Existence of Laplace Transforms

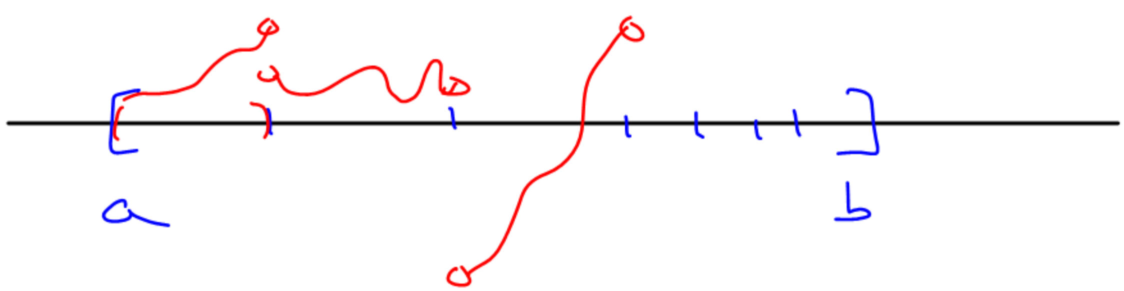
$F(s)$

Before continuing our use of Laplace transforms for solving DEs, it is worth digressing through a quick investigation of which functions actually *have* a Laplace transform.

A function f is ***piecewise continuous*** on an interval $t \in [a, b]$ if the interval can be partitioned by a finite number of points $a = t_0 < t_1 < \cdots < t_n = b$ such that

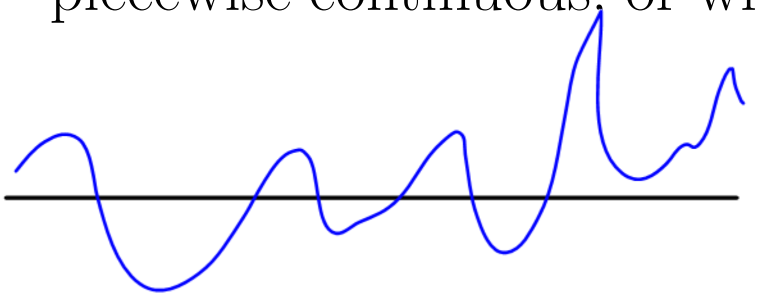
- f is continuous on each open subinterval (t_{i-1}, t_i).
- f approaches a finite limit as the endpoints of each subinterval are approached from within the subinterval.

In other words, f is continuous on $[a, b]$ except for a finite number of jump discontinuities. A function is piecewise continuous on $[0, \infty)$ if $f(t)$ is piecewise continuous on $[0, N]$ for all $N > 0$.

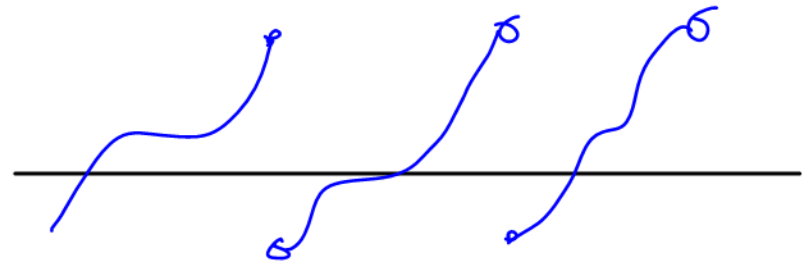


$\int_0^{\infty}$

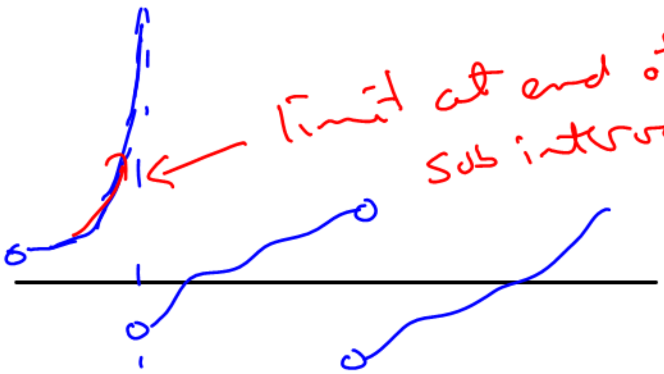
Problem. Draw examples of functions which are continuous and piecewise continuous, or which have different kinds of discontinuities.



piecewise cont's



piece wise cont's



limit at end of sub interval is not finite.

not be piecewise cont's

One of the requirements for a function having a Laplace transform is that it be piecewise continuous. Classify the graphs above based on this criteria.

Another requirement of the Laplace transform is that the integral $\int_0^\infty e^{-st} f(t)\, dt$ *converges* for at least some values of s. To help determine this, we introduce a generally useful idea for comparing functions, "Big-O notation".

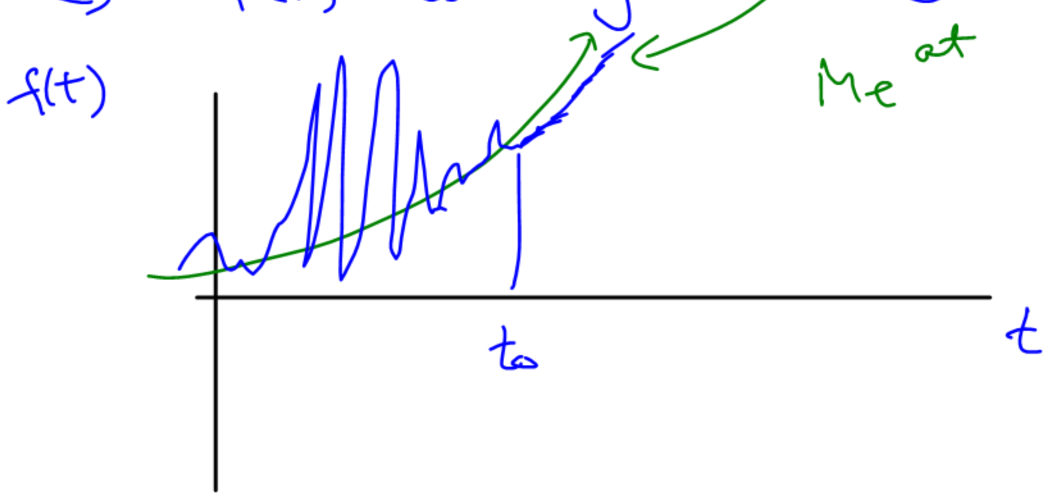
"OL"

