LINEAR PROGRAMMING BOUNDS ON THE UNION PROBABILITY

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Abstract. Lower and upper bounds on the union probability for N events are derived in terms of the individual and pairwise event probabilities by solving a linear program with $\frac{(N-1)^3+N+3}{2}$ variables. The bounds, which can be efficiently determined, are shown to be optimal when $N \leq 5$ and are always sharper than recent optimal bounds which use slightly less information. Their competitive sharpness is also illustrated via numerical comparisons with state-of-the-art bounds in the literature.

Keywords: Probabbility of a finite union, Lower and upper bounds, Optimal bounds, Linear programming.

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1. Introduction

The derivation of sharp and computationally efficient bounds on the probability of a finite union of events when partial information is available about the events joint probabilities is a longstanding actively researched problem (Gallot 1966; Dawson and Sankoff 1967; Kounias 1968; Galambos and Simonelli 1996; De Caen 1997; Kuai, Alajaji, and Takahara 2000a; Vizvári 2004; Prékopa and Gao 2005; Feng, Li, and Shen 2010; Frolov 2012; Boros et al. 2014; Yang, Alajaji, and Takahara 2014; Yang, Alajaji, and Takahara 2016a, 2016b) and has pertinent applications in several areas including among others, probability theory (e.g., Erdős and Rényi 1959; Feng, Li, and Shen 2009; Feng and Li 2013; Frolov 2017), statistics (Owen, Maximov, and Chertkov 2017), operations research (Ahmed and Papageorgiou 2013), and information theory and statistical communications (e.g., Seguin 1998; Kuai, Alajaji, and Takahara 2000b; Behnamfar, Alajaji, and Linder 2007).

Optimal lower/upper bounds of $P\left(\bigcup_{i=1}^{N} A_i\right)$ in terms of the individual event probabilities $P(A_i)$'s and the pairwise event probabilities $P(A_i \cap A_j)$'s can be seen as special cases of the Boolean probability bounding problem (Boros et al. 2014; Vizvári 2004), which can be solved numerically via a linear programming (LP) problem involving $2^N - 1$ variables. Unfortunately, the number of variables for Boolean probability bounding problems increases exponentially with N, which makes finding the solution impractical. Therefore, some suboptimal numerical bounds were proposed (Galambos and Simonelli 1996; Vizvári 2004; Prékopa and Gao 2005; Boros et al. 2014) in order to reduce the complexity of the LP problem. However, among the existing works, very few can provide optimality guarantees in any sense. Recently, lower/upper bounds shown to be optimal within the class of bounds that use the individual $P(A_i)$'s and the *sums* of pairwise event probabilities, $\sum_j P(A_i \cap A_j)$'s were proposed via numerical algorithms solving a linear programming (LP) problem with $N^2 - N + 1$ variables (Yang, Alajaji, and Takahara 2014; Yang, Alajaji, and Takahara 2016a).

In this paper, we consider exploiting the full knowledge of $\{P(A_i)\}$ and $\{P(A_i \cap A_j)\}$. Specifically, we establish new numerical lower/upper bounds by solving an LP problem with $\frac{(N-1)^3+N+3}{2}$ variables. These bounds are optimal among all bounds having knowledge of $\{P(A_i)\}$ and $\{P(A_i \cap A_j)\}$ when $N \leq 5$. They are also guaranteed to be sharper than the optimal bounds in (Yang, Alajaji, and Takahara 2014; Yang, Alajaji, and Takahara 2016a) which employ $\{P(A_i)\}$ and $\{\sum_j P(A_i \cap A_j)\}$. Finally, numerical comparisons with state-of-the-art bounds, including bounds that utilize more information, illustrate the competitive sharpness of the new bounds.

2. Main Results

Consider a finite family of N events $\{A_1, \dots, A_N\}$ in a finite probability space $(\Omega, \mathscr{F}, P)$ where N is a fixed positive integer. Note that there are only finitely many Boolean atoms specified by the A_i's.

