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CODIMENSION TWO INTEGRAL POINTS ON SOME RATIONALLY CONNECTED THREEFOLDS ARE POTENTIALLY DENSE

DAVID MCKINNON AND MIKE ROTH

Abstract

Let X be a smooth, projective, rationally connected variety, defined over a number field k, and let $Z \subset X$ be a closed subset of codimension at least two. In this paper, for certain choices of X, we prove that the set of Z-integral points is potentially Zariski dense, in the sense that there is a finite extension K of k such that the set of points $P \in X(K)$ that are Z-integral is Zariski dense in X. This gives a positive answer to a question of Hassett and Tschinkel from 2001.

1. Introduction

In [HT], as Problem 2.13 ("The Arithmetic Puncturing Problem"), Hassett and Tschinkel ask the following question:

Question 1.1. Let X be a projective variety with canonical singularities and Z a Zariski closed subset of codimension at least two,[1] all defined over a number field k. Assume that rational points on X are potentially dense. Are integral points on (X, Z) potentially dense?

Of course, the hypothesis that Z has codimension at least two cannot be removed, as there are countless well known examples of varieties with a dense set of rational points but a degenerate set of integral points if Z is a divisor.

In [HT] Hassett and Tschinkel provide positive answers to Question 1.1 in various cases, including toric varieties and products of elliptic curves. In contrast, a recent preprint of Levin [L] gives an example of a singular fourfold X for which the answer to Question 1.1 is negative. The purpose of this paper

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[1] I.e., each irreducible component of Z has codimension at least two.

is to give a positive answer to Question 1.1 for a large number of examples in dimension up to three.

The case for curves seems vacuous, but if one views a curve defined over a number field as an arithmetic surface, then one can choose Z to be an arithmetic zero-cycle, in which case there is something to prove. This is Lemma 2.3, and is a crucial technical tool for the paper.

For surfaces, the situation is more complicated, as it is unknown which surfaces have a potentially dense set of rational points. If the Kodaira dimension is negative, however – which is believed to be the case in which rational points are most plentiful – we give a positive answer to Question 1.1 in Theorem 3.1.

In the central part of the paper, X will be a smooth, projective, rationally connected threefold. It is a well known result (see Theorem 1.4.1 of [IP]) that there is a birational map $f\colon X \dashrightarrow V$, where V is a normal projective threefold with only $\mathbb{Q}$-factorial and terminal singularities with a morphism $\pi\colon V \to Y$ of one of the following three types:

(a) The variety Y is a normal projective surface with at most rational singularities, and π makes V a conic bundle over Y.
(b) The variety Y is isomorphic to $\mathbb{P}^1$, and a generic fibre of the morphism Y is a smooth del Pezzo surface.
(c) The variety Y is a point, and $\mathrm{Pic}(V) \cong \mathbb{Z}$.

This list provides a natural set of examples on which to test Question 1.1. Indeed, in light of Lemma 2.2, a positive answer to Question 1.1 for the varieties listed above will provide a positive answer for any blowup of such varieties, which constitutes a huge proportion of all smooth, rationally connected threefolds. In this paper, we will deal with examples from cases (b) and (c). Specifically, we prove the following results:

Theorem 4.1. *Let X be a complex Fano threefold of Picard rank one and index at least two. Assume that X is defined over a number field k, and let Z be a Zariski closed subset of X of codimension at least two. If X is a hypersurface of degree 6 in the weighted projective space $\mathbb{P}(1,1,1,2,3)$, then we make the further assumption that Z does not contain the unique basepoint of the square root of the anticanonical linear system. Then the Z-integral points of X are potentially Zariski dense.*

Theorem 6.1. *Let k be a number field. Let X be a smooth threefold with a map $\pi\colon X \to \mathbb{P}^1$ whose generic fibre is a del Pezzo surface of degree at least three, all defined over k. Let $Z \subset X$ be a Zariski closed subset of codimension at least two. Then the Z-integral points of X are potentially Zariski dense.*

The rest of the paper is structured as follows. In section 2, we make some preliminary definitions, and prove two useful results, including Lemma 2.3.

Section 3 gives a positive answer to Question 1.1 for rational and ruled surfaces. Sections 4 and 6 are the heart of the paper, giving a positive answer to Question 1.1 for a wide range of rationally connected threefolds: section 4 deals with Fano threefolds with Picard rank one, and section 6 with del Pezzo fibrations. In section 5, given a fibration $\pi\colon X \to \mathbb{P}^1$, we prove the existence of sections avoiding a subset of codimension at least two, which may be of independent interest. Section 7 gives some applications of these results to integral points on families of curves and surfaces, including some classical cases of points integral with respect to a divisor. Finally, section 8 proves that the indeterminacy locus of a birational map to $\mathbb{P}^2$ may be chosen to avoid any fixed proper Zariski closed subset, a result needed in the proof of Theorem 3.3.

2. Preliminaries

We first fix some notation and definitions. Let X be a projective algebraic variety, $Z \subset X$ a Zariski closed subset, both defined over a number field k. By "defined over k", we mean that X is a variety over $\operatorname{Spec}(\overline{k})$, and that there is a variety X_0 over $\operatorname{Spec}(k)$ along with an isomorphism $X \cong X_0 \times_k \overline{k}$. For varieties X and Y defined over k, a morphism $f\colon X \to Y$ is defined over k if there is a morphism $f_0\colon X_0 \to Y_0$ such that the diagram

$$\begin{tikzcd} X \arrow[r] \arrow[d, "f"'] \arrow[dr, phantom, "\square"] & X_0 \arrow[d, "f_0"] \\ Y \arrow[r] & Y_0 \end{tikzcd}$$

is cartesian. I.e., f is defined over k if it is the base change to $\operatorname{Spec}(\overline{k})$ of a morphism $f_0\colon X_0 \to Y_0$ over $\operatorname{Spec}(k)$.

Let M_k be the set of places of k. Definition 2.1 is essentially Definition 1.4.3 in [Vo]:

Definition 2.1. Let S be a finite set of places of k containing all the archimedean places. A subset $R \subset X(k) - Z(k)$ is called (Z, S)-*integralizable* if and only if there are global Weil functions $\lambda_{Z,v}$ and non-negative real numbers n_v such that $n_v = 0$ for all but finitely many places v, and such that

$$\lambda_{Z,v}(P) \leq n_v$$

for all $v \in M_k - S$ and $P \in R$.

If the $\lambda_{Z,v}$ and n_v are fixed, we will say that a k-rational point P is (Z, S)-*integral* (or Z-*integral*, if S is understood) if and only if $\lambda_{Z,v}(P) \leq n_v$ for all $v \in M_k - S$.

This definition is somewhat involved, and for the sake of brevity, we refer the reader to section 1.4 of [Vo] for a more detailed discussion, including the definition and elementary properties of Weil functions, and a discussion of integralizable sets of points.

We begin with a lemma.

Lemma 2.2. *Let $f\colon X \to Y$ be a birational morphism between irreducible varieties. Assume that f, X, and Y are all defined over the same number field k. If Question* 1.1 *has a positive answer for every Zariski closed subset $Z \subset Y$ of codimension at least two, then it has a positive answer for every Zariski closed subset $Z \subset X$ of codimension at least two.*

Proof. Let $Z \subset X$ be a Zariski closed subset of codimension at least two. Then the closure W of $f(Z)$ is a Zariski closed subset of Y of codimension at least two, so by hypothesis the W-integral points of Y are potentially Zariski dense. But then the $f^{-1}(W)$-integral points of X are potentially Zariski dense as well, so *a fortiori* the Z-integral points of X are also potentially Zariski dense. □

Lemma 2.3 is a generalization of Theorems 1 and 2 of [Sh], and also appears as Theorem 3.1 in [MZ], but the proof given in the latter is slightly different. The idea behind this lemma is to show that if a curve has infinitely many integral points on it, then deleting an arithmetic zero-cycle from it will either delete all the integral points or else leave an infinite set of integral points. (If the curve C is projective, then "integral points" refers to rational points.)

In the statement of the lemma, C is the curve we're considering, and Z is the "locus at infinity" – that is, we assume that C has an infinite set of Z-integral points. We then delete a further set N which is assumed to intersect C in an arithmetic zero-cycle, and the assumption is that C contains at least one $(Z \cup N)$-integral point. Lemma 2.3 then says that C must still contain an infinite set of $(Z \cup N)$-integral points. In other words, Question 1.1 has a positive answer for curves, considered as arithmetic surfaces.

Lemma 2.3. *Let V be an algebraic variety defined over a number field k, and let C be an irreducible curve on V. Let Z and N be Zariski closed subsets of V. Let $L = Z \cup N$, and let S be a set of places of k that contains all the archimedean places of k.*

For every place v of k, let n_v be a non-negative real number such that $n_v = 0$ for all but finitely many places v. Choose Weil functions $\lambda_{L,v}$ for each place v. Assume that there is a point $P \in C(k)$ satisfying

$$\lambda_{L,v}(P) \leq n_v$$

for every place $v \notin S$.

If $N \cap C = \emptyset$ and C contains an infinite set of (Z, S)-integral points, then there are infinitely many points $Q \in C(k)$ satisfying

$$\lambda_{L,v}(Q) \le n_v$$

for every place $v \notin S$.

Proof. Since $C(k)$ is infinite, it follows by a theorem of Faltings (Satz 7 of [Fa]) that C must have geometric genus zero or one. The condition that C contain a dense set of Z-integral points implies that C must intersect Z in at most two places of C (places in the sense of points of the normalization of C), and that $C \cap Z = \emptyset$ if C has genus one.

We first assume that $C \cap Z = \emptyset$. Note that without loss of generality, we may assume that S is precisely the set of archimedean places of k, as increasing S only makes the lemma easier to prove.

Let S' be the set of places v of k such that either v is archimedean or $\lambda_{N,v}(Q) \ne 0$ for some $Q \in C(k_v)$, where k_v is the completion of k at v. Note that S' is finite because $N \cap C = \emptyset$.

For each $v \notin S'$, $\lambda_{N,v}(Q) = 0$ for all $Q \in C$, so we may restrict our attention to $v \in S'$.

If v is finite with corresponding prime π of $\mathcal{O}_k$, then the condition $\lambda_{N,v}(P) < n_v$ depends only on the residue class of P modulo a suitable power of π. (See for example subsection 2.2.2 of [BG].) Thus, the collection of all $Q \in C(k)$ satisfying $\lambda_{N,v}(Q) \le n_v$ for all finite v contains the set of points Q such that $Q \equiv P \pmod{M}$ for some suitable nonzero $M \in \mathcal{O}_k$.

There are now two cases: either the geometric genus of C is zero or one.

If the geometric genus of C is zero, then Lemma 2.3 follows immediately from the Weak Approximation Theorem for $\mathbb{P}^1$.

Weak Approximation does not hold for curves of genus one, however, so we must work a bit harder. The set B of points Q such that $Q \equiv P \pmod{M}$ for some nonzero $M \in \mathcal{O}_k$ contains the image on C of a coset of a finite index subgroup A of the Mordell-Weil group of the normalization $\tilde{C}$ of C over k. Since the set of rational points of C is infinite, the group A is infinite, and so we are done with the case $C \cap N = \emptyset$.

