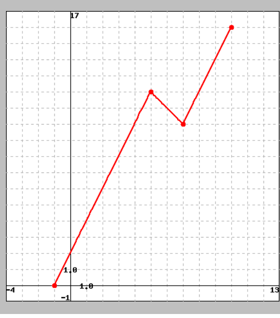
Unit #1 - Transformation of Functions, Exponentials and Logarithms

Some problems and solutions selected or adapted from Hughes-Hallett Calculus.

Note: This unit, being review of pre-calculus has **substantially more practice problems** than later units. Don't be intimidated by the length, and just do as many problems as you need to refresh your skills.

Functions

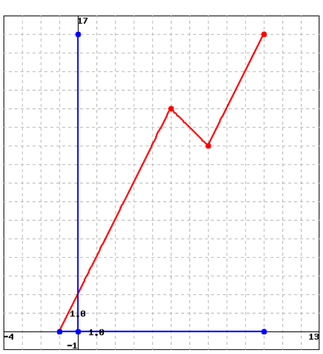
1. What are the domain and range of $f(x)$ shown in the graph below?



- The domain is $[-1, 10]$ (component of the graph covered in the x direction).

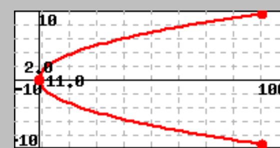
- The range is $[0, 16]$ (component of the graph covered in the y direction).

The graph below shows the domain and range with blue lines.

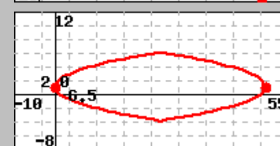


2. Which of the following graphs is the graph of a function?

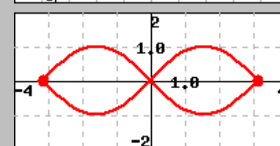
A



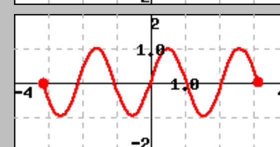
B



C



D



(D) is the graph of a function. **(C)** , **(A)** , and **(B)** all fail the vertical line test.

Remember that, for a given x value, a function can only have **one single** y value.

3. Consider the function $g(x) = \dfrac{(4 - 4x^2)}{(4x^2 + 3x)}$.

   (a) Find the domain of $g(x)$.
   (b) Find the values of x that satisfy $g(x) = 0$.

(a) The domain is determined for this function by values that avoid a zero denominator. Setting the denominator equal to zero, we find the $x = 0$ and $x = -0.75$ make the denominator zero. Since all other values of x will produce valid outputs for $g(x)$, the domain is

$$(-\infty, -0.75)U(-0.75, 0)U(0, \infty).$$

(b) The values that satisfy $g(x) = 0$ can be found by solving the equation

$$\frac{(4 - 4x^2)}{(4x^2 + 3x)} = 0$$

The numerator must equal zero, so the solutions come from

$$4 - 4x^2 = 0$$

which gives $x = -1$ and $x = 1$ as solution.

4. Let $f(t) = \sqrt{t^2 - 36}$.

   a) Find all values of t for which $f(t)$ is a real number.
   b) Find the value(s) of t such that $f(t) = 8$.

(a) The domain is $(-\infty, -6]U[6, \infty)$. We can only use t values that lead to a 0 or positive numbers under the square root.

(b) The solutions to $f(t) = 8$ are $t = -10$ and $t = 10$. Square both sides of $\sqrt{t^2 - 36} = 8$ and then solve for t.

5. Give the domain and range of the function $y = \dfrac{1}{x^2+5}$.

(a) The domain is $(-\infty, \infty)$. No matter what x values are used, the denominator will always be positive/non-zero, so there are no 'bad' x values.

(b) The range is $(0, 0.2]$. This take a little more work to figure out, but if you start plotting the graph, you'll see that as x moves away from $x = 0$, y will always be decreasing.

   - At $x = 0$, the value of $y = 1/5 = 0.2$.
   - As x moves away from zero, the denominator gets larger and larger, so $y \to 0$, though it can never actually equal zero.

   This gives the range of $0 < y \leq 0.2$, or in interval notation, $y \in (0, 0.2]$.

6. Give the domain and range of the function $y = x^2 + 1$.

(a) The domain is $(-\infty, \infty)$. The usual reasons for a break in the domain, a divide by zero, square root of a negative number, or a logarithm, are all absent, so any x value will produce a real y value.

(b) The range is $[1, \infty)$. This can be inferred either from the graph (a parabola that opens up, shifted up by 1), or by noting that y is smallest when x^2 is smallest, i.e. when $x = 0$, and increases as x moves away from $x = 0$.

7. Determine the domain of the function $f(x) = \dfrac{1}{x+1}$

The domain of a rational function $\frac{P(x)}{Q(x)}$ is the set of numbers x such that $Q(x) \neq 0$.
In this case, we need $x + 1 \neq 0$, or $x \neq -1$. Therefore the domain is $(-\infty, -1) \cup (-1, \infty)$.

8. The monthly charge for a waste collection service is 1475 dollars for 100 kg of waste and 1895 dollars for 130 kg of waste.

   (a) Find a linear model for the cost, C, of waste collection as a function of the number of kilograms, w.

   (b) What is the slope of the line found in part (a)?

   Give units for the units for the slope.

   (c) What is the value of the vertical intercept of the line found in part (a)? Give the units of the intercept.