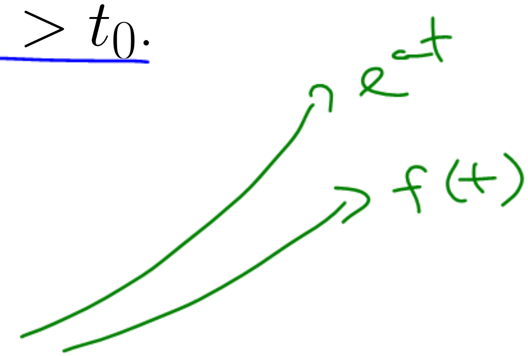
"on the order of"

Big-O notation

We write $f(t) = O\left(e^{at}\right)$ as $t \to \infty$ and say f is ***of exponential order*** a (as $t \to \infty$) if there exists a positive real number M and a real number t_0 such that $|f(t)| \le Me^{at}$ for all $t > t_0$.

"f(t) is big O of e^{at}"





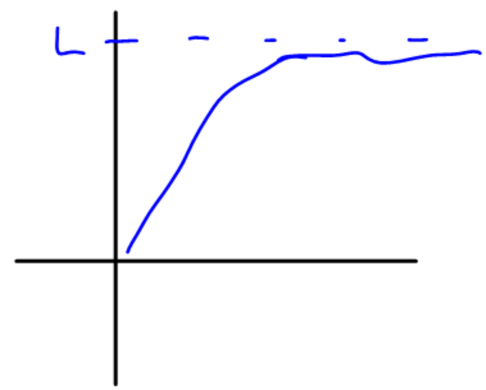
Lemma. *Assume* $\lim_{t\to\infty} \frac{|f(t)|}{e^{at}}$ *exists. Then*

$$\lim_{t\to\infty} \frac{|f(t)|}{e^{at}} < \infty \qquad \text{finite}$$

if and only if $f(t) = O\big(e^{at}\big)$ *as* $t \to \infty$. $\square$

Problem. Show that bounded functions and polynomials are of exponential order a for all $a > 0$.

bounded f: $f \le L$ for all t



finite

$$\lim_{t\to\infty} \frac{|f(t)|}{e^{at}} \le \lim_{t\to\infty} \frac{L}{\underbrace{e^{at}}_{\to\infty}} = 0$$

poly's: $\lim_{t\to\infty} \frac{\text{poly'l}}{e^{at}} = 0$

l'Hopital

both bounded and poly'l f's are of exp'l order.

Problem. Show that $e^{(t^2)}$ does **not** have exponential order.

$$\lim_{t\to\infty} \frac{e^{t^2}}{e^{at}} = \lim_{t\to\infty} e^{(t^2-at)} \to e^{\infty} \to \infty$$

e^{t^2} not of exp'l order

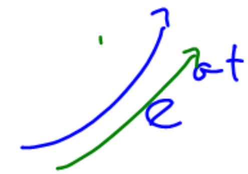
Problem. Are all the functions we have seen so far in our DE solutions of exponential order? Yes!

sin, cos : bounded ✓

poly's ✓

e^{at} ✓

The final reveal: what kinds of functions have Laplace transforms?



Proposition. *If f is*

- *piecewise continuous on $[0, \infty)$ and*
- *of exponential order a,*

then the Laplace transform $\mathcal{L}\{f(t)\}(s)$ exists for $s > a$.

The proof is based the comparison test for improper integrals.

Laplace Transform of Piecewise Functions

In our earlier DE solution techniques, we could not directly solve non-homogeneous DEs that involved piecewise functions. Laplace transforms will give us a method for handling piecewise functions.

$$y'' + ay' + by = \underbrace{\begin{matrix} \sin(at) & \cos(at) \\ e^{bt} & \text{poly} \end{matrix}}$$

$$= \begin{cases} \sin(t) & 0 < t < 1 \\ \vdots & \end{cases}$$

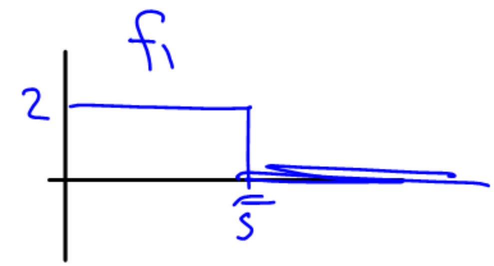
Problem. Use the definition to determine the Laplace transform of

$$f(t) = \begin{cases} 2 & 0 < t \le 5\,, \\ 0 & 5 < t \le 10\,, \\ e^{4t} & 10 < t\,. \end{cases}$$

$\to f_1 = 0$ for $t \geq 5$

$\to f_2$

$$\mathcal{L}(f(t)) = \int_0^\infty e^{-st} \cdot f(t)\, dt$$



$$= \int_{t=0}^{t=\infty} e^{-st} f_1\, dt + \int_0^\infty e^{-st} \cdot f_2\, dt$$

