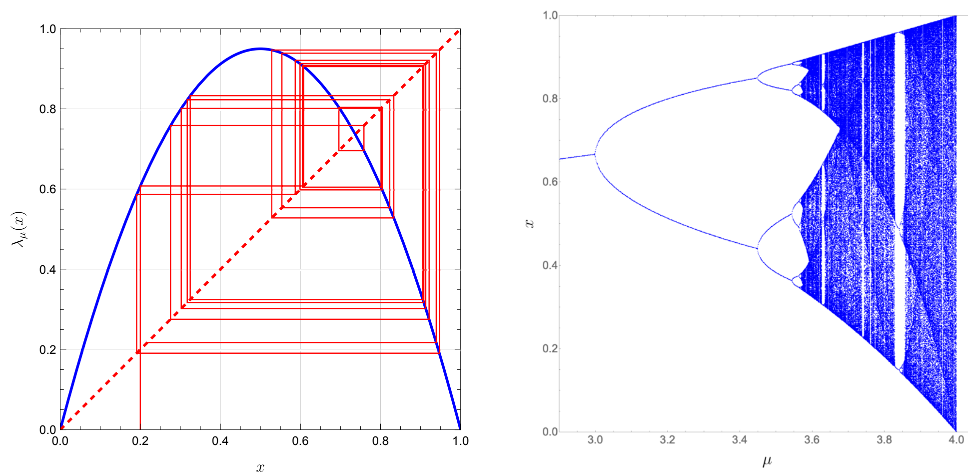

Figure 2. Cobweb plot with $\mu = 3.9$ (left) and bifurcation diagram (right) for logistic map

for the logistic map for a fixed value of μ makes for an interesting picture of seemingly chaotic dynamics, some of the most interesting features of this dynamical system arise from studying its behaviour as μ gets near 4. One is interested in the long time behaviour of orbits. To do this, one chooses an initial state and then iterates it many times, then chopping off the first segment of iterates of some long length. What remains are the points visited along a "steady-state" orbit. This process is carried out and displayed on the right in Figure 2. For values of μ at the left side of the plot, one sees that there is a single stable fixed point to which initial states eventually evolve. As one increases μ, there are two states that branch off and to which initial states eventually evolve; this is a period 2 orbit. There then arises more and more periodic orbits with the period doubling each time. This *period doubling* accumulates at $\mu \approx 3.56995$, and beyond this parameter value, the dynamics are typically *chaotic*. At $\mu = 1 + \sqrt{8}$, one has a period 3 orbit. We shall see in Section 3.2 that periodic orbits play an important rôle in characterising chaotic dynamics. A study of the *bifurcation diagram* we show above is carried out in [Devaney 2003, pages 31-50].

The tent map. The logistic map is widely studied and some of its interesting features can be incorporated into an undergraduate dynamical systems course. We shall also consider here the interval map known as the tent map, which is the discrete-time dynamical system with state space $[0,1]$ and defined by the mapping

$$\begin{aligned}\tau_\alpha\colon [0,1] &\to [0,1]\\ x &\mapsto \alpha\min\{2x, 2-2x\},\end{aligned}$$

where the parameter α takes values in $[0,1]$. In Figure 3 we show a cobweb plot and the

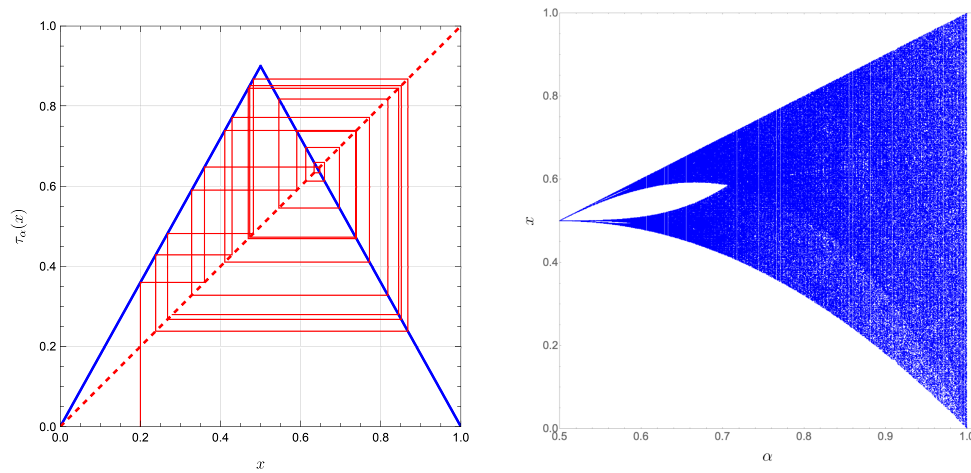

Figure 3. Cobweb plot with $\alpha = 0.9$ (left) and bifurcation diagram (right) for tent map

bifurcation diagram for the tent map.

We shall principally consider the case where $\alpha = 1$, and will use the tent map in this case as an example of a dynamical system that can be explicitly be shown to exhibit the characteristics of chaotic dynamics. For this reason, let us understand tent map better in this case, following the presentation of Crampin and Heal [1994]. To most easily understand the tent map, we represent $x \in [0,1]$ by its binary decimal expansion:

$$x = \sum_{j=0}^{\infty} \frac{a_j}{2^j},$$

where $a_j \in \{0,1\}$. We call this the ***bicimal expansion*** of x and will write $x = 0.a_1a_2a_3\cdots$ as one does for decimal expansions. We then note that τ_1 is defined by one of two operations: (1) multiplication by 2 (if $x \in [0,\frac{1}{2}]$) or (2) subtraction from 1 followed by multiplication by 2 (if $x \in [\frac{1}{2},1]$). One then easily ascertains that

$$\underbrace{\tau_1 \circ \cdots \circ \tau_1}_{k \text{ times}}(0.a_1a_2a_3\cdots) = \begin{cases} 0.a_{k+1}a_{k+2}a_{k+3}\cdots, & a_k = 0,\\ 0.a'_{k+1}a'_{k+2}a'_{k+3}\cdots, & a_k = 1,\end{cases}$$

where $a \mapsto a'$ is the "bit flip" mapping, swapping 0 with 1 and vice versa.

The logistic map and the tent map are examples of interval maps. We refer to [Collet and Eckmann 2009] for a general treatment of the dynamics of interval maps.