Let $\mathscr{B}$ denote the collection of all non-empty subsets of $\{1, 2, \dots, N\}$. For simplicity, for given $B \in \mathscr{B}$, we let ω_B denote the atom in the union $\cup_{i=1}^N A_i$ such that for $i = 1, \dots, N$, $\omega_B \subset A_i$ if $i \in B$ and $\omega_B \not\subset A_i$ if $i \notin B$ (note that some of these "atoms" may be the empty set). For ease of notation, for a singleton $\omega \in \Omega$, we denote $P(\{\omega\})$ by $p(\omega)$ and $P(\omega_B)$ by p_B. Since $\{\omega_B : i \in B\}$ is the collection of all the atoms in A_i, we have $P(A_i \cap A_j) = \sum_{\omega \in A_i \cap A_j} p(\omega) = \sum_{B \in \mathscr{B}: i,j \in B} p_B$, and

$$P\left(\bigcup_{i=1}^{N} A_i\right) = \sum_{B \in \mathscr{B}} p_B. \tag{1}$$

If we consider the p_B's in (1) as variables, the following (exhaustive) LP problem with $2^N - 1$ variables gives the optimal lower/upper bound established using the sets of probabilities $\{P(A_i)\}$ and $\{P(A_i \cap A_j)\}$:

$$\begin{aligned} &\min_{\{p_B, B \in \mathscr{B}\}} \Big/ \max_{\{p_B, B \in \mathscr{B}\}} \sum_{B \in \mathscr{B}} p_B \\ &\text{s.t.} \quad \sum_{B \in \mathscr{B}: i,j \in B} p_B = P(A_i \cap A_j), \quad i, j \in \{1, \ldots, N\}, \\ &\qquad\quad p_B \geq 0, B \in \mathscr{B}. \end{aligned} \tag{2}$$

The optimality of (2) can be readily proved by showing its achievability (see (Yang, Alajaji, and Takahara 2016a, Lemma 3)): for each p_B, construct an atom ω_B such that $P(\omega_B) = p_B$ and let $\omega_B \in A_i, \forall i \in B$. However, the computational complexity of the optimal lower/upper bound in Eq. (2) is exponential in N.

Defining $\mathscr{B}_k := \{B \in \mathscr{B} : |B| = k\}$, we can write $\mathscr{B}$ as the disjoint union of $\{\mathscr{B}_k, k = 1, \ldots, N\}$, i.e., $\mathscr{B} = \bigcup_{k=1}^{N} \mathscr{B}_k$. Furthermore, we define $\mathscr{A}_k$ to be the set of all ordered subsets of $\{1, \cdots, N\}$ of size k.

Next, we consider the following relaxed version of (2):

(3)

$$\begin{aligned}
&\min_{\{p_B, B\in\mathscr{B}\}} / \max_{\{p_B, B\in\mathscr{B}\}} \sum_{B\in\mathscr{B}} p_B, \\
\text{s.t.} \quad &\sum_{B\in\mathscr{B}: i,j\in B} p_B = P(A_i\cap A_j), \quad i,j\in\{1,\dots,N\}, \\
&\sum_{B\in\mathscr{B}_k: i,j\in B} p_B \geq 0, \quad (i,j)\in\mathscr{A}_2, \quad k=\overline{1,N}, \\
&\sum_{B\in\mathscr{B}_k: i,j\notin B} p_B \geq 0, \quad (i,j)\in\mathscr{A}_2, \quad k=\overline{1,N}, \\
&\sum_{B\in\mathscr{B}_k: i\in B, j\notin B} p_B \geq 0, \quad (i,j)\in\mathscr{A}_2, \quad k=\overline{1,N}, \\
&\sum_{\substack{B\in\mathscr{B}_k:\\ i,j,l\in B}} p_B + \sum_{\substack{B\in\mathscr{B}_k:\\ i,j,l\notin B}} p_B \geq 0, \quad (i,j,l)\in\mathscr{A}_3, \quad k=\overline{1,N}, \\
&\sum_{\substack{B\in\mathscr{B}_k:\\ i,j\in B, l\notin B}} p_B + \sum_{\substack{B\in\mathscr{B}_k:\\ l\in B, i,j\notin B}} p_B \geq 0, \quad (i,j,l)\in\mathscr{A}_3, \quad k=\overline{1,N},
\end{aligned}$$

where $k=\overline{1,N}$ is short for $k=1,\dots,N$. Note that several of the constraints above are identical for every ordering of $(i,j)\in\mathscr{A}_2$ or $(i,j,l)\in\mathscr{A}_3$, but we leave this redundancy in the constraints in the interest of notational simplicity.

Lemma 2.1. *The solution of problem* (3) *coincides the optimal lower/upper bound in* (2) *when* $N\leq 5$.

Proof. It is straightforward to verify that the last five constraints of (3) reduce to $p_B\geq 0, B\in\mathscr{B}$ when $N\leq 5$. □

We next show that the solution of (3), which yields a lower/upper bound for the union probability $P\left(\bigcup_{i=1}^N A_i\right)$, can actually be determined by solving an LP problem with only $\frac{(N-1)^3+N+3}{2}$ variables.