$f_1 = 2$ on $t = 0..5$

$$= \int_0^{\boxed{5}} e^{-st} \cdot 2\, dt + \int_{10}^\infty e^{-st} \cdot e^{4t}\, dt$$

e^{-st+4t}

shortcut improper

$$= 2 \cdot \frac{e^{-st}}{-s} \Bigg|_0^5 + \frac{e^{-st+4t}}{(-s+4)} \Bigg|_{10}^\infty$$

$$= \frac{2}{-s}\left(e^{-st}\right)\Big|_0^5 + \frac{1}{-s+4}\, e^{-st+4t}\Big|_{t=10}^{t=\infty}$$

$$f(t) = \begin{cases} 2 & 0 < t \le 5\,, \\ 0 & 5 < t \le 10\,, \\ e^{4t} & 10 < t\,. \end{cases}$$

$$= \frac{2}{-s}\left(e^{-5s} - 1\right) + \frac{1}{-s+4}\left(0 - e^{-s\cdot 10 + 4\cdot 10}\right)$$

$$= \frac{2}{s} - \frac{2e^{-5s}}{s} + \underbrace{\frac{1}{s-4}}_{\mathcal{L}(e^{4t})} e^{-10s}\cdot e^{40}$$

$\leftarrow$ e^{-ns} are indicators of $f(t)$ being piecewise.

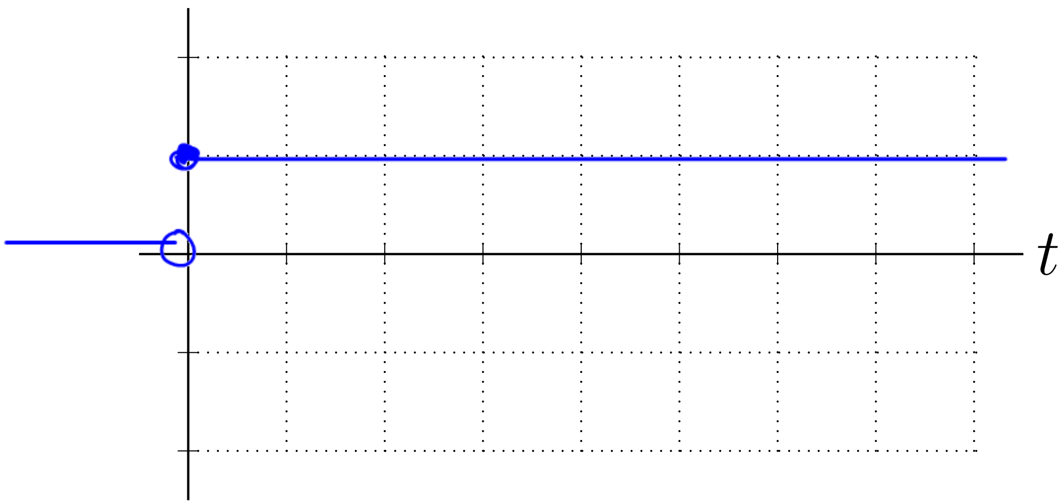
similar to previous

We would like avoid having to use the Laplace definition integral if there is an easier alternative. A new notation tool will help to simplify the transform process.

The ***Heaviside step function*** or ***unit step function*** is defined by

$$u(t) := \begin{cases} 0 & \text{for } t < 0, \qquad \text{off} \\ 1 & \text{for } t \geq 0. \qquad \text{on} \end{cases}$$

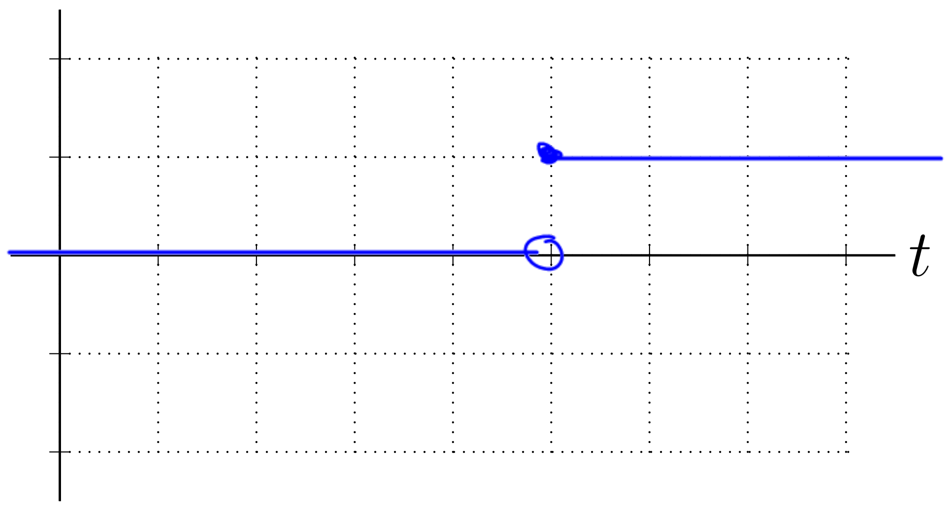
Problem. Sketch the graph of $u(t)$.



$$u(t) := \begin{cases} 0 & \text{for } t < 0, \\ 1 & \text{for } t \geq 0. \end{cases}$$

Problem. Sketch the graph of $u(t-5)$.

shift graph right by 5



Laplace Transform Using Step Functions

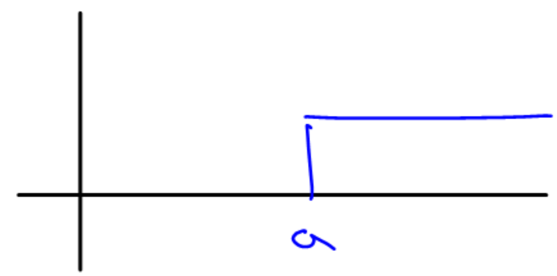
Problem. For $a > 0$, compute the Laplace transform of

$$u(t-a) = \begin{cases} 0 & \text{for } t < a, \\ 1 & \text{for } t \geq a. \end{cases}$$

$$\mathcal{L}(u(t-a)) \underset{\text{defn}}{=} \int_0^\infty e^{-st}\, u(t-a)\, dt$$