Hyperbolic toral automorphism. The next dynamical system we consider is again discrete-time, this time having the state space of the 2-torus, $\mathbb{T}^2 = \mathbb{R}^2/\mathbb{Z}^2$. To define the mapping $\mathbb{T}^2 \to \mathbb{T}^2$ that prescribes this dynamical system, consider the 2×2 matrix

$$\boldsymbol{A} = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}.$$

We note that, regarded as an invertible linear mapping of $\mathbb{R}^2$, $\boldsymbol{A}$ maps the integer lattice onto itself: $\boldsymbol{A}(\mathbb{Z}^2) \subseteq \mathbb{Z}^2$. This implies that $\boldsymbol{A}$ descends to a mapping $\phi_{\boldsymbol{A}} \colon \mathbb{T}^2 \to \mathbb{T}^2$. This mapping is sometimes called the "cat map" because it is illustrated in [Arnol′d and Avez 1968, Example 1.16] by showing what the mapping does to an image of a cat. The easiest way to come to an understanding of what this mapping does is to depict the torus as a square with the left and right, and the top and bottom edges identified. In Figure 4 we

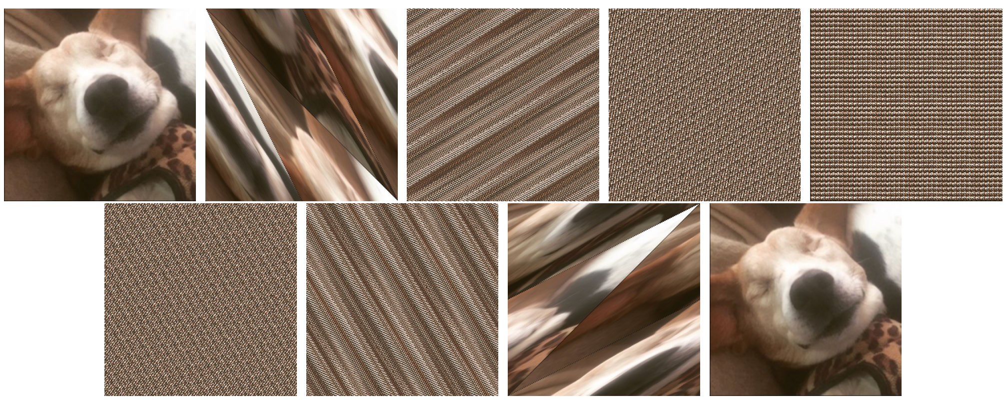

Figure 4. The map $\phi_{\boldsymbol{A}}$ applied $0, 1, 10, 20, 30, 40, 50, 59, 60$ times to the square representation of a torus

show the effects of repeated iterations of $\phi_{\boldsymbol{A}}$ on an image within the square torus.

Smale horseshoe. A common mechanism for arriving at invariant sets on which the dynamics is chaotic is to show that there are subsets where the dynamics is *conjugate* to those of the so-called ***Smale horseshoe map***. This is a discrete-time dynamical system introduced by Smale [1967] to understand certain behaviour in dynamical systems; we will describe the 2-dimensional version of this mapping here. The mapping is an homeomorphism of $\mathbb{R}^2$ that behaves in a prescribed manner on the square $S = [0, 1]^2$. At the top in Figure 5 we depict what the mapping does to the square after two iterates. (To really understand this mapping, the reader should anticipate devoting a little time.) The main features of the mapping are its stretching and folding behaviour. On the bottom in the same figure, we look at the set of points that remain in S after one, and then two, forward and backward iterates of the mapping. As can be easily seen from the construction, the set of points in S that remain after all forward and backward iterates will be a *Cantor set*. A compact discussion of the Smale horseshoe, and how it arises from other dynamical systems, can be found in [Teschl 2012, Chapter 13].

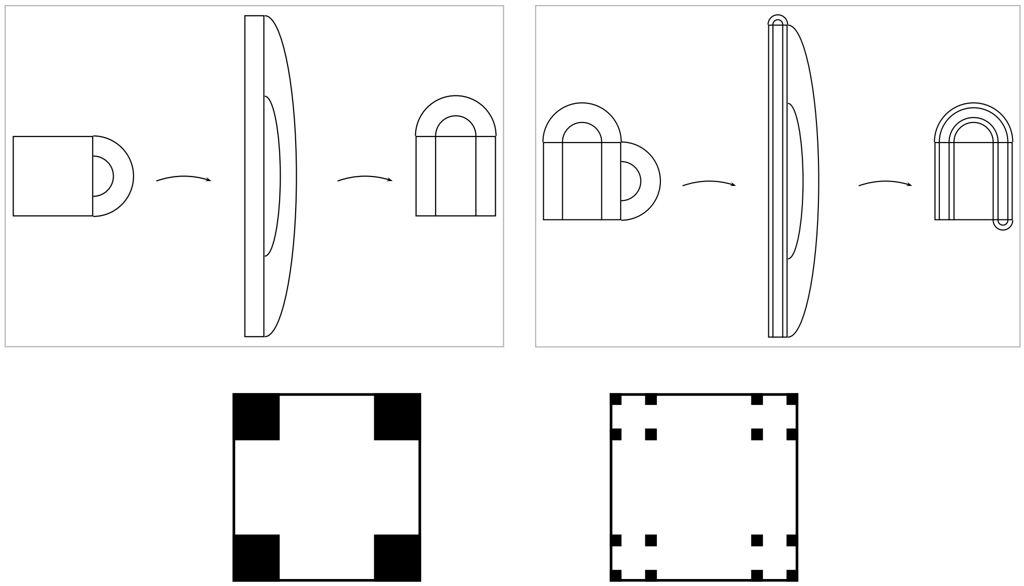

Figure 5. The first (top right) and second (top left) iterate of the Smale horseshoe map, and the sets $\cap_{n=-1}^{1}\phi^n(S)$ and $\cap_{n=-2}^{2}\phi^n(S)$ (bottom)

Winding flow on the torus. The final three examples we give are continuous-time semiflows coming from flows of vector fields, as in Example 2.2–1. For such systems defined on 2-dimensional manifolds, chaotic dynamics is not possible within a compact invariant set. For systems defined on open subsets of $\mathbb{R}^2$, this is a consequence of the Poincaré–Bendixson Theorem. For other 2-dimensional manifolds, there are other theorems that govern the long time behaviour of the dynamics. Thus, for these continuous-time semiflows, one requires a state space of dimension at least three to get dynamics that one would consider to be chaotic.