Theorem 2.2. *Defining* $a_{ij}(k) = \sum_{B\in\mathscr{B}_k:i,j\in B} p_B$*, the LP problem* (3) *can be reformulated as an LP in* $\{a_{ij}(k)\}$*; i.e., with* N^3 *variables. The number of variables can further be reduced from* N^3 *to* $\frac{(N-1)^3+N+3}{2}$*.*

Proof. Define $a(k) = \sum_{B\in\mathscr{B}_k} p_B$ and $a_i(k) = \sum_{B\in\mathscr{B}_k:i\in B} p_B$. Then it can be shown that $a(k) = \sum_{i=1}^N \frac{a_i(k)}{k}$ and $a_i(k) = \sum_{j=1}^N \frac{a_{ij}(k)}{k}$. Therefore, both $a(k)$ and $a_i(k)$ are linear functions of $\{a_{ij}(k)\}$.

We next demonstrate that the number of variables can be reduced from N^3 to $\frac{(N-1)^3+N+3}{2}$. Note that according to the definition of $a_{ij}(k)$, the following hold:

i) For $\forall i \neq j$, we have $a_{ij}(1) = P\left(\{x \in A_i \cap A_j, \deg(x) = 1\}\right) = 0$;
ii) For any i and j, we have $a_{ij}(k) = a_{ji}(k)$;
iii) Since $a_{ij}(N) = P\left(\bigcap_{l=1}^N A_l\right)$ for any i and j, then the $a_{ij}(N)$'s are all equal.

Therefore, the number of variables for different values of k can be reduced to

$$\begin{cases} N & \text{if } k = 1 \\ \frac{N(N-1)}{2} & \text{if } k = 2, \ldots, N-1 \\ 1 & \text{if } k = N \end{cases} \tag{4}$$

Thus, the total number of variables is $N + \frac{N(N-1)(N-2)}{2} + 1$.

Now it suffices to show that the objective function and all the constraints in (3) can be written as functions of $a_{ij}(k)$. Indeed the following identities hold.

The objective function and the first constraint of (3) can be written as

$$\begin{aligned} &\sum_k \sum_i \sum_j \frac{a_{ij}(k)}{k^2} = \sum_{B \in \mathscr{B}} p_B, \\ &\sum_k a_{ij}(k) = \sum_{B \in \mathscr{B}: i,j \in B} p_B = P(A_i \cap A_j), \quad \forall i, j. \end{aligned} \tag{5}$$

Finally, for all $i, j, l, k \in \{1, \dots, N\}$, the sums in the other constraints of (3) as functions of $\{p_B\}$ can be written as functions of $\{a_{ij}(k)\}$ as follows:

$$\begin{aligned} &a_{ij}(k) = \sum_{B \in \mathscr{B}_k: i,j \in B} p_B, \\ &a(k) - a_i(k) - a_j(k) + a_{ij}(k) = \sum_{B \in \mathscr{B}_k: i,j \notin B} p_B, \\ &a_i(k) - a_{ij}(k) = \sum_{B \in \mathscr{B}_k: i \in B, j \notin B} p_B \\ &a(k) - a_l(k) - a_i(k) - a_j(k) + a_{ij}(k) + a_{il}(k) + a_{jl}(k) \\ &\qquad = \sum_{B \in \mathscr{B}_k: i,j,l \in B} p_B + \sum_{B \in \mathscr{B}_k: i,j,l \notin B} p_B, \\ &a_l(k) + a_{ij}(k) - a_{il}(k) - a_{jl}(k) \\ &\qquad = \sum_{B \in \mathscr{B}_k: i,j \in B, l \notin B} p_B + \sum_{B \in \mathscr{B}_k: l \in B, i,j \notin B} p_B. \end{aligned} \tag{6}$$

Therefore, the lower/upper bounds of (7) can be solved by an LP with $\frac{(N-1)^3+N+3}{2}$ variables. $\square$

Remark 2.3. The proposed lower/upper bounds are sharper than the numerical bounds in (Yang, Alajaji, and Takahara 2014; Yang, Alajaji, and Takahara 2016a), which have been shown to be optimal within the class of lower/upper bounds that employ $\{P(A_i)\}$ and $\{\sum_j P(A_i \cap A_j)\}$. This can be proven by observing that the LP problem of the numerical bounds in (Yang, Alajaji, and Takahara 2016a, Theorem 1) is a relaxed version of (3). ◁