$u = 0$ for $t < a$

$$= \int_{\underbrace{a}_{t \geq a}}^\infty e^{-st} \cdot 1\, dt$$



$$= \frac{e^{-st}}{-s}\Bigg|_{a=t}^{\infty=t}$$

$\mathcal{L}\{1\}$

$\downarrow$

$$= \left(0 - \frac{e^{-s\cdot a}}{-s}\right) = \frac{1}{s} e^{-as}$$

$\swarrow$ step function

Laplace Transform of Step Functions

$$\mathcal{L}(u_a(t)f(t-a)) = e^{-as}F(s)$$

An alternate (and more directly useful form) is

$$\mathcal{L}(u_a(t)f(t)) = e^{-as}\mathcal{L}(f(t+a))$$

Notation: $u(t-a)$

$= u_a(t)$

$$\mathcal{L}(u_a(t)f(t)) = e^{-as}\mathcal{L}(f(t+a))$$

t's replaced by t+a

Problem. Find $\mathcal{L}(u_2)$.

no t's to replace

$$\mathcal{L}\{u_2 \cdot 1\} = e^{-2s}\,\mathcal{L}\{1\}$$

$$= e^{-2s} \cdot \frac{1}{s}$$

Problem. Find $\mathcal{L}(u_\pi)$.

$$\mathcal{L}\{u_\pi\} = e^{-\pi s} \cdot \mathcal{L}\{1\}$$

$$= e^{-\pi s} \cdot \frac{1}{s}$$

$$\mathcal{L}(u_a(t)f(t)) = e^{-as}\mathcal{L}(f(t+a))$$

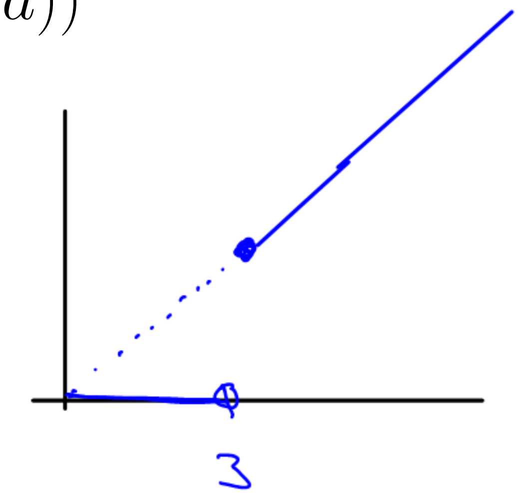
Problem. Find $\mathcal{L}(tu_3)$.

$\mathcal{L}\{3\} = \frac{3}{s}$

$\mathcal{L}(u_3 \cdot t) = e^{-3s}\mathcal{L}\{t+3\}$

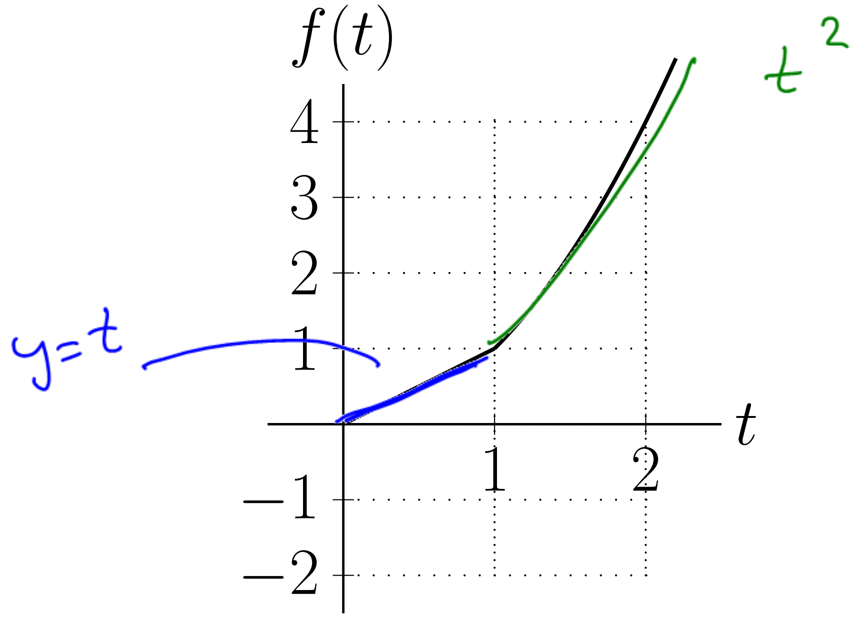
$\mathcal{L}(t^1) = \frac{1!}{s^{1+1}}$

replace t's by t+3



$$= e^{-3s}\left(\frac{1}{s^2} + \frac{3}{s}\right)$$

Problem. Here is a more complicated function made up of $f = t$ and $f = t^2$.