This being said, our first continuous-time example has state space being the 2-torus. As with the toral automorphism of the preceding section, we consider first a dynamical systems on $\mathbb{R}^2$, that associated with the simple ordinary differential equation

$$\begin{aligned}\dot{x}_1(t) &= \omega_1 x_1(t),\\ \dot{x}_2(t) &= \omega_2 x_2(t),\end{aligned}$$

for $\omega_1, \omega_2 \in \mathbb{R} \setminus \{0\}$. We note that the associated vector field

$$\overline{F} = \omega_1 \frac{\partial}{\partial x_1} + \omega_2 \frac{\partial}{\partial x_2}$$

projects to a well-defined vector field F on $\mathbb{T}^2$, i.e., if (x_1, x_2) and (y_1, y_2) project to the same point on $\mathbb{T}^2$, then $\overline{F}(x_1, x_2)$ and $\overline{F}(y_1, y_2)$ project to the same tangent vector on $\mathbb{T}^2$. It follows from [Lee 2003, Lemma 18.4] that the semiflow associated to $\overline{F}$ descends to the semiflow associated with F. The resulting semiflow we denote by Φ^F, as per Example 2.2–1. In Figure 6 we depict orbits of this semiflow in cases where ω_1 and ω_2 are rationally and

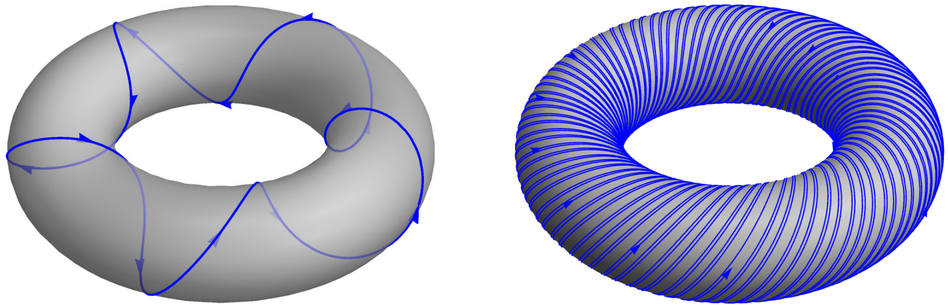

Figure 6. Orbits for the winding flow on the torus for $\frac{\omega_1}{\omega_2}$ rational (left) and irrational (right)

irrationally related. We see that, in the rational case, the orbits are embedded copies of $\mathbb{S}^1$, while, in the irrational case, the orbits are immersed copies of $\mathbb{R}$. Said differently, in the rational case all orbits are periodic, while in the irrational case none of the orbits are periodic.

Note that there is a natural discrete-time dynamical system that one can associate with this continuous-time system. If one considers a planar cross-section of the torus where the plane intersects the torus in a circle, then one arrives at a mapping of the circle by considering where the orbit in $\mathbb{T}^2$ through a point on the circle next intersects the plane. The resulting mapping of the circle will be a rotation, just as considered in our example above. Thus we see that there are reasons, apart from their intrinsic interest, to be concerned with discrete-time dynamical systems, even when one is studying continuous-time dynamical systems.

The forced Duffing equation. In Figure 7 we depict a system with a hanging metallic strip clamped at one end and free at the other. At the free end, magnets render the straight down position unstable and create a pair of stable equilibrium configurations. The assembly is shaken by a sinusoidal force. If the horizontal displacement of the tip of the strip is denoted by x, an approximate model of the dynamics is

$$\ddot{x}(t) + \delta\dot{x}(t) - \alpha x(t) + \beta x(t)^3 = \gamma\cos(\omega t),$$

for positive constants δ, α, β, γ, and ω. In this form, this differential equation does not fit into our dynamical system framework because the differential equation is not autonomous. However, we can introduce time as a state variable, and as well use the displacement x and its velocity as state variables, i.e., $(x_1, x_2, x_3) = (x, \dot{x}, t)$. The resulting

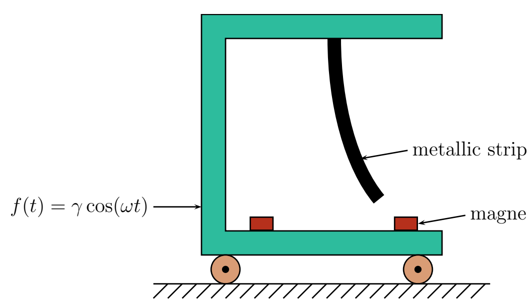

Figure 7. A physical model of the Duffing equation

differential equation is

$$\begin{aligned}\dot{x}_1(t) &= x_2(t),\\ \dot{x}_2(t) &= \alpha x_1(t) - \beta x_1(t)^3 - \delta x_2(t) + \gamma\cos(\omega x_3),\\ \dot{x}_3(t) &= 1.\end{aligned} \tag{2.1}$$

While introducing time as a state is a nice dodge, it has the unsatisfactory consequence of losing touch with the important state variables (x_1, x_2). This can be rectified by defining a mapping on $\mathbb{R}^2$ as follows. Let $(x_{10}, x_{20}) \in \mathbb{R}^2$ and consider the solution

$$t \mapsto (x_1(t), x_2(t), t)$$

to the differential equations (2.1) with the initial condition (x_{10}, x_{20}, t_0). We then define

$$\phi_{t_0}(x_{10}, x_{20}) = \left(x_1\left(\tfrac{2\pi}{\omega}\right), x_2\left(\tfrac{2\pi}{\omega}\right)\right).$$

The idea is that the mapping ϕ_{t_0} takes a snapshot of the states (x_1, x_2) after one period of the forcing. This is, then, another instance of how one can arrive at a discrete-time dynamical system from a continuous-time dynamical system. A fairly detailed consideration of the Duffing equation, with many references, is undertaken by Ueda [1991].

The Lorenz equation. The dynamical system we consider here was introduced by Lorenz [1963] as a simplified model for atmospheric convention. The system is determined by the three state differential equation

$$\begin{aligned}\dot{x}_1(t) &= -\sigma(x_1(t) - x_2(t)),\\ \dot{x}_2(t) &= r x_1(t) - x_2(t) - x_1(t)x_3(t),\\ \dot{x}_3(t) &= x_1(t)x_2(t) - b x_3(t)\end{aligned}$$

for parameters $\sigma, b, r \in \mathbb{R}_{>0}$. A relatively recent overview of some of the methods for understanding the dynamics of the Lorenz equation is given by Viana [2000], with an emphasis on understanding its *strange attractor*.

3. The ingredients of chaotic dynamics

The term *chaos* seems to have originated in the mid-1970's, with two early papers using the terminology being [Li and Yorke 1975] and [May 1975]. Both papers work with discrete-time dynamical systems whose state space is an interval, where it is simpler to provide definitions of what is meant by complex dynamics. The definitions of chaos used in these early papers are not the same, and indeed no attempt is made to give a universal definition. The property that seems to be interpreted by Li and Yorke as being chaos is the existence of an uncountable set of initial states where the lim sup of distances between orbits with different initial states is positive. The idea is that there is no simple steady state behaviour of the dynamics. On the other hand, May shows the existence of infinitely many aperiodic bounded orbits.

