Binary Signaling of Correlated Sources 
Over Orthogonal Multiple Access Channels

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Abstract—The optimal energy allocations for minimizing the 
joint symbol-error rate for binary signaling of two correlated 
sources over the orthogonal multiple access Gaussian channel 
(OMAGC) under joint maximum a priori (MAP) detection are 
determined. An exact expression for the system’s probability 
of joint symbol error, as well as its union bound, is derived. 
Analytic minimization of the union bound reveals that the optimal 
energy allocation coincides with that of non-uniform binary 
signaling over the single-user additive white Gaussian noise 
(AWGN) channel. It is also shown numerically that the optimal 
energies which minimize the union bound also minimize the 
exact probability of error. Lastly, it is shown via simulations for 
strongly biased sources that the use of joint MAP detection over 
two independent single-user systems leads to significant gains.

Index Terms—Error analysis, joint source-channel coding, 
MAP detection, multiple access channels, non-uniform sources.

I. INTRODUCTION

We consider the joint source-channel coding problem of 
designing an optimal source-matched modulation scheme for 
the reliable transmission of binary correlated sources over 
the orthogonal multiple access Gaussian channel (OMAGC) 
without the explicit use of data compression or error-correcting 
codes. The construction of such low-delay, low-complexity 
signaling schemes has pertinent applications to wireless sensor 
networks where multiple sensors observe correlated data and 
relay them in real-time to the base station [1]. In such 
wireless systems, delay, power, complexity, and scalability 
constraints often restrict the use of powerful coding and signal 
processing methods, and thus each sensor often performs a 
simple signaling function when sending the data it senses.

In a related prior work, considering orthogonal and 
nonorthogonal signaling separately, Korn et. al. [2] derive 
onimal energy allocations for uncoded single-user binary 
communications over a point-to-point AWGN channel with 
non-equal symbol probabilities. In a follow-up work, Ipatov 
a simplified treatment. Motivated by applications to sensor 
networks and the fact that nonuniform sources are good 
models for many types of data (e.g., see [4]), in this letter 
we extend the coherent detection results from [2], to the two-
user OMAGC and a two-dimensional nonuniform correlated 
source. In other related works, various signal set designs for 
systems with non-equal symbol probabilities were studied in 
[5]–[9] (to name a few); see also [10]–[12] for works on the 
error analysis of such systems.

The contributions of this work are organized as follows. 
In Section II, the system set-up is presented. In Section III, 
an exact expression for the system’s joint symbol error rate 
under joint MAP decoding and the union upper bound on the 
symbol error rate are established. Making use of its simplified 
form, the union bound is analytically minimized over the set of 
signaling energies. It is then shown numerically that the signaling 
energies that optimize the union bound also minimize the 
exact symbol error rate and are hence optimal. Performance 
results illustrating the benefits of using the optimal energy 
signaling schemes under joint MAP detection are presented in 
Section IV. Section V concludes the letter.

II. SYSTEM MODEL

The system consists of Transmitters 1 and 2, two independent 
AWGN channels, and a joint MAP detector. Transmitter i 
(with i = 1, 2) can be described as follows: let

\[ s_{i0}(t) = \sqrt{E_{i0}}\psi_1^{(i)}(t) \] and \[ s_{i1}(t) = \sqrt{E_{i1}}\psi_2^{(i)}(t) \]

be arbitrary binary transmission signals with energies \( E_{i0} \) and 
\( E_{i1} \), respectively, such that \( s_{i0} \) has probability \( p_i \) with 0 ≤ 
\( p_i \leq 0.5 \) and \( s_{i1} \) has probability \( 1-p_i \), for \( i = 1, 2 \), and where 
\( \psi_j^{(i)}: [0, T] \rightarrow \mathbb{R} \) has unit energy for \( i, j = 1, 2 \). Further, \( \psi_1^{(i)} \) 
and \( \psi_2^{(i)} \) have correlation

\[ \gamma_i = \int_0^T \psi_1^{(i)}(t)\psi_2^{(i)}(t)dt. \]

Thus, Transmitter i will have an average energy per symbol of

\[ E_i = E_{i0}p_i + E_{i1}(1-p_i), \] (1)