Write the function in piecewise form, and again using step functions.

t "on" at $t=0$

t "on" at $t=1$

$$f(t) = \begin{cases} t & 0 \le t \le 1 \\ t^2 & 1 < t \end{cases} = t(u_0 - u_1) + t^2 \cdot u_1$$

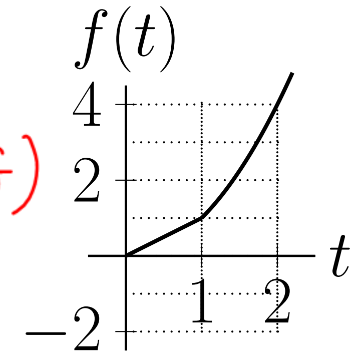
$t > 1$, $t \cdot (1-1)$

$= t \cdot 0$

$y = ?$

write w/ u's

Problem. Find $\mathcal{L}(t(u_0 - u_1) + t^2 u_1)$.



$$= \mathcal{L}(t \cdot u_0) - \mathcal{L}(t \cdot u_1) + \mathcal{L}(t^2 \cdot u_1)$$

$$\mathcal{L}(u_a \cdot f) = e^{-as} \cdot \mathcal{L}(f(t+a))$$

$$= \overset{1}{\cancel{e^{-0s}}} \cdot \mathcal{L}(t+0) - e^{-1s}\mathcal{L}(t+1) + e^{-1s}\mathcal{L}((t+1)^2)$$

expand

$$= \frac{1}{s^2} - e^{-s}\left(\frac{1}{s^2} + \frac{1}{s}\right) + e^{-s}\mathcal{L}(t^2 + 2t + 1)$$

$$= \frac{1}{s^2} \overbrace{- e^{-s}\left(\frac{1}{s^2} + \cancel{\frac{1}{s}}\right) + e^{-s}}^{\text{common}} \left(\frac{2}{s^3} + \cancel{2}\frac{1}{s^2} + \cancel{\frac{1}{s}}\right)$$

$$= \frac{1}{s} + e^{-s}\left(\frac{2}{s^3} + \frac{1}{s^2}\right)$$

Inverse Laplace Transform of Step Functions

a first in table

$$\mathcal{L}^{-1}\left\{e^{-as}F(s)\right\} = f(t-a)u_a$$

t's replaced by t−a

Problem. Find $\mathcal{L}^{-1}\left\{\dfrac{e^{-2s}}{s^2}\right\}$

$F(s) = \dfrac{1}{s^2}$

$$= \mathcal{L}^{-1}\left\{e^{-2s} \cdot \frac{1}{s^2}\right\}$$

$\mathcal{L}(t)$

$$= u_2(t)\,[t-2]$$

step "on" at $t=2$

$$\mathcal{L}^{-1}\left\{e^{-as}F(s)\right\} = f(t-a)u_a$$

Problem. Find $\mathcal{L}^{-1}\left\{\dfrac{e^{-3s}}{s-4}\right\}$

$$= \mathcal{L}^{-1}\left\{ e^{-3s}\,\frac{1}{s-4} \right\}$$

$$\uparrow$$

$$\mathcal{L}(e^{4t})$$

$$= u_3(t)\cdot e^{4(t-3)}$$

$$\mathcal{L}^{-1}\left\{e^{-as}F(s)\right\} = f(t-a)u_a$$

Problem. Which of the following equals $f(t) = \mathcal{L}^{-1}\left\{\dfrac{e^{-\pi s}}{s^2+4}\right\}$?

1. $\dfrac{1}{s}\cos(\pi t)u_\pi$
2. $\dfrac{1}{\pi s}\cos(\pi(t-\pi))u_\pi$
3. $\dfrac{1}{2}\sin(2(t-\pi))u_\pi$
4. $\dfrac{1}{\pi}\sin(2(t-\pi))u_\pi$

$$\mathcal{L}^{-1}\left\{e^{-\pi s}\cdot\frac{1}{2}\,\frac{\times 2}{s^2+4}\right\}$$

$$= \frac{1}{2}\,u_\pi(t)\,\sin(2(t-\pi))$$

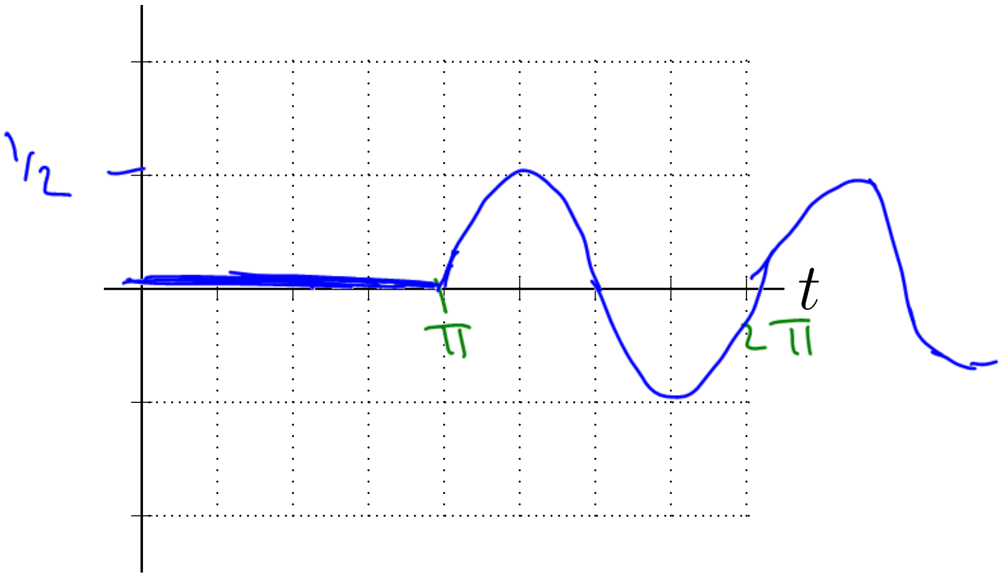
$$\mathcal{L}^{-1}\left\{\frac{k}{s^2+k^2}\right\} = \sin(kt)$$