The only cases that remain are when C is a genus zero curve with either one or two places supported on Z. Let $\pi\colon \tilde{C} \to C$ be the normalization map over k. The set of (π^*Z, S)-integral points R of $\tilde{C}$ with $\pi(R) \notin Z$ are the integral points of the principal homogeneous space $C - Z$ for an arithmetic group ($\mathbb{G}_a$ if there is one place of C on Z, and $\mathbb{G}_m$ if there are two places), so by choosing a (π^*Z, S)-integral point R_0 on $\tilde{C}$, we can give the (π^*Z, S)-integral points of $\tilde{C}$ the structure of an arithmetic group G. The set B of points Q of C such that $Q \equiv P \pmod{N}$ for some nonzero $N \in \mathcal{O}_k$ – which, as before, is contained in the set of points Q satisfying $\lambda_{L,v}(Q) \leq n_v$ for all v not in S – contains the image on C of a coset of a finite index subgroup A of G, and is therefore infinite, as desired. □

The statement of Lemma 2.3 is somewhat technical, so we include a weaker version that is much simpler to state. It follows immediately from Lemma 2.3.

Lemma 2.4. *Let k be a number field with ring of integers $\mathcal{O}_k$, and let $\mathcal{C}$ be a scheme over* $\mathrm{Spec}(\mathcal{O}_k)$ *with generic fibre $C = \mathcal{C} \times \mathrm{Spec}\, k$. Assume that C is a geometrically integral curve.*

Let $\mathcal{N} \subset \mathcal{C}$ be a subscheme such that $\mathcal{N} \times \mathrm{Spec}(k) = \emptyset$ – that is, $\mathcal{N}$ is supported over a finite set of primes.

Assume that there is an infinite set of k-rational points on C. Assume further that there is a single point P satisfying $\overline{P} \cap \mathcal{N} = \emptyset$, where $\overline{P}$ denotes the closure of P over $\mathrm{Spec}(\mathcal{O}_k)$. *Then there is an infinite set of points Q on C satisfying $\overline{Q} \cap \mathcal{N} = \emptyset$.*

3. Surfaces

Lemma 2.3 gives a positive answer to Question 1.1 for curves, in an arithmetic sense. The next natural question is to ask if it has a positive answer for surfaces. This is as yet unknown in general, but there are nevertheless a great many cases in which it is known. For example, Question 1.1 has a positive answer for every toric variety, by Corollary 4.2 in [HT]. In fact, we can prove much more.

Theorem 3.1. *Let X be a complex surface with negative Kodaira dimension, defined over a number field k. Then Question* 1.1 *has a positive answer for X.*

Proof. Every surface of negative Kodaira dimension is, after possibly a finite extension of the field k, the blowup of a Hirzebruch surface, the projective plane, or a ruled surface. By Lemma 2.2, then, it suffices to assume that X is one of these three.

If X is a projective plane or Hirzebruch surface, then it is, in particular, a toric variety, and therefore Question 1.1 has a positive answer.

If X is a ruled surface, then there is a fibration $f\colon X \to C$ for some smooth curve C. If C has genus at least two, then Question 1.1 has a vacuously positive answer for X, because the rational points on X are not potentially dense. If the genus of C is zero, then X is a Hirzebruch surface and Question 1.1 has a non-vacuously positive answer for X, as just noted.

Thus, assume that C has genus 1, and let Z be a Zariski closed subset of codimension at least two – that is, let Z be a finite set of points of X. By a finite extension of the field of definition k, we may assume that there is a Z-integral point P on X defined over k, and that C has an infinite number of k-rational points. By, for example, Theorem V.2.17.(c) of [Ha], there is a very ample divisor class V on X whose elements are sections of f, and therefore have infinitely many rational points. Let Y_1 be a curve in the class V that contains the point P, but does not intersect Z. By Lemma 2.3, Y_1 has a Zariski dense set of Z-integral points, and in particular, there are an infinite number of fibres F of f for which $Y_1 \cap F$ is a Z-integral point, and for which $F \cap Z = \emptyset$. By Lemma 2.3 again, this means that F has a dense set of Z-integral points, implying that the set of Z-integral points is dense, and that Question 1.1 has a positive answer for X. □

Lemma 3.2. *Let X be a smooth projective surface defined over k, $Z \subset X$ a proper subvariety, and $R \subset X(k)$ a finite set of points. Suppose that there is a birational map $X \dashrightarrow \mathbb{P}^2$ defined over k. Then there exists a birational map $f\colon X \dashrightarrow \mathbb{P}^2$, also defined over k, such that f is defined in a neighbourhood of Z and R, and such that f is an isomorphism in a neighbourhood of R.*

Proof. By Theorem 8.1(d) there is a birational map $g\colon X \dashrightarrow \mathbb{P}^2$ defined over k such that neither R nor Z are in the indeterminacy locus of g. Equivalently, resolving g, there are a smooth projective surface Y_1, and birational morphisms $\pi_1\colon Y_1 \longrightarrow X$, $h_1\colon Y_1 \longrightarrow \mathbb{P}^2$, all defined over k, such that π_1 is an isomorphism in a neighbourhood of Z and R. Via π_1, we may consider Z and R to be subsets of Y_1.

Applying Proposition 8.6 to Y_1, we obtain a smooth projective surface Y_2, and birational morphisms $\pi_2\colon Y_2 \longrightarrow Y_1$, and $h_2\colon Y_2 \longrightarrow \mathbb{P}^2$ all defined over k, such that π_2 is an isomorphism in a neighbourhood of Z and R, and such that h_2^{-1} is defined at $h_2(\pi_2^{-1}(Q))$, for each $Q \in R$. The map $f := h_2 \circ (\pi_1 \circ \pi_2)^{-1}$ then satisfies the conditions of the lemma. □

Remarks.

(1) The notation used in Lemma 3.2 and Proposition 8.6 do not quite match up. In applying Proposition 8.6 in the proof above, for the W of the proposition one uses the Z of this lemma, and for the Z of the proposition one uses the R of this lemma.

(2) The proofs of Theorem 8.1 and Proposition 8.6 do not depend on any other results in this paper.

The following analogue of Lemma 2.3 will be used in the next section.

Theorem 3.3. *Let k be a number field and $\mathcal{X} \to \operatorname{Spec}(\mathcal{O}_k)$ an arithmetic threefold. Let $X \to \operatorname{Spec}(k)$ be the generic fibre of $\mathcal{X}$, and assume that X is birational to $\mathbb{P}^2_k$ over k. Fix an effective arithmetic 1-cycle $\mathcal{Z}$ on $\mathcal{X}$, with $\mathcal{Z}$ defined over $\mathcal{O}_k$. (Note that $\mathcal{Z}$ is not assumed to be flat over $\operatorname{Spec}(\mathcal{O}_K)$.) Let*

S be a finite set of places of k including all the archimedean places. If there is a $\mathcal{Z}$-integral point on X, then the set of $\mathcal{Z}$-integral points is Zariski dense.

Proof. Let $Z = \mathcal{Z} \times \operatorname{Spec}(k)$ be the generic part of $\mathcal{Z}$, and let $Q_X \in X$ be the given $\mathcal{Z}$-integral point. By hypothesis X is birational to $\mathbb{P}^2$ over k, and hence by Lemma 3.2 there is a birational map $f\colon X \dashrightarrow \mathbb{P}^2$ also defined over k, which is defined in a neighbourhood of Z, and an isomorphism in a neighbourhood of Q_X.

The map f can be extended to a rational map $F\colon \mathcal{X} \dashrightarrow \mathbb{P}^2_{\mathcal{O}_k}$. Let $\mathcal{Y} \subset \mathcal{X} \times_{\mathcal{O}_k} \mathbb{P}^2_{\mathcal{O}_k}$ be the closure of the graph of F, and $\pi_1 : \mathcal{Y} \to \mathcal{X}$ and $\pi_2 : \mathcal{Y} \to \mathbb{P}^2_{\mathcal{O}_k}$ the two projections. (We will use π_i to refer also to the restriction to the generic fibre of $\mathcal{Y}$.)

Let $Q_Y = \pi_1^{-1}(Q_X)$; by the hypothesis on f, this is a single k-rational point. Then Q_Y is a $\mathcal{Z}'$-integral point on Y, where $\mathcal{Z}' = \pi_1^{-1}(\mathcal{Z})$.

Set $Q = \pi_2(Q_Y)$. Let L be a k-rational line in $\mathbb{P}^2_k$ which contains Q and which avoids both the finite set $\pi_2(\pi_1^{-1}(Z))$ and the finite set of points of indeterminacy of f^{-1}.

Let $\mathcal{C}$ be the closure of $\pi_2^{-1}(L)$ in $\mathcal{Y}$, with generic fibre $C = \mathcal{C} \times \operatorname{Spec}(k)$. Then C is an irreducible curve with a dense set of k-rational points, and at least one $\mathcal{Z}'$-integral point, namely Q_Y. Since $C \cap Z' = \emptyset$ (where $Z' = \mathcal{Z}' \times \operatorname{Spec}(k)$), Lemma 2.3 implies that C contains a dense set of $\mathcal{Z}'$-integral points.

The set of such curves C is dense on Y, so we immediately deduce that Y contains a dense set of $\mathcal{Z}'$-integral points, and therefore that X contains a dense set of $\mathcal{Z}$-integral points, as desired. $\square$

For surfaces with non-negative Kodaira dimension, the situation is more complex, and indeed it is still not known which of these surfaces have a Zariski dense set of rational points, never mind integral ones. (Of course, there are many particular examples in which the answer is known – abelian surfaces, bielliptic surfaces, many $K3$ and Enriques surfaces, for example – but the general classification is not yet complete.) We will therefore move on to threefolds.

4. Fano threefolds

For the purposes of this paper, a Fano threefold is a smooth, three-dimensional, projective algebraic variety X whose anticanonical sheaf $-K_X$ is ample. If X has Picard rank one – that is, if the Picard group of X is isomorphic to $\mathbb{Z}$ – then there is a unique ample generator H of the Picard group. The index of a Fano threefold is the unique integer r such that $-K_X = rH$. The main theorem of this section is the following:

Theorem 4.1. *Let X be a complex Fano threefold of Picard rank one and index at least two, and let Z be a Zariski closed subset of X of codimension at least two. Assume that both X and Z are defined over a number field k. If X is a hypersurface of degree 6 in the weighted projective space $\mathbb{P}(1,1,2,3)$, then we make the further assumption that Z does not contain the unique basepoint of the square root of the anticanonical linear system. Then the Z-integral points of X are potentially Zariski dense.*

Remark 4.2. Note that Theorem 4.1 includes – with its one caveat – all del Pezzo threefolds.

Proof. The proof relies crucially on the classification of Fano threefolds of Picard rank one, found (for example) in [IP]. Section 12.2 of [IP] gives the following list of Fano threefolds of Picard rank one and index at least two:

(a) $\mathbb{P}^3$.
(b) A smooth quadric in $\mathbb{P}^4$.
(c) A smooth linear section of the Plücker-embedded Grassmannian $\mathrm{Gr}(2,5)$.
(d) A smooth intersection of two quadrics in $\mathbb{P}^5$.
(e) A smooth cubic hypersurface in $\mathbb{P}^4$.
(f) A double cover of $\mathbb{P}^3$, branched on a smooth quartic surface.
(g) A smooth hypersurface of degree 6 in the weighted projective space $\mathbb{P}(1,1,1,2,3)$.

