On Bounding the Union Probability

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Abstract—We present new results on bounding the probability of a finite union of events, \( P\left(\bigcup_{i=1}^{N} A_i\right)\) for a fixed positive integer \( N\), using partial information on the events joint probabilities. We first consider bounds that are established in terms of \( \{P(A_i)\}\) and \( \left\{\sum_{j=1}^{N} c_j P(A_i \cap A_j)\right\}\) where \( c_1, \ldots, c_N\) are given weights. We derive a new class of lower bounds of at most pseudo-polynomial computational complexity. This class of lower bounds generalizes the recent bounds in [1], [2] and can be tighter in some cases than the Gallot-Kounias [3]–[5] and Prékopa-Gao [6] bounds which require more information on the events probabilities. We next consider bounds that fully exploit knowledge of \( \{P(A_i)\}\) and \( \{P(A_i \cap A_j)\}\). We establish new numerical lower/upper bounds on the union probability by solving a linear programming problem with \( \frac{(N-1)^2}{2} + N + 3 \) variables. These bounds coincide with the optimal lower/upper bounds when \( N \leq 7\) and are guaranteed to be sharper than the optimal lower/upper bounds of [1], [2] that use \( \{P(A_i)\}\) and \( \{\sum_{j=1}^{N} P(A_i \cap A_j)\}\).

Index Terms—Union probability, lower and upper bounds, linear programming, probability of error analysis, communication systems.

I. INTRODUCTION

Lower/upper bounds on the union probability \( P\left(\bigcup_{i=1}^{N} A_i\right)\) in terms of the individual event probabilities \( P(A_i)\)'s and the pairwise event probabilities \( P(A_i \cap A_j)\)'s were actively investigated in the recent past. The optimal bounds can be obtained numerically by solving linear programming (LP) problems with \( 2^N \) variables [6], [7]. Since the number of variables is exponential in the number of events, \( N\), some suboptimal but numerically efficient bounds were proposed, such as the bounds in [8] that employ the dual basic feasible solutions to reduce the complexity of the LP problem, and the algorithmic Bonferroni-type lower/upper bounds in [9], [10].

Among the established analytical bounds is the Kuai-Alajaji-Takahara lower bound (for convenience, hereafter referred to as the KAT bound) [11] that was shown to be better than the Dawson-Sankoff (DS) [12] and the D. de Caen (DC) [13] bounds. Noting that the KAT bound is expressed in terms of \( \{P(A_i)\}\) and only the sums of the pairwise event probabilities, i.e., \( \left\{\sum_{j \neq i} P(A_i \cap A_j)\right\}\), in order to fully exploit all pairwise event probabilities, it is observed in [14]–[16] that the analytical bounds can be further improved algorithmically by optimizing over subsets. Furthermore, in [6], the KAT bound is extended by using additional partial information such as the sums of joint probabilities of three events, i.e., \( \left\{\sum_{j \neq 1} P(A_i \cap A_j)\right\} \) for \( i = 1, \ldots, N\). Recently, using the same partial information as the KAT bound, i.e., \( \{P(A_i)\}\) and \( \left\{\sum_{j \neq i} P(A_i \cap A_j)\right\}\), the optimal lower/upper bound as well as a new analytical bound which is sharper than the KAT bound were developed in [1], [2].