Problem. Sketch the graph of

$$f(t) = \mathcal{L}^{-1}\left\{\frac{e^{-\pi s}}{s^2+4}\right\} = \frac{1}{2}\sin(2(t-\pi))u_\pi$$

freq → period $= \frac{2\pi}{2} = \pi$

"on" at $t = \pi$



$f(t)$

Problem. Find $\mathcal{L}^{-1}\left\{\dfrac{e^{-2s}}{(s-1)(s-2)}\right\}$

$$\mathcal{L}^{-1}\{e^{-as}\cdot F(s)\} = u_a \cdot f(t-a)$$

$$= \mathcal{L}^{-1}\left\{e^{-2s}\cdot\left[\frac{1}{(s-1)(s-2)}\right]\right\}$$

↓ need part frac

$$\frac{1}{(s-1)(s-2)} = \frac{A}{s-1} + \frac{B}{s-2}$$

$$1 = A(s-2) + B(s-1)$$

$s=2$ $\quad 1 = B(2-1) \quad \boxed{B=1}$

$s=1$ $\quad 1 = A(1-2) \quad \boxed{A=-1}$

$$\mathcal{L}^{-1}\left\{e^{-2s}\left[\left(\frac{-1}{s-1}\right)+\left(\frac{1}{s-2}\right)\right]\right\}$$

$$= -\mathcal{L}^{-1}\left\{e^{-2s}\frac{1}{s-1}\right\} + \mathcal{L}^{-1}\left\{e^{-2s}\frac{1}{s-2}\right\} = -u_2\, e^{1(t-2)} + u_2\, e^{2(t-2)}$$

Tips for Inverse Laplace With Step/Piecewise Functions

- Separate/group all terms by their e^{-as} factor.
- Complete any partial fractions **leaving the** e^{-as} **out front** of the term.
  - The e^{-as} only affects final inverse step.
  - Partial fraction decomposition only works for polynomial numerators.

$$\frac{10(e^{-10s})}{(s-1)(s-2)} = \frac{A}{s-1} + \frac{B}{s-2}$$

$$= \left(\frac{A}{s-1} + \frac{B}{s-2}\right) \cdot e^{-10s}$$

The reason Laplace transforms can be helpful in solving differential equations is because there is a (relatively simple) transform rule for derivatives of functions.

Proposition (Differentiation). *If f is continuous on $[0, \infty)$, $f'(t)$ is piecewise continuous on $[0, \infty)$, and both functions are of exponential order a, then for $s > a$, we have*

initial condition

$$\mathcal{L}\{f'(t)\}(s) = s\mathcal{L}\{f\}(s) - f(0)$$

s multiplier $\mathcal{L}$ of orig'l

Problem. Confirm the transform table entry for $\mathcal{L}\{\cos(kt)\}$ with the help of the transform derivative rule and the transform of $\sin(kt)$.

$$\mathcal{L}\{\cos(kt)\} = \frac{s}{s^2+k^2}$$

$$\mathcal{L}\{\cos(kt)\} = \mathcal{L}\left\{\frac{d}{dt}\frac{1}{k}\sin(kt)\right\}$$

$$= \mathcal{L}\left\{\frac{1}{k}(\sin(kt))'\right\} \xrightarrow{\mathcal{L}\text{ for deriv}} = \frac{1}{k}\left[s\,\mathcal{L}\{\sin(kt)\} - \sin(0)\right]$$

$$= \frac{1}{\cancel{k}}\, s\left[\frac{\cancel{k}}{s^2+k^2} - 0\right]$$

$$= \frac{s}{s^2+k^2}$$

We can generalize this rule to the transform of higher derivatives of a function.

Theorem (General Differentiation). *If $f(t), f'(t), \ldots, f^{(n-1)}(t)$ are continuous on $[0,\infty)$, $f^{(n)}(t)$ is piecewise continuous on $[0,\infty)$, and all of these functions are of exponential order a, then for $s > a$, we have*

s power decr → f deriv incr

$$\mathcal{L}\{f^{(n)}(t)\}(s) = s^n\mathcal{L}\{f\}(s) - \underbrace{s^{n-1}f(0) - s^{n-2}f'(0) - \cdots - f^{(n-1)}(0)}_{\text{series of initial cond's}}.$$

Can be proven using integration by parts n times.

n^{th} deriv

F(S)

s^n · Transform of orig'l f

Most commonly in this course, we will need specifically the transform of the second derivative of a function.

Corollary (Second Differentiation)**.** *If $f(t)$ and $f'(t)$ are continuous on $[0, \infty)$, $f''(t)$ is piecewise continuous on $[0, \infty)$, and all of these functions are of exponential order a, then for $s > a$, we have*

2 derivs $\quad$ s^2 $\quad$ $\longrightarrow$ s power dec

$$\mathcal{L}\{f''(t)\}(s) = s^2 \mathcal{L}\{f\}(s) - sf(0) - f'(0)\,.$$

$\longrightarrow$

f deriv incr

Solving Initial Value Problems with Laplace Transforms

Problem. Sketch the general method.

$\mathcal{L}(y)$, unknown

DE $\quad y'' + ay' + by = f(t) \xrightarrow{\mathcal{L}} s^2 Y(s) + \ldots \qquad Y(s) = \mathcal{L}(f(t))$

no derivs

y_c, y_{NH}

solve for $Y(s)$

algebraic eq'n

solution $\quad y = \ldots\ldots \xleftarrow{\mathcal{L}^{-1}} Y(s) = \ldots\ldots$

Problem. Find the Laplace transform of the entire DE

$$\mathcal{L}\left(x' + x\right) = \mathcal{L}\left(\cos(2t)\right), x(0) = 0$$

$\mathcal{L}(x(t)) = X(s)$

$$\underbrace{\left[sX(s) - x(0)\right]}_{\mathcal{L}(x')} + X(s) = \frac{s}{s^2+2^2}$$

$$sX(s) + X(s) = \frac{s}{s^2+4}$$

Problem. Note the form of the equation now: are there any derivatives left?

No!

