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Series Expansion for Functions Bandlimited to a Ball

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Abstract

An expansion related to the sampling theorem is derived for functions with Fourier transforms that vanish outside a ball in d dimensions. Such functions are determined by weighted averages of their values on a sequence of spheres in $\mathbb{R}^d$. The number of measurements per unit volume is equal to the Nyquist-Landau density. Fourier transforms that vanish outside ellipsoids and outside Cartesian products of balls are also considered.

Key words and phrases: Nyquist-Landau density, Nyquist rate, multidimensional sampling theorems, bandlimited functions

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1 Introduction

The Nyquist rate is one of the fundamental constraints on the processing of bandlimited functions. The author [2] has previously shown that the Nyquist rate in 2 and 3 dimensions is capable of an interpretation beyond the usual one of sampling rate. In this note we extend this result to any number of dimensions.

Let $G \in L^2(-W, W)$ and let

$$g(t) = \int_{-W}^{W} e^{i2\pi ft}\, G(f)\, df.$$

Then

$$g(t) = \sum_{n=-\infty}^{\infty} g(\frac{n}{2W}) \frac{\sin \pi(2Wt - n)}{\pi(2Wt - n)}.$$

Thus, if g is bandlimited to the interval $[-W, W]$, then g is determined completely by its sample values $g(\frac{n}{2W})$. The function g is sampled at the Nyquist rate $2W$ samples per unit time. For the history of this theorem and for many extensions of the theorem; see books by Zayed [12] and Higgins [6].

The extension of the theorem to functions bandlimited to a d-dimensional rectangle is immediate. Let R be the rectangle defined by

$$R := \{\mathbf{y} = (y_1, y_2, \dots, y_d) : -W_i \le y_i \le W_i,\ i = 1, 2, \dots, d\},$$

let $G \in L^2(R)$, let $\mathbf{x} \cdot \mathbf{y} = \sum_1^d x_k y_k$, and let

$$g(\mathbf{x}) = \int_R e^{i2\pi \mathbf{x}\cdot\mathbf{y}}\, G(\mathbf{y})\, d\mathbf{y}.$$

Then

$$g(\mathbf{x}) = \sum_{n_1,\dots,n_d=-\infty}^{\infty} g\left(\frac{n_1}{2W_1}, \dots, \frac{n_d}{2W_d}\right) \prod_{i=1}^{d} \frac{\sin \pi(2W_i x_i - n_i)}{\pi(2W_i x_i - n_i)}.$$

Thus, if g is bandlimited to the rectangle R in the sense described above, then g is determined completely by its sample values $g\left(\frac{n_1}{2W_1}, \dots, \frac{n_d}{2W_d}\right)$. The number of sample points per unit volume, $2^d W_1 \dots W_d$, is equal to the volume of the rectangle R. For some further results on multidimensional sampling theorems and for a more complete bibliography concerning such theorems, see the recent papers by Annaby [1] and Zayed [13].

Landau [7, 8] has shown quite generally that if f is any function on $\mathbb{R}^d$ whose Fourier transform has support set C, then the minimal sampling density (sample values per unit volume) for stable recovery of f is $m(C)$, the Lebesgue measure of C. This density is often called the Nyquist-Landau density. The shape of the supporting set determines whether sampling of f at the Nyquist-Landau density is sufficient to recover f or whether a larger density is needed. (For details, see Higgins [6, Chapter 14].) For example, if C is a ball, sampling at the Nyquist-Landau density is not sufficient. We can, of course, enclose the ball in a rectangle and sample at the (larger) density associated with the rectangle. In this note we show that an alternative is to measure certain other quantities instead of sample values. Specifically, for the case that the Fourier transform of f is supported by a ball, we show that f is completely determined by the values of certain integrals of f evaluated on a sequence of spheres. The density of these measurements is equal to the Nyquist-Landau density. This result is extended in the final section to deal with the case that the support is an ellipsoid or the Cartesian product of two balls.

An earlier paper [2] used special properties of Bessel functions and Legendre polynomials to deduce these results for $d = 2$ and $d = 3$. Here, we use some general properties of spherical harmonics [9] to deal with all dimensions d.