(a) We find the slope m and intercept b in the linear equation $C = b + mw$. To find the slope m, we use
$m = \dfrac{\Delta C}{\Delta w} = \dfrac{1895 - 1475}{130 - 100} = 14.$
We substitute to find b:
$1895 = b + (14)(130)$
so that $b = 75$.
The linear formula is $C = 14w + 75$.

(b) The slope is 14 dollars per kilogram. Each additional kilogram of waste costs 14 dollars.

(c) The intercept is 75 dollars. The flat monthly fee to subscribe to the waste collection is 75 dollars, even if there is no waste.

9. For tax purposes, you may have to report the value of your assets, such as cars or refrigerators. The value you report drops with time. "Straight-line depreciation" assumes that the value is a linear function of time. If a 1100 dollar refrigerator depreciates completely in 14 years, find a formula for its value as a function of time, x, in years.

We are looking for a linear function $y = f(x)$ that, given a time x in years, gives a value y in dollars for the value of the refrigerator. We know that when $x = 0$, that is, when the refrigerator is new, $y = 1100$, and when $x = 14$, the refrigerator is worthless, so $y = 0$. Thus $(0, 1100)$ and $(14, 0)$ are on the line that we are looking for. The slope is then given by

$$m = \frac{1100}{-14} = -\frac{550}{7}$$

It is negative, indicating that the value decreases as time passes. Having found the slope, we can take the point $(14, 0)$ and use the point-slope formula:

$$y - y_1 = m(x - x_1)$$

So,

$$y - 0 = -\frac{550}{7}(x - 14),$$

and the equation for the value is $y = 1100 - \frac{550}{7}x$.

10. Residents of the town of Maple Grove who are connected to the municipal water supply are billed a fixed amount monthly plus a charge for each cubic foot of water used. A household using 1500 cubic feet was billed \$55, while one using 2100 cubic feet was billed \$73.

(a) What is the charge per cubic foot?

(b) Write an equation for the total cost, C, of a resident's water as a function of cubic feet, x, of water used.

(c) How many cubic feet of water used would lead to a bill of \$100?

(a) The charge per cubic foot is 0.03 dollars.

(b) The equation for the cost, using x cubic feet, is $C(x) = 0.03x + 10$.

(c) We solve the equation $C(x) = 100$, and get $x = (100 - 10)/0.03 = 3000$ cubic feet of water would cost \$100.

11. The value of a car, $V = f(a)$, in thousands of dollars, is a function of the age of the car, a, in years.

(a) Interpret the statement $f(5) = 6$.

(b) Sketch a possible graph of V against a. Is f an increasing or decreasing function?

(c) In terms of the value of the car, what does the horizontal intercept signify? What does the vertical intercept signify?

(a) The car has a value of 6,000 dollars after 5 years.

(b) The value of the car will be decreasing over time.

(c) The horizontal intercept represents when $V = 0$, or when the car will have a value of zero dollars. The vertical intercept represents when $t = 0$, or the value of the car when it is new (assuming $t = 0$ represents the purchase time).

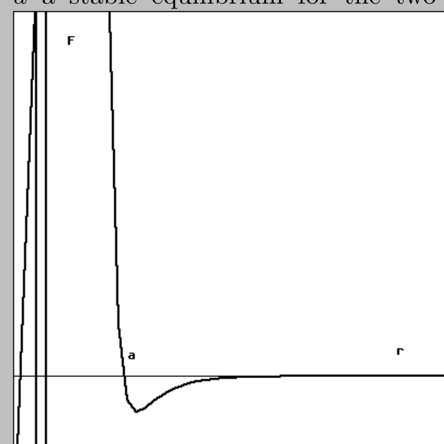
12. The force, F, between two atoms depends on the distance r separating them. See the graph below. A positive F represents a repulsive force; a negative F represents an attractive force.

(a) What happens to the force if the atoms start with $r = a$ and are:

(i) Pulled slightly further apart?

(ii) Pushed slightly closer together?

(b) The atoms are said to be in stable equilibrium if the force between them is zero and the atoms tend to return to the equilibrium after a minor disturbance. Is the distance $r = a$ a stable equilibrium for the two particles?



(a) (i) Moving from $r = a$, if r is increased, the force goes negative, and so there is an attractive force between the particles.
(ii) Moving from $r = a$, if r is *decreased*, the force goes *positive*, and so there is a repulsive force between the particles.

(b) Yes, the distance $r = a$ is a stable equilibrium. If the atoms are pulled apart, a force attracts them back together, and if they are pushed closer together, they repel away.

13. The volume of a sphere, V, is proportional to the cube of its radius, r. Write a formula representing V in terms of r.

$V = k\, r^3$

14. The average velocity, v, for a trip over a fixed distance, d, is inversely proportional to the time of travel, t. Write a formula representing v in terms of t and d.

average velocity $= \frac{d}{t}$

15. The strength, S, of a beam is proportional to the square of its thickness, h. Write a formula representing S in terms of h.

$S = k\ h^2$

16. The energy, E, expended by a swimming dolphin is proportional to the cube of the speed, v, of the dolphin. Write a formula representing E in terms of v.

$E = k\ v^3$

17. The number of animal species, N, of a certain body length, l, is inversely proportional to the square of l. Write a formula representing N in terms of l.

$N = k\ \frac{1}{l^2}$

18. When Galileo was formulating the laws of motion, he considered the motion of a body starting from rest and falling under gravity. He originally thought that the velocity of such a falling body was proportional to the distance it had fallen.

    Below are the results from two experiments dropping an object, with negligible air resistance.