In this paper, we first establish a new class of lower bounds on \( P\left(\bigcup_{i=1}^{N} A_i\right)\) using \( \{P(A_i)\}\) and \( \{\sum_{j=1}^{N} c_j P(A_i \cap A_j)\}\) for a given weight or parameter vector \( c = (c_1, \ldots, c_N)\)\(^T\). These lower bounds are shown to have at most pseudo-polynomial computational complexity and to be sharper in certain cases than the existing Gallot-Kounias (GK) [3]–[5] and Prékopa-Gao (PG) [6] bounds, although the later bounds employ more information on the events joint probabilities. Furthermore, for bounds on \( P\left(\bigcup_{i=1}^{N} A_i\right)\) that fully exploit knowledge of \( \{P(A_i)\}\) and \( \{P(A_i \cap A_j)\}\), a new numerical lower/upper bound is proposed by solving an LP problem with \( \frac{(N-1)^2}{2} + N + 3 \) variables. This numerical lower/upper bound is proven to be an optimal lower/upper bound when \( N \leq 7\) and to be always better than the optimal lower/upper bound which uses \( \{P(A_i)\}\) and \( \{\sum_{j=1}^{N} P(A_i \cap A_j)\}\). Finally, we should note that these general union probability bounds can be applied to effectively estimate and analyze the error performance of a variety of coded or uncoded communication systems (e.g., see [2], [9], [10], [14], [17]–[22]).

II. NEW BOUNDS USING \( \{P(A_i)\}\) AND \( \{\sum_{j=1}^{N} c_j P(A_i \cap A_j)\}\)

For simplicity, and without loss of generality, we assume the events \( \{A_1, \ldots, A_N\} \) are in a finite probability space \( (\Omega, \mathcal{F}, P)\), where \( N \) is a fixed positive integer. Let \( \mathcal{B} \) denote the collection of all non-empty subsets of \( \{1, 2, \ldots, N\}\). Given \( B \in \mathcal{B}\), we let \( \omega_B \) denote the atom in the union \( \bigcup_{i=1}^{N} A_i \) such that for all \( i = 1, \cdots, N\), \( \omega_B \in A_i\) if \( i \in B\) and \( \omega_B \notin A_i\) if \( i \notin B\) (note that some of these “atoms” may be the empty set). For ease of notation, for a singleton \( \omega \in \Omega\), we denote \( P(\{\omega\}) \) by \( p(\omega)\) and \( p(\omega_B)\) by \( p_B\). Since \( \{p_B : i \in B\} \) is the collection of all the atoms in \( A_i\), we have \( P(A_i) = \sum_{\omega \in A_i} p(\omega) = \sum_{B \in \mathcal{B}, \omega_B \in B} p_B\). and
\[ P \left( \bigcup_{i=1}^{N} A_i \right) = \sum_{B \subseteq \mathcal{B}} p_B. \] (1)

Suppose there are \( N \) functions \( f_i(B), i = 1, \ldots, N \) such that \( \sum_{i=1}^{N} f_i(B) = 1 \) for any \( B \in \mathcal{B} \) (i.e., for any atom \( \omega_B \)). If we further assume that \( f_i(B) = 0 \) if \( i \notin B \) (i.e., \( \omega_B \notin A_i \)), we can write
\[ P \left( \bigcup_{i=1}^{N} A_i \right) = \sum_{B \subseteq \mathcal{B}} \left( \sum_{i=1}^{N} f_i(B) \right) p_B = \sum_{i=1}^{N} \sum_{B : i \in B} f_i(B) p_B. \] (2)

Note that if we define
\[ f_i(B) = \begin{cases} \frac{1}{|B|} & \text{if } i \in B \\ 0 & \text{if } i \notin B \end{cases} \] (3)
where the degree of \( \omega \), \( \deg(\omega) \), is the number of \( A_i \)'s that contain \( \omega \), then \( \sum_{i=1}^{N} f_i(B) = 1 \) is satisfied and (2) becomes
\[ P \left( \bigcup_{i=1}^{N} A_i \right) = \sum_{i=1}^{N} \sum_{\omega \in A_i} p(\omega) / \deg(\omega). \] (4)

Note that many of the existing bounds, such as the DC bound [13] and KAT bound [11] and the bounds in [1] [2], are based on (4).

In the following lemma, we propose a generalized expression of (4). To the best of our knowledge this lemma is novel.