Remark 2.4. We conjecture that problem (3) shares the same solution with the following LP:

$$
\begin{aligned}
&\min_{\{p_B, B\in\mathscr{B}\}} \Big/ \max_{\{p_B, B\in\mathscr{B}\}} \sum_{B\in\mathscr{B}} p_B, \\
\text{s.t.} \quad & \sum_{B\in\mathscr{B}: i,j\in B} p_B = P(A_i \cap A_j), \quad i,j \in \{1,\ldots,N\}, \\
& \sum_{B\in\mathscr{B}_k: i,j,l\in B} p_B \geq 0, \quad (i,j,l)\in\mathscr{A}_3, \quad k=\overline{1,N}, \\
& \sum_{B\in\mathscr{B}_k: i,j\in B, l\notin B} p_B \geq 0, \quad (i,j,l)\in\mathscr{A}_3, \quad k=\overline{1,N}, \\
& \sum_{B\in\mathscr{B}_k: i\in B, j,l\notin B} p_B \geq 0, \quad (i,j,l)\in\mathscr{A}_3, \quad k=\overline{1,N}, \\
& \sum_{B\in\mathscr{B}_k: i,j,l\notin B} p_B \geq 0, \quad (i,j,l)\in\mathscr{A}_3, \quad k=\overline{1,N}.
\end{aligned} \tag{7}
$$

Since it can be shown that the solution of problem (7) coincides the optimal lower/upper bound in (2) when $N \leq 7$, we therefore conjecture that our new bounds (3) coincide with the optimal bounds when $N \leq 7$. ◁

3. Numerical Examples

In this section, we investigate existing lower and upper bounds for comparison with the proposed new bounds. For lower bounds, we include bounds that utilize $\{P(A_i)\}$ and $\{\sum_j P(A_i \cap A_j), i = 1, \ldots, N\}$, such as the Kuai-Alajaji-Takahara (KAT) lower bound (Kuai, Alajaji, and Takahara 2000a), the numerical Yang-Alajaji-Takahara (YAT-I) lower bound in (Yang, Alajaji, and Takahara 2016a; Yang, Alajaji, and Takahara 2014), and the analytical optimal lower bound (YAT-II) in this class (Yang, Alajaji, and Takahara 2014; Yang, Alajaji, and Takahara 2016a). Note that in this class, the YAT-I bound is known to be optimal. Also, both YAT-I and YAT-II are sharper than KAT (see Yang, Alajaji, and Takahara 2016a for details). We do not include the Dawson-Sankoff (DS) bound since the KAT bound is always sharper than the DS bound (see Kuai, Alajaji, and Takahara 2000a). We also include the Gallot-Kounias (GK) lower bound (Gallot 1966; Kounias 1968; Feng, Li, and Shen 2010) and the stepwise lower bound (Kuai, Alajaji, and Takahara 2000b), which fully exploits $\{P(A_i)\}$ and $\{P(A_i \cap A_j)\}$. We do not show the Chung-Erdős bound (Chung and Erdős 1952) as it is a special case of the GK bound (see Feng, Li, and Shen 2010). Another recent lower bound in (Yang, Alajaji, and Takahara 2016b, Theorem 2), denoted as YAT-III, which uses $\{P(A_i)\}$ and $\{\sum_j c_j P(A_i \cap A_j)\}$ for any given positive weight vector $\boldsymbol{c} = (c_1, \ldots, c_N)^T$ is also included. We randomly generate $100,000$ weight vectors $\boldsymbol{c}$ and select the best possible result (Yang, Alajaji, and Takahara 2016b). Note that it is known that, although not always, the YAT-III bound can be sharper than the GK bound under certain conditions (see Yang, Alajaji, and Takahara 2016b for details).

For upper bounds, we include the widely-used union upper bound

$$P\left(\bigcup_{i=1}^{N} A_i\right) \le \sum_i P(A_i), \tag{8}$$

as well as a new analytical upper bound using $\{P(A_i)\}$ and $\{\sum_j P(A_i \cap A_j)\}$ that is sharper than the union bound (Yang, Alajaji, and Takahara 2016b, Corollary 2):

$$\begin{aligned} P\left(\bigcup_{i=1}^{N} A_i\right) \;\; \le \;\; & \sum_i P(A_i) - \frac{1}{N-1} \sum_{j:j\neq i} P(A_i \cap A_j) \\ & + \frac{1}{N-1} \min_i \{\sum_{j:j\neq i} P(A_i \cap A_j)\}. \end{aligned} \tag{9}$$

Note that (9) is one example of the class of new upper bounds in (Yang, Alajaji, and Takahara 2016b). It is conjectured in (Yang, Alajaji, and Takahara 2016b, Corollary 2) to be the best upper bound in that class. Furthermore, we include the numerical optimal upper bound (YAT-I) in the class using $\{P(A_i)\}$ and $\{\sum_j P(A_i \cap A_j)\}$ (Yang, Alajaji, and Takahara 2016a; Yang, Alajaji, and Takahara 2014) and the algorithmic Greedy upper bound (Kuai, Alajaji, and Takahara 2000b) which fully uses $\{P(A_i)\}$ and $\{P(A_i \cap A_j)\}$.