Table A					
Distance (ft)	0	1	2	3	4
Velocity (ft/sec)	0	8	11.3	13.9	16

Table B					
Time (sec)	0	1	2	3	4
Velocity (ft/sec)	0	32	64	96	128

a) What does the experimental data in **Table A** tell you about Galileo's hypothesis?

b) What alternative hypothesis is suggested by the two sets of data in **Table A** and **Table B**?

(a) The data indicates that the hypothesis is **incorrect**. If it were true, then adding 1 to the distance should add a fixed amount to the velocity as well. We see in the table though that the velocity increases by different amounts for each fixed increase in the distance.

(b) Table B shows a relationship that **is** proportional: in particular, the velocity is proportional to the time in free fall. (Each second leads to 32 more feet per second in the velocity).

Exponentials

19. A town has a population of 1600 people at time $t = 0$. In each of the following cases, write a formula for the population, P, of the town as a function of year t.

    a) The population increases by 62 people per year.
    b) The population increases by 9% per year.

(a) $P(t) = 1600 + 62t$.

(b) $P(t) = 1600(1 + 0.09)^t$ or $1600(1.09)^t$.

20. *a*) Transform the function $P = P_0e^{0.8t}$ into the form $P = P_0a^t$.

    b) Does the function represent exponential growth or exponential decay?

(a) We can convert by equating a^t and $e^{0.8t}$, or ignoring the common t value,
$a = e^{0.8} \approx 2.2255$ so
$a^t = e^{0.8t} \approx 2.2255^t$
Thus, $P = P_0(2.2255)^t$ is equivalent to $P = P_0e^{0.8t}$.

(b) It represents exponential growth. The base is larger than 1, so taking higher powers leads to larger numbers.

21. The table below shows some values of a linear function f and an exponential function g. Fill in exact values (not decimal approximations) for each of the missing entries.

x	0	1	2	3	4
$f(x)$	15			225	
$g(x)$	15			225	

Since f is linear, its slope is a constant m, where $m = \frac{225-15}{3} = 70$. Thus $f(x) = 70 \cdot x + 15$, and we have

$$f(1) = 70{+}15, \quad f(2) = 70{\cdot}2{+}15, \quad \text{and} \quad f(4) = 70{\cdot}4{+}15.$$

Since g is exponential, its growth factor is constant. Writing $g(x) = ab^x$, we have $g(0) = a = 15$, so $g(x) = 15b^x$. Since $g(3) = 15b^3 = 225$, we have $b^3 = \frac{225}{15} = 15$, so $b = (15)^{1/3}$. Thus $g(x) = 15(15)^{x/3}$, and we have

$$g(1) = 15(15)^{\frac{1}{3}}, \quad g(2) = 15(15)^{\frac{2}{3}}, \quad \text{and} \quad g(4) = 15(15)^{\frac{4}{3}}.$$

22. Match the functions $f(x)$, $g(x)$ and $h(x)$, whose values are given in the table below, with the formulas

$$y = a(1.3)^x, \qquad y = b(1.08)^x, \qquad y = c(1.02)^x,$$

assuming a, b and c are constants. Note that the function values in the table have been rounded to two decimal places.

x	$f(x)$	x	$g(x)$	x	$h(x)$
2	3.38	0	4	1	1.08
3	4.39	1	4.08	2	1.17
4	5.71	2	4.16	3	1.26
5	7.43	3	4.24	4	1.36
6	9.65	4	4.33	5	1.47

$y = a(1.3)^x$ is the function $y = b(1.08)^x$ is the function $y = c(1.02)^x$ is the function

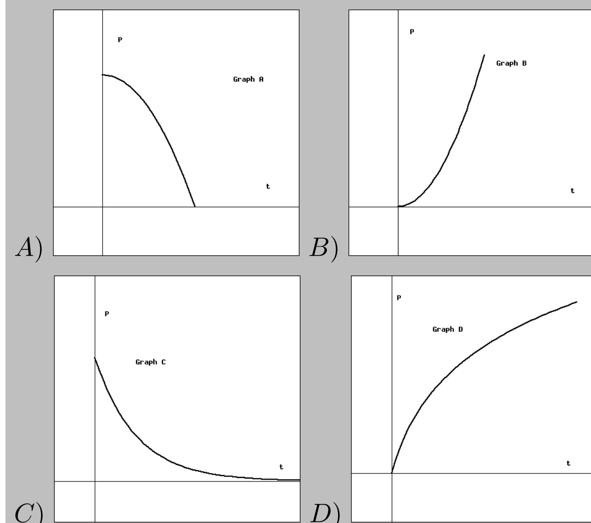
We can match the functions by taking ratios of the successive terms to determine the base of each of the exponential functions in the data.

For $f(x)$, we have $\frac{4.39}{3.38} = 1.3$, $\frac{5.71}{4.39} = 1.3$, $\frac{7.43}{5.71} = 1.3$, etc., so that it appears that this must be the function $y = a(1.3)^x$.

Similarly, $g(x)$ is $y = c(1.02)^x$ and $h(x)$ is $y = b(1.08)^x$

23. When a new product is advertised, more and more people try it. However, the rate at which new people try it slows as time, t, goes on.

a) Which graph best reflects the above situation?



b) What do you know about the concavity of the graph?