In the years following these initial forays into attempts to characterise chaotic dynamics, there were numerous attempts to pin down precisely what this means. While there is no consensus about this, a commonly used characterisation is due to Devaney [2003], and in this section we will outline the ingredients of this characterisation. These ingredients are (1) sensitive dependence on initial conditions, (2) denseness of periodic points, and (3) a transitivity condition. Sometimes, versions of these conditions are interconnected, e.g., [Banks, Brooks, Cairns, Davis, and Stacey 1992]. This notwithstanding, we will give precise definitions for all of these phenomenon, and discuss when these ingredients are present in our examples of Section 2.3.

3.1. Sensitive dependence on initial conditions. The idea of sensitive dependence on initial conditions, as we indicated at the end of Section 1.1, is one that is present in any popular account of chaotic dynamics. However, to make the notion precise is not as easy as it is sometimes made out to be. For example, exponential divergence of the distance between trajectories is often used to characterise this. However, this becomes meaningless when the state space is compact; this is made more meaningless by the fact that many instances of chaotic dynamics arising in practice occur on compact state spaces or on compact subsets of state spaces. Therefore, a better definition should be adopted. We give the definition of [Hasselblatt and Katok 2003, Definition 7.2.11].

3.1 Definition: (Sensitive dependence on initial conditions) Let $(\mathcal{M}, \mathrm{d})$ be a metric space. A C^0-semiflow $\Phi\colon \mathbb{T}\times\mathcal{M}\to\mathcal{M}$ has ***sensitive dependence on initial conditions*** if there exists $\Delta\in\mathbb{R}_{>0}$ such that, for every $x\in\mathcal{M}$ and $\epsilon\in\mathbb{R}_{>0}$, there exists $y\in\mathcal{M}$ and $T\in\mathbb{T}$ such that

(i) $\mathrm{d}(x,y)<\epsilon$ and

(ii) $\mathrm{d}(\Phi(T,x),\Phi(T,y))\geq\Delta$. •

The idea is that there is some distance Δ such that, no matter how close one chooses two initial conditions, the trajectories through them will differ by at least Δ at some point in time; this is rather like the notion used by [Li and Yorke 1975]. Let us examine this notion in some of the examples from Section 2.3.

3.2 Examples: (Sensitive dependence on initial conditions)

1. If one considers the simple exponential growth equations

$$\dot{x}(t)=\alpha x(t)\ \ (\text{resp. } x(n+1)=\alpha x(n))$$

in continuous-time (resp. discrete time) with $\alpha>0$ (resp. $\alpha>1$), then, of course, one has sensitive dependence on initial conditions and, moreover, the constant Δ in the definition can be taken to be arbitrarily large. The inference to make is that sensitive dependence on initial conditions, by itself, is not a good indicator of chaotic dynamics.

2. The rotation of the circle and the winding flow on the torus do not have sensitive dependence on initial conditions. Indeed, in both cases, the distance between trajectories is constant as a function of time.

3. For some values of the parameter μ, the logistic map does have sensitive dependence on initial conditions. To rigorously prove this, however, is something that goes beyond what we can do here. One thing that can be said fairly easily, however, is that the mapping

$$x\mapsto\tfrac{1}{2}(1-\cos(2\pi x))$$

establishes a *conjugacy* between the logistic map with $\mu = 4$ and the tent map with $\alpha = 1$. As we shall see in the next example, the tent map with $\alpha = 1$ has sensitive dependence on initial conditions.

4. We claim that the tent map $\phi = \tau_1$ with $\alpha = 1$ has sensitive dependence on initial conditions. We show this as follows. Let $\epsilon \in \mathbb{R}_{>0}$ and let

$$y_1 = 0.b_1b_2b_3\cdots, \quad y_2 = 0.c_1c_2c_3\cdots$$

be bicimal representations of two points in $[0, 1]$. Now let

$$x_1 = 0.a_1\cdots a_k0b_1b_2b_3\cdots, \quad x_2 = 0.a_1\cdots a_k0c_1c_2c_2\cdots$$

be two initial conditions. If k is large enough, these two initial conditions will differ by at most ϵ. Moreover, $\Phi^\phi(k+1, x_1) = y_1$ and $\Phi^\phi(k+1, x_2) = y_2$. That is to say, we can find two initial conditions differing by as small a quantity as we want, but which will end up after time $k+1$ at any two specified points. It is easy to see that this observation implies sensitive dependence on initial conditions.

5. The hyperbolic toral automorphism has sensitive dependence on initial conditions, although this requires a more detailed analysis than we can perform here. We can say, however, that the easiest way to understand the dynamics of hyperbolic toral automorphisms is to use symbolic dynamics [Lind and Marcus 1995, Example 6.5.10].

6. The dynamics of the Smale horseshoe map on the invariant set inferred in our discussion of this dynamical system in Section 2.3 can be shown to have sensitive dependence on initial conditions. This is most easily proved using symbolic dynamics.

7. For some values of parameters, there are invariant subsets of the state space for both the forced Duffing equation and the Lorenz equation on which the dynamics exhibits sensitive dependence on initial conditions. This assertion, however, is well beyond what we can show here, and indeed a rigorous justification has taken concerted effort over the span of many years. For the Duffing equation, we refer the reader to [Ueda 1991] and, for the Lorenz equation, we refer the reader to [Tucker 2002]. •

In general, explicitly ascertaining sensitive dependence on initial conditions in a given example will be a challenge. However, we shall provide some related techniques in Section 4.

3.2. Denseness of periodic points. In various places in our discussion above, we made reference to periodic orbits. Let us provide a definition here.

3.3 Definition: (Periodic orbit, periodic point) Let X be a set and let $\Phi\colon \mathbb{T} \times X \to X$ be a semiflow. If $x \in X$, the orbit $\mathrm{Orb}(x)$ is a ***periodic orbit*** for Φ if there exists $T > 0$ such that $\Phi(T, x) = x$. In this case, x is a ***periodic point***. We denote by $\mathrm{Per}(\Phi)$ the set of periodic points. •

The attribute of chaotic dynamics we consider in this section is the density of periodic points. The meaning of this is pretty much clear from the wording, but let us give a definition in any case. We denote by cl the topological closure operator.

3.4 Definition: (Denseness of periodic points) Let $\mathcal{S}$ be a topological space and let $\Phi\colon \mathbb{T}\times\mathcal{S} \to \mathcal{S}$ be a C^0-semiflow. Then Φ has ***denseness of periodic points*** if $\mathrm{cl}(\mathrm{Per}(\Phi)) = \mathcal{S}$. •

Let us consider denseness of periodic points for the examples from Section 2.3.