The Prékopa-Gao (PG) lower/upper bounds (Prékopa and Gao 2005), which extend the KAT bound by using $\{P(A_i)\}$, $\{\sum_j P(A_i \cap A_j)\}$ and $\{\sum_{j,l} P(A_i \cap A_j \cap A_l)\}$, is also investigated in the examples. Note that the PG lower/upper bounds can be sharper than the optimal bounds (2) since they utilize probabilities of intersections of three events. In all numerical examples, we use *italics* to denote the PG bound if it is sharper than the optimal bound (2).

First, the same eight systems as in (Yang, Alajaji, and Takahara 2016a) are used and the corresponding results are shown in Tables 1 and 2. From Table 1, one remarks that the PG bound which uses sums of joint probabilities of three events, may be even poorer (e.g., see Systems I and VI) than the numerical bound YAT-I of (Yang, Alajaji, and Takahara 2016a; Yang, Alajaji, and Takahara 2014) which utilizes less information but is optimal in the class of lower bounds using $\{P(A_i)\}$ and $\{\sum_j P(A_i \cap A_j)\}$. It is also weaker than some other tested bounds in several cases (see Systems I-IV). The proposed numerical bound (3) is always sharper than the other tested bounds which use individual event probabilities and pairwise event probabilities, and coincides with the optimal bound (2) with exponential complexity in N when $N \leq 7$, thus agreeing with our conjecture in (2.4). Similar performance trends for the upper bounds can be seen from Table 2. For example, in Systems VI and VII, the PG upper bound is weaker than YAT-I, the Greedy upper bound as well as the proposed upper bound in (3) even though it utilizes more information.

We next provide additional examples for $N > 7$ to illustrate cases where the proposed lower/upper bounds do not coincide with the optimal lower/upper bounds of (2). System IX (with $N = 8$) and System X (with $N = 10$) are given in Appendix A, while System XI (with $N = 15$ and System XII (with $N = 17$) are not included due to space limitations. We generated these four systems randomly as follows. First we randomly generated the number of atoms N_0 in the union uniformly form 1 to $\min\{2^N - 1, N^4\}$. We chose a relatively smaller number of atoms for large values of N (by constraining it via N^4) due to the computational load. For example, Systems IX, X, XI and XII had 68, 84, 2975 and 4292 randomly generated atoms, respectively. Then we constructed an indicator matrix of size $N_0 \times N$, each element of which belongs to $\{0, 1\}$, to indicate which events the atom belongs to. Finally, we assigned probabilities for all atoms under a proper scaling to guarantee that the true union probability is less than 1.

The lower and upper bounds for these four systems are given in Tables 3 and 4, respectively (note that "N/A" means "not available" as the optimal bound (2) is exponentially complex in N). The results indicate that the new bounds (3) do not coincide with the optimal bounds (2). However, they are still sharper than all other tested bounds that do not use more information and can be computed in polynomial time. The PG bounds, which utilize the additional knowledge of joint probabilities of three events, are mostly better than (2) and (3), except for Systems XI and XII in Table 3. Overall, these numerical examples (and others) demonstrate a competitive effectiveness for the new bounds.

Finally, we give a comment on the computational complexity of the proposed bounds. The proposed upper/lower bounds require solving numerically an LP problem with $\frac{(N-1)^3+N+3}{2}$ variables. Therefore, although tighter than the existing numerical bound YAT-I, the proposed bounds have a slightly higher computational complexity than YAT-I, which is an LP problem with $N^2 - N + 1$ variables (see Yang, Alajaji, and Takahara 2016a). Furthermore, although analytical bounds are in general weaker than numerical bounds, they usually have lower computational complexity. For example, the analytical lower bounds YAT-II, KAT and DS are looser in comparison with YAT-I and the new numerical bound in (3); however, they are much easier to calculate particularly in large data sets applications involving a high number of events.

Table 1. Comparison of lower bounds (in the table, $P(\bigcup A_i)$ is short for $P\left(\bigcup_{i=1}^N A_i\right)$, a bold number indicates coincidence with the optimal bound (2) of exponential complexity in N, and an italic number for the PG bound indicates that this bound is sharper than the optimal bound (2) by using more information).