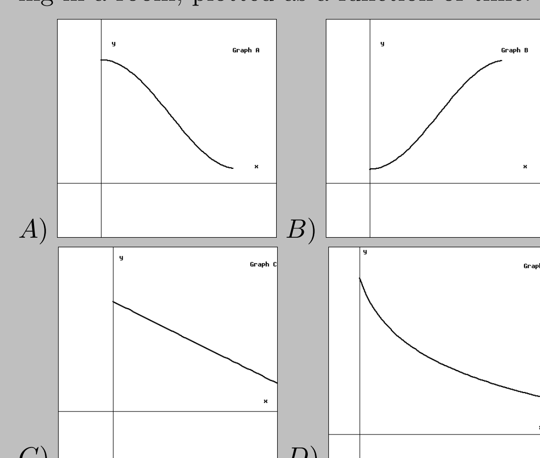
(a) Graph D

(b) Graph D is concave down.

24. Choose the graph that best reflects the given situation:

a) The total revenue generated by a car rental business, plotted against the amount spent on advertising.

b) The temperature of a cup of hot coffee standing in a room, plotted as a function of time.



(a) Graph B

(b) Graph D

25. When the Olympic Games were held outside Mexico City in 1968, there was much discussion about the effect the high altitude (7340 feet) would have on the athletes. Assuming air pressure decays exponentially by 0.4% every 100 feet, by what percentage is air pressure reduced by moving from sea level to Mexico City?

The air pressure is reduced by 25.49%.

From the "0.4% reduction every 100 feet", we immediately arrive at the model $P(h) = (1 - 0.004)^{t/100}$ where t is measured in feet. (When $t = 100$, we'll get $P(100) = (1 - 0.004)^1 = 0.996$, which is a 0.4% reduction.) In Mexico city, $h = 7340$, so $P(7340) = 0.7451$, or 74.51% of sea-level pressure, for a drop of $1 - 0.7451$ or 25.49%

26. During April 2006, Zimbabwe's inflation rate averaged 0.67% a **day**. This means that, on average, prices went up by 0.67% from one day to the next.

(a) By what percentage did prices in Zimbabwe increase in April of 2006?
(b) Assuming the same rate all year, what was Zimbabwe's annual inflation rate during 2006?

(a) The prices increased to $(1 + 0.0067)^{30} = 1.22181$, which indicates an increase of 22.181%.

(b) The prices increased to $(1+0.0067)^{365} = 11.44261$, which indicates an increase of 1044.261%.

27. (a) The half-life of radium-226 is 1620 years. Write a formula for the quantity, Q, of radium left after t years, if the initial quantity is q.

(b) What percentage of the original amount of radium is left after 600 years?

(a) $Q(t) = q(1/2)^{(t/1620)}$

(b) At $t = 600$, $Q(600) = q \cdot 0.7739$, so 77.39% of the original amount is left.

28. In the early 1960s, radioactive strontium-90 was released during atmospheric testing of nuclear weapons and was absorbed into the bones of people alive at the time. If the half-life of strontium-90 is 29 years, what fraction of the strontium-90 absorbed in 1960 remained in people's bones in 2002?

The fraction of strontium-90 left is given by the model $S(t) = (1/2)^{t/29}$ if t is measured in years. Therefore, between 2002 and 1960 (42 years), the fraction will have dropped to
$S(42) = (1/2)^{((2002-1960)/29)} \approx 0.3665$, so 36.65% of the original amount will be present still in 2002.

Function Transformations

29. Are the following functions invertible?

(a) $f(t)$ is the total accumulated rainfall in inches t minutes into a sudden rainstorm in July, 2005.

(b) $g(w)$ is the cost of mailing a letter weighing w grams.

(a) $f(t)$ is invertible. Given an amount of rain that has fallen, say $f(t) = 1.3$ inches, there can be only one time t at which that amount of rain had fallen.

(b) $g(w)$ is not invertible. Given a cost of mailing, say $g(w) = 1.4$ dollars, there is a minimum weight that will cost 1.4 dollars to send, and any letter whose weight w is less than a gram more than that will cost the same amount. So we do not know what weight the letter has.

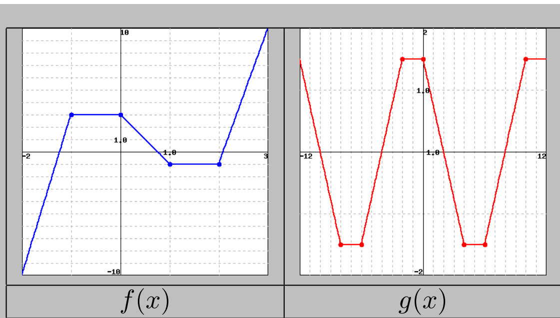
30. Let $f(x)$ be the temperature ($°C$) when the column of mercury in a particular thermometer is x inches long. What is the meaning of $f^{-1}(22)$ in practical terms?

The length of the column of mercury when the temperature is 22 degrees Celsius.

31. For the function $f(x) = 2e^x$ and $g(x) = x^6$, find the following:
(a) $f(g(1))$
(b) $g(f(1))$
(c) $f(g(x))$
(d) $g(f(x))$
(e) $f(t)g(t)$

(a) $f(g(1)) = f(1^6) = 2e^{(1^6)}$
(b) $g(f(1)) = g(2e^1) = (2e^1)^6$
(c) $f(g(x)) = f(x^6) = 2e^{(x^6)}$
(d) $g(f(x)) = g(2e^x) = (2e^x)^6$
(e) $f(t)g(t) = 2e^t t^6$

32.



Use the figures above to find the following values. Note that you can find exact values.
(a) $f(g(3))$
(b) $g(f(2))$
(c) $g(g(9))$

(a) $f(g(3)) = f(-0.75)$ which is 3.

(b) $g(f(2)) = g(-1)$ which is 1.5.

(c) $g(g(9)) = g(0.75)$ which is 0.9375.