3.5 Examples: (Denseness of periodic points)

1. For the rotation mapping of the circle, either all points are periodic points (when α is rational) or there are no periodic points (when α is rational). Thus, when α is rational, this semiflow has denseness of periodic points. What one should deduce from this is that denseness of periodic points is, by itself, not an indicator of chaotic dynamics.
2. As was the case for sensitive dependence on initial conditions, for some parameter values, the logistic map has denseness of periodic points. To rigorously prove this takes some effort. However, in the case of $\mu = 4$, we again can use the conjugacy with the tent map (considered next) to give denseness of periodic points.
3. For the tent map $\phi = \tau_1$ with $\alpha = 1$, one can show that $x \in \mathrm{Per}(\Phi^\phi)$ if and only if $x = \frac{k}{q}$ for an even nonnegative integer k and an odd positive integer q with $k < q$. To prove this, one does the following [Crampin and Heal 1994].
   (a) Define the mapping $\psi(x) = 2x \mathrm{mod} 1$. The mapping ψ, when applied to the bicimal representation of numbers in $[0, 1]$, is simply a shift to the left by one decimal place.
   (b) For the discrete-time semiflow Φ^ψ, one can show that $x \in \mathrm{Per}(\Phi^\psi)$ if and only if $x = \frac{k}{q}$ for $k \in \mathbb{Z}_{\geq 0}$ and q an odd positive integer with $k < q$. This can be shown noting that periodic points have repeating bicimal representations, then using with the characterisation of ψ from (a).
   (c) Using the directly verified fact that $\phi \circ \psi = \phi \circ \phi$, one shows inductively that
   $$\phi \circ \underbrace{\psi \circ \cdots \circ \psi}_{n \text{ times}} = \underbrace{\phi \circ \cdots \circ \phi}_{n+1 \text{ times}}$$
   for $n \in \mathbb{Z}_{>0}$.
   (d) If $x = \frac{k}{q}$ has the claimed form to be in $\mathrm{Per}(\Phi^\phi)$, then one notes that $\frac{x}{2} \in \mathrm{Per}(\Phi^\psi)$ by (b). Thus $\Phi^\psi(n, \frac{x}{2}) = \frac{x}{2}$ for some $n \in \mathbb{Z}_{>0}$. Using (c), one shows that $\Phi^\phi(n, x) = x$ and so $x \in \mathrm{Per}(\Phi^\phi)$.
   (e) If $x \in \mathrm{Per}(\Phi^\phi)$, then $\Phi^\phi(n, x) = x$ for some $n \in \mathbb{Z}_{>0}$. Using (c), one shows that
   $$\phi(\tfrac{x}{2}) = \phi(\Phi^\psi(n, \tfrac{x}{2})),$$
   which implies that either (i) $\Phi^\psi(n, \frac{x}{2}) = \frac{x}{2}$ or (ii) $\Phi^\psi(n, \frac{x}{2}) = 1 - \frac{x}{2}$. In each case, we use the definition of ψ to show that $\frac{x}{2}$ has the form for periodic points of Φ^ψ from (b). Thus x has the asserted form for periodic points of Φ^ϕ.

   It is now easy to show that $\mathrm{cl}(\mathrm{Per}(\Phi^\phi)) = [0, 1]$.
4. The hyperbolic toral automorphism has denseness of periodic points. As with sensitive dependence on initial conditions, this is most easily demonstrated using symbolic dynamics [Lind and Marcus 1995, Example 6.5.10].
5. On the invariant set for the Smale horseshoe map discussed in Section 2.3, the set of periodic points is dense. This is proved most easily using symbolic dynamics.
6. The situation concerning denseness of periodic points for the winding flow on the torus is rather the same as that for rotation of the circle: either all points are periodic points or no points are periodic points.

7. For the forced Duffing equation and the Lorenz equation, for some parameter values there are invariant subsets of the state space on which the restricted dynamics have denseness of periodic points. It is a difficult matter to rigorously prove such assertions, and we refer to the references given in Example 3.2–7 above. •

3.3. Transitivity or indecomposability. The final ingredient to chaotic dynamics that we consider is that the dynamics should, in a sense that must be made precise, "bind together" the state space. There are various precise versions of such properties, and we give three such.

3.6 Definition: (Minimal, topologically transitive, dense orbit) Let $\mathcal{S}$ be a topological space and let $\Phi\colon \mathbb{T}\times\mathcal{S}\to\mathcal{S}$ be a C^0-semiflow in $\mathcal{S}$. Then:

(i) Φ is ***minimal*** if the only nonempty, closed, and invariant subset of $\mathcal{S}$ is $\mathcal{S}$ itself;

(ii) Φ is ***topologically transitive*** if, for open sets $\mathcal{U},\mathcal{V}\subseteq\mathcal{S}$, there exists $T>0$ such that $\Phi_T(\mathcal{U})\cap\Phi_T(\mathcal{V})\neq\varnothing$;

(iii) Φ has a ***dense orbit*** if there exists $x\in\mathcal{S}$ such that $\mathrm{cl}(\mathrm{Orb}(x))=\mathcal{X}$. •

These notions are not independent, in general. Relationships between them are discussed in the discrete-time case in [de Vries 2014] and in the continuous-time case in [Alongi and Nelson 2007].

Let us explore these notions for our examples from Section 2.3. In order to not wander off in too many directions, let us fix upon the possession of a dense orbit as the transitivity notion we will consider.

3.7 Examples: (Dense orbits)

1. For the rotation of the circle, there is a dense orbit if and only if α is irrational, in which case all orbits are dense. One should infer from this that the possession of a dense orbit is not enough, by itself, to make conclusions about chaotic dynamics.
2. As with the other identifiers of chaotic dynamics, for certain parameter values, the logistic map possesses a dense orbit. In the case of the parameter value $\mu=4$, this follows from the fact (that we shall see next) that the tent map possesses a dense orbit for the parameter value $\alpha=1$.
3. We claim that the tent map $\phi=\tau_1$ for the parameter value $\alpha=1$ has a dense orbit. First let $\tilde{x}_0$ be the number in $[0,1]$ with bicimal representation obtained by concatenating all finite sequences of 0's and 1's according to the rules: (1) shorter sequences precede longer ones; (2) sequences of the same length are ordered lexicographically. Thus the first few terms in the bicimal expansion of $\tilde{x}_0$ are

$$0.0|1|00|01|10|11|000|001|010|011|100|101|110|111|\cdots.$$

We now construct x_0 by applying the following rule. Starting from the bicimal point and working right, if we encounter a finite sequence that ends in 1, we flip the bits of the next finite sequence. Thus the first few terms in the bicimal expansion for x_0 are

$$0.0|1|11|10|10|11|111|110|010|011|011|010|110|111|\cdots.$$

One can then show that $\mathrm{cl}(\mathrm{Orb}(x_0))=[0,1]$ by using the characterisation of ϕ on bicimal representations.