2 Spherical harmonics

Let B denote the unit ball in $\mathbb{R}^d$, $d \geq 2$, and let Σ denote its surface, i.e.,

$$B := \{\mathbf{x} : \|\mathbf{x}\|^2 := \sum_{k=1}^{d} x_k^2 \leq 1\}, \quad \Sigma := \{\mathbf{x} : \|\mathbf{x}\|^2 = 1\}.$$

Let Δ denote the Laplace operator,

$$\Delta u := \sum_{k=1}^{d} \frac{\partial^2 u}{\partial x_k^2}.$$

Our objective is to find a series expansion for functions f bandlimited to the unit ball in the sense that

$$f(\mathbf{x}) = \int_B e^{i2\pi \mathbf{x}\cdot\mathbf{y}}\, F(\mathbf{y})\, d\mathbf{y}, \tag{1}$$

where $F \in L^2(B)$. The method will be to expand the functions in the integrand as series of eigenfunctions of the boundary-value problem:

$$\Delta u + \lambda^2 u = 0 \quad \text{for } \mathbf{x} \in B, \quad u = 0 \quad \text{for } \mathbf{x} \in \Sigma. \tag{2}$$

We shall use a number of properties of spherical harmonics and of eigenfunctions of the Laplace operator on the unit ball. Most of these properties are developed in [9, Chapter 4] and in [5, Chapter 11]. Homogeneous polynomials of degree n that satisfy $\Delta u = 0$ are called solid spherical harmonics of degree n. The restriction of a solid spherical harmonic to Σ is called a surface spherical harmonic of degree n. Surface spherical harmonics of different degrees are orthogonal on Σ. The space of surface spherical harmonics of degree n has dimension

$$h(n,d) := \begin{cases} 1, & \text{if } n = 0 \\ d, & \text{if } n = 1 \\ \binom{d+n-1}{n} - \binom{d+n-3}{n-2} & \text{if } n \geq 2. \end{cases} \tag{3}$$

Note that the complex conjugate of a surface spherical harmonic is also a surface spherical harmonic of the same degree. The surface spherical harmonics can be orthogonalized and normalized to form a basis for $L^2(\Sigma)$. We use the notation $Y_n^m(\mathbf{x})$ $(n = 0, 1, 2, \ldots; m = 1, 2, \ldots, h(n,d))$ to denote these orthonormal surface spherical harmonics, so that

$$\int_\Sigma Y_n^m(\mathbf{x})\overline{Y_{n'}^{m'}(\mathbf{x})}\, d\sigma(\mathbf{x}) = \delta_{mm'}\delta_{nn'}, \tag{4}$$

where $d\sigma(\mathbf{x})$ denotes Lebesgue measure on Σ. The corresponding solid spherical harmonics are $\|\mathbf{x}\|^n Y_n^m(\mathbf{x}/\|\mathbf{x}\|)$.

Let the positive zeros of the Bessel function $J_{\frac{d}{2}+n-1}(z)$ be denoted $\lambda(k,n)$ for $k = 1, 2, \dots$. Then the eigenvalues and normalized eigenfunctions of the boundary-value problem (2) are [3, p. 194] respectively $[\lambda(k,n)]^2$ and

$$\psi_{kmn}(\mathbf{x}) := \frac{\sqrt{2} J_{\frac{d}{2}+n-1}(\lambda(k,n)\|\mathbf{x}\|)}{J_{\frac{d}{2}+n}(\lambda(k,n))\|\mathbf{x}\|^{\frac{d}{2}-1}} Y_n^m\left(\frac{\mathbf{x}}{\|\mathbf{x}\|}\right), \tag{5}$$

where the eigenvalue $[\lambda(k,n)]^2$ has multiplicity $h(n,d)$. These eigenfunctions form an orthonormal basis for $L^2(B)$.

The following property of spherical harmonics will be used:

Lemma *Let Σ be the unit sphere in $\mathbb{R}^d$ and let $\mathbf{x} \in \Sigma$. Let Y_n be any surface spherical harmonic of degree n. Then*

$$\int_\Sigma e^{-i\xi\,\mathbf{x}\cdot\mathbf{y}}\, Y_n(\mathbf{y})\, d\sigma(\mathbf{y}) = i^{-n}(2\pi)^{\frac{d}{2}}\xi^{1-\frac{d}{2}} J_{\frac{d}{2}+n-1}(\xi) Y_n(\mathbf{x}). \tag{6}$$

This lemma was proved by Erdélyi [4, eq. (4.4) and (4.5)] in a slightly different notation. A different proof can be developed using Theorem 3.10 of [9].