33. Calculate the composite functions $f \circ g$ and $g \circ f$ given that $f(x) = \cos(x)\,, g(x) = 6x^3 + 9x^2 - 3$

$$f(g(x)) = f(6x^3 + 9x^2 - 3) = \cos\big(6x^3 + 9x^2 - 3\big)\,;$$
$$g(f(x)) = g(\cos(x)) = 6\cos^3(x) + 9\cos^2(x) - 3.$$

34. Calculate the composite functions $f \circ g$ and $g \circ f$, given that $f(x) = 6^x, g(x) = x^7$

$$f(g(x)) = f(x^7) = 6^{x^7};$$
$$g(f(x)) = g(6^x) = 6^{7x}.$$

35. Relative to the graph of

$$y = x^2$$

the graphs of the following equations have been changed in what way?
(a) $y = (5x)^2$
(b) $y = (x + 5)^2$
(c) $y = (x - 5)^2$
(d) $y = (x/5)^2$

(a) Compressed horizontally by the factor 5.

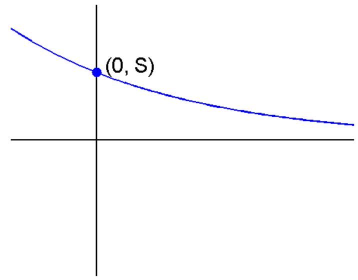
(b) Shifted 5 units left.

(c) Shifted 5 units right.

(d) Stretched horizontally by the factor 5.

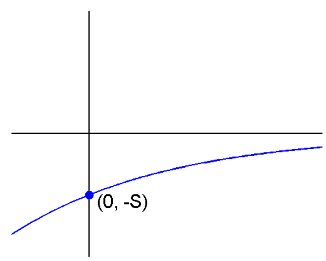
36. How does the graph of $Q = S(1 - e^{-kt})$ relate to the graph of the exponential decay function, $y = Se^{-kt}$?

The original graph, $y = Se^{-kt}$ is an exponential decay curve, with a t intercept at $t = 0$, $y = S$.

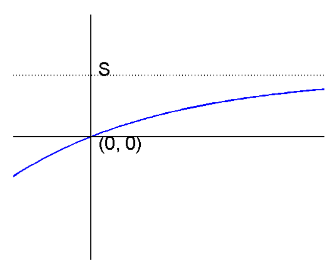


To obtain the graph of Q, we need two further transformations:

- $y = -Se^{-kt}$, which is the previous graph flipped vertically across $y = 0$. This new graph is increasing, with the t intercept at $t = 0$, $y = -S$, and a horizontal asymptote at $y = 0$.

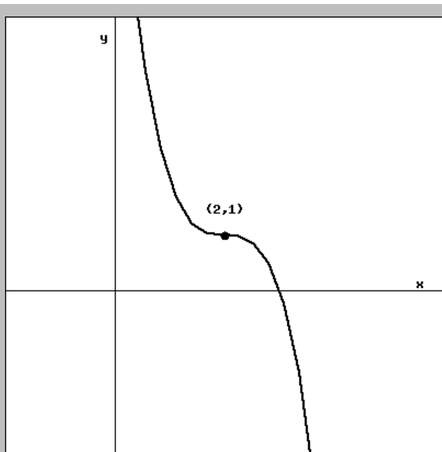


- $Q = -Se^{-kt} + S$, which is another way to write the goal function of $Q = S(1 - e^{-kt})$. By adding S, we shift the graph vertically up by S. The t intercept is now at $y = 0$, and the horizontal asymptote is



at $y = S$.

37.



Give a possible formula for the function shown in the graph using shifts of x^2 or x^3.

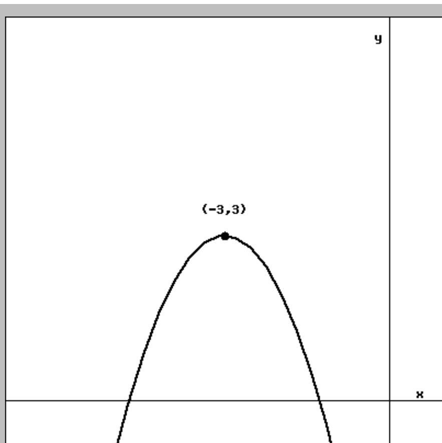
From experience, we know this is a transformation of the cubic $y = x^3$, as it is *not* a parabola ($y = x^2$). The following transformations have been applied:

- It has been shifted **up** by 1: include "+1" at the end.
- It has been shifted **right** by 2: change "x" to "$(x-2)$".
- It has been flipped vertically: change "x^3" to "$-x^3$".

Final function: $y = -1(x-2)^3 + 1$

Confirmation: at $x = 2$, we get $y = -1(2-2)^3 + 1 = 1$, so $(2,1)$ is on the graph. As as $x \to \infty$, $y \to -\infty$, also as shown on the graph.

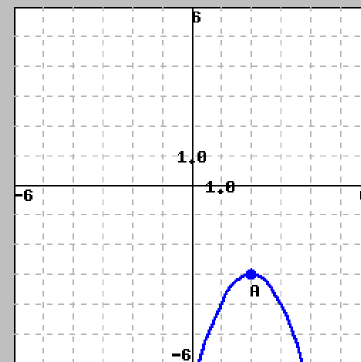
38.



Give a possible formula for the function shown in the graph using shifts of x^2 or x^3.

$-1(x-(-3))^2 + 3$

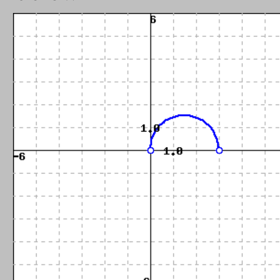
39. The graph of a function $f(x)$ is given below.