**Lemma 1:** Suppose \( \{ \omega_B, B \in \mathcal{B} \} \) are all the \( 2^N - 1 \) atoms in \( \bigcup A_i \). If \( c = (c_1, \ldots, c_N)^T \in \mathbb{R}^N \) satisfies
\[ \sum_{B \subseteq \mathcal{B}} c_k 
eq 0, \quad \text{for all } B \in \mathcal{B} \] (5)
then we have
\[ P \left( \bigcup_{i=1}^{N} A_i \right) = \sum_{i=1}^{N} \sum_{B : i \in B} \frac{c_i p_B}{c_k} = \sum_{i=1}^{N} \sum_{\omega \in A_i} \frac{c_i p(\omega)}{\sum_{k : \omega \in A_k} c_k}. \] (6)

**Proof:** If we define
\[ f_i(B) = \begin{cases} \frac{c_i}{\sum_{k : \omega \in A_k} c_k} & \text{if } i \in B \\ 0 & \text{if } i \notin B \end{cases} \] (7)
where the parameter vector \( c = (c_1, c_2, \ldots, c_N)^T \) satisfies \( \sum_{k : B \subseteq \mathcal{B}} c_k \neq 0 \) for all \( B \in \mathcal{B} \) (therefore \( c_i \neq 0 \), \( i = 1, \ldots, N \)), then \( \sum_{i} f_i(B) = 1 \) holds and we can get (6) from (2).

Note that (6) holds for any \( c \) that satisfies (5) and is clearly a generalized expression of (4).

**A. Relation to the Cohen-Merhav bound [19]**

Let \( m_i(\omega_B) \) be non-negative functions. Then by the Cauchy-Schwarz inequality,
\[ \left( \sum_{B : i \in B} f_i(B) p_B \right) \left( \sum_{B : i \in B} \frac{p_B}{f_i(B)} m_i(\omega_B) \right) \geq \left( \sum_{B : i \in B} \frac{p_B m_i(\omega_B)}{f_i(B)} \right)^2. \] (8)
Thus, using (2), we have
\[ P \left( \bigcup_{i=1}^{N} A_i \right) \geq \sum_{i=1}^{N} \left[ \sum_{B : i \in B} P_B m_i(\omega_B) \right]^2. \] (9)
If we define \( f_i(B) \) by (3), then (9) reduces to
\[ P \left( \bigcup_{i=1}^{N} A_i \right) \geq \sum_{i} \left[ \sum_{\omega \in A_i} P(\omega) m_i(\omega) \right]^2, \] (10)
which is the Cohen-Merhav lower bound in [19, Theorem 2.1]; note that equality in (10) holds when \( m_i(\omega) = \frac{1}{\deg(\omega)} \) (i.e., \( m_i(\omega_B) = \frac{1}{|B|} \)).

**B. Relation to the GK Bound [3], [4]**

In this subsection, we assume that the elements of \( c \) are positive, i.e., \( c \in \mathbb{R}_+^N \), and connect the GK bound [3] [4] with (6). The GK bound was recently revisited in [5] where it is reformulated as
\[ \ell_{\text{GK}} = \max_{c \in \mathbb{R}^N} \sum_{i} c_i P(A_i)^2 / \sum_{i} c_i \sum_{k : i \in A_k} c_k P(A_i \cap A_k), \] (11)
and the optimal \( c \) for (11), denoted by \( \tilde{c} \), can be computed by
\[ \tilde{c} = \Sigma^{-1} \alpha, \] (12)
where \( \alpha = \left( P(A_1), \ldots, P(A_N) \right)^T \) and \( \Sigma \) is the \( N \times N \) matrix whose \((i, j)\)-th element is \( P(A_i \cap A_j) \).