\( i = 1, 2 \). At each transmission instance, Transmitters 1 and 2 
send \( S_1 \in S_1 = \{s_{10},s_{11}\} \) and \( S_2 \in S_2 = \{s_{20},s_{21}\} \), 
respectively, over the independent AWGN channels, where 
the random source pair \( (S_1,S_2) \) has joint probability mass 
function given by

\[ p_{S_1,S_2}(s_{10},s_{20}) = 1 - (1-p_1) - (1-p_2) + p_{11} \]
\[ p_{S_1,S_2}(s_{10},s_{21}) = (1-p_2) - p_{11} \]
\[ p_{S_1,S_2}(s_{11},s_{20}) = (1-p_1) - p_{11} \]
\[ p_{S_1,S_2}(s_{11},s_{21}) = p_{11}. \]

Note that \( p_{11} \) can be expressed as

\[ p_{11} = \rho \sqrt{p_1(1-p_1)p_2(1-p_2)} + (1-p_1)(1-p_2) \]

where \( \rho \) is the correlation coefficient of the underlying jointly 
distributed binary source pair \( (U_1,U_2) \in \{0,1\}^2 \) defined by 
letting \( U_i = 0 \) if \( S_i = s_{i0} \) and \( U_i = 1 \) if \( S_i = s_{i1} \) for \( i = 1,2 \).
The received information at the output of the matched filter is given by the random pair \((R_1, R_2)\) such that \(R_i = s_i + N_i\), where \(s_i \in S_i\) and \(N_i\) is zero-mean Gaussian noise with variance \(\sigma_i^2\) for \(i = 1, 2\), such that \(N_1\) and \(N_2\) are independent. The joint MAP detector, which is optimal in terms of minimizing the probability of joint symbol error, receives \((r_1, r_2)\) — the realizations of \((R_1, R_2)\) — and implements the following MAP decision rule

\[
\hat{s} = \arg \max_{(s_1, s_2) \in S_1 \times S_2} P(S_1 = s_1, S_2 = s_2 | R_1 = r_1, R_2 = r_2)
\]

where \(s = (s_1, s_2)\) and

\[
h(s) = \ln p_{S_1, S_2}(s_1, s_2) - \frac{s_1^2 - 2 r_1 s_1}{2 \sigma_1^2} - \frac{s_2^2 - 2 r_2 s_2}{2 \sigma_2^2}
\]

(2)

so that \(\hat{s} = (\hat{s}_1, \hat{s}_2) \in S_1 \times S_2\) is the MAP estimate of the transmitted pair. Note that \(s_1^2 = E_{s_1}\) and \(s_2^2 = E_{s_2}\) are the energies of the respective signals, where \(i, j \in \{0, 1\}\).

III. PROBABILITY OF SYMBOL ERROR ANALYSIS

A. Probability of Symbol Error

A symbol error event, \(e\), occurs when \(s \neq \hat{s}\), where \(s\) and \(\hat{s}\) are defined above. Setting \(\{00\} = (s_{10}, s_{20}), \{01\} = (s_{11}, s_{21}), \{10\} = (s_{10}, s_{21})\), and \(\{11\} = (s_{11}, s_{20})\), the probability of error, \(P(e)\), can be expressed as

\[
P(e) = \sum_{k \in \{0,1\}^2} P(\hat{s} \neq s_k | k) P(s_k)
\]

(3)

where \(P(\hat{s} = s_k | k)\) is the probability of correct detection given that \(s_k \triangleq (s_{1k}, s_{2k})\), where \(k \triangleq (k_1, k_2) \in \{0,1\}^2\), was transmitted, and \(P(s_k)\) is the probability that \(s_k\) was transmitted. Let

\[
h(s) = \ln p_{S_1, S_2}(s_1, s_2) - \frac{s_1^2 - 2 R_1 s_1}{2 \sigma_1^2} - \frac{s_2^2 - 2 R_2 s_2}{2 \sigma_2^2}
\]