Problem. Solve for $X(s)$.

$$sX(s) + X(s) = \frac{s}{s^2+4}$$

$$(s+1)X(s) = \frac{s}{s^2+4}$$

$$X(s) = \frac{s}{(s^2+4)(s+1)}$$

$$X(s) = \frac{s}{(s^2+4)(s+1)}$$

Problem. Put $X(s)$ in a form so that you can find its inverse transform. → partial fractions

$$\frac{s}{(s^2+4)(s+1)} = \frac{As+B}{s^2+4} + \frac{C}{s+1}$$

$$s = (As+B)(s+1) + C(s^2+4)$$

$s=-1$ $\quad -1 = C(1+4) \qquad \boxed{C = -1/5}$

equate s^2 $\quad 0 = A + C \qquad A = -C \qquad \boxed{A = 1/5}$

const coff's $\quad 0 = B + 4C \qquad B = -4C \qquad \boxed{B = 4/5}$

$$X(s) = \frac{1}{5}\frac{s}{s^2+4} + \frac{4}{5}\frac{1}{s^2+4} + \frac{-1}{5}\frac{1}{s+1}$$

Problem. Find $x(t)$ by taking the inverse transform.

$$X(s) = \frac{1}{5}\,\frac{s}{s^2+4} + \frac{4}{5}\,\frac{1}{2}\,\frac{1\times 2}{s^2+4} - \frac{1}{5}\,\frac{1}{s+1}$$

$$\Big\downarrow \mathcal{L}^{-1}$$

$$\frac{2}{s^2+2^2} \qquad \frac{1}{s-a}$$

$$\boxed{x(t) = \frac{1}{5}\cos(2t) + \frac{2}{5}\sin(2t) - \frac{1}{5}e^{-t}}$$

particular sol'n

b/c we used initial condition $x(0)=0$

Problem. Confirm that the function you found is a solution to the differential equation $x' + x = \cos(2t)$.

proposed sol'n

$$x(t) = \frac{1}{5}\cos(2t) + \frac{2}{5}\sin(2t) - \frac{1}{5}e^{-t}$$

$\downarrow$ need x'

$$x' = -\frac{2}{5}\sin(2t) + \frac{4}{5}\cos(2t) + \frac{1}{5}e^{-t}$$

Sub into LHS of eq'n:

$$\underbrace{\left(-\frac{2}{5}\cancel{\sin(2t)} + \frac{4}{5}\cos(2t) + \cancel{\frac{1}{5}e^{-t}}\right)}_{x'} \underbrace{+}_{+} \underbrace{\left(\frac{1}{5}\cos(2t) + \frac{2}{5}\cancel{\sin(2t)} - \cancel{\frac{1}{5}e^{-t}}\right)}_{x}$$

$$= \cos(2t) = \text{RHS of DE } \checkmark$$

Problem. Solve $y'' + y = \sin(2t)$, $y(0) = 2$, and $y'(0) = 1$.

$\mathcal{L}$ of whole DE:

$\mathcal{L}(y(t)) = Y(s)$

$$\left[s^2 Y(s) - s(2) - 1\right] + Y(s) = \frac{2}{s^2+4}$$

$s^2 \mathcal{L}(y) - s\,y(0) - y'(0)$

Solve for $Y(s)$

$$(s^2+1)\,Y(s) - 2s - 1 = \frac{2}{s^2+4}$$

$$(s^2+1)\,Y(s) = \frac{2}{s^2+4} + 2s + 1$$

$$Y(s) = \frac{2}{(s^2+4)(s^2+1)} + \frac{2s+1}{s^2+1}$$

$$y'' + y = \sin(2t),\ y(0) = 2,\ \text{and}\ y'(0) = 1.$$

$$Y(s) = \frac{2}{(s^2+4)(s^2+1)} + \frac{2s+1}{(s^2+1)}$$

$\downarrow$ part frac

$$\frac{2}{(s^2+4)(s^2+1)} = \frac{As+B}{s^2+4} + \frac{Cs+D}{s^2+1}$$

$$2 = (As+B)(s^2+1) + (Cs+D)(s^2+4)$$

$s=0$ $\quad 2 = B + 4D$ ① $\qquad$ ①−③ $\quad 2 = 3D$ $\quad \boxed{D = \frac{2}{3}}$

s^3: $\quad 0 = A + C$ ② $\qquad$ ③ $\to$ $\boxed{B = -D = \frac{-2}{3}}$

s^2: $\quad 0 = B + D$ ③

s: $\quad 0 = A + 4C$ ④ $\qquad$ ②−④ $\quad 0 = -3C$ $\quad \boxed{C = 0}$

④ $\quad \boxed{A = 0}$

$$Y(s) = \overset{B}{\overset{\downarrow}{-\frac{2}{3}}}\,\frac{1}{2}\,\frac{\times 2}{s^2+4} + \overset{D}{\overset{\downarrow}{\frac{2}{3}}}\,\frac{1}{s^2+1} + \frac{2s+1}{s^2+1}$$

$$= -\frac{1}{3}\,\frac{2}{s^2+4} + 2\,\frac{s}{s^2+1} + \frac{5}{3}\,\frac{1}{s^2+1}$$

$\mathcal{L}^{-1} \downarrow$

$$y(t) = -\frac{1}{3}\sin(2t) + 2\cos(t) + \frac{5}{3}\sin(t)$$

particular solution.

$$\underline{y'' + y} = \sin(2t),\ y(0) = 2, \text{ and } y'(0) = 1.$$

Problem. Confirm your solution is correct.

$$y = \overbrace{-\tfrac{1}{3}\sin(2t)}^{y_{NH}} + \underbrace{2\cos(t) + \tfrac{5}{3}\sin(t)}_{y_c}$$