First, consider \( c \in \mathbb{R}^N_+ \) fixed. Then, by the Cauchy-Schwarz inequality, we have
\[ \left[ \sum_{B : i \in B} \frac{c_i p_B}{c_k} \right] \left[ \sum_{B : i \in B} \frac{c_k P(A_k \cap A_i)}{c_i} \right] \geq P(A_i)^2. \] (13)
Note that
\[ \sum_{B : i \in B} \left( \frac{c_k P(A_k \cap A_i)}{c_i} \right) = \frac{1}{c_i} \sum_{k = 1}^{N} \sum_{B : i \in B, k \in B} c_k p_B = \sum_{k} c_k P(A_i \cap A_k) / c_i. \] (14)
Then for all \( i \),
\[ \sum_{B : i \in B} c_i p_B \geq c_i \sum_{k} c_k P(A_i \cap A_k) \] (15)
By summing (15) over \( i \), we get another new lower bound:
\[ P \left( \bigcup_{i=1}^{N} A_i \right) \geq \sum_{i} \left[ \sum_{k} c_k P(A_i \cap A_k) \right] / c_i. \] (16)
Note that we can use Cauchy-Schwarz Inequality again:
\[ \left[ \sum_{i} c_i \sum_{k} c_k P(A_i \cap A_k) \right] \left[ \sum_{i} c_i \sum_{k} c_k P(A_i \cap A_k) \right] \geq \left[ \sum_{i} c_i \left( P(A_i) \right)^2 \right]. \] (17)
Since the above inequality holds for any positive $c$, we have

$$P \left( \bigcup_{i=1}^{N} A_i \right) \geq \max_{c \in \mathbb{R}_+} \left\{ \sum_{i=1}^{N} c_i^2 P(A_i \cap A_k) \right\}$$

$$\geq \max_{c \in \mathbb{R}_+} \left\{ \sum_{i=1}^{N} \sum_{c} c_i P(A_i \cap A_k) \right\}$$

(18)

Note that the lower bounds in (18) are weaker than the GK bound (11), however, if the optimal $c$ of (11), $\tilde{c}$, happen to satisfy $\tilde{c} \in \mathbb{R}_+$, then the bounds in (18) coincide with the GK bound (11).

C. New Class of Lower Bounds

We only consider $c \in \mathbb{R}_+$ in this subsection. A new class of lower bounds is given in the following theorem.

Theorem 1: Defining $\mathcal{B}^{-} = \mathcal{B} \setminus \{1, \ldots, N\}$, $\tilde{\gamma}_i := \sum_k c_k P(A_i \cap A_k)$, $\tilde{\alpha}_i := P(A_i)$ and

$$\tilde{\delta} := \max_i \left( \tilde{\gamma}_i - \left( \sum_k c_k - \min_k c_k \right) \tilde{\alpha}_i \right)$$

(19)

where $c \in \mathbb{R}_+$, a class of lower bounds is given by

$$P \left( \bigcup_{i=1}^{N} A_i \right) \geq \tilde{\delta} + \sum_{i=1}^{N} \ell_i(c, \tilde{\delta})$$

(20)

where

$$\ell_i(c, x) = \left[ P(A_i) - x \right] \left( \sum_{k \in B_i} c_k \right) + \left( \sum_{k \in B_i} c_k \right) - \left( \sum_{k \in B_i} c_k \right) \frac{\left( \sum_{k \in B_i} c_k \right)}{\left( \sum_{k \in B_i} c_k \right)}$$

(21)

and

$$B_1^{(i)} = \arg \max_{\{B \in \mathcal{B}^{-} : \alpha \in B\}} \left\{ \sum_{k \in B} c_k \right\}$$

s.t. $\sum_{k \in B} c_k \leq \sum_{k \in B} c_k [P(A_i \cap A_k) - x]$, $\sum_{k \in B} c_k \leq \sum_{k \in B} c_k [P(A_i) - x]$, $\sum_{k \in B} c_k \geq \sum_{k \in B} c_k [P(A_i \cap A_k) - x]$, $\sum_{k \in B} c_k \geq \sum_{k \in B} c_k [P(A_i) - x]$. $i = 1, \ldots, N$.