be our MAP detection metric based on (2). Then, we can write

\[
P(\hat{s} = s_k | k) = P\left(\hat{h}(s_k) = \max_{j \in \{0,1\}^2} \hat{h}(s_j) | k\right)
\]

\[
= P\left(\bigcap_{j \neq k} \{\hat{h}(s_k) > \hat{h}(s_j)\} | k\right)
\]

\[
= P\left(\bigcap_{j \neq k} \{V_j^{(k)} < 0\} | k\right)
\]

where

\[
V_j^{(k)} = \hat{h}(s_j) - \hat{h}(s_k)
\]

for \(j, k \in \{0,1\}\). Each \(V_j^{(k)}\) is a random variable. Moreover, since for \(i = 1, 2\), \(R_i\) is Gaussian given \(k\), each \(V_j^{(k)}\) is Gaussian given \(k\). Further, since \(N_1\) and \(N_2\) are independent, \(R_1\) and \(R_2\) are independent, given \(k\). Now define \(k' \in \{0,1\}\) such that \(k' = k \oplus (1, 1)\), where \(\oplus\) denotes component-wise modulo 2 addition. Then, we have

\[
V_j^{(k')} = \alpha_k + \sum_{j \neq k, k'} V_j^{(k)}
\]

(5)

where

\[
\alpha_k = \begin{cases} 
\ln \frac{P(s_k)}{P(s_{k1})P(s_{k1})} & \text{if } k = (0,0), (1,1) \\
\ln \frac{P(s_{k1})P(s_{k1})}{P(s_{k1})P(s_{k1})} & \text{if } k = (0,1), (1,0).
\end{cases}
\]

Thus, using (5), we can write (4) as

\[
P\left(\bigcap_{j \neq k, k'} \{V_j^{(k)} < 0\} | k\right).
\]

(6)

For example if \(k = (0,0)\), then (6) is given by

\[
P(V_{10}^{(00)} < 0, V_{01}^{(00)} < 0, V_{10}^{(00)} + V_{01}^{(00)} + \alpha_{00} < 0 | k = (0,0)).
\]

By independence of \(N_1\) and \(N_2\), for \(j \neq k, k'\) we have

\[
\mu_j^{(k)} = E[V_j^{(k)} | k]
\]

\[
= \ln \frac{P(s_j)}{P(s_k)} + \frac{s_{1k_j}^2 - s_{1j1}^2 + s_{2k_j}^2 - s_{2j2}^2}{2 \sigma_j^2}
\]

\[
+ \frac{s_{1j1} - s_{1k_j}}{2 \sigma_1^2} \frac{1}{\sigma_1^2} + \frac{s_{2j2} - s_{2k_j}}{2 \sigma_2^2} \frac{1}{\sigma_2^2}
\]

and

\[
\sigma_j^{(k)} = \text{Var}(V_j^{(k)} | k) = \frac{(s_{1k_j} - s_{1j1})^2}{\sigma_1^2} + \frac{(s_{2k_j} - s_{2j2})^2}{\sigma_2^2}.
\]

It can be shown that (6) can be calculated as

\[
\prod_{j \neq k, k'} \frac{Q\left(\frac{\mu_j^{(k)}}{\sigma_j^{(k)}}, \Delta_k\right)}{Q\left(\frac{\mu_j^{(k)}}{\sigma_j^{(k)}}, \Delta_k\right)} - \Delta_k
\]

where

\[
Q(x) = \frac{1}{\sqrt{2\pi}} \int x e^{-t^2/2} dt
\]

is the Gaussian Q-function and \(\Delta_k\) given by

\[
\Delta_k = \int_{-\alpha_k}^{0} \int_{-\alpha_k}^{0} f_{V_k}(v_1, v_2) dv_1 dv_2
\]

if \(\alpha_k > 0\) and zero otherwise, where

\[
f_{V_k}(v_1, v_2) = \frac{1}{2\pi \prod_{j \neq k, k'} \sigma_j^{(k)}} \exp\left\{-\frac{1}{2} \sum_{j \neq k, k'} \frac{(v_i - \mu_j^{(k)})^2}{\sigma_j^{(k)}^2}\right\}
\]

where

\[
i = \begin{cases} 
1 & \text{if } j \oplus k = (1,0) \\
2 & \text{if } j \oplus k = (0,1)
\end{cases}
\]