System	I	II	III	IV	V	VI	VII	VIII
N	6	6	6	7	3	4	4	4
$P(\bigcup A_i)$	.7890	.6740	.7890	.9687	.3900	.3252	.5346	.5854
KAT	.7247	.6227	.7222	.8909	.3833	.2769	.4434	.5412
GK	.7601	.6510	.7508	.9231	.3813	.2972	.4750	.5390
PG	.7443	.6434	.7556	.9148	**.3900**	.3240	*.5281*	*.5726*
YAT-II	.7247	.6227	.7222	.8909	**.3900**	.3205	.4562	.5464
YAT-I	.7487	.6398	.7427	.9044	**.3900**	**.3252**	**.5090**	.5531
YAT-III	.7783	.6633	.7810	.9501	**.3900**	.3203	.4992	.5666
Stepwise	**.7890**	**.6740**	**.7890**	**.9687**	**.3900**	.3027	.5009	**.5673**
Bound (3)	**.7890**	**.6740**	**.7890**	**.9687**	**.3900**	**.3252**	**.5090**	**.5673**

Table 2. Comparison of upper bounds (in the table, $P(\bigcup A_i)$ is short for $P\left(\bigcup_{i=1}^N A_i\right)$, a bold number indicates coincidence with the optimal bound (2) of exponential complexity in N, and an italic number for the PG bound indicates that this bound is sharper than the optimal bound (2) by using more information). Upper bounds are not truncated by 1.

System	I	II*	III*	IV	V	VI	VII	VIII*
N	6	6	6	7	3	4	4	4
$P(\bigcup A_i)$	.7890	.6740	.7890	.9687	.3900	.3252	.5346	.5854
Union (8)	2.07	1.716	1.916	2.418	.5	.5612	.8022	1.574
(9)	1.374	1.136	1.282	1.776	.4125	.3814	.5642	.7813
YAT-I	1.28	.9310	1.133	1.505	**.3900**	**.3252**	**.5346**	**.7070**
Greedy	1.144	.8910	1.078	1.457	.4450	**.3252**	**.5346**	**.7070**
PG	*.8222*	*.6847*	*.8038*	*.9959*	**.3900**	.3413	.5494	*.6132*
Bound (3)	**.8550**	**.8130**	**.9550**	**1.070**	**.3900**	**.3252**	**.5346**	**.7070**

TABLE 3. Comparison of lower bounds: Additional systems for $N > 7$ (a bold number indicates coincidence with the optimal bound (2) of exponential complexity in N, and an italic number for the PG bound indicates that this bound is sharper than the optimal bound (2) by using more information).

System	IX	X	XI	XII
N	8	10	15	17
$P\left(\bigcup_{i=1}^{N} A_i\right)$	.5051	.5261	.1678	.1684
KAT	.4391	.4847	.1581	.1593
GK	.4381	.4852	.1581	.1592
PG	*.4747*	*.5027*	.1630	.1638
YAT-II	.4391	.4847	.1581	.1593
YAT-I	.4393	.4847	.1582	.1593
YAT-III	.4381	.4852	.1581	.1592
Stepwise	.4349	.4758	.1316	.1301
Bound (3)	.4400	.4874	.1646	.1641
Optimal Bound (2)	**.4421**	**.4878**	N/A	N/A

TABLE 4. Comparison of upper bounds: Additional systems for $N > 7$ (a bold number indicates coincidence with the optimal bound (2) of exponential complexity in N, and an italic number for the PG bound indicates that this bound is sharper than the optimal bound (2) by using more information). Upper bounds are not truncated by 1.

System	IX	X	XI	XII
N	8	10	15	17
$P\left(\bigcup_{i=1}^{N} A_i\right)$	.5051	.5261	.1678	.1684
Union (8)	2.110	2.749	1.262	1.429
(9)	1.070	1.418	.6729	.7569
YAT-I	.9566	1.265	.6588	.7396
Greedy	.8735	1.158	.6465	.7325
PG	*.6387*	*.6415*	.2157	.2213
Bound (3)	.7362	.9887	.5906	.6466
Optimal Bound (2)	**.7359**	**.9831**	N/A	N/A

A. Additional Examples: Systems IX and X

For simplicity and without loss of generality, we assume $\omega_i = i$ for all i. Then each event is a set of integers. We use $p_i := p(\omega_i)$ to denote the probability of the i-th atom.