Note that our list is in the reverse order of that in [IP], and in particular, item (g) is the same as the unnamed threefold in [IP] with $-K_X^3 = 8$ and $h^{1,2} = 21$.

We now proceed by cases. Case (a) is the easiest, as the answer to Question 1.1 is well known to be positive for $\mathbb{P}^3$ – see for example Corollary 5.2 of [HT].

Case (b). X is a smooth quadric hypersurface in $\mathbb{P}^4$.

Let P be a k-rational point of $X - Z$, and let $\pi\colon X \to \mathbb{P}^3$ be the linear projection from P. Then $\pi(Z)$ is a Zariski closed subset of $\mathbb{P}^3$ of codimension at least two, and so the $\pi(Z)$-integral points are potentially Zariski dense. It therefore follows immediately that the Z-integral points of X are also potentially Zariski dense.

Case (c). X is a smooth linear section of the Plücker-embedded Grassmannian $\mathrm{Gr}(2,5)$.

X can be obtained by blowing up a smooth quadric threefold $V \subset \mathbb{P}^4$ along a smooth rational curve of degree three, and then contracting the strict transform of a smooth quadric surface S. If $\pi\colon Y \to V$ is the blowup, and $\phi\colon Y \to X$ is the contraction, then a Z-integral point on X corresponds to a $\phi^* Z$-integral point on Y. Any $\pi(\phi^* Z)$-integral point of V pulls back to a

ϕ^*Z-integral point of Y, so it suffices to show that the $\pi(\phi^*Z)$-integral points of V are potentially Zariski dense.

The scheme $\pi(\phi^*Z)$ is contained in the union of S and a subset of V of codimension at least two. It therefore suffices to prove that the Z-integral points of V are potentially Zariski dense, where Z is the union of S and a subset W of codimension at least two.

After a finite extension of the base field k, we may assume that there is a Z-integral point P on Q, and that $\mathbb{G}_m$ over k has infinitely many integral points. Let $T \subset \mathbb{P}^4$ be a 2-plane containing P, but with $T \cap S$ finite, $T \cap V$ irreducible and smooth at P, and $T \cap W = \emptyset$. Then $(T \cap V) - (T \cap S)$ is a rational curve with at most two places deleted, and has a Z-integral point. Therefore, Lemma 2.3 implies that $T \cap V$ contains infinitely many Z-integral points.

For each such point P', we can find another 2-plane T' such that $P' \in T'$, $T' \cap S$ finite, $T' \cap V$ irreducible and smooth at P', $T' \cap W = \emptyset$, and $T' \neq T$. This, via Lemma 2.3, yields a set of Z-integral points whose Zariski closure Y has dimension at least 2. If $Y \neq V$, then for each Z-integral point P_i'' on Y, we can find a 2-plane T_i'' such that $P_i'' \in T_i''$, $T_i'' \cap S$ finite, $T_i'' \cap V$ irreducible and smooth at P_i'', $T_i'' \cap W = \emptyset$, and $T_i'' \cap V \not\subseteq Y \cup T_1'' \cup \ldots \cup T_{i-1}''$. By Lemma 2.3, we obtain a set of Z-integral points that is Zariski dense in V.

Case (d). X is a smooth intersection of two quadrics in $\mathbb{P}^5$.

After a finite extension of the base field k, we can choose a Z-integral point P that is not contained in Z, and such that the singular locus of the linear projection of X from P is not contained in the image of Z. Let $\pi_1 \colon X \dashrightarrow \mathbb{P}^4$ be the projection from P, and let W be the closure of $\pi_1(X)$. Then W is a singular cubic threefold, and if P' is a singular point of W that is not contained in $\pi_1(Z)$, then the projection $\pi_2 \colon W \dashrightarrow \mathbb{P}^3$ of W away from P' induces a birational map $\phi \colon X \dashrightarrow \mathbb{P}^3$ such that $\phi(Z)$ is a Zariski closed subset of $\mathbb{P}^3$ of codimension at least two. Since $\phi(Z)$-integral points are potentially Zariski dense in $\mathbb{P}^3$, it follows that Z-integral points on X are also potentially Zariski dense, as desired.

Case (e). X is a smooth cubic threefold in $\mathbb{P}^4$.

Let ℓ be a line on X with $\ell \cap Z = \emptyset$, and let $\pi \colon Y \to X$ be the blowing up of X along ℓ with exceptional divisor S. Then Y admits the structure of a conic bundle $\phi \colon Y \to \mathbb{P}^2$. Note that S is a rational surface and a double section of ϕ.

After a fixed extension of k, we may assume that ℓ, Y, π, and ϕ are all defined over k, and that S has a dense set of k-rational points, including a point P that is also Z-integral. By Theorem 3.3, this means that S has a Zariski dense set of Z-integral points as well.

The dimension of Z is at most one, so there is a dense set A of points of S such that for all $Q \in A$, the fibre of ϕ through Q does not meet Z. By Lemma 2.3, each such fibre has an infinite set of Z-integral points, and so the Z-integral points of X are dense.

Case (f). X is a double cover of $\mathbb{P}^3$ branched on a smooth quartic surface.

The threefold X is known to be geometrically unirational (see for example [IP, Example 10.1.3.(iii)]), so we may extend the number field k to ensure that X has a Zariski dense set S of rational points. Let $\pi\colon X \to \mathbb{P}^3$ be the double cover. The set $\pi(S)$ is Zariski dense in $\mathbb{P}^3$, and by extending the field k again we may assume that at least one point P of $\pi(S)$ is $\pi(Z)$-integral.

Any line ℓ in $\mathbb{P}^3$ lifts to a curve of geometric genus at most one on X. The net N of lines through P induces an elliptic threefold structure (fibred over a rational surface) on a blowup $\tilde{X}$ of X. Lemma 4.3 is helpful here:

Lemma 4.3. *Let $\pi\colon X \to Y$ be a morphism of smooth projective varieties whose generic fibre is a smooth curve of genus 1, with a section S of π that makes it an elliptic fibration. Assume that π, S, X, and Y are all defined over a number field k. Then there is a proper Zariski closed subset G of X such that every k-rational point P that is a torsion point in its fibre $\pi^{-1}(\pi(P))$ satisfies $P \in G$.*

Proof. By a theorem of Merel ("Théorème" of [Me]), there is a positive integer N such that for any elliptic curve defined over k, and any k-rational point P of finite order, the order of P divides N. Therefore, the set of k-rational points of finite order in their fibre is contained in a finite number of multisections of π, and in particular is not Zariski dense in X. $\square$

Let G be the set given by Lemma 4.3 for $\tilde{X}$. We further enlarge G to contain all the singular fibres of $\tilde{X}$. Let U be the complement of the image of G in $\mathbb{P}^3$. Then $U \cap \pi(S)$ consists entirely of k-rational points Q of $\mathbb{P}^3$ whose preimages on X are also k-rational, and such that the elliptic curve lying over the line joining the point Q to P has positive rank. (The point P is viewed as the identity element.)

Since Z has codimension at least two, there is a Zariski dense set of points Q in U such that the line ℓ joining P to Q is disjoint from Z. In each such case, the elliptic curve E lying over ℓ has positive Mordell-Weil rank (because Q is non-torsion with respect to P), and so by Lemma 2.3, E contains an infinite set of Z-integral points. Since the set of such E is Zariski dense, the theorem follows.

Case (g). X is a smooth hypersurface of degree 6 in the weighted projective space $\mathbb{P}(1, 1, 1, 2, 3)$, and the subset Z does not contain the unique basepoint P of the square root of the anticanonical linear system.

In this case, X is a double cover of the cone V in $\mathbb{P}^6$ over the Veronese embedding of $\mathbb{P}^2$ in $\mathbb{P}^5$; denote the cover by $\pi\colon X \to V$. Note that P is the preimage $P = \pi^{-1}(v)$ of the vertex v of the cone V.

Blowing up v on V yields a smooth threefold V^*, which admits the structure of a $\mathbb{P}^1$-bundle $f\colon V^* \to \mathbb{P}^2$ over $\mathbb{P}^2$, where the fibres of the bundle are the strict transforms of the lines of the ruling of V. The corresponding blowup of X yields a biregular map $g\colon X^* \to X$ and a double cover $\pi^*\colon X^* \to V^*$, where X^* inherits the structure of an elliptic fibration over $\mathbb{P}^2$ via $\phi = f \circ \pi^*$.

$$\begin{tikzcd}
X^* \arrow[r, "\pi^*", "2:1"'] \arrow[d, "g"] \arrow[rr, bend left, "\phi"] & V^* \arrow[r, "f", "\mathbb{P}^1"'] \arrow[d] & \mathbb{P}^2 \\
X \arrow[r, "\pi", "2:1"'] & V &
\end{tikzcd}$$

Since $P \notin Z$, it follows that $Z^* = g^{-1}(Z)$ is a subset of X^* of codimension at least two. In Section 4 of [BT], the authors show that there is a two-dimensional family of double sections of ϕ that are singular, but birational to $K3$ surfaces. After a possible finite field extension, we may assume that one of those double sections, which we will call S, satisfies the following properties:

- S intersects Z properly.
- S contains a singular point s which is Z^*-integral.
- S contains a Zariski dense set of rational points.

To see that such a choice is possible, note that [BT] proves the Zariski density of the rational points, and allows for a two-dimensional linear system full of such double sections S. (The extra field extension is necessary for the existence of the Z^*-integral singular point.)

Given such an S, we blow up the singular locus with $h\colon S^* \to S$ to obtain an elliptically fibred, smooth $K3$ surface S^*. The exceptional divisor over s is a (-2)-curve on S^* with a dense set of rational points, each of which is $h^{-1}(Z^*)$-integral. Thus, every smooth elliptic curve in the elliptic fibration on S^* contains at least one $h^{-1}(Z^*)$-integral point. The density of rational points on S^* implies that there are infinitely many such curves with positive Mordell-Weil rank. Therefore, since $Z^* \cap S$ is of codimension at least two, we conclude by Lemma 2.3 that the set of $h^{-1}(Z^*)$-integral points on S^* is Zariski dense, and therefore that the Z^*-integral points on S are also Zariski dense on S.

For any Z^*-integral point x on X^*, Lemma 2.3 again shows that the Z^*-integral points are Zariski dense on the fibre of ϕ through x, provided that its Mordell-Weil rank is positive. Since [BT] proves that the set of rational points on X^* are Zariski dense, there is a Zariski dense set of fibres with positive Mordell-Weil rank. Therefore, the set of Z^*-integral points on X^* is Zariski dense. This immediately implies that the set of Z-integral points on X is Zariski dense, as desired. $\square$

As far as the authors are aware, the exceptional case in (g) (i.e., the case where $P \in Z$) is still open.

5. Sections avoiding given subsets

In this section we prove a lemma guaranteeing the existence of a section of a rationally connected fibration over a curve, such that the section avoids (respectively fails to be contained in) a given subset of codimension ≥ 2 (respectively ≥ 1). The result is well-known to experts on families of curves on varieties, but we include a proof for lack of a reference. We begin by recalling background material.

Let X be a smooth projective variety, and set $n = \dim(X)$.