(7)

Thus, we have

\[
P(e) = 1 - \sum_{k \in \{0,1\}^2} \left[\prod_{j \neq k, k'} \frac{Q\left(\frac{\mu_j^{(k)}}{\sigma_j^{(k)}}, \Delta_k\right)}{Q\left(\frac{\mu_j^{(k)}}{\sigma_j^{(k)}}, \Delta_k\right)} - \Delta_k\right] P(s_k).
\]

(8)

B. Analytic Optimization of the Union Bound

From (3) the union bound on \(P(e)\), which we denote by \(P_{UB}(e)\), is given by

\[
P(e) \leq P_{UB}(e) = \sum_{k \in \{0,1\}^2} P(s_k) \sum_{j \neq k} P(e_{jk})
\]

where \(e_{jk}\) is the event that \(s_j\) has a higher MAP metric than \(s_k\), and \(P(e_{jk}) = P(\hat{h}(s_j) > \hat{h}(s_k)) = 1 - P(V_j^{(k)} < 0)\). Further,

\[
1 - P(V_j^{(k)} < 0) = 1 - Q\left(\frac{\mu_j^{(k)}}{\sigma_j^{(k)}}\right).
\]
Thus, the union bound on the probability of symbol error becomes

\[ P_{UB}(e) = \sum_{k \in \{0,1\}^2} P(s_k) \left[ \sum_{j \neq k} 1 - Q \left( \frac{\mu_j^{(k)}}{\sigma_j^{(k)}} \right) \right] \quad (9) \]

In (9), when \( j = k' \), we have that \( \mu_j^{(k)} \) is the conditional expectation of (5) given \( k \) and satisfies

\[ \mu_j^{(k)} = E \left[ \alpha_k + \sum_{j \neq k, k'} \mu_j^{(k)} \right] = \alpha_k + \sum_{j \neq k, k'} \mu_j^{(k)} \]

and \( (\sigma_j^{(k)})^2 \) is the conditional variance of (5) given \( k \), satisfying

\[ (\sigma_j^{(k)})^2 = \text{Var} \left( \sigma_j^{(k)} | k \right) = \sum_{j \neq k, k'} (\sigma_j^{(k)})^2. \]

Observe that when \( j \neq k' \), in (9) we have

\[ \frac{\mu_j^{(k)}}{\sigma_j^{(k)}} = \frac{\sigma_i \ln[P(s_j)/P(s_k)]}{\sqrt{(s_{i0} - s_{i0})^2}} + \frac{s_{i0} - s_{i0}}{2\sigma_i \sqrt{(s_{i0} - s_{i0})^2}} + \frac{s_{i0} - s_{i0}}{\sigma_i \sqrt{(s_{i0} - s_{i0})^2}} s_{i0} \quad (10) \]

where \( i \) is determined by (7). Now, letting

\[ A_i = \frac{s_{i0}^2 + s_{i0} - 2\gamma_i \sqrt{s_{i0} s_{i0}}}{\sigma_i^2} \quad (11) \]

for \( i = 1, 2 \), (10) can be written as

\[ \frac{\mu_j^{(k)}}{\sigma_j^{(k)}} = \frac{\ln[P(s_j)/P(s_k)]}{\sqrt{A_i}} - \frac{\sqrt{A_i}}{2} \quad (12) \]

where \( i \) is determined, again, by (7). Further, when \( j = k' \) in (9) we have that

\[ \frac{\mu_j^{(k)}}{\sigma_j^{(k)}} = \frac{\alpha_k + \sum_{j \neq k, k'} \ln[P(s_j)/P(s_k)]}{\sqrt{A_1 + A_2}} - \frac{\sqrt{A_1 + A_2}}{2}. \quad (13) \]

Hence, each term of the inner sum of (9) can be written in terms of \( A_1 \) and \( A_2 \). More specifically, (9) is decreasing in both \( A_1 \) and \( A_2 \). Thus, to minimize (9) we must maximize \( A_1 \) and \( A_2 \). Hence, to determine the optimal energy allocations \( E_{10} \) and \( E_{20} \), we can maximize \( A_1 \) and \( A_2 \) under the average energy constraint given by (1).