A.1. System IX. System IX consists of 69 atoms with probabilities:

$$
\begin{aligned}
&\{p_i, i = 1, \ldots, 69\} \\
&= \{0.0100, 0.0019, 0.0066, 0.0069, 0.0054, 0.0137, 0.0136, 0.0120, 0.0129, 0.0021, \\
&\quad 0.0106, 0.0036, 0.0062, 0.0010, 0.0043, 0.0081, 0.0120, 0.0102, 0.0008, 0.0075, \\
&\quad 0.0026, 0.0087, 0.0129, 0.0074, 0.0012, 0.0070, 0.0063, 0.0091, 0.0058, 0.0073, \\
&\quad 0.0080, 0.0034, 0.0030, 0.0017, 0.0011, 0.0128, 0.0141, 0.0119, 0.0087, 0.0108, \\
&\quad 0.0070, 0.0015, 0.0127, 0.0084, 0.0038, 0.0004, 0.0060, 0.0089, 0.0032, 0.0030, \\
&\quad 0.0026, 0.0083, 0.0136, 0.0135, 0.0101, 0.0034, 0.0091, 0.0062, 0.0055, 0.0117, \\
&\quad 0.0023, 0.0014, 0.0054, 0.0143, 0.0036, 0.0137, 0.0070, 0.0112, 0.0141\}
\end{aligned}
$$

The $N = 8$ events $A_1, \ldots, A_8$ are as follows:

$$
\begin{aligned}
A_1 =& \{2, 5, 9, 11, 12, 15, 17, 18, 21, 23, 24, 25, 26, 27, 32, 33, 34, \\
& 36, 37, 40, 43, 44, 45, 46, 47, 52, 55, 58, 59, 61, 62, 63, 67\}
\end{aligned}
$$

$$
\begin{aligned}
A_2 =& \{1, 2, 3, 5, 6, 8, 9, 10, 12, 16, 18, 19, 22, 23, 24, 26, 28, 29, 30, 33, 35, \\
& 36, 38, 39, 40, 41, 43, 46, 48, 50, 55, 57, 58, 60, 62, 63, 65, 68, 69\}
\end{aligned}
$$

$$
\begin{aligned}
A_3 =& \{5, 8, 9, 15, 18, 21, 23, 25, 26, 32, 34, 35, 36, 37, 38, 39, \\
& 40, 42, 43, 44, 46, 50, 51, 52, 53, 55, 56, 57, 58, 59, 63, 64, 69\}
\end{aligned}
$$

$$
\begin{aligned}
A_4 =& \{3, 6, 7, 8, 9, 10, 12, 13, 14, 15, 16, 17, 19, 24, 25, 26, 27, 28, 29, 30, 31, \\
& 32, 33, 36, 37, 40, 42, 47, 49, 50, 51, 52, 55, 57, 58, 61, 62, 63, 64, 67, 69\}
\end{aligned}
$$

$$A_5 =\{1, 2, 4, 5, 7, 8, 9, 11, 12, 13, 14, 15, 16, 19, 21, 24, 26, 27, 28, 30, 31,$$
$$32, 36, 38, 40, 42, 43, 44, 46, 47, 50, 51, 55, 57, 58, 59, 60, 62, 64, 66, 68\}$$

$$A_6 =\{4, 6, 8, 11, 12, 15, 16, 19, 22, 24, 25, 27, 28, 32, 33, 38,$$
$$40, 41, 43, 44, 46, 48, 50, 54, 56, 57, 59, 60, 61, 63, 66\}$$

$$A_7 =\{3, 6, 7, 8, 9, 13, 14, 15, 16, 19, 20, 22, 24, 27, 28, 30,$$
$$32, 35, 36, 42, 47, 48, 51, 53, 55, 56, 57, 58, 63, 66, 67, 68\}$$

$$A_8 =\{2, 5, 6, 9, 11, 12, 15, 18, 19, 20, 21, 23, 24, 25, 26, 28, 30, 32, 36, 39,$$
$$b40, 41, 43, 45, 46, 48, 50, 51, 53, 54, 55, 56, 58, 61, 62, 63, 67, 68, 69\}$$

A.2. **System X.** System X consists of 85 atoms with probabilities:

$\{p_i, i = 1, \ldots, 85\}$

$= \{0.0045, 0.0048, 0.0014, 0.0057, 0.0099, 0.0006, 0.0033, 0.0003, 0.0072, 0.0064,$
$0.0103, 0.0074, 0.0091, 0.0057, 0.0111, 0.0071, 0.0117, 0.0105, 0.0099, 0.0033,$
$0.0058, 0.0072, 0.0110, 0.0041, 0.0057, 0.0061, 0.0089, 0.0055, 0.0025, 0.0058,$
$0.0026, 0.0050, 0.0028, 0.0085, 0.0080, 0.0101, 0.0075, 0.0066, 0.0019, 0.0109,$
$0.0067, 0.0058, 0.0051, 0.0111, 0.0097, 0.0115, 0.0080, 0.0042, 0.0065, 0.0010,$
$0.0063, 0.0041, 0.0018, 0.0065, 0.0101, 0.0109, 0.0028, 0.0082, 0.0014, 0.0026,$
$0.0039, 0.0056, 0.0106, 0.0073, 0.0024, 0.0049, 0.0015, 0.0112, 0.0073, 0.0032,$
$0.0035, 0.0091, 0.0049, 0.0063, 0.0062, 0.0096, 0.0012, 0.0111, 0.0062, 0.0069,$
$0.0017, 0.0110, 0.0091, 0.0031, 0.0077\}$

The $N = 10$ events $A_1, \ldots, A_{10}$ are as follows:

$$A_1 =\{1, 4, 6, 7, 8, 12, 14, 17, 20, 22, 23, 24, 25, 27, 31, 32, 35, 36, 37, 39, 43, 48, 49,$$
$$52, 54, 55, 56, 57, 58, 61, 64, 65, 66, 67, 68, 69, 70, 71, 72, 73, 77, 80, 82, 84, 85\}$$

$$A_2 = \{8, 10, 11, 12, 21, 22, 25, 30, 33, 35, 36, 38, 39, 40, 44, 46, 48, 49,$$
$$53, 54, 55, 57, 58, 59, 60, 65, 67, 68, 69, 70, 74, 75, 77, 79, 84\}$$
$$A_3 = \{4, 6, 7, 9, 12, 13, 14, 19, 20, 21, 23, 24, 26, 30, 31, 32, 34, 35, 37, 38, 39, 44,$$
$$45, 47, 48, 50, 52, 57, 58, 59, 60, 62, 64, 68, 69, 70, 74, 75, 78, 80, 82, 83, 84\}$$
$$A_4 = \{1, 2, 3, 4, 6, 7, 9, 12, 13, 14, 15, 17, 18, 20, 22, 23, 24, 26, 27, 31, 32, 34, 37, 38, 44, 45,$$
$$47, 48, 49, 50, 52, 53, 54, 55, 56, 59, 60, 61, 62, 64, 66, 67, 69, 71, 74, 77, 79, 80, 82, 83\}$$
$$A_5 = \{1, 3, 8, 9, 10, 11, 13, 14, 15, 16, 19, 20, 21, 22, 27, 32, 35, 36, 37, 38, 41,$$
$$42, 43, 44, 45, 46, 49, 52, 55, 56, 58, 60, 62, 64, 66, 67, 69, 72, 76, 77\}$$
$$A_6 = \{3, 8, 11, 13, 15, 16, 19, 20, 22, 23, 26, 27, 29, 30, 32, 33, 34, 35, 39, 40,$$
$$47, 48, 50, 51, 52, 53, 54, 55, 58, 60, 61, 62, 64, 65, 67, 73, 74, 75, 77, 79, 84\}$$
$$A_7 = \{3, 4, 6, 7, 8, 9, 11, 12, 13, 14, 15, 17, 18, 20, 21, 22, 25, 26, 27, 30, 31, 32, 34, 35, 36, 37,$$
$$40, 42, 43, 47, 48, 49, 51, 52, 53, 56, 57, 58, 59, 61, 64, 72, 73, 76, 78, 79, 80, 82, 83, 84\}$$
$$A_8 = \{1, 2, 5, 6, 7, 9, 12, 13, 14, 15, 17, 20, 21, 23, 26, 29, 33, 34, 36, 37, 38, 40, 41, 42, 44,$$
$$45, 47, 49, 50, 52, 54, 55, 58, 66, 67, 68, 69, 70, 71, 73, 76, 79, 82, 84, 85\}$$
$$A_9 = \{3, 4, 5, 7, 8, 10, 11, 12, 13, 14, 16, 19, 20, 21, 24, 26, 27, 28, 31, 32, 33, 34, 35,$$
$$36, 39, 41, 42, 43, 44, 46, 48, 49, 50, 51, 52, 53, 62, 64, 66, 72, 76, 81, 82, 85\}$$
$$A_{10} = \{1, 5, 12, 14, 17, 19, 22, 23, 24, 26, 27, 30, 31, 33, 34, 36, 39, 41, 43, 44, 45, 46, 48,$$
$$49, 50, 54, 55, 58, 60, 61, 62, 65, 67, 68, 70, 72, 74, 75, 76, 77, 78, 79, 80, 83, 85\}$$

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