Proof: Let $x = p_{\{1, 2, \ldots, N\}}$ and consider $\sum_i \ell_i(c, x) + x$ as a new lower bound where where $\ell_i(c, x)$ equals to the objective value of the problem

The solution of (23) exists if and only if

$$\min_{k} c_k \leq \frac{\gamma_i - (\sum_k c_k) x}{\alpha_i - x} \leq \sum_k c_k - \min_k c_k$$

(24)

Therefore, the new lower bound can be written as

$$\min_{x} \left[ x + \sum_{i=1}^{N} \ell_i(c, \tilde{\delta}) \right] \text{ s.t.} \left[ \frac{\gamma_i - (\sum_k c_k - \min_k c_k) \alpha_i}{\min_k c_k} \right] \leq x \leq \frac{\gamma_i - (\sum_k c_k) \alpha_i}{\min_k c_k}$$

(25)

We can prove that the objective function of (25) is non-decreasing with $x$. Therefore, defining $\tilde{\delta}$ as in (19), the new lower bound can be written as (25) where $\ell_i(c, \tilde{\delta})$ can be obtained by solving (23), which is given in (21).

Remark 1: Note that the problems in (21) are exactly the 0/1 knapsack problem with mass equals to $23$, which can be computed in pseudo-polynomial time, and can be arbitrarily closely approximated by an algorithm running in polynomial time $\mathcal{O}(N)$.

Remark 2: It can readily be shown that if $c = \kappa \mathbf{1}$ for any non-zero constant $\kappa$ with $\mathbf{1}$ being the all-one vector of length $N$, the new lower bound reduces to the analytical lower bound in [1], [2], which is sharper than the KAT bound. It can also be shown that if the optimal $\tilde{c}$ of the GK bound satisfies $\tilde{c} \in \mathbb{R}_+$, then the new lower bound is sharper than the GK bound.

III. New Bounds Using $\{P(A_i)\}$ and $\{P(A_i \cap A_j)\}$

In this section, we derive new numerical lower/upper bounds for $P \left( \bigcup_{i=1}^{N} A_i \right)$ using $\{P(A_i)\}$ and $\{P(A_i \cap A_j)\}$. First, consider the $p_B$'s in (1) as variables. Then the following (exhaustive) LP problem with $2^N$ variables gives the optimal lower/upper bound established using $\{P(A_i)\}$ and $\{P(A_i \cap A_j)\}$:

$$\min_{\{p_B, B \in \mathcal{B}\}} \max_{\{p_B, B \in \mathcal{B}\}} \sum_B p_B$$

s.t. $\sum_{i,j \in B, B \in \mathcal{B}} p_B = P(A_i \cap A_j)$, $i, j \in \{1, \ldots, N\}$,

$$p_B \geq 0, B \in \mathcal{B}$$

(26)

The optimality of (26) can be easily proved by showing its achievability: for each $p_B$, construct an atom $\omega_B$ such that $p(\omega_B) = p_B$ and let $\omega_B \in A_i, \forall i \in B$. However, the computational complexity of the optimal lower/upper bound
in (26) is exponential. Next, we consider a relaxed problem of (26), which is given in the following:

\[
\min_{\{p_B, B \in \mathcal{B}\}} \sum_{B \in \mathcal{B}} p_B, \quad \max_{\{p_B, B \in \mathcal{B}\}} \sum_{B \in \mathcal{B}} p_B \\
\text{s.t. } \sum_{i,j} p_B = P(A_i \cap A_j), \quad i, j \in \{1, \ldots, N\}, \\
\sum_{B:i,j \in B, |B|=k} p_B \geq 0, \quad \sum_{B:i,j \in B, |B|=k} p_B \geq 0, \\
\forall i, j, k \in \{1, \ldots, N\}. 
\]

(27)

Since the solution of (27) is a lower/upper bound for the union probability \(P\left(\bigcup_{i=1}^N A_i\right)\), we next show that the solution of (27) can be obtained by solving an LP problem with \(\frac{(N-1)^3 + N^3}{2}\) variables, which coincides with the optimal lower/upper bounds when \(N \leq 7\). The main results are in the following.