We also notice that (11) is analogous to [2, Eq. (7)] and, with the appropriate modifications, to [3, Eq. (1)]. Thus, to minimize (9) one can use the methods from [2], [3] to show that the optimal energy allocation of Transmitter \( i \) is given by

\[ E_{i0} = \frac{E_i}{2p_i} \left[ 1 + \frac{1 - 2p_i}{\sqrt{1 - 4p_i(1 - p_i)(1 - \gamma_i^2)}} \right] \quad (14) \]

\[ E_{i1} = \frac{E_i}{2(1 - p_i)} \left[ 1 - \frac{1 - 2p_i}{\sqrt{1 - 4p_i(1 - p_i)(1 - \gamma_i^2)}} \right] \quad (15) \]

when \( \gamma_i \in [-1, 0) \) and

\[ E_{i0} = E_i/p_i \quad \text{and} \quad E_{i1} = 0 \quad (16) \]

when \( \gamma_i \in [0, 1] \).

C. Comparison of the Union Bound and the Error Probability

There appears to be no simple way for analytically optimizing the probability of symbol error \( P(e) \) given in (8). We can however show numerically that the values which minimize the union bound \( P_{UB}(e) \) in (9), also minimize (8).

In Figs. 1 and 2, we use \( E_1 = E_2 = 10 \) and \( 2E_1 = E_2 = 20 \), respectively. Also, \( p_1 = p_2 = 0.1 \), \( \rho = 0.9 \), and \( \sigma_1 = \sigma_2 = 4 \) and \( \gamma_1 = \gamma_2 \). In these plots the ‘*’ point shows the location of the numerically determined minimum achieved on the interval \([0, E_i/p_i]\). For these test values we conclude that the optimal energy allocations seen in (14)–(16), which minimize (9), also minimize (8). It should be noted that the numerical optimization was also performed for many other values of the system parameters including when \( \gamma_1 \neq \gamma_2 \).

In all cases, we obtained that the optimal energy allocation determined by minimizing the union bound also minimizes the probability of error.

Fig. 1. Numerical analysis under the average energy constraint of \( E_1 = E_2 = 10 \). Here, \( p_1 = p_2 = 0.1 \), \( \rho = 0.9 \), and \( \sigma_1 = \sigma_2 = 4 \). Note that in this plot the grey and black lines overlap exactly.
We have shown that the optimal energy allocations for binary non-uniform signaling over a single-user system and over the OMAGC coincide. In this section, we assess the benefits, via simulation, of the OMAGC system over independent single-user systems. For the simulations we denote by Scheme 1 and Scheme 2 the two independent single-user systems and the OMAGC system, respectively. Note that both schemes implement the optimal energy allocations; thus, the difference between the two schemes is that the decoder of Scheme 2 exploits the dependence in the source components. In particular, with the true underlying source correlation set at \( \rho = 0.9 \), Schemes 1 and 2 use a knowledge of \( \rho \) for their decoders given by \( \hat{\rho}_1 = 0 \) and \( \hat{\rho}_2 = 0.9 \), respectively (i.e., Scheme 1 uses two independent MAP detectors, while Scheme 2 employs joint MAP detection). The following parameters are used in the simulation: \( p_1 = p_2 = 0.1 \), and \( \sigma_1 = \sigma_2 = 2 \). In Fig. 3 we use \( E_1 = 2 \) and in Fig. 4 we use \( 2E_1 = E_2 \). We choose to show the results for \( 2E_1 = E_2 \) since it has similar performance to \( E_1 = E_2 \) (not shown here). The value \( E_1 \) is held constant in Fig. 3 since as the signal-to-noise ratio (SNR), defined by \((E_1 + E_2)/(\sigma_1^2 + \sigma_2^2)\), increases, the difference in average energy between the two transmitters also increases, and thus this setting gives a boundary case at high SNRs. Note that choosing a value for \( E_1 \) that is larger than 2 provides similar performance trends with an error floor forming at higher SNRs than in the case of \( E_1 = 2 \). In summary, when \( E_2 = 2E_1 \), as seen in Fig. 4, Scheme 2 realizes over Scheme 1 a gain ranging from 0.67 to 0.8 dB (depending on the value of \( \gamma \)) at high SNRs. Similar gains are achievable when \( E_1 = E_2 \). However, when \( E_2 \gg E_1 \), as seen in Fig. 3, the performance improvement is quite large, with a minimal gain of 5.8 dB for \( \gamma = -1 \) and \( P(e) \approx 10^{-1.2} \); the gains are much larger for smaller \( P(e) \) target values.