3 Series expansion

The main theorem of this note is the following:

Theorem *Let f satisfy* (1)*, where $F \in L^2(B)$. Then*

$$f(\mathbf{x}) = \sum_{n=0}^{\infty} \sum_{m=1}^{h(n,d)} \sum_{k=1}^{\infty} A_{kmn}(f) W_{kmn}(\mathbf{x}), \tag{7}$$

where

$$A_{kmn}(f) := \int_\Sigma f\left(\frac{\lambda(k,n)}{2\pi}\mathbf{x}\right) \overline{Y}_n^m(\mathbf{x})\, d\sigma(\mathbf{x}), \tag{8}$$

and

$$W_{kmn}(\mathbf{x}) := \frac{2[\lambda(k,n)]^{\frac{d}{2}} J_{\frac{d}{2}+n-1}(2\pi\|\mathbf{x}\|)\, Y_n^m\left(\frac{\mathbf{x}}{\|\mathbf{x}\|}\right)}{(2\pi\|\mathbf{x}\|)^{\frac{d}{2}-1} J_{\frac{d}{2}+n}(\lambda(k,n))[\lambda(k,n)^2 - 4\pi^2\|\mathbf{x}\|^2]}. \tag{9}$$

The series (7) *converges uniformly and absolutely on $\mathbb{R}^d$.*

Remark. The coefficient $A_{kmn}(f)$ in (7) is a weighted average of f on the sphere of radius $\lambda(k,n)/2\pi$. There are $h(n,d)$ different weighted averages on this sphere. The theorem shows that functions bandlimited to the unit ball in

$\mathbb{R}^d$ are determined completely by certain weighted averages on the sequence of spheres with radii $\lambda(k,n)/2\pi$.

Proof. It will be convenient to use radial coordinates and coordinates on the unit sphere, defined by

$$r := \|\mathbf{x}\|, \quad \rho := \|\mathbf{y}\|, \quad \mathbf{x}' := \mathbf{x}/r, \quad \mathbf{y}' := \mathbf{y}/\rho. \tag{10}$$

(Note that $\mathbf{x}'$ and $\mathbf{y}'$ are coordinates of points on Σ and are independent of r and ρ, respectively.) The notation $\langle u, v\rangle$ will be used for the usual inner product on $L^2(B)$. We rewrite (1) as

$$f(\mathbf{x}) = \langle F(\mathbf{y}), e^{-i2\pi\mathbf{x}\cdot\mathbf{y}}\rangle, \tag{11}$$

where the notation implies that the inner product is formed by fixing $\mathbf{x}$ and integrating with respect to $\mathbf{y}$. We expand the two terms in the inner product in a series of eigenfunctions (5) and use Parseval's theorem.

From (8), (1), and an interchange in the order of integration, we have

$$A_{kmn}(f) = \int_B F(\mathbf{y}) \left(\int_\Sigma \overline{Y}_n^m(\mathbf{x}) e^{i\lambda(k,n)\mathbf{x}\cdot\mathbf{y}}\, d\sigma(\mathbf{x}) \right) d\mathbf{y}.$$

Using the Lemma, we have

$$A_{kmn}(f) = \int_B F(\mathbf{y}) i^n (2\pi)^{\frac{d}{2}} [\lambda(k,n)\|\mathbf{y}\|]^{1-\frac{d}{2}} J_{\frac{d}{2}+n-1}(\lambda(k,n)\|\mathbf{y}\|) \overline{Y}_n^m \left(\frac{\mathbf{y}}{\|\mathbf{y}\|} \right) d\mathbf{y}.$$

A comparison with (5) shows that

$$\langle F, \psi_{kmn}\rangle = i^{-n}\sqrt{2}(2\pi)^{-\frac{d}{2}} \frac{[\lambda(k,n)]^{\frac{d}{2}-1}}{J_{\frac{d}{2}+n}(\lambda(k,n))} A_{kmn}(f). \tag{12}$$