Recall that a *rational curve* in X is a nonconstant map $f\colon \mathbb{P}^1 \longrightarrow X$. The rational curve is said to be free if $f^*T_X = \oplus_{i=1}^n \mathcal{O}_{\mathbb{P}^1}(a_i)$ with each $a_i \geq 0$. Fix an ample line bundle L on X. Then for each $d \geq 1$ there is a quasi-projective variety $\mathrm{Hom}(\mathbb{P}^1, X)_d$ parametrizing maps $f\colon \mathbb{P}^1 \longrightarrow X$ such that $\deg(f^*L) = d$. (This is a special case of the general construction of [Ko, 1.10] constructing parameter spaces $\mathrm{Hom}(Y, X)$ for any projective varieties Y and X. Any morphism $Y \longrightarrow X$ can be identified with its graph, a subset of $Y \times X$, and the spaces $\mathrm{Hom}(Y, X)$ are then realized as the open subscheme of $\mathrm{Hilb}(Y \times X)$ parametrizing such graphs. The restriction $\deg(f^*L) = d$ is used to fix the Hilbert polynomial of the graph.)

For a map $f\colon \mathbb{P}^1 \longrightarrow X$, with $\deg(f^*L) = d$, we denote by $[f]$ the corresponding point of $\mathrm{Hom}(\mathbb{P}^1, X)_d$. One also has an *evaluation map*

$$\begin{array}{rccc} \mathrm{ev} & : & \mathrm{Hom}(\mathbb{P}^1, X)_d \times \mathbb{P}^1 & \longrightarrow & X, \\ & & ([f], p) & \longmapsto & f(p). \end{array}$$

Let $\mathrm{Hom}(\mathbb{P}^1, X)_d^\circ$ denote the subset of $\mathrm{Hom}(\mathbb{P}^1, X)_d$ consisting of those $[f]$ such that f is free. By [Ko, II.3.5.4, p. 115], $\mathrm{Hom}(\mathbb{P}^1, X)_d^\circ$ is an open subset of $\mathrm{Hom}(\mathbb{P}^1, X)_d$, and the evaluation map

$$\mathrm{Hom}(\mathbb{P}^1, X)_d^\circ \times \mathbb{P}^1 \xrightarrow{\mathrm{ev}} X$$

is smooth. (Thus $\mathrm{Hom}(\mathbb{P}^1, X)_d^\circ$ is also smooth, although one can see this last point directly by computing the tangent space to the Hilbert scheme).

It is a result of Kollár, Miyaoka, and Mori (see [KMM, Corollary 2.5] or [Ko, Theorem 3.11, p. 205]) that if $X \longrightarrow S$ is a smooth proper morphism in characteristic zero with S connected, then if one fibre is rationally connected, all fibres are rationally connected. In Lemma 5.1 and Corollary 5.2 we will consider surjective maps π from a smooth projective variety X to a smooth curve. Since X is projective, π is automatically proper, and π is smooth away from finitely many points of the base. When we say that "the general fibre is rationally connected", we mean that at least one, and hence all, of the smooth fibres is rationally connected.

Lemma 5.1.

(a) *Let X be a smooth irreducible projective variety defined over an algebraically closed field of characteristic zero, $\pi\colon X \longrightarrow \mathbb{P}^1$ a surjective map whose general fibre is rationally connected, $Z \subset X$ a subvariety of codimension ≥ 2, and $T \subset X$ a subvariety of codimension ≥ 1. Then there exists a section of π which is not contained in T, and which does not meet Z.*

(b) *If X, and π, Z, and T are defined over a field k of characteristic zero, and if the general fibre of π over $\overline{k}$ is rationally connected, then there exists such a section defined over a finite extension k' of k.*

Proof. We first prove (a). By [GHS, Theorem 1.1] there is a map $g\colon \mathbb{P}^1 \longrightarrow X$ which is a section of π. Furthermore, by [KMM, Theorem 2.13], given that such a section exists, and given any point q on a smooth fibre of π, there exists a curve $f'\colon \mathbb{P}^1 \longrightarrow X$ which is a free curve, a section of π, and passes through q (i.e., so that q is in the image of f').

Choose any point q in a smooth fibre, and not in Z or T, and let f' be a free curve and section passing through q provided by those theorems. Set $d = \deg((f')^*L)$, and let V be the irreducible component of $\mathrm{Hom}(\mathbb{P}^1, X)^\circ_d$ containing $[f']$. We consider the diagram

$$\begin{array}{ccc} V \times \mathbb{P}^1 & \xrightarrow{\text{ev}} & X \\ \Big\downarrow{\scriptstyle pr_1} & & \\ V & & \end{array}$$

The property that f is a section of π is equivalent to $\deg(f^*\pi^*\mathcal{O}_{\mathbb{P}^1}(1)) = 1$, i.e., the degree of $\mathrm{ev}^*\,\pi^*\mathcal{O}_{\mathbb{P}^1}(1)$ on the fibre $pr_1^{-1}([f])$ is 1. Since pr_1 is flat, the degree of $\mathrm{ev}^*\,\pi^*\mathcal{O}_{\mathbb{P}^1}(1)$ is constant on the fibres of pr_1 and it follows that every $[f] \in V$ is also a section of π.

The map pr_1 is proper, and by [Ko, II.3.5.4.2, p. 115], ev is smooth. The set of $[f] \in V$ such that $f(\mathbb{P}^1)$ is contained in T is the locus where the map $\mathrm{ev}^{-1}(T) \xrightarrow{pr_1} V$ has 1-dimensional fibres. By upper semicontinuity of fibre dimension [EGA IV$_3$, Cor. (13.1.5)] this locus is a closed subset of V. Let U' be its complement. The set U' is nonempty since $[f'] \in U'$. Every point $[f] \in U'$ is now a section of π not contained in T. To prove part (a) we just need to find such an $[f]$ so that $f(\mathbb{P}^1) \cap Z = \emptyset$.

Set $N = \dim(V)$. Since Z is of codimension ≥ 2, and ev flat, $\mathrm{ev}^{-1}(Z)$ also has codimension ≥ 2, and hence has dimension at most $N + 1 - 2 = N - 1$. Thus $pr_1(\mathrm{ev}^{-1}(Z))$ has dimension $\leq N - 1$ and so is a proper subset of V. Let U'' be its complement. Any $[f] \in U''$ satisfies $f(\mathbb{P}^1) \cap Z = \emptyset$. Since V is irreducible, $U := U' \cap U'' \neq \emptyset$, giving (a).

For (b), we are assuming that X, π, Z and T are all defined over k. Following the notational convention in section 2, we let X_0, π_0, Z_0, and T_0 be the corresponding varieties and morphism over $\mathrm{Spec}(k)$ whose base changes to $\mathrm{Spec}(\overline{k})$ give X, π, Z and T.

We then follow the construction as in part (a), but working over $\mathrm{Spec}(k)$. Specifically, we fix an ample line bundle L_0 on X_0, and using L_0 to define the degree, for each $d \in \mathbb{N}$ look at the k-scheme $\mathrm{Hom}_k(\mathbb{P}^1, X_0)_d$. We then restrict to the open set $\mathrm{Hom}_k(\mathbb{P}^1, X_0)^\circ_d$ parametrizing free morphisms, and then further to the open subset where the morphisms are sections of π_0 (this is open for the same reason as before: it is a condition on the degree of $ev^*\pi_0^*\mathcal{O}_{\mathbb{P}^1}(1)$ on the fibres of the projection $pr_1\colon \mathrm{Hom}_k(\mathbb{P}^1, X_0)^\circ_d \times \mathbb{P}^1 \longrightarrow \mathrm{Hom}_k(\mathbb{P}^1, X_0)^\circ_d)$.

The locus where the fibre dimension of $\mathrm{ev}^{-1}(T_0) \longrightarrow \mathrm{Hom}(\mathbb{P}^1, X_0)^\circ_d$ is 1 is again closed, as is the subset $pr_1(\mathrm{ev}^{-1}(Z_0))$. Intersecting the complements of these closed sets with the open set in the previous paragraph, for each d we obtain a (possibly empty) open subscheme $U_d \subset \mathrm{Hom}_k(\mathbb{P}^1, X_0)^\circ_d$. For any extension k'/k, the k'-points of U_d parameterize maps $g\colon \mathbb{P}^1 \longrightarrow X'$ over $\mathrm{Spec}(k')$ which are sections of π' avoiding Z' and not contained in T', and with $\deg(g^*L') = d$. Here the prime denotes the base change of the respective object from k to k'.

By part (a), there is some d for which U_d has a $\overline{k}$-point, and thus, for this d, U_d is nonempty. Since $\mathrm{Hom}_k(\mathbb{P}^1, X_0)_d$ is of finite type, the residue field of any closed point is finite over k. Thus taking any closed point $[f_0] \in U_d$, and letting k' be its residue field, we obtain a morphism whose base change to $\overline{k}$ is a section $f\colon \mathbb{P}^1 \longrightarrow X$ defined over k', avoiding Z, and not contained in T. □

Using an idea from [GHS] due to Aise Johan de Jong, one can extend Lemma 5.1 to the case where the base curve has arbitrary genus. We will not need this extension, but record the statement and the idea of its proof.

Corollary 5.2.

(a) *Let X be a smooth irreducible projective variety defined over an algebraically closed field of characteristic zero, $\pi\colon X \longrightarrow C$ a surjective map to a smooth curve C such that the general fibre of π is rationally connected, $Z \subset X$ a subvariety of codimension ≥ 2, and $T \subset X$ a subvariety of codimension ≥ 1. Then there exists a section of π which is not contained in T, and which does not meet Z.*

(b) *If X, and π, Z, and T are defined over a field k of characteristic zero, and if the general fibre of π over $\overline{k}$ is rationally connected, then there exists such a section defined over a finite extension k' of k.*

Proof. We repeat the argument of de Jong from [GHS, §3.2]. To prove (a), given $\pi\colon X \longrightarrow C$ choose any finite map $g\colon C \longrightarrow \mathbb{P}^1$, and then form the "norm" of X. This is a variety and map $\varphi\colon Y \longrightarrow \mathbb{P}^1$ (well defined up to birational equivalence) whose fibre over a general point $p \in \mathbb{P}^1$ is the product $\prod_{q\in g^{-1}(p)} \pi^{-1}(q)$. The utility of the norm construction is that sections of φ give sections of π. Given a section σ of φ, for each $p \in \mathbb{P}^1$, σ gives a point of $\prod_{q\in g^{-1}(p)} \pi^{-1}(q)$, and thus for each point $q \in C$, setting $p = g(q)$, σ gives a point in the fibre $\pi^{-1}(q)$.

To ensure that the resulting section of π misses Z and is not contained in T, we define appropriate subsets of Y. Let $\tilde{Z} \subset Y$ be the subset

$$\tilde{Z} = \left\{ y \in Y \;\middle|\; \begin{array}{l} \text{at least one of the coordinates of } y \in \\ \varphi^{-1}(\varphi(y)) = \prod_{q\in g^{-1}(\varphi(y))} \pi^{-1}(q) \text{ is in } Z \end{array} \right\}$$

and similarly define $\tilde{T}$.