4. The hyperbolic toral automorphism has a dense orbit. As with the verifications of the other determiners of chaotic dynamics for hyperbolic toral automorphisms, one can use symbolic dynamics to verify the existence of a dense orbit.
5. The existence of a dense periodic on the invariant set for the Smale horseshoe map can be shown using symbolic dynamics.
6. For the winding flow on the torus, as with the rotation of the circle, either there are no dense orbits or all orbits are dense.
7. For the forced Duffing and the Lorenz equations, for some parameter values there are invariant subsets of the state spaces on which the dynamics possesses a dense orbit. Showing this is challenge, and we refer to references in Example 3.2–7 above. •

3.4. Strange attractors. Above we were able to demonstrate, with a little effort, that the simple tent map, as a dynamical system, possesses all of the ingredients of chaotic dynamics. However, the situation of the tent map is not typical in one important sense. Normally a dynamical system is not chaotic in its entire state space. Rather, it is more common for there to be invariant subsets of the state space on which the restricted dynamics will have the attributes from Sections 3.1–3.3 required for chaotic dynamics.

A common way of arriving at such invariant sets is to make use of the Smale horseshoe map which can be applied when there are so-called *transverse homoclinic points*. While the Smale horseshoe is a useful way to understand how chaotic dynamics can arise in some classes of dynamical systems, the resulting invariant sets are not stable.[1] Thus one would be unlikely to observe these exact dynamics in real world manifestations of these systems or in numerical simulation. What one is really interested in in chaotic dynamics are *attractors* on which the dynamics are chaotic. Such attractors are called ***strange attractors***, a term coined by Ruelle and Takens [1971]. As with so many concepts surrounding chaotic dynamics, there is no uniformly agreed upon definition of a strange attractor. A fairly recent discussion of the concept is undertaken by Ruelle [2006].

We shall give the definition of an attractor, a notion on which there is some common understanding of what it is (but see [Milnor 1985]). In the definition, int denotes the topological interior operator.

3.8 Definition: (Trapping region, attracting set, attractor) Let $\mathcal{S}$ be a topological space and let $\Phi\colon \mathbb{T} \times \mathcal{S} \to \mathcal{S}$ be a C^0-semiflow.

(i) An nonempty invariant subset $\mathcal{T} \subseteq \mathcal{S}$ is a ***trapping region*** for Φ if there exists $T > 0$ such that $\mathrm{cl}(\Phi_T(\mathcal{T})) \subseteq \mathrm{int}(\mathcal{T})$.

(ii) An ***attracting set*** for Φ is a subset A such that there exists a trapping region $\mathcal{T}$ for which $A = \cap_{t\in\mathbb{T}} \Phi_t(\mathcal{T})$.

(iii) An ***attractor*** for Φ is a nonempty, topologically transitive, attracting set. •

Sometimes different transitivity notions are used in the definition of an attractor. For example, in the Fundamental Theorem of Dynamical Systems of Conley [1978], the notion of *chain transitivity* is used.

Often it is possible to locate a trapping region, either analytically or numerically, and this necessarily gives rise to an attracting set, which may be a candidate for being an

[1] Precisely, they are hyperbolic, meaning they have both stable and unstable manifolds; this is a direct consequence of the stretching and folding character of the map.

attractor. However, it is typically a substantial challenge to (1) understand what the attractor looks like and (2) prove that the attractor is a strange attractor, if indeed it is. As we have indicated in our discussions above, some significant effort was expended to prove the existence of a strange attractor for certain parameter values for the forced Duffing equation [Ueda 1991] and for the Lorenz equation [Tucker 2002]. Here we content ourselves with displaying an orbit for each of these dynamical systems in Figure 8. Once one proves

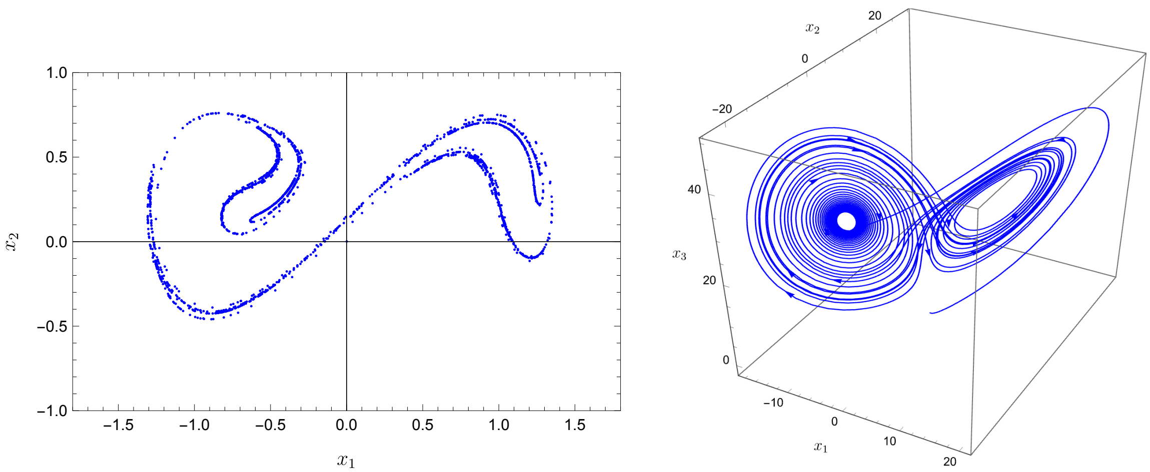

Figure 8. An orbit for each of the forced Duffing equation (left) and the Lorenz equation (right)

the presence of a strange attractor within a trapping region, the orbit through an initial condition in the trapping region will approach the attractor, and will asymptotically give some idea as to the behaviour of the dynamics of the attractor.

Another problem concerns the topological nature of strange attractors. As we intimated when discussing the Smale horseshoe, the invariant subset on which the dynamics is chaotic in that example is a Cantor set. A Cantor set is an example of what is known as a *fractal set*. We will not carefully define what is meant by a fractal set, but will content ourselves with saying that it is a set whose dimension, suitably defined, is not an integer. An examination of the relationship between fractals and strange attractors is undertaken in [Hentschel and Procaccia 1983]. Related work is that of Grebogi, Ott, and Yorke [1987].

4. Detecting chaotic dynamics

The tools for explicitly determining the presence of chaotic dynamics, according to the prescription for such as laid out in Section 3, are quite limited. One can use perturbation methods, as was done initially by Poincaré, or one can hope that the system succumbs to explicit analysis, such as we have done above with the tent map. Typically, however, these methods are inapplicable, and it is not possible to show the presence of chaotic dynamics via an analytical, or conclusive numerical, proof. For example, Smale [1998] gives the proof of the Lorenz attractor being a strange attractor as one of his eighteen important mathematical problems at the turn of the twenty first century. In recognising the reality that chaotic dynamics were more far reaching than the few instances where it had been proved to exist, Eckmann and Ruelle [1985] proposed new tools for characterising complex dynamics. A recent survey of these subjects can be found in the article of Buzzi [2023].