Let

$$b_{kmn}(\mathbf{x}) := \langle e^{-i2\pi\mathbf{x}\cdot\mathbf{y}}, \psi_{kmn}(\mathbf{y})\rangle = \int_B e^{-i2\pi\mathbf{x}\cdot\mathbf{y}} \overline{\psi}_{kmn}(\mathbf{y})\, d\mathbf{y}.$$

Then using (5) and the notation of (10),

$$b_{kmn}(\mathbf{x}) = \int_0^1 \sqrt{2}\, \rho^{\frac{d}{2}} \frac{J_{\frac{d}{2}+n-1}(\lambda(k,n)\rho)}{J_{\frac{d}{2}+n}(\lambda(k,n))} \left(\int_\Sigma e^{-i2\pi r\rho\mathbf{x}'\cdot\mathbf{y}'} \overline{Y}_n^m(\mathbf{y}')\, d\sigma(\mathbf{y}') \right) d\rho.$$

Since $\overline{Y}_n^m$ is a surface spherical harmonic, we can use the Lemma to evaluate the inner integral. The result, after some algebraic simplification, is

$$\begin{aligned} b_{kmn}(\mathbf{x}) &= i^{-n} 2\pi\sqrt{2}\, \frac{r^{1-\frac{d}{2}} \overline{Y}_n^m(\mathbf{x}')}{J_{\frac{d}{2}+n}(\lambda(k,n))} \int_0^1 \rho J_{\frac{d}{2}+n-1}(\lambda(k,n)\rho) J_{\frac{d}{2}+n-1}(2\pi r\rho)\, d\rho \\ &= i^{-n} 2\pi\sqrt{2}\, \frac{\lambda(k,n) r^{1-\frac{d}{2}} J_{\frac{d}{2}+n-1}(2\pi r)}{[\lambda(k,n)]^2 - 4\pi^2 r^2} \overline{Y}_n^m(\mathbf{x}'). \end{aligned} \tag{13}$$

The integral in the last calculation was evaluated using [11, p. 134, eq. (8)] and the fact that $\lambda(k,n)$ is a zero of $J_{n+\frac{d}{2}-1}(z)$.

Finally, an application of Parseval's theorem to the inner product in (11), together with (12) and (13) yields (7). The absolute convergence of the series in (7) follows from viewing this series as an inner product in ℓ^2.

To see the uniform convergence, we proceed as in [2]. Let

$$S_{KN}(\mathbf{x}) := \sum_{n=0}^{N} \sum_{m=1}^{h(n,d)} \sum_{k=1}^{K} A_{kmn}(f) W_{kmn}(\mathbf{x}),$$

and let

$$P_{KN}(\mathbf{y}) := \sum_{n=0}^{N} \sum_{m=1}^{h(n,d)} \sum_{k=1}^{K} \langle F, \psi_{kmn} \rangle \psi_{kmn}(\mathbf{y}).$$

Note that, by Parseval's theorem,

$$\langle P_{KN}(\mathbf{y}), e^{-i2\pi \mathbf{x}\cdot\mathbf{y}} \rangle = \sum_{n=0}^{N} \sum_{m=1}^{h(n,d)} \sum_{k=1}^{K} \langle F, \psi_{kmn} \rangle \bar{b}_{kmn}(\mathbf{x}) = S_{KN}(\mathbf{x}).$$

Then for each $\mathbf{x} \in \mathbb{R}^d$,

$$|f(\mathbf{x}) - S_{KN}(\mathbf{x})| = |\langle F(\mathbf{y}) - P_{KN}(\mathbf{y}), e^{-i2\pi \mathbf{x}\cdot\mathbf{y}} \rangle| \le \|F - P_{KN}\| \, \|e^{-i2\pi \mathbf{x}\cdot\mathbf{y}}\|.$$

But $\|e^{-i2\pi \mathbf{x}\cdot\mathbf{y}}\|^2 = m(B)$ for each $\mathbf{x}$, while $\|F - P_{KN}\|$ is independent of $\mathbf{x}$ and approaches zero as $K, N \to \infty$. Thus, $S_{KN} \to f$ uniformly on $\mathbb{R}^d$.