Sections σ of φ which do not meet $\tilde{Z}$ and are not contained in $\tilde{T}$ induce sections of π similarly missing Z and not contained in T. The codimension of $\tilde{Z}$ in Y is equal to the codimension of Z in X, and similarly $\operatorname{codim}(\tilde{T}, Y) = \operatorname{codim}(T, X)$.

Since the product of rationally connected varieties is rationally connected, the general fibre of Y is rationally connected, and so we can apply Lemma 5.1(a), proving (a) of the corollary.

To prove (b), supposing everything defined over k, if we choose our map $g\colon C \longrightarrow \mathbb{P}^1$ to be defined over k, then so are Y, $\tilde{Z}$, and $\tilde{T}$. Thus applying Lemma 5.1(b), we obtain a section of φ defined over a finite extension k' missing $\tilde{Z}$ and not contained in $\tilde{T}$. This then induces a section of π, also defined over k', with the desired properties. $\square$

6. Del Pezzo fibrations

In this section, we prove the potential density of integral points for del Pezzo fibrations, provided that the degree of the (generic) del Pezzo surface is at least three.

Let $\pi\colon X \to Y$ be a morphism, where X is a smooth, rationally connected, projective threefold, and Y is a smooth curve. Since X is rationally connected, Y must be isomorphic to $\mathbb{P}^1$ over k. (This may require a finite extension of k.) We further assume that a generic fibre of π is a del Pezzo surface. We will show that in many cases, the Z-integral points of X are potentially Zariski dense.

Theorem 6.1. *Let k be a number field. Let X be a smooth threefold with a map $\pi\colon X \to \mathbb{P}^1$ whose generic fibre is a del Pezzo surface of degree at least three, all defined over k. Let $Z \subset X$ be a Zariski closed subset of codimension at least two. Then the Z-integral points of X are potentially Zariski dense.*

Proof. Let T be the union of the (-1)-curves in the fibres of π. Applying Lemma 5.1, after at most a finite field extension – which we continue to call k – we obtain a k-rational section $\sigma\colon \mathbb{P}^1 \to X$ of π whose image is a smooth rational curve $C \subset X$, and disjoint from Z, and meeting T in only finitely many points (i.e., only finitely many points of C are contained in (-1)-curves of the fibres of π). Furthermore, after blowing up, we may decrease the degree of the generic fibre of π to three without changing the hypothesis or conclusion of Theorem 6.1. (We choose the blowup locus to be disjoint from Z.) Let $S \subset \mathbb{P}^1(k)$ be the finite subset of points p where either $\pi^{-1}(p)$ contains a 1-dimensional component of Z or $\pi^{-1}(p)$ intersects C in a point on a (-1)-curve of the fibre.

We will finish the proof by applying Lemma 6.2 to the fibres $\pi^{-1}(p)$, with $p \in \mathbb{P}^1(k) - S$. (Note that Lemma 6.2 is not implied by Theorem 3.3 because a del Pezzo surface need not be birational to $\mathbb{P}^2$ over k.)

Lemma 6.2. *Let V be a del Pezzo surface of degree three defined over a number field k. Let $Z \subset V$ be a Zariski closed subset of codimension at least two. Assume that there is a k-rational point $P \in V(k)$ that is Z-integral, and that does not lie on a (-1)-curve of V. Then the Z-integral points are Zariski dense.*

Proof. A generic member of the linear system $|-K_V|$ is a smooth curve of genus one, and $|-K_V|$ is basepoint free because V is del Pezzo. Consider the linear subsystem of $|-K_V|$ consisting of curves containing P. This subsystem has dimension three, so we can choose a pencil H of curves defined over k such that every curve in H contains P, a generic curve in H is smooth, and the base locus consists of three points $\{P, Q, R\}$, none of which lie in Z. (Note that $P \notin Z$ trivially.)

Since H is defined over k, so is the triple $\{P, Q, R\}$, and we may therefore blow it up to obtain a surface $\tilde{V}$ defined over k, with a morphism $\psi\colon \tilde{V} \to V$ whose fibres are precisely the (strict transforms of) curves in H. The morphism ψ makes $\tilde{V}$ into an elliptic surface, with a section $\mathcal{O}$ given by the exceptional curve lying over P. Note that $\mathcal{O}$ is disjoint from Z.

The class $-K_V$ embeds V in $\mathbb{P}^3$ as a smooth cubic surface. We may therefore consider the curve T defined by the intersection of the tangent plane T_P with the embedded surface V. Note that T is irreducible because P does not lie on any (-1)-curves, so T is an irreducible plane cubic curve. Moreover, T is singular at P, so it has geometric genus zero. Indeed, T is birational to $\mathbb{P}^1$ over k via projection from P in the plane T_P, so T has a dense set of rational points.

For each rational point A of T, the intersection of T_A with the embedded surface V is again a cubic curve with a singularity at A, albeit possibly reducible. At most finitely many A correspond to reducible curves in this way (there are only finitely many intersections of T with lines), so there are infinitely many A whose tangent curves are birational to $\mathbb{P}^1$ over k, and therefore have a dense set of rational points. We therefore deduce that the rational points of V are Zariski dense.

By Lemma 4.3, this means that there is a dense set of k-rational points on V each lying on a smooth fibre of ψ and having infinite order in that fibre. In particular, there are an infinite number of genus one curves on V that contain an infinite set of k-rational points, one of which is the Z-integral point P. By Lemma 2.3, each of those curves contains an infinite set of Z-integral points. We conclude that the Z-integral points on V are dense. $\square$

We now finish the proof of Theorem 6.1. The curve C is disjoint from Z over the generic fibre, so after a further finite extension of k – which we stubbornly persist in calling k – we may assume that C contains a Z-integral point. To see this, note that over $\mathrm{Spec}(\mathcal{O}_k)$, $\mathcal{N} = \mathcal{C} \cap \mathcal{Z}$ is an arithmetic zero-cycle supported on finitely many places of k. After a suitably chosen finite extension of k, we may assume that for every place v of k, there is a point p_v of $\mathcal{C}$ lying over v that does not lie in the support of $\mathcal{N}$. By the Chinese Remainder theorem – since C is a rational curve – there is some k-rational point P of C such that for all v over which $\mathcal{N}$ is supported, $P \equiv p_v \mod v$. This P is the Z-integral point that we seek.

Since C is a rational curve, Lemma 2.3 implies that C contains an infinite number of Z-integral points. For each such point, the corresponding fibre, by

Lemma 6.2, contains a dense set of Z-integral points. It therefore follows that the Z-integral points of V are Zariski dense, as desired. □

7. Application to integral points in families

We can apply the theorems of the previous sections to families of curves and surfaces on surfaces and threefolds, to get results about integral points in the classical sense – that is, integral points with respect to a divisor. The proof of Theorem 7.1 is trivial:

Theorem 7.1. *Let X be a smooth, projective variety, defined over a number field k, and let $\mathcal{P}$ be a family of cycles on X. We further assume that on some dense open subset U of X, every point $P \in U$ lies on exactly one element of $\mathcal{P}$.*

Let Z be a subset of X of codimension at least 2. If the Z-integral points of X are Zariski dense, then there is a Zariski dense set of k-rational cycles in $\mathcal{P}$ with at least one Z-integral point defined over k.

In full generality, this theorem applies to a huge range of examples, and even though the codimension of Z in X is at least 2, the codimension of $Z \cap A$ in A (where A is a member of the family) can be anywhere from 0 to $\dim A$ for particular A – indeed $Z \cap A$ could even be empty for most A in $\mathcal{P}$.

Of perhaps greatest interest is the case in which the codimension of $Z \cap A$ in A is 1 – the classical case of integral points. This happens, for example, if $\mathcal{P}$ is a pencil of hypersurfaces in X with no fixed component. Or in the case where $\mathcal{P}$ is the locus of lines through a fixed point of $\mathbb{P}^n$. Note that in these two cases the existence of the open set U is automatic.

Another particularly interesting case is when $\mathcal{P}$ is a family of curves, and Z intersects every element of $\mathcal{P}$ nontrivially. For any subset $Z' \subset Z$, any Z-integral point is automatically also Z'-integral. Theorem 7.1 therefore implies that over a fixed finite extension, infinitely many curves in $\mathcal{P}$ acquire a Z'-integral point, where Z' is any chosen point of Z. Indeed, Theorem 7.1 even allows us to conclude that, for any function $f\colon \mathcal{P} \to Z$, there are infinitely many curves C in $\mathcal{P}$ with an $\{f(C)\}$-integral point!

The assumption that every point $P \in U$ lies on *exactly* one element of $\mathcal{P}$ can be relaxed to "*at least* one element of $\mathcal{P}$", but the cycles in the conclusion might not be defined over k.

The results of this paper show that the hypotheses of Theorem 7.1 are satisfied when X is any rational or ruled surface, or any rationally connected threefold of a type considered in sections 4 or 5, whenever the rational points are dense or potentially dense. Specifically, we have the following corollaries of Theorem 7.1.

Corollary 7.2. *Let X be a smooth surface that is uniruled over a number field k. Let $\mathcal{P}$ be a pencil of curves on X with zero-dimensional base locus Z. Assume that X has a Zariski dense set of k-rational points. Then there is a finite extension k'/k such that there is an infinite set of curves in $\mathcal{P}$ that are defined over k' and contain at least one Z-integral point.*

Corollary 7.3. *Let X be a smooth Fano threefold defined over a number field k, with geometric Picard rank one and index at least two. Let $\mathcal{P}$ be a pencil of surfaces on X with one-dimensional base locus Z that, if X is isomorphic to a smooth hypersurface of degree 6 in the weighted projective space $\mathbb{P}(1,1,1,2,3)$, does not contain the unique basepoint of the square root of the anticanonical linear system. Assume that X has a Zariski dense set of k-rational points. Then there is a finite extension k'/k such that there is an infinite set of surfaces in $\mathcal{P}$ that are defined over k' and contain at least one Z-integral point.*

Corollary 7.4. *Let X be a smooth Fano threefold defined over a number field k, with geometric Picard rank one and index at least two. Let $\mathcal{N}$ be a two-dimensional family of curves on X with common intersection locus Z that, if X is isomorphic to a smooth hypersurface of degree 6 in the weighted projective space $\mathbb{P}(1,1,1,2,3)$, does not contain the unique basepoint of the square root of the anticanonical linear system. Assume that X has a Zariski dense set of k-rational points, and that there is a Zariski dense open subset U of X such that every $P \in U$ lies on exactly one curve in $\mathcal{N}$. Then there is a finite extension k'/k such that there is a Zariski dense set of curves in $\mathcal{N}$ that are defined over k' and contain at least one Z-integral point.*

Note also that in cases where the potential density of rational points is already known, the hypothesis on the existence of a dense set of rational points is unnecessary.

8. Moving the indeterminacy locus of a map to $\mathbb{PP}^2$

Theorem 8.1 addresses a technical point in the proof of Theorem 3.3. Its proof does not rely on any other results in this paper.

Theorem 8.1. *Let X be a smooth projective surface and $f : X \dashrightarrow \mathbb{P}^2$ a birational map. Set n to be the number of points (over $\overline{k}$) in the indeterminacy locus of f. Then:*

(a) *For any general collection $I = \{q_1, q_2, \ldots, q_n\}$ of n points in X, there is a birational map $f_I : X \dashrightarrow \mathbb{P}^2$ whose indeterminacy locus is I.*

(b) *In particular, given any proper closed subset $Z \subset X$, there is always such a birational map defined at all points of Z.*

If f is defined over k, then the k-points of X are Zariski dense, and hence the k-points of X^n are Zariski dense.