**Lemma 2**: The solution of problem (27) coincides with the optimal lower/upper bound in (26) when \(N \leq 7\).

**Lemma 3**: The problem (27) shares the same solution with the following LP:

\[
\min_{\{p_B, B \in \mathcal{B}\}} \sum_{B \in \mathcal{B}} p_B, \quad \max_{\{p_B, B \in \mathcal{B}\}} \sum_{B \in \mathcal{B}} p_B \\
\text{s.t. } \sum_{i,j} p_B = P(A_i \cap A_j), \quad i, j \in \{1, \ldots, N\}, \\
\sum_{B:i,j \in B, |B|=k} p_B \geq 0, \quad \sum_{B:i,j \in B, |B|=k} p_B \geq 0, \\
\forall i, j, k \in \{1, \ldots, N\}. 
\]

(28)

**Theorem 2**: Defining \(a_{ij}(k) = \sum_{i,j \in B, |B|=k} p_B\), the LP problem (28) can be reformulated as an LP of \(\{a_{ij}(k)\}\) (i.e., \(N^3\) variables). The number of variables can hence be reduced from \(N^3\) to \(\frac{(N-1)^3 + N^3}{2}\).

**Proof**: Define \(a(k) = \sum_{B:|B|=k} p_B\) and \(a_1(k) = \sum_{i \in B \in [N]} a_{ij}(k)\), then it can be readily shown that \(a(k) = \sum_{i=1}^N a_i(k)\) and \(a_i(k) = \sum_{j=1}^N a_{ij}(k)\). Therefore, both \(a(k)\) and \(a_i(k)\) are linear functions of \(\{a_{ij}(k)\}\).

We next demonstrate that the number of variables can be reduced from \(N^3\) to \(\frac{(N-1)^3 + N^3}{2}\). Note that according to the definition of \(a_{ij}(k)\), we have: i) \(a_{ij}(1) = P\{x \in A_i \cap A_j, \deg(x) = 1\} = 0, \forall i \neq j\); ii) \(a_{ij}(k) = a_{ij}(k)\); iii) \(a_{ij}(N) = P\left(\bigcap_{i=1}^N A_i\right)\) for any \(i\) and \(j\). Therefore, the number of variables for different values of \(k\) can be reduced to

\[
\left\{\begin{array}{ll}
\frac{N}{2} & \text{if } k = 1 \\
\frac{N(N-1)}{2} & \text{if } k = 2, \ldots, N - 1 \\
1 & \text{if } k = N
\end{array}\right.
\]

(29)

Thus, the total number of variables is \(N + \frac{N(N-1)(N-2)}{2} + 1\).

Now it is suffices to show that the objective function and all the constraints in (28) can be written as functions of \(a_{ij}(k)\) so that all \(\{p_B\}\) can be replaced using \(a_{ij}(k)\). In the following, we directly give the results, which one can easily verify.

The objective function and the first constraint of (28) can be written as

\[
\sum_k \sum_i \sum_j \frac{a_{ij}(k)}{k^2} = \sum_{B \in \mathcal{B}} p_B, \\
\sum_k a_{ij}(k) = \sum_{i,j} p_B = P(A_i \cap A_j), \quad \forall i, j.
\]

(30)

Finally, for all \(i, j, k \in \{1, \ldots, N\}\), the other constraints of (28) as functions of \(\{p_B\}\) can be written as functions of \(\{a_{ij}(k)\}\) as follows:

\[
a_{ij}(k) = \sum_{B:i,j \in B, |B|=k} p_B + \sum_{B:i,j \notin B, |B|=k} p_B, \\
a(k) - a_i(k) - a_j(k) + a_{ij}(k) = \sum_{B:i \in B, |B|=k} p_B + \sum_{B:j \in B, |B|=k} p_B, \\
a(k) - a_i(k) - a_j(k) - a_{ij}(k) + a_{ij}(k) = \sum_{B:i \in B, |B|=k} p_B + \sum_{B:j \in B, |B|=k} p_B, \\
a_i(k) = a_{ij}(k) - a_{ik}(k) - a_{jk}(k) = \sum_{B:i,j \in B, |B|=k} p_B + \sum_{B:i,j \notin B, |B|=k} p_B, \\
a_i(k) - a_{ij}(k) = \sum_{B:i,j \in B, |B|=k} p_B + \sum_{B:i,j \notin B, |B|=k} p_B.
\]