Need y', y''

$$y' = -\tfrac{2}{3}\cos(2t) - 2\sin(t) + \tfrac{5}{3}\cos(t)$$

d/dt

$$y'' = +\tfrac{4}{3}\sin(2t) - 2\cos(t) - \tfrac{5}{3}\sin(t)$$

$$\text{LHS} = \underbrace{\left[\tfrac{4}{3}\sin(2t) - \cancel{2\cos(t)} - \cancel{\tfrac{5}{3}\sin(t)}\right]}_{y''} \underbrace{+}_{+} \underbrace{\left[-\tfrac{1}{3}\sin(2t) + \cancel{2\cos(t)} + \cancel{\tfrac{5}{3}\sin(t)}\right]}_{y}$$

$$= 1\sin(2t) = \text{RHS}$$

Problem. Solve $\underbrace{y'' - 2y' + 5y}_{2^{nd}\text{ order}} = -8e^{-t}$, $y(0) = 2$, and $y'(0) = 12$.

non-homog's

[could use y_{NH}, y_c]

$\mathcal{L}$ of DE

$$\underbrace{\left[s^2 Y(s) - s\,y(0) - y'(0)\right]}_{\mathcal{L}(y'')} - 2\underbrace{\left[s\,Y(s) - y(0)\right]}_{\mathcal{L}(y')} + 5\,Y(s) = -8\,\frac{1}{s+1}$$

Gather $Y(s)$ terms

$$Y(s)\left[s^2 - 2s + 5\right] \underbrace{- s\cdot 2 - 12 - 2(-2)} = -8\,\frac{1}{s+1}$$

$$Y(s)\left[s^2 - 2s + 5\right] = \frac{-8}{s+1} + 2s + 8$$

$$Y(s) = \frac{-8}{(s+1)(s^2-2s+5)} + \frac{2s+8}{(s^2-2s+5)}$$

$$y'' - 2y' + 5y = -8e^{-t},\ y(0) = 2, \text{ and } y'(0) = 12.$$

$$Y(s) = \frac{-8}{(s+1)(s^2-2s+5)} + \frac{2s+8}{(s^2-2s+5)}$$

$$\frac{-8}{(s+1)(s^2-2s+5)} = \frac{A}{s+1} + \frac{Bs+C}{s^2-2s+5}$$

$$-8 = A(s^2-2s+5) + (Bs+C)(s+1)$$

$s=-1$ $\qquad -8 = A(1+2+5)$ $\qquad \boxed{A=-1}$

s^2 coeff $\qquad 0 = A + B$ $\qquad \boxed{B=1}$

const $\qquad -8 = 5A + C$ $\qquad \boxed{C = -8-5A = -3}$

$$Y(s) = \frac{-1}{s+1} + \frac{s}{s^2-2s+5} - 3\frac{1}{s^2-2s+5} + \frac{2s+8}{s^2-2s+5}$$

$$= \frac{-1}{s+1} + 3\frac{s}{s^2-2s+5} + 5\frac{1}{s^2-2s+5}$$

$$Y(s) = \frac{-1}{s+1} + 3\,\frac{s}{s^2-2s+5} + 5\,\frac{1}{s^2-2s+5}$$

Goal find $\mathcal{L}^{-1}(Y(s))$

$$Y(s) = \frac{-1}{s+1} + 3\,\frac{s}{(\underbrace{s^2-2s+1}_{(s-1)^2})-1+5} + 5\,\frac{1}{(s^2-2s+1)-1+5}$$

$$Y(s) = \frac{-1}{s+1} + 3\,\frac{(s-1)+1}{(s-1)^2+4} + 5\,\frac{1}{(s-1)^2+4}$$

(2)

(2) $\mathcal{L}^{-1}\left(\frac{s}{s^2+2^2}\right)$

$= \cos(2t)$

$$Y(s) = \frac{-1}{s+1} + 3\,\frac{(s-1)}{(s-1)^2+2^2} + 8\,\frac{1}{2}\,\frac{\cancel{1}\,2}{(s-1)^2+2^2}$$

$\mathcal{L}^{-1}(F(s-a))$

$= e^{at}\,\mathcal{L}^{-1}(F(s))$

$\downarrow \mathcal{L}^{-1}$

$$\boxed{y(t) = -e^{-t} + 3e^{t}\cos(2t) + 4e^{t}\sin(2t)}$$

$$y'' - 2y' + 5y = -8e^{-t},\ y(0) = 2, \text{ and } y'(0) = 12.$$

Problem. Confirm your solution is correct.

$y(t) = -e^{-t} + 3e^{t}\cos(2t) + 4e^{t}\sin(2t)$ d/dt

Need y', y'' for LHS:

$y' = e^{-t} + 3e^{t}\cos(2t) - 6e^{t}\sin(2t) + 4e^{t}\sin(2t) + 8e^{t}\cos(2t)$

$= e^{-t} + 11e^{t}\cos(2t) - 2e^{t}\sin(2t)$ d/dt

$y'' = -e^{-t} + \underbrace{11e^{t}\cos(2t) - \underbrace{22e^{t}\sin(2t) - 2e^{t}\sin(2t)}_{-24} - 4e^{t}\cos(2t)}_{=7}$

LHS: $\left(-e^{-t} + 7e^{t}\cos(2t) - 24e^{t}\sin(2t)\right)$

$-2\left(e^{-t} + 11e^{t}\cos(2t) - 2e^{t}\sin(2t)\right)$

$+5\left(-e^{-t} + 3e^{t}\cos(2t) + 4e^{t}\sin(2t)\right)$

$= -8e^{-t}$ = RHS ✓

e^{-t}'s: $-1 - 2 - 5 = -8e^{-t}$

$e^{t}\cos(2t)$: $7 - 22 + 15 \overset{?}{=} 0$

$e^{t}\sin(2t)$: $-24 + 4 + 20 \overset{?}{=} 0$