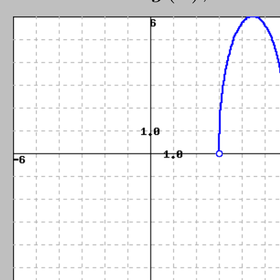
Find a possible formula for $f(x)$ whose graph is illustrated above by using a shift of either x^2 or x^3. Point A has coordinates $(2, -3)$.

This graph looks like a shift of $y = -x^2$. The graph is shifted to the right by 2 units and down by 3 units, so a possible formula is $f(x) = -1(x-2)^2 - 3$.

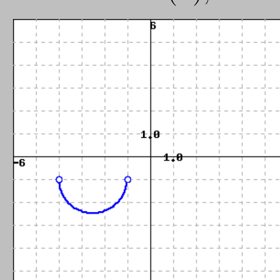
40. The function $f(x) = \sqrt{3x - x^2}$ is given graphed below:



(a) Starting with the formula for $f(x)$, find a formula for $g(x)$, which is graphed below:



(b) Starting with the formula for $f(x)$, find a formula for $h(x)$, which is graphed below:



(a) The vertical scale is magnified by 4 so we multiply outside the function 4; and
the graph is shifted horizontally to the right by 3, so we replace every x with $x-3$:
New graph: $y = 4\sqrt{3(x-3)-(x-3)^2}$

(b) The graph is flipped vertically (multiply outside by -1),
shifted down vertically by one (add a -1 at the end of the function), and
moved left by 4 (replacing each x with $x+4$).
New graph: $y = -1\left(\sqrt{3(x+4)-(x+4)^2}\right) - 1$

41. A spherical balloon is growing with radius $r = 3t+4$, in centimeters, for time t in seconds. Find the volume of the balloon at 3 seconds.

At $t = 3$, we have a radius of $3 \cdot 3 + 4 = 13$ cm, so the volume is $\frac{4}{3}\pi r^3 \approx 9202.77$ cm^3.

42. A tree of height y meters has, on average, B branches, where $B = y - 1$. Each branch has, on average, n leaves where $n(B) = 2B^2 - B$. Find the average number of leaves of a tree as a function of height.

Let's define L as number of leaves. We are looking for L, as function of height y.

At a height y,

- there will be $B = y - 1$ branches, and
- **each** branch will have $n = 2B^2 - B = 2(y-1)^2 - (y-1)$ leaves.

The total number of leaves will then be (number of branches) × (number of leaves per branch):
$$L(y) = \underbrace{(y-1)}_{\#\text{ branches}}\ \underbrace{(2(y-1)^2-(y-1))}_{\#\text{ leaves/branch}}$$

43. The cost of producing q articles is given by the function $C = f(q) = 125 + 3q$.

(a) Find a formula for the inverse function, $f^{-1}(C)$. (b) In practical terms, what does the inverse function tell you?

(a) Solving for q, $q = (C - 125)/3$

(b) The function $q(C)$ gives the number of articles produced at given cost.

44. A kilogram weighs about 2.2 pounds.

(a) Write a formula for the function, f, which gives an object's mass in kilograms, k, as a function of its weight in pounds, p:

(b) Find a formula for the inverse function of f:

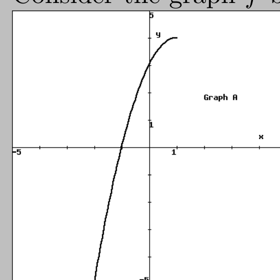
(c) In practical terms, what does inverse function tell you?

(a) $k = p/2.2$

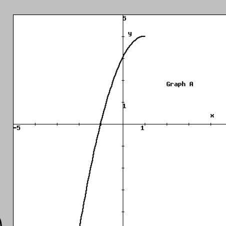
(b) $p = 2.2k$

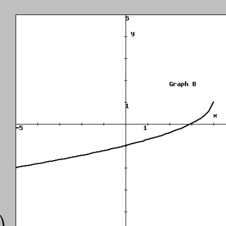
(c) The inverse here tells us the object's mass in pounds (p) as a function of kilograms (k).

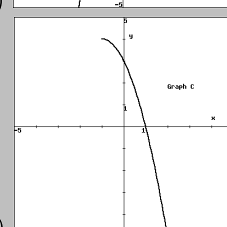
45. Consider the graph f below:

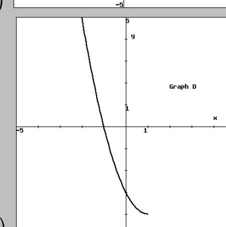


a) Which graph could be a representation of f^{-1}?

A) 

B) 

C) 

D) 

b) Using your graph choice from part a), estimate $f^{-1}(1)$.

(a) B

(b) From the graph B, $f^{-1}(1) = -1$.

Logarithms

46. Find the exact value for each expression:

(a) $\log_{3125} 5$.
(b) $\ln(e^{\sqrt{9}})$.

To simplify a logarithm value, you need to express the inside (5) as a power of the base. E.g. $\log_{10}(100) = 2$ because $\log_{10}(100) = \log_{10}(10^2) = 2$.

(a) If we look at powers of 5, we note that $5^5 = 3125$, which means $3125^{1/5} = 5$.
Thus $\log_{3125} 5 = \log_{3125}(3125^{1/5}) = \frac{1}{5}$.