(c) *The set of k-points $(q_1, \ldots, q_n) \in X^n$ with the property that there exists a birational map $f_I \colon X \dashrightarrow \mathbb{P}^2$ defined over k and with indeterminacy locus $I = \{q_1, \ldots, q_n\}$ is Zariski dense in X^n.*

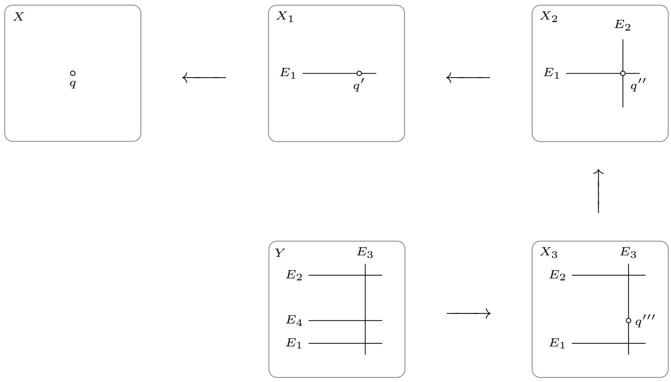
(d) *In particular, if f is defined over k then given any proper closed subset $Z \subset X$ there is always such a birational map f_I defined over k, and defined at all points of Z.*

It is clear that (a) implies (b), and that (c) implies (d). Part (c) also follows fairly easily from (a) by keeping track of the field of definition of the constructions. Most of the work is in proving (a).

The idea of the proof of (a) is simple: The birational map f may be resolved by a sequence of blowups, starting with the points of the indeterminacy locus. On the resulting surface Y, the induced birational morphism $Y \longrightarrow \mathbb{P}^2$ is given by a base point free line bundle with three sections. This line bundle may be written as a line bundle pulled back from X and twisted by the exceptional divisors. If we vary the initial points of the blowup in a family, the theorems on cohomology, base change, and semicontinuity imply that for a general choice of n initial points the resulting line bundle on the new blowup will continue to give a birational morphism to $\mathbb{P}^2$.

Although simple, writing out this argument in detail is unfortunately somewhat long, with most of the length being taken up in constructing the parameter spaces on which to apply the above named theorems. We start by constructing these spaces, then record some consequences of the theorems on base change. The proof of Theorem 8.1, using these results, appears after the proof of Lemma 8.5.

The spaces we want to construct parameterize not only the n points on X where we first blow up, but also the points of further blowups on further exceptional divisors. How the further blowups are to be continued is described by discrete data. To explain this terminology, consider the sequence of blowups below, which starts with blowing up a single point q in X. (In the diagram, and in further discussion, we use the name E_i to denote the exceptional divisor of the i-th blowup, as well as its proper transform in further blowups.)



In this example, we start by blowing up at q, then blow up at a point on E_1, then at the intersection of E_1 and E_2, and finally blow up at a point on E_3 which is not on E_1 or E_2.

The discrete portion of this data is the information of which exceptional divisor each successive blowup occurs on, including the possibility that the point to be blown up is on the intersection of two exceptional divisors. The portion which may vary in a family is the choice of which point on each exceptional divisor to blow up, unless one is supposed to blow up on the intersection of two exceptional divisors, in which case there is no choice at all.

Formally, for use below, the discrete portion of the data is a finite list of instructions, whose first instruction is "blow up at a (variable) point q on X", and where successive elements of the list are either of the form "blow up at a point on E_i (but not on any of the other exceptional divisors)" or of the form "blow up at the intersection of E_i and E_j". We implicitly assume that all instructions of the second type are possible, e.g., E_i does intersect E_j. In general we will want to start blowing up at n distinct points $q_1, \ldots, q_n$ of X, with further blowups given by discrete data describing the pattern of blowups over each of the points (the pattern of blowups may be different over each point). It is useful to separate the case $n = 1$, where we start by blowing up at a single point q on X, from the general case.

Proposition 8.2 (Existence and properties of the parameter space, $n = 1$)**.** *Let X be a smooth projective surface. Given discrete data with s steps describing the pattern of blowups over a single point, there exists:*

- *A quasi projective variety B, with a morphism $\varphi\colon B \longrightarrow X$.*

B is the parameter space for "choices of blowup with the given discrete data". For $\overline{q} \in B$, $\varphi(\overline{q}) \in X$ is the point where we first blow up.

- *A quasi projective variety $\mathcal{Y}$ with morphism $\Pi\colon \mathcal{Y} \longrightarrow B \times X$.*

The morphism Π *is the "universal blow down map". Setting* $\psi := pr_1 \circ \Pi$, *the morphism* $\psi\colon \mathcal{Y} \longrightarrow B$ *is the universal family of blown up surfaces. For a point* $\overline{q} \in B$, *we denote by* $\mathcal{Y}_{\overline{q}} := \psi^{-1}(\overline{q})$ *the fibre of* $\mathcal{Y}$ *over* $\overline{q}$, *and* $\pi_{\overline{q}}\colon \mathcal{Y}_{\overline{q}} \longrightarrow X$ *the restriction of* Π *over* $\overline{q}$.

- *For each* i, $i = 1,\ldots s$, *a closed subscheme* $\mathcal{E}_i \subset \mathcal{Y}$.

The $\mathcal{E}_i$ *are the relative families of exceptional divisors. The commutative diagram below summarizes some of this data.*

$$\begin{tikzcd}
\mathcal{E}_i \arrow[r, hook] \arrow[dr, "\mathbb{P}^1"'] & \mathcal{Y} \arrow[r, "\Pi"] \arrow[d, "\psi"] & B \times X \arrow[d, "pr_1"] \\
& B \arrow[r, equal] & B
\end{tikzcd}$$

These varieties and morphisms satisfy the properties the descriptions above promise, namely:

(a) ψ *is a smooth proper morphism of relative dimension* 2;
(b) *Via* ψ, *each* $\mathcal{E}_i$ *is a* $\mathbb{P}^1$*-bundle over* B;
(c) φ *is a smooth surjective morphism, and* B *is smooth and irreducible;*
(d) *For each* $\overline{q} \in B$, *the morphism* $\pi_{\overline{q}}\colon \mathcal{Y}_{\overline{q}} \longrightarrow X$ *is a blow up of* X *starting at* $\varphi(\overline{q})$ *in* X, *with further blowups following the pattern described by the given discrete data;*
(e) *Conversely, given a birational map* $\pi\colon Y \longrightarrow X$ *which is obtained by blowing up* X *at a single point* $q \in X$, *and then continuing in the pattern described by the discrete data, there exists* $\overline{q} \in B$ *such that* $\mathcal{Y}_{\overline{q}} = Y$ *and* $\pi_{\overline{q}} = \pi$ *as objects and morphisms over* X.
(f) *If* X *is defined over* k, *then* B, $\mathcal{Y}$, *the* $\mathcal{E}_i$, *the morphisms* Π, ψ, φ, *and the inclusion morphisms* $\mathcal{E}_i \hookrightarrow \mathcal{Y}$ *are defined over* k.
(g) *If, in addition, the* k*-points of* X *are Zariski dense, then the* k*-points of* B *are Zariski dense, and the map* $B(k) \longrightarrow X(k)$ *induced by* φ *is surjective;*
(h) Π *is a proper map, and is an isomorphism over the complement of* Γ_φ, *where* $\Gamma_\varphi \subset B \times X$ *denotes the graph of* φ.

Remarks.

(1) The point $\overline{q}$ in (d) is unique, although we will not need this detail.
(2) $\mathcal{Y}$ and each $\mathcal{E}_i$ are smooth and irreducible, as follows by combining (a), (b), (c), and (d).

Proof. The construction is inductive, following the steps of the discrete information describing the blowups. We indicate with a superscript t (e.g., B^t, $\mathcal{Y}^t$, $\mathcal{E}_i^t$, Π^t) the varieties and maps constructed after the t-th step. We do not need to explicitly construct ψ^t, since it is always the composition $pr_1 \circ \Pi^t$, where pr_1 is projection onto the first factor.

For the base case we start with $B^1 := X$ and set $\Pi^1 \colon \mathcal{Y}^1 \longrightarrow B^1 \times X$ to be the blowup of $B^1 \times X = X \times X$ along the diagonal. We set $\mathcal{E}_1^1 \subset \mathcal{Y}^1$ to be the exceptional divisor of Π^1 and $\varphi^1 \colon B^1 \longrightarrow X$ to be the identity map.

For the inductive step, we assume that we have constructed B^t, $\mathcal{Y}^t$, the morphisms $\Pi^t \colon \mathcal{Y}^t \longrightarrow B^t \times X$ and φ^t, and divisors $\mathcal{E}_i^t \subset \mathcal{Y}^t$, $i = 1,\ldots t$. In describing the construction we will use properties (e.g., "$\mathcal{E}_i^t$ is a $\mathbb{P}^1$-bundle over B^t") which, strictly speaking, we will only show later when proving (a)–(h). However, each of those arguments is also inductive, and thus we may assume their validity for the t-th step when making the construction for the $(t+1)$-st.

By our working definition of "discrete data", the $(t+1)$-st step must be of one of the following two operations:

(i) Blow up at a point on an exceptional divisor E_i, but which is not on any other exceptional divisor, or
(ii) Blow up at a point on the intersection of E_i and E_j (assuming that E_i and E_j do intersect).

In case (i): We set $B^{t+1} := \mathcal{E}_i^t - \bigcup_{j\neq i} \mathcal{E}_j^t$. Thus B^{t+1} parameterizes points which lie on the i-th exceptional divisor, but not on any others. We have a natural surjective map $B^{t+1} \longrightarrow B^t$ given by the composition of the open immersion $B^{t+1} \hookrightarrow \mathcal{E}_i^t$ and the fibration $\mathcal{E}_i^t \longrightarrow B^t$. (The fibres of $B^{t+1} \longrightarrow B^t$ are $\mathbb{P}^1$'s minus a finite number of points.) Let W be the fibre product $W := B^{t+1} \times_{B^t} \mathcal{Y}^t$. The inclusion morphism $B^{t+1} \hookrightarrow \mathcal{E}_i^t \hookrightarrow \mathcal{Y}^t$ induces a section σ of W over B^{t+1}:

$$\begin{tikzcd} W \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & \mathcal{Y}^t \arrow[d] \\ B^{t+1} \arrow[r] \arrow[u, bend left, "\sigma"] & B^t \end{tikzcd}$$

Let $\gamma \colon \mathcal{Y}^{t+1} \longrightarrow W$ be the blowup of W along $\sigma(B^{t+1})$ (i.e., we define $\mathcal{Y}^{t+1}$ to be the blowup, and use γ for the blowdown map), and define $\mathcal{E}_{t+1}^{t+1}$ to be the exceptional divisor of γ.