4.1. Lyapunov exponents. The notion of a Lyapunov exponent is designed explicitly to measure sensitive dependence on initial condition, as outlined in Section 3.1. We shall give a concrete construction in the setting of semiflows on manifolds. A more general construction is possible, even advisable, although it is far more complicated, e.g., [Katok and Hasselblatt 1995, §S.2]. To make our constructions, let M be a smooth manifold and let $\Phi\colon \mathbb{T}\times\mathsf{M}\to\mathsf{M}$ be a C^1-semiflow. Let $x_0\in\mathsf{M}$ be an initial state and let $v_0\in\mathsf{T}_{x_0}\mathsf{M}$ be a tangent vector at x. Fixing $t\in\mathbb{T}$, the derivative $T_{x_0}\Phi_t(v_0)$ in the direction of v_0 gives the variation in the state trajectory $\Phi_t(x_0)$ if one varies the initial condition x_0 in the direction v_0.

To numerically measure the data from the preceding paragraph, we suppose that M possesses a Riemannian metric $\mathbb{G}$ and denote the norm on tangent spaces by $\|\cdot\|_{\mathbb{G}}$. We then define the ***Lyapunov exponent*** in the direction v_0 by

$$\lambda^{\Phi}(v_0)=\lim_{t\to\infty}\frac{\log\|T_{x_0}\Phi_t(x_0)\|_{\mathbb{G}}}{t}, \tag{4.1}$$

provided this limit exists. The idea is that a positive Lyapunov exponent for some v_0 suggests chaotic dynamics as it suggests exponential divergence of trajectories with nearby initial conditions.

There are multiple issues arising from this definition.

1. It is not always the case that the limit in the above definition exists.
2. It is not clear that the construction is independent of Riemannian metric. Indeed, generally it is *not* independent of Riemannian metric.
3. If one wishes to draw rigorous conclusions about dynamics using Lyapunov exponents, one should be prepared to learn just what conclusions the existing theory permits. These may not be the conclusions one would like to make.

These issues aside, one of the advantages of Lyapunov exponents is that it can be computed easily in elementary cases (in order to get some insight into its meaning) and it can be computed numerically in complicated examples (where it can be used as a tool to make inferences about complex dynamics). In Figure 9 we show the Lyapunov exponent as a function of the parameter for the logistic equation (on the left). In the same figure on the right, we show the time evolution of the expression in (4.1) whose limit is being taken.

4.2. Topological entropy. The notion of entropy arises in a variety of settings in mathematics, physics, and engineering. We will give a cursory discussion of topological entropy in dynamical systems; there is also a notion of *measure theoretic entropy* that is important in dynamical systems.

We shall consider a discrete-time semiflow Φ on a compact metric space $(\mathcal{M},\mathrm{d})$. We define a sequence d_k^{Φ}, $k\in\mathbb{Z}_{>0}$, of metrics on $\mathcal{M}$ by

$$\mathrm{d}_k^{\Phi}(x,y)=\max\{\mathrm{d}(x,y),\mathrm{d}(\phi(x),\phi(y)),\dots,\mathrm{d}(\phi^{k-1}(x),\phi^{k-1}(y))\}.$$

If the dynamics is "expansive" in the sense that distance is increased as we iterate the mapping, then the balls in these metrics are smaller than the balls for the original metric.

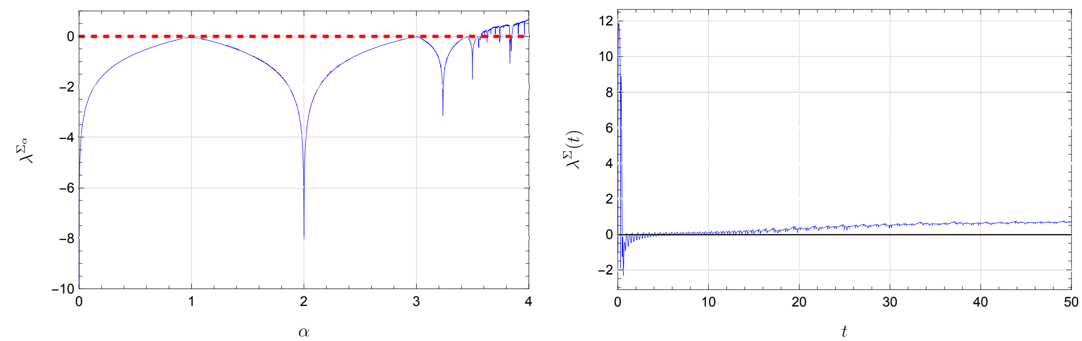

Figure 9. Lyapunov exponent for the logistic map as a function of α (left) and the function of time whose limit is the Lyapunov exponent for the Lorenz equation (right)

Next we fix a radius r and denote by $N(k, r, \Phi)$ the maximum cardinality of a set each of whose points is a distance at least r from any other point in the metric d_k^{Φ}; this number is finite by compactness of $\mathcal{M}$. The idea of $N(k, r, \Phi)$ is that it measures the maximum number of orbits that can be distinguished after k iterations of Φ at a resolution r. We then define

$$h(r, \Phi) = \limsup_{k\to\infty} \frac{\log(N(k, r, \Phi))}{k}.$$

This should be thought of as the exponential growth rate of $N(k, r, \Phi)$. If Φ is not expansive, then we have $h(r, \Phi) = 0$; different orbits will not get further apart. The ***topological entropy*** eliminates the dependence on r by defining it to be

$$h(\Phi) = \lim_{r\to 0} h(r, \Phi).$$

The idea is that a larger topological entropy means more complicated dynamics because one loses the ability to distinguish different orbits by arbitrarily fine knowledge of their initial conditions.

The notion of topological entropy can seem conceptually tantalising. However, it is difficult to compute in any but the most elementary examples. For example, with some effort, one can show that the topological entropy for the tent map τ_α is $\log \alpha$ [Katok and Hasselblatt 1995, page 499]. Sometimes there are relationships between topological entropy and Lyapunov exponents, e.g., the *Margulis–Ruelle inequality* [Ruelle 1978]. Also see [Katok 1980].

4.3. Ergodic theory. Alongside the "butterfly effect," one of the popular impressions of chaotic dynamics is that "random dynamics can arise from deterministic models." The mathematical development of this idea is *ergodic theory* [Walters 1982].