(31)

Therefore, the lower/upper bounds of (27) can be solved by an LP with \(\frac{(N-1)^3 + N^3}{2}\) variables.

**Remark 3**: According to Lemma 2, the new numerical lower/upper bound coincides with the optimal lower/upper bounds in (26) when \(N \leq 7\). Furthermore, we can show that the new numerical lower/upper bounds are sharper than the numerical bounds in [1], [2], which have been proved to be the optimal lower/upper bounds in terms of \(\{P(A_i)\}\) and \(\{\sum_j P(A_i \cap A_j)\}\).

**IV. Numerical Examples**

Due to the space limitation, we only present lower bounds in this section. The same eight systems as in [1] are used and the corresponding results are shown in Table I. For comparison, we include bounds that utilize \(\{P(A_i)\}\) and
\[
\{ \sum_{j} (P(A_i \cap A_j), i = 1, \ldots, N) \}, \text{ such as the KAT bound [11], the analytical bound in [1], [2], and the numerical optimal bound in this class [1], [2]. We also include the GK bound [3], [4] and the stepwise bound [9], which fully exploit \{ P(A_i) \} and \{ P(A_i \cap A_j) \}. The PG lower bound [6], which extends the KAT bound by using \{ P(A_i) \}, \{ \sum_{j} P(A_i \cap A_j) \} and \{ \sum_{j,k} P(A_i \cap A_j \cap A_k) \}, is also investigated in the examples. The Cohen-Merhav bound (10) [19] is not included since it is not clear how to choose the function \( m_i(\epsilon) \) in our examples.
\]

For the proposed bound (20) we consider two cases for choosing \( c \). The first choice for \( c \), denoted by \( \tilde{c}^+ \), has components \( \tilde{c}_i^+ = \max(\tilde{c}_i, \epsilon) \) with \( \tilde{c} \) given in (12) and \( \epsilon > 0 \) close to zero. Therefore, if \( \tilde{c} \in \mathbb{R}^N_+ \) then \( \tilde{c}^+ = \tilde{c} \), so that in this case the new bound (20) is guaranteed to be sharper than the GK bound. If \( \tilde{c} \notin \mathbb{R}^N_+ \), on the other hand, we still have \( \tilde{c}^+ \in \mathbb{R}^N_+ \). The second choice of \( c \) is to randomly generate \( \tilde{c} \in \mathbb{R}^N_+ \) and compute (20). In the examples, we generate 1000 values for \( c \) and show the largest obtained value for (20).

From Table I, one remarks that for Systems II, III and VIII we have \( \tilde{c} \in \mathbb{R}^N_+ \), that the new bound (20) with \( c = \tilde{c} \) is sharper than the GK bound, as expected. Also, the new bound (20) can be further improved by randomly generating additional \( c \) values as shown in the table. Furthermore, the PG bound which uses sums of joint probabilities of three events, may be even poorer (e.g., see Systems I and VI) than the numerical bound in [1], [2] which utilizes less information but is optimal in the class of lower bounds using \{ P(A_i) \} and \{ \sum_j P(A_i \cap A_j) \}. It is also weaker than (20) in several cases (see Systems I-IV). Finally, our numerical bound (27) is always sharper than the other tested bounds, and coincides with the optimal bound (26) with exponential complexity in \( N \) since \( N < 7 \) holds for these examples.

\section*{References}