For each ℓ, $\ell = 1,\ldots, t$, the base change of $\mathcal{E}_\ell^t \hookrightarrow \mathcal{Y}^t$ to B^{t+1} gives a divisor $\mathcal{F}_\ell \hookrightarrow W$, a $\mathbb{P}^1$-fibration over B^{t+1}. For $\ell \neq i$, $\mathcal{F}_\ell$ is disjoint from $\sigma(B^{t+1})$, and is thus contained in the open set where γ^{-1} is an isomorphism. For $\ell \neq i$ we set $\mathcal{E}_\ell^{t+1} := \gamma^{-1}(\mathcal{F}_\ell) \subset \mathcal{Y}^{t+1}$. For $\ell = i$, $\sigma(B^{t+1})$ is a section of the $\mathbb{P}^1$-fibration $\mathcal{F}_i \longrightarrow B^{t+1}$. Since this section is a Cartier divisor in $\mathcal{F}_i$, blowing up along the section leaves $\mathcal{F}_i$ unchanged. We define $\mathcal{E}_i^{t+1}$ to be this blowup, i.e., to be the proper transform of $\mathcal{F}_i$ in $\mathcal{Y}^{t+1}$. Equivalently, $\mathcal{E}_i^{t+1}$ is the divisor

$\gamma^*(\mathcal{F}_i)(-\mathcal{E}_{t+1}^{t+1})$. We note that for each $\ell = 1, \ldots, t$, $\mathcal{E}_\ell^{t+1}$, is isomorphic to $\mathcal{F}_\ell$ as a scheme over B^{t+1}.

Finally we define $\Pi^{t+1}\colon B^{t+1} \times X$ as the composition of the blowdown map γ with the morphism $W \longrightarrow B^{t+1} \times X$ obtained as the base change of $\Pi^t\colon \mathcal{Y}^t \longrightarrow B^t$ to B^{t+1}, and define $\varphi^{t+1}\colon B^{t+1} \longrightarrow X$ as the composition of the surjective map $B^{t+1} \longrightarrow B^t$ with $\varphi^t\colon B^t \longrightarrow X$.

This completes the inductive step of the construction in case (i).

In case (ii): The intersection $\mathcal{E}_i^t \cap \mathcal{E}_j^t$ is a section of $\mathcal{Y}^t \longrightarrow B^t$, which we again call σ. We set $B^{t+1} := B^t$, define $\mathcal{Y}^{t+1}$ to be the blowup of $\mathcal{Y}^t$ along σ, and define $\mathcal{E}_{t+1}^{t+1}$ to be the exceptional divisor of the blowup. We define $\Pi^{t+1}\colon \mathcal{Y}^{t+1} \longrightarrow B^{t+1} \times X$ to be the composition of the blowdown map $\gamma\colon \mathcal{Y}^{t+1} \longrightarrow \mathcal{Y}^t$ with Π^t, and set $\varphi^{t+1} = \varphi^t$ (a map $B^{t+1} = B^t \longrightarrow X$).

For $\ell \notin \{i, j\}$, $\mathcal{E}_\ell^t \subset \mathcal{Y}^t$ is disjoint from the section blown up, and we set $\mathcal{E}_\ell^{t+1} := \gamma^{-1}(\mathcal{E}_\ell^t)$. For $\ell \in \{i, j\}$ the section is, as in case (a), a section of the $\mathbb{P}^1$-fibration $\mathcal{E}_\ell^t \longrightarrow B^t$, and a Cartier divisor in $\mathcal{E}_\ell^t$. We set $\mathcal{E}_\ell^{t+1}$ to be the proper transform of $\mathcal{E}_\ell^t$ in $\mathcal{Y}^{t+1}$. As before, for $\ell = 1, \ldots, t$, each $\mathcal{E}_\ell^{t+1}$ is isomorphic, as a scheme over B^{t+1}, to the base change of $\mathcal{E}_\ell^t$ to B^{t+1}.

This completes the inductive step of the construction in case (ii), and thus the inductive step overall.

We define $\mathcal{Y}$, B, Π, ψ, φ and $\mathcal{E}_1, \ldots, \mathcal{E}_s$ to be the end result of the inductive steps (i.e., the result on the s-th step).

The properties (a)–(h) are deduced fairly easily by following the steps of the inductive construction.

(a) As part of the construction we have a sequence of maps

$$B = B^s \xrightarrow{\tau_s} B^{s-1} \xrightarrow{\tau_{s-1}} \cdots \xrightarrow{\tau_3} B^2 \xrightarrow{\tau_2} B^1 = X,$$

where we use τ_t for the map $\tau_t\colon B^t \longrightarrow B^{t-1}$. In both cases (i) and (ii), $\mathcal{Y}^{t+1}$ and $\mathcal{Y}^t$ are related by the diagram

$$\begin{tikzcd} \mathcal{Y}^{t+1} \arrow[r, "\gamma"] \arrow[d, "\psi^{t+1}"] & \mathcal{Y}^t \times_{B^t} B^{t+1} \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & \mathcal{Y}^t \arrow[d, "\psi^t"] \\ B^{t+1} \arrow[r, equal] & B^{t+1} \arrow[r, "\tau_{t+1}"] \arrow[u, bend left, "\sigma"] & B^t \end{tikzcd} \tag{8.2.1}$$

where γ is the blowup of $\mathcal{Y}^t \times_{B^t} B^{t+1}$ along the section σ. (We omit calling them γ_{t+1} and σ_{t+1} to slightly simplify notation.)

Setting $\mathcal{Y}^0$ to be the trivial family $\mathcal{Y}^0 := X \times X \xrightarrow{pr_1} X =: B^0$, τ_1 to be the identity map, and σ the diagonal section, then (8.2.1) with $t = 0$ also describes the base case of the construction.

Each ψ^{t+1} is thus obtained from ψ^t by base change and then blowing up along a section. If ψ^t is smooth and proper of relative dimension 2 we conclude that the same is therefore true of ψ^{t+1}. Since X is smooth and proper the hypothesis is valid for $\psi^0 = pr_1$, and so by induction for all ψ^t.

(b) In step $t+1$ (including the base case, $t = 0$), $\mathcal{E}^{t+1}_{t+1}$ is the exceptional divisor of the blowup γ in (8.2.1). Thus, by (8.2.1) and the inductive argument in (a), γ is the blowup of a family of smooth surfaces along a section and therefore $\mathcal{E}^{t+1}_{t+1}$ is a $\mathbb{P}^1$ bundle over B^{t+1}. On the other hand, as remarked in the construction, for each $\ell = 1, \ldots, t$, $\mathcal{E}^{t+1}_{\ell}$ is isomorphic, as a scheme over B^{t+1}, to the base change of $\mathcal{E}^t_{\ell}$ to B^{t+1}. Since (inductively) each $\mathcal{E}^t_{\ell}$ is a $\mathbb{P}^1$-bundle over B^t, each $\mathcal{E}^{t+1}_{\ell}$ is a $\mathbb{P}^1$-bundle over B^{t+1}.

(c) The map φ is the composition $\varphi = \tau_2 \circ \tau_3 \circ \cdots \circ \tau_s$. In case (ii) τ_{t+1} is the identity map, while in case (i) τ_{t+1} is a fibration, whose fibres (by (b)) are $\mathbb{P}^1$'s minus a finite number of points. Since each of the τ_{t+1} is a smooth surjective morphism, so is φ.

Each B^{t+1} is either equal to B^t (case (ii)) or an open subset of a $\mathbb{P}^1$-bundle over B^t (case (i)). Starting with $B^1 = X$ smooth and irreducible, we conclude that each B^t, and hence B, is also smooth and irreducible. (The fact that B is smooth also follows from fact that φ is a smooth morphism, and X a smooth variety.)

(d) Given $\overline{q} \in B$, let $\overline{q}^t$ be the image of $\overline{q}$ in B^t under the composition $\tau_{t+1} \circ \tau_{t+2} \circ \cdots \circ \tau_s$. Taking the fibre product of (8.2.1) over $\{\overline{q}^{t+1}\} = \{\overline{q}^{t+1}\} \xrightarrow{\sim} \{\overline{q}^t\}$ (sitting inside $B^{t+1} = B^{t+1} \longrightarrow B^t$), we obtain

$$\mathcal{Y}^{t+1}_{\overline{q}^{t+1}} \longrightarrow \mathcal{Y}^t_{\overline{q}^t} = \mathcal{Y}^t_{\overline{q}^t},$$

expressing $\mathcal{Y}^{t+1}_{\overline{q}^{t+1}}$ as the blow up of $\mathcal{Y}^t_{\overline{q}^t}$ at the point $\sigma(\overline{q}^{t+1})$.

Starting with $q = \varphi(\overline{q})$ in X, $\mathcal{Y}^1_{\overline{q}^1}$ is the blow up of $\mathcal{Y}^0_q = X$ at the point q (the section in this case being the diagonal map). For subsequent blowups, in case (i), where the step is "blow up along E_i", the fibre of τ_{t+1} over $\overline{q}^t$ is the fibre of $\mathcal{E}^t_i \setminus \cup_{j\neq i}\mathcal{E}^t_j$ over $\overline{q}^t$, i.e., points of the exceptional divisor E_i in $\mathcal{Y}^t_{\overline{q}^t}$ which do not lie on any other exceptional divisor E_j. Given a point $\overline{q}^{t+1}$ on this fibre, $\sigma(\overline{q}^{t+1})$ is that point in $\mathcal{Y}^t_{\overline{q}^t}$, so that $\mathcal{Y}^{t+1}_{\overline{q}^{t+1}}$ is the blow up of $\mathcal{Y}^t_{\overline{q}^t}$ at a point on E_i, but not on any other exceptional divisor.

In case (ii), where the step is "blow up on the intersection of E_i and E_j, the map τ_{t+1} is the identity map (so $\overline{q}^{t+1} = \overline{q}^t$), and $\sigma(\overline{q}^{t+1})$ is the point of $\mathcal{Y}^t_{\overline{q}^t}$ on the intersection of E_i and E_j. Thus $\mathcal{Y}^{t+1}_{\overline{q}^{t+1}}$ is the blow up of $\mathcal{Y}^t_{\overline{q}^t}$ at the intersection of E_i and E_j.

Continuing to $\mathcal{Y}_{\overline{q}} = \mathcal{Y}^{s}_{\overline{q}^s}$, we conclude that $\mathcal{Y}_{\overline{q}}$ is obtained from X by blowing up at $q = \varphi(\overline{q})$, and continuing in the pattern dictated by the discrete instructions.

(e) The above argument is reversible. If Y is a surface obtained from X by blowing up at a point q, and then continuing following the given pattern of the discrete instructions, each step of the blowing up process tells us how to pick a point $\overline{q}^{t+1}$ in the fibre of τ_{t+1} over the previously chosen $\overline{q}^t$. By the description above, each $\mathcal{Y}^{t+1}_{\overline{q}^{t+1}}$ is then the surface obtained by following those instructions up to the $(t+1)$-st step. Setting $\overline{q} = \overline{q}^s$, we conclude that $Y = \mathcal{Y}_{\overline{q}}$ and that the blowdown map is $\pi_{\overline{q}}$.

(f) The fibre product of varieties defined over k (via morphisms defined over k) is again defined over k, and the blow up of a variety defined over k along a subvariety defined over k is again defined over k. The varieties $\mathcal{Y}$, B, the $\mathcal{E}_i$, and the resulting morphisms are all constructed iteratively from X following the essentially combinatorial instructions in steps (i) or (ii) describing which fibre products or blowups to make. It follows that if X is defined over k, so are the varieties and maps produced by the proposition.