(b) This can be evaluated directly, using the fact that $\ln(x)$ and e^x are inverse functions: for **any** value x, $\ln(e^x) = x$.
Here, $\ln(e^{\sqrt{9}}) = \sqrt{9}$, which we can optionally simplify to $\sqrt{9} = 3$.

47. Solve each equation for x:
(a) Solve $\ln(8x - 1) = -2$ for x.
(b) Solve $e^{(4x+3)} = 6$ for x.

(a)

$$\begin{aligned} \ln(8x-1) &= -2 \\ \text{Insert } e \text{ as a base: } e^{\ln(8x-1)} &= e^{-2} \\ \ln, e \text{ inverse: } 8x - 1 &= e^{-2} \\ \text{solve normally: } 8x &= e^{-2} + 1 \\ x &= \frac{e^{-2}+1}{8} \approx 0.1419 \end{aligned}$$

(b)

$$\begin{aligned} e^{(4x+3)} &= 6 \\ \text{ln of both sides: } \ln(e^{(4x+3)}) &= \ln(6) \\ \ln, e \text{ inverse: } 4x + 3 &= \ln(6) \\ 4x &= \ln(6) - 3 \\ x &= \frac{\ln(6)-3}{4} \approx -0.3021 \end{aligned}$$

48. Classify each of the following as either *True* or *False*.
(a) $(\ln a^b) = b \ln a$
(b) $\ln \frac{P}{Q} = \ln P - \ln Q$
(c) $\ln ab = b \ln a$
(d) $\log_2 PQR = \log_2 P + \log_2 Q + \log_2 R$

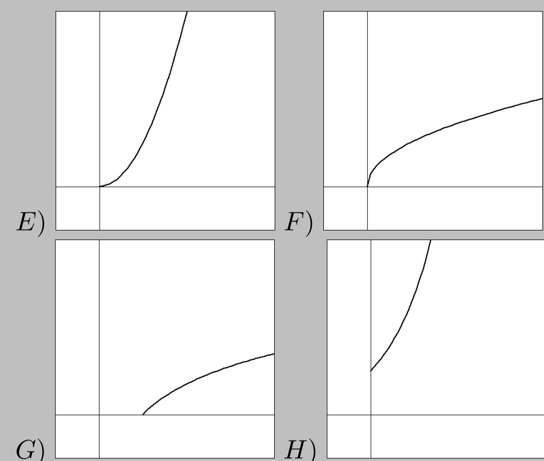
(a) True - “power inside” turns into “multiplication outside”.

(b) True - “division inside” turns into “difference/subtraction outside”.

(c) False - This is an incorrect mix of power and multiplication rules.

(d) True - This is the “multiplication inside” turns into “addition outside” rule, applied several times.

49. Without a calculator or computer, match the functions e^x, $\ln x$, x^2, and $x^{1/2}$ to their graphs.
(a) e^x
(b) $\ln x$
(c) x^2
(d) $x^{1/2}$



(a) $y = e^x$ is graph H

(b) $y = \ln(x)$ is graph G

(c) $y = x^2$ is graph E

(d) $y = x^{1/2}$ is graph F

50. The exponential function $y(x) = Ce^{\alpha x}$ satisfies the conditions $y(0) = 3$ and $y(1) = 5$.
a) Find the constant C.
b) Find the constant α.
c) What is $y(2)$?

(a) Since $y(0) = 3$, $3 = Ce^{\alpha \cdot 0} = C \cdot 1$, so $C = 3$.

(b) Since $y(1) = 5$, $5 = 3e^{\alpha \cdot 1}$.

$$\begin{aligned} \frac{5}{3} &= e^{\alpha} \\ \ln\left(\frac{5}{3}\right) &= \alpha \\ \text{so } \alpha &\approx 0.51083 \end{aligned}$$

(c) $y(2) = 3e^{0.51083 \cdot 2} \approx 8.334$

51. If $h(x) = \ln(x + a)$, where $a > 0$, what is the effect of increasing a on:
a) The y-intercept?
b) The x-intercept?

(a) It increases the y intercept.

(b) It moves the x intercept to the left.

52. If $g(x) = \ln(ax+2)$, where a is a positive quantity, what is the effect of increasing a on:

(a) The y-intercept?

(b) The x-intercept?

(a) It has no effect on the y intercept, because $g(0)$ will always equal $\ln(2)$ regardless of a.

(b) It moves the x intercept to the right.

53. If $f(x) = a\ln(x+2)$, what is the effect of increasing a on the vertical asymptote?

It will have no effect, because a changes the vertical scaling, but the asymptote of the graph will remain at $x = -2$.

54. There are currently 21 frogs in a (large) pond. The frog population grows exponentially, tripling every 7 days.

(a) How long will it take (in days) for there to be 210 frogs in the pond?

(b) The pond's ecosystem can support 1900 frogs. How long until the situation becomes critical?

(a) 14.7 days to reach 210 frogs.

Based on the information in the problem, one model for the population that can be constructed immediately is $P(t) = 21(3)^{t/7}$. We then solve the equation $210 = 213^{t/7}$ for t.