![Fig. 3](image-url)  
**Fig. 3.** Performance comparison of Schemes 1 and 2 when \( E_1 = 2 \).

**IV. Performance Analysis**

We have shown that the optimal energy allocations for binary non-uniform signaling over a single-user system and the OMAGC coincide. In this section, we assess the benefits, via simulation, of the OMAGC system over independent single-user systems. For the simulations we denote by Scheme 1 and Scheme 2 the two independent single-user systems and the OMAGC system, respectively. Note that both schemes implement the optimal energy allocations; thus, the difference between the two schemes is that the decoder of Scheme 2 exploits the dependence in the source components. In particular, with the true underlying source correlation set at \( \rho = 0.9 \), Schemes 1 and 2 use a knowledge of \( \rho \) for their decoders given by \( \hat{\rho}_1 = 0 \) and \( \hat{\rho}_2 = 0.9 \), respectively (i.e., Scheme 1 uses two independent MAP detectors, while Scheme 2 employs joint MAP detection). The following parameters are used in the simulation: \( p_1 = p_2 = 0.1 \), and \( \sigma_1 = \sigma_2 = 2 \). In Fig. 3 we use \( E_1 = 2 \) and in Fig. 4 we use \( 2E_1 = E_2 \). We choose to show the results for \( 2E_1 = E_2 \) since it has similar performance to \( E_1 = E_2 \) (not shown here). The value \( E_1 \) is held constant in Fig. 3 since as the signal-to-noise ratio (SNR), defined by \((E_1 + E_2)/(\sigma_1^2 + \sigma_2^2)\), increases, the difference in average energy between the two transmitters also increases, and thus this setting gives a boundary case at high SNRs. Note that choosing a value for \( E_1 \) that is larger than 2 provides similar performance trends with an error floor forming at higher SNRs than in the case of \( E_1 = 2 \). In summary, when \( E_2 = 2E_1 \), as seen in Fig. 4, Scheme 2 realizes over Scheme 1 a gain ranging from 0.67 to 0.8 dB (depending on the value of \( \gamma \)) at high SNRs. Similar gains are achievable when \( E_1 = E_2 \). However, when \( E_2 \gg E_1 \), as seen in Fig. 3, the performance improvement is quite large, with a minimal gain of 5.8 dB for \( \gamma = -1 \) and \( P(e) \approx 10^{-1.2} \); the gains are much larger for smaller \( P(e) \) target values.

![Fig. 4](image-url)  
**Fig. 4.** Performance comparison of Schemes 1 and 2 when \( 2E_1 = E_2 \).

**V. Conclusion**

Based on the analysis of the probability of joint symbol error and the corresponding union error bound derived for the OMAGC channel, we have shown that the optimal energy allocations coincide with the single-user case from \([2], [3]\). This result can be intuitively inferred by noting that as the two transmitters have no capacity to communicate with each other, the non-interfering (orthogonal) nature of the channel makes the optimal energy allocation dependent only on the marginal distribution of the source. Also, through simulation we have shown that the use of joint detection as implemented in the OMAGC is everywhere better that two independent single-user systems. In particular, when \( E_2 = 2E_1 \) or \( E_1 = E_2 \), we have a performance gain of up to 0.8 dB at high SNRs. When \( E_2 \gg E_1 \), the performance gain is drastically increased with a smallest realizable gain of 5.8 dB. Future work will include similar analysis to that seen in this letter on the traditional multiple access Gaussian channel where the two users interfere with each other.

**References**