Let us briefly describe the approach. We consider a topological space $\mathcal{S}$ with a continuous semiflow $\Phi\colon \mathbb{T} \times \mathcal{S} \to \mathcal{S}$, which may be continuous- or discrete-time. A Borel probability measure μ on $\mathcal{S}$ is ***Φ-invariant*** if $\mu(\Phi_t^{-1}(B)) = \mu(B)$ for every $t \in \mathbb{T}$ and for every Borel set $B \subseteq \mathcal{S}$. One is interested in finding such Φ-invariant measures as these should give some information about the dynamics. As a very coarse instance, the support of a Φ-invariant will be an invariant subset, as per Definition 2.4. A Φ-invariant measure μ is ***ergodic*** if, for any subset $A \subseteq \mathcal{S}$ for which $\Phi_t^{-1}(A) = A$, it holds that either $\mu = 0$ or $\mu = 1$. The idea of an ergodic measure is that it indicates that the trajectory of a "typical" point will visit every point in a suitably uniform and randomly distributed manner, with randomness determined by the measure.

With this cursory outline of ergodic theory, we will illustrate it by a numerical experiment. For the logistic map, we iterate many times a large collection of random initial conditions for the mapping. The terminal states are then binned into small subintervals of equal lengths. The frequency of the terminal states in bins are then displayed and normalised so the largest frequency is 1. The results are shown in Figure 10 for a few values

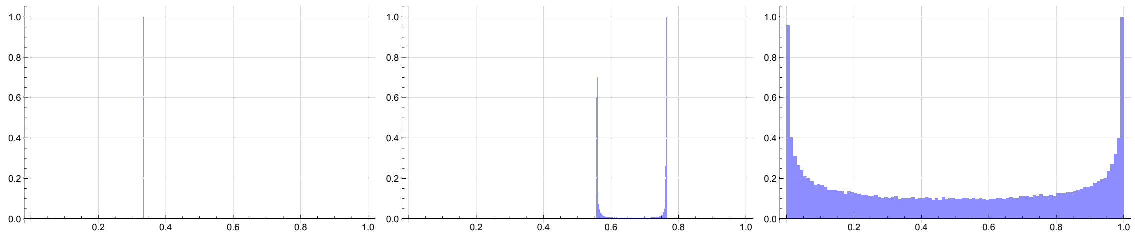

Figure 10. Terminal states for logistic map with $\alpha \in \{1.5, 3.1, 4\}$

of the parameter μ. We can see that, when there are stable periodic orbits of period k, almost all terminal states land in one of the k states determined by the periodic orbit. At the extreme value of the parameter μ, the distribution of terminal states appears to be continuous, not discrete.

In Figure 11 we show the results of the same sort of experiment applied to the tent map.

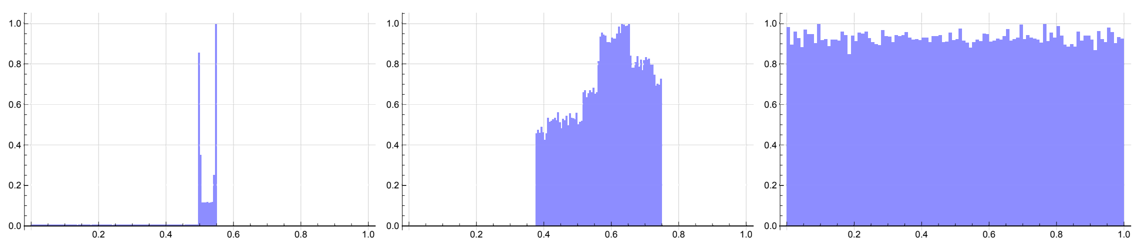

Figure 11. Terminal states for tent map with $\alpha \in \{0.55, 0.75, 1\}$

In this case, for the three parameter values displayed, the distribution of terminal states is continuous, a fact that can be proved analytically, cf. [Lasota and Yorke 1973].

5. Ongoing lines of investigation in chaotic dynamics

It is probably fair to say that the period of "hype" that was enjoyed by chaotic dynamics in the 1980's and 1990's has subsided, and that serious interest in chaotic dynamics has merged with the active field of *dynamical systems*. The field of dynamical systems does not admit many well-polished open questions; the topics and the approaches are too broad and boundaryless for this. Rather, there are areas where things are poorly understood, but where concerted effort may lead to better understanding. In this section, and with no pretence to authority, we list a few such areas. We encourage readers to explore the subject and decide for themselves which areas are most interesting or most likely to be profitable.

5.1. Definitions of chaotic dynamics. In Section 3 we gave one fairly clear definition of what is meant by chaotic dynamics. And we saw, via examples, that chaotic dynamics can arise in at least three ways: (1) in dynamical systems for which the dynamics are globally chaotic, e.g., the tent map; (2) as invariant subsets where the restricted dynamics are chaotic, but for which the subset is not attractive, e.g., the Smale horseshoe; (3) as strange attractors, e.g., the Lorenz equation. However, if one peruses the volumes of a journal such as *Physica D* (to name one), one will see a large number of papers discussing "chaos." In most cases, one does not see a careful bookkeeping of the precise definitions of chaotic dynamics, but rather a numerical investigation of some sort (perhaps involving Lyapunov exponents). It is interesting to think about whether some, most, or all of these instances of chaotic dynamics fall within the setting of Section 3 (this seems unlikely), or whether there are other definitions of what is entailed by chaotic dynamics that is adhered to by some, most, or all of these (as seems more likely).

5.2. Genericity of chaotic dynamics. *Genericity* in dynamical systems concerns what behaviours are observed if one reaches into the bag of dynamical systems and pulls out a "typical" dynamical system. When formulated precisely, genericity is often characterised in topological terms by the notion of a *residual set*, i.e., one that is a countable intersection of open and dense sets. Thus one wonders whether, now making reference to Section 5.1, there is a good definition of chaotic dynamics for which this definition being verified to hold is residual in a class of dynamical systems carrying a certain topology. Said a little more colloquially, should one expect to typically encounter chaotic dynamics, by some reasonable notions of "typical" and "chaotic dynamics"?

5.3. Robustness of chaotic dynamics. A related question to that of genericity is *robustness*. Here one wishes to understand which properties of dynamical systems are shared by those dynamical systems that are nearby, in a suitable sense. This can be made precise by asking whether the set of dynamical systems with a certain property is open, with respect to some topology. Similarly to (but not the same as) genericity, one can ask whether there is a good definition of chaotic dynamics such that, if a system has chaotic dynamics, then nearby dynamical systems also have chaotic dynamics. Of course, there should also be a mechanism that connects the chaotic dynamics of a dynamical system to those that are nearby. A common theme in investigations of robustness properties is *hyperbolicity*, which features prominently in the influential paper of [Smale 1967]. For example, the situations giving rise to Smale horseshoes possess hyperbolic dynamics.

5.4. Chaotic dynamics and turbulence. In their early paper discussing what we would call "chaotic dynamics," Ruelle and Takens [1971] also discuss turbulence in the Navier–Stokes equation as being representative of the same sort of phenomenon. However, there remains no understanding of whether there are common mathematical (or possibly physical) mechanisms shared by our rather narrow idea of chaotic dynamics and the turbulent motion of a fluid.

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