If building models like that isn't in your experience, you can always start with a default base-e exponential model:

$$P(t) = P_0 e^{kt}$$

This model has two values to be determined, P_0 and k. We can solve for them using the given information:

$P(0) = 21$ (initial population)
$P(7) = 3 \times 21 = 63$ (triple population after one week)

Using those values in the $P_0 e^{kt}$ form gives:

$$\begin{aligned} 21 &= P_0 e^0 \text{ so } P_0 = 21 \\ 63 &= (21)(e^{k\cdot 7}) \\ \frac{63}{21} &= e^{7k} \\ 3 &= e^{7k} \\ \ln(3) &= 7k \\ k &= \frac{\ln(3)}{7} \approx 0.15694 \end{aligned}$$

This gives a model of $P(t) = 21e^{0.15694t}$.
To answer the question given, how long does it take to reach 210 frogs, we set $P = 210$ and solve for t:

$$\begin{aligned} 210 &= 21e^{0.15694t} \\ 10 &= e^{0.15694t} \\ \ln(10) &= 0.15694t \\ t &= \frac{\ln(10)}{0.15694} \approx 14.7 \text{ days} \end{aligned}$$

(b) 28.7 days to reach 1900 frogs.
We can use either of the models discussed in part (a) to solve this. We just change the target population in the last step to 1900 frogs, instead of 210 frogs.

55. At time t hours after taking the cough suppressant hydrocodone bitartrate, the amount, A, in mg, remaining in the body is given by $A = 10(0.83)^t$.

(a) What was the initial amount taken?

(b) What percentage of the drug leaves the body each hour?

(c) How much of the drug is left in the body 8 hours after the dose is administered?

(d) How long is it until only 1 mg of the drug remains in the body?

(a) 10 mg were taken, because $A(0) = 10$.

(b) 17%, because after one hour there is 0.83 times as much, or a loss of 1-0.83 = 0.17 or 17%.

(c) $A(8) = 2.2523$ mg.

(d) Solving for $A(t) = 1$, we find $t = 12.358$ hours.

56. A cup of coffee contains 110 mg of caffeine, which leaves the body at a **continuous** rate of 18% per hour.

(a) Write a formula for the amount, A mg, of caffeine in the body t hours after drinking a cup of coffee.
(b) Use logarithms to find the half-life of caffeine.

(a) $A(t) = 110e^{-0.18t}$

(b) Setting $A = (0.5)110$ (half the initial amount of 110 mg), and then solving for t gives $t = 3.851$ hours as the half-life.

57. In 2000, there were about 212 million vehicles (cars and trucks) and about 283 million people in the US. The number of vehicles has been growing at 6% a year, while the population has been growing at 1% a year.

If the growth rates remain constant, when is there, to the nearest year, one vehicle per person?

Since both rates would be interpreted as "after one year, there 6% more cars, and 1% more people", we can build functions that represent both quantities using the $a_0(1+r)^t$ growth formulas (which guarantee exactly r% growth in one time interval).

$$C(t) = 212(1+0.06)^t$$
$$P(t) = 283(1+0.01)^t$$

Setting these two quantities equal to each other, to find the t when (# of cars) = (# of people). (Note that there are multiple ways to manipulate the following equations and solve for t: we just show one option here.)

$$\begin{aligned}
& 212(1.06)^t = 283(1.01)^t \\
\text{ln both sides:}\quad & \ln(212(1.06)^t) = \ln(283(1.01)^t) \\
\text{Using ln rules:}\quad & \ln(212) + t\ln(1.06) = \ln(283) + t\ln(1.01) \\
& t\ln(1.06) - t\ln(1.01) = \ln(283) - \ln(212) \\
\text{factor out } t\text{:}\quad & t(\ln(1.06) - \ln(1.01)) = \ln(283) - \ln(212) \\
& t = \frac{\ln(283) - \ln(212)}{(\ln(1.06) - \ln(1.01))} \\
& \approx 5.98
\end{aligned}$$

Since the 'initial' populations were measured in the year 2000, $t \approx 5.98$ or 6 would indicate that the car and people populations would equal each other in the year 2006.

58. The air in a factory is being filtered so that the quantity of a pollutant, P (in mg/liter), is decreasing according to the function $P = P_0e^{-kt}$, where t is time in hours. If 15% of the pollution is removed in the first 6 hours:

(a) What percentage of the pollution is left after 12 hours?
(b) How long is it before the pollution is reduced to 70% of its original level?

(a) 72.25% will be left after 12 hours.
(b) $P(t) = 0.70$ when $t = 13.168$ hours.

59. A scientist places 20 cells in a petri dish. She knows the cells grow at an exponential rate, doubling in number every hour.

(a) How long will it take (in hours) for there to be 1900 cells in the dish?
(a) How long will it take to reach 230 cells?

(a) 6.57 hours.

(b) 3.52 hours.

60. A picture supposedly painted by Vermeer (1632-1675) contains 99.5% of its carbon-14 (half-life 5730 years).

(a) What percentage of the painting's original carbon would be left today if the picture had been created in 1675?
(b) From this information, decide whether the picture is a fake.

(a) A general formula for the percentage of carbon-14 left in the painting would be given by

$$P(t) = 100e^{-kt}$$

and then use the half-life information to solve for k, knowing that $P(5730) = 50$ percent. Solving this would give $k = \frac{-\ln(0.5)}{5730} \approx 0.00012097$.

Alternatively, because we were given information specifically about the half-life, we can use a formula tailored for that:

$$P(t) = 100\left(\frac{1}{2}\right)^{(t/5730)}$$

We will use this (1/2) based formula, just because it doesn't involve nasty decimal values; the $100e^{-kt}$ formula will produce the same answer though.

Rounding, we are in year 2020, so between now and 1675 is 345 years.

The amount of carbon-14 left will be given by

$$P(345) = 100\left(\frac{1}{2}\right)^{(345/5730)} \approx 95.9\%$$

This tells us that we should expect close to 96% carbon-14 would be left if the painting were in fact made in 1675.

(b) Fake. The amount of C-14 in the sample is 99.5%, but it should be around 96% if the painting were the age it was reported to be.