4 Density of measurements

Instead of measuring function values at a sequence of points, as in the sampling theorem, we must measure the averages $A_{kmn}(f)$ on a sequence of spheres in order to reconstruct the function using (7). Since these measurements are concentrated on a sequence of spheres, it is possible to define the density of these measurements. Recall that the eigenvalue $\lambda(k,n)^2$ has multiplicity $h(n,d)$ and that the number of measurements on the sphere of radius $\lambda(k,n)/2\pi$ is also $h(n,d)$. It follows that each measurement can be associated with one eigenvalue, provided that an eigenvalue of multiplicity $h(n,d)$ is regarded as $h(n,d)$ eigenvalues. It is known [10, p. 169] that the number of eigenvalues of (2) less than Λ, $N(\Lambda)$ say, is asymptotically

$$N(\Lambda) \sim \frac{\Lambda^{\frac{d}{2}}}{2^d [\Gamma(\frac{d}{2}+1)]^2} .$$

Thus, the number of measurements inside a sphere of radius R is the number of eigenvalues for which $\lambda(k,n)/2\pi < R$, or $N(4\pi^2R^2)$ and, since the volume of the ball of radius R is

$$\frac{\pi^{\frac{d}{2}}R^d}{\Gamma(\frac{d}{2}+1)},$$

the density of measurements is

$$\lim_{R\to\infty}\frac{N(4\pi^2R^2)\Gamma(\frac{d}{2}+1)}{\pi^{\frac{d}{2}}R^d}=\frac{\pi^{\frac{d}{2}}}{\Gamma(\frac{d}{2}+1)}=m(B).$$

That is, the density of measurements is equal to the Nyquist-Landau density for B. Thus, the Nyquist-Landau density has a significance beyond that associated with point sampling.

5 Extensions

The results above are easily extended to functions bandlimited to the interior of an ellipsoid. Let $a_i > 0$ for each i, let E be the set

$$E := \left\{\mathbf{y} : \sum_{i=1}^{d}\left(\frac{y_i}{a_i}\right)^2 \le 1\right\},$$

and let

$$f(\mathbf{x}) = \int_E e^{i2\pi\mathbf{x}\cdot\mathbf{y}}\,F(\mathbf{y})\,d\mathbf{y}.$$

Then the changes of variables $x_i = \xi_i/a_i,\ y_i = a_i\eta_i$ puts the expression in a form such that the Theorem can be employed. The result is that $f(\mathbf{x})$ is determined by weighted averages on the sequence of ellipsoids with equations

$$\sum_{i=1}^{d}(a_ix_i)^2 = \frac{[\lambda(k,n)]^2}{4\pi^2}.$$

Moreover, the density of measurements is equal to the Nyquist-Landau density. We omit the details.

Another possible extension is to functions bandlimited to the Cartesian product of two (or more) balls. Let B_1 be the unit ball in $\mathbb{R}^{d_1}$ and let B_2 be the unit ball in $\mathbb{R}^{d_2}$. If $\mathbf{x} \in \mathbb{R}^{d_1+d_2}$ and

$$f(\mathbf{x}) = \int_{B_1\times B_2} e^{i2\pi\mathbf{x}\cdot\mathbf{y}}\,F(\mathbf{y})\,d\mathbf{y},$$

then f is determined by its weighted averages on a sequence of sets of the form $S_1 \times S_2$, where S_1 is a sphere in $\mathbb{R}^{d_1}$ and S_2 is a sphere in $\mathbb{R}^{d_2}$. Thus, each set has dimension $d_1 + d_2 - 2$. The derivation involves applying the linear operator A_{kmn} of (8) successively to the first d_1 coordinates and then to the remaining d_2 coordinates. We omit the details. An example of this type is to be found in [2], where the function is bandlimited to a solid cylinder $\{(x, y, z) : x^2 + y^2 \leq a^2, |z| \leq b\}$. This set is the Cartesian product of a ball in $\mathbb{R}^2$ and a ball in $\mathbb{R}^1$. Since the boundary of the ball in $\mathbb{R}^1$ consists of just two points, the "integral" over this boundary is absorbed in the summation. The remaining integrals are on a sequence of circles in $\mathbb{R}^3$.

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