(g) As remarked in (c), each $\tau^{t+1} \colon B^{t+1} \longrightarrow B^t$ is either the identity map (case (ii)) or a fibration which is an open subset of a $\mathbb{P}^1$ bundle, i.e., a fibration whose fibres are each $\mathbb{P}^1$ minus finitely many points (case (i)). It follows immediately that if the k-points of X are Zariski dense, then so are the k-points of B, and that the induced map $B(k) \longrightarrow X(k)$ is surjective.

(h) Using the factorization $\psi^t = pr_1 \circ \Pi^t$ we can insert the maps Π^t and Π^{t+1} into (8.2.1), the result being the diagram below.

$$\begin{tikzcd}
\mathcal{Y}^{t+1} \arrow[r, "\gamma"] \arrow[d, "\Pi^{t+1}"] & \mathcal{Y}^t \times_{B^t} B^{t+1} \arrow[r, "\alpha"] \arrow[d, "\beta"] \arrow[dr, phantom, "\square"] & \mathcal{Y}^t \arrow[d, "\Pi^t"] \\
B^{t+1} \times X \arrow[r, equal] \arrow[d, "pr_1"] & B^{t+1} \times X \arrow[r, "\theta"] \arrow[d, "pr_1"] \arrow[dr, phantom, "\square"] & B^t \times X \arrow[d, "pr_1"] \\
B^{t+1} \arrow[r, equal] & B^{t+1} \arrow[r, "\tau_{t+1}"] \arrow[uu, bend left, "\sigma"] & B^t
\end{tikzcd} \tag{8.2.2}$$

For use in the argument we have introduced names α, β, θ for some of the maps in the diagram. (These maps already have longer names, for instance $\theta = \tau_{t+1} \times \mathrm{Id}_X$, and β is the base change of Π^t via θ.) As in (a) we omit subscripts from these maps to slightly simplify notation.

If Π^t is proper, then its base change β is proper, and therefore so is the composition $\Pi^{t+1} = \beta \circ \gamma$, since γ, being a blowup, is proper. Starting with the (proper) identity map $\Pi^0 \colon X \times X \longrightarrow X \times X$, we conclude by induction that all the Π^t are proper, and hence so is $\Pi = \Pi^s$.

For each t, $t = 1, \ldots, s$, let $\varphi^t \colon B^t \longrightarrow X$ be the map $\varphi^t := \tau_1 \circ \tau_2 \circ \cdots \circ \tau_t$, so that $\varphi = \varphi^s$. Here, as in (a), we use τ_1 for the identity map from B^1 to $B^0 = X$. We will prove by induction that each Π^t is an isomorphism over the complement of Γ_{φ^t}, where again $\Gamma_{\varphi^t} \subset B^t \times X$ denotes the graph of φ^t.

By definition $\Pi^1 \colon \mathcal{Y}^1 \longrightarrow B^1 \times X = X \times X$ is the blowup along the diagonal. Since $\varphi^1 = \tau_1$ is the identity map, $\Gamma_{\varphi^1} \subset B^1 \times X$ is the diagonal. Thus the claim holds in the base case.

To show the inductive step (that Π^{t+1} is an isomorphism over the complement of $\Gamma_{\varphi^{t+1}}$) it suffices to establish

(8.2.3) $$\theta^{-1}(\Gamma_{\varphi^t}) = \Gamma_{\varphi^{t+1}}$$

and

(8.2.4) $$\operatorname{Im}(\beta \circ \sigma) \subseteq \Gamma_{\varphi^{t+1}}.$$

To see why these are sufficient, we note that since $\Pi^{t+1} = \beta \circ \gamma$, it is enough to prove that each of β and γ is an isomorphism over the complement of $\Gamma_{\varphi^{t+1}}$.

By the inductive hypothesis Π^t is an isomorphism over the complement of Γ_{φ^t}; hence by the top right fibre square in (8.2.2), β is an isomorphism over the complement of

$$\theta^{-1}(\Gamma_{\varphi^t}) \overset{(8.2.3)}{=} \Gamma_{\varphi^{t+1}}.$$

On the other hand, by definition γ is the blowup along $\sigma(B^{t+1})$, and so γ is an isomorphism over the complement of $\beta(\sigma(B^{t+1}))$, which, by (8.2.4), contains the complement of $\Gamma_{\varphi^{t+1}}$.

To show (8.2.3) we recall that $\theta = \tau_{t+1} \times \mathrm{Id}_X$. Starting with

$$\Gamma_{\varphi^t} = \big\{(b, x) \mid x = \varphi^t(b)\big\} \subset B^t \times X,$$

we therefore have

$$\theta^{-1}(\Gamma_{\varphi^t}) = \big\{(b', x) \mid x = \varphi^t(\tau_{t+1}(b'))\big\} \subset B^{t+1} \times X.$$

Since $\varphi^{t+1} = \varphi^t \circ \tau_{t+1}$, we conclude that $\theta^{-1}(\Gamma_{\varphi^t}) = \Gamma_{\varphi^{t+1}}$.

We will establish (8.2.4) through a series of reductions. By (8.2.3) to show (8.2.4) it is equivalent to show that $\operatorname{Im}(\theta \circ \beta \circ \sigma) \subseteq \Gamma_{\varphi^t}$, which, by the commutativity of (8.2.2), is the same as showing that

$\mathrm{Im}(\Pi^t \circ \alpha \circ \sigma) \subseteq \Gamma_{\varphi^t}$. By the inductive hypothesis (that Π^t is an isomorphism over the complement of Γ_{φ^t}) this last statement is equivalent to showing that $\mathrm{Im}(\alpha \circ \sigma)$ is contained in the locus where Π^t is not an isomorphism.

In each of cases (i) and (ii) the section σ was constructed by giving a map $B^{t+1} \longrightarrow \mathcal{Y}^t$, and of course $\alpha \circ \sigma$ is this map. In case (i) B^{t+1} is an open subset of $\mathcal{E}_i^t$ for some i, and $\alpha \circ \sigma$ is the composition of inclusions $B^{t+1} \hookrightarrow \mathcal{E}_i^t \hookrightarrow \mathcal{Y}^t$. In case (ii), $B^{t+1} = B^t$, and there are i and j such that the map $\alpha \circ \sigma$ is given by sending a point of B^{t+1} to the unique point on the intersection of $\mathcal{E}_i^t \cap \mathcal{E}_j^t$ above it. Thus in either case, there is an ℓ such that $\mathrm{Im}(\alpha \circ \sigma) \subseteq \mathcal{E}_\ell^t$, and so to establish (8.2.4) we are reduced to proving that Π^t is not a local isomorphism at any point of $\mathcal{E}_\ell^t \subset \mathcal{Y}^t$, for any $\ell = 1, \ldots, t$.

Let y be any point of $\mathcal{E}_\ell^t$, and set $\overline{q}^t := \psi^t(y) \in B^t$. To show that Π^t is not an isomorphism at y, it suffices to show that the map $\pi_{\overline{q}^t}^t : \mathcal{Y}_{\overline{q}^t}^t \longrightarrow X$ obtained by restricting Π^t over $\overline{q}^t$ is not an isomorphism at $y \in \mathcal{Y}_{\overline{q}^t}^t$.

Set $\overline{q}^{t-1} := \tau_t(\overline{q}^t)$. Taking the fibre product of (8.2.2) (shifted to be a diagram relating $\mathcal{Y}^t$ and $\mathcal{Y}^{t-1}$, instead of $\mathcal{Y}^{t+1}$ and $\mathcal{Y}^t$) over $\{\overline{q}^t\} = \{\overline{q}^t\} \xrightarrow{\sim} \{\overline{q}^{t-1}\}$ we obtain, similarly to the argument in (d), a commutative diagram

$$\begin{tikzcd}
\mathcal{Y}_{\overline{q}^t}^t \arrow[r] \arrow[d, "\pi_{\overline{q}^t}^t"] & \mathcal{Y}_{\overline{q}^{t-1}}^{t-1} \arrow[r, equal] \arrow[d, "\pi_{\overline{q}^{t-1}}^{t-1}"] & \mathcal{Y}_{\overline{q}^{t-1}}^{t-1} \arrow[d, "\pi_{\overline{q}^{t-1}}^{t-1}"] \\
X \arrow[r, equal] & X \arrow[r, equal] & X
\end{tikzcd}$$

Since this is a commutative diagram of smooth surfaces, to show that $\pi_{\overline{q}^t}^t$ is not an isomorphism at y, it suffices to show that $\pi_{\overline{q}^{t-1}}^{t-1}$ is not an isomorphism at the image of y in $\mathcal{Y}_{\overline{q}^{t-1}}^{t-1}$. Continuing, we are reduced to showing that $\pi_q^1 : \mathcal{Y}_q^1 \longrightarrow X$ is not an isomorphism at the image of y in $\mathcal{Y}_q^1$, where $q = \varphi^t(\overline{q}^t)$.

Under the map $\mathcal{Y}^t \longrightarrow \mathcal{Y}^{t-1}$, if $\ell < t$ then $\mathcal{E}_\ell^t \subset \mathcal{Y}^t$ is sent to $\mathcal{E}_\ell^{t-1} \subset \mathcal{Y}^{t-1}$, while, if $\ell = t$, then there is some $i \leq t-1$ such that the image of $\mathcal{E}_t^t$ is contained in $\mathcal{E}_i^{t-1}$. Iterating this argument, we conclude that the image of y in $\mathcal{Y}_q^1$ lies in the restriction of $\mathcal{E}_1^1$ to $\mathcal{Y}_q^1$. This restriction is the exceptional divisor of π_q^1, the blowup of X at q, which is precisely the locus where π_q^1 is not an isomorphism.

This establishes that Π^t is not an isomorphism at any point of $\mathcal{E}_\ell^t$, and completes the proofs of (8.2.4) and (h).

Note: Since $\beta \circ \sigma$ is a section of pr_1 (i.e., $pr_1 \circ \beta \circ \sigma = \mathrm{Id}_{B^{t+1}}$), it follows that $\mathrm{Im}(\beta \circ \sigma)$ is the graph of a morphism $B^{t+1} \longrightarrow X$, necessarily the morphism $pr_2 \circ \beta \circ \sigma$. The inclusion (8.2.4) therefore implies the equality $\mathrm{Im}(\beta\circ\sigma) = \Gamma_{\varphi^{t+1}}$, or equivalently, that $pr_2\circ\beta\circ\sigma = \varphi^{t+1}$.

This finishes the proof of Proposition 8.2. □

We now construct the parameter space for general n. This is essentially done by taking fibre products over the constructions for $n = 1$.

Proposition 8.3 (Existence and properties of the parameter space, general n)**.** *Let X be a smooth projective surface. Given $n \geq 1$, suppose that for each j, $j = 1, \ldots, n$, we are given discrete data with s_j steps describing the pattern of blowups over a single point. Let $V \subset X^n$ be the open subset which is the complement of all the pairwise diagonals in X^n, so that*

$$V = \left\{ (q_1, q_2, \ldots, q_n) \in X^n \;\middle|\; q_i \neq q_j \text{ if } i \neq j \right\}.$$

Then there exists:

- *A quasi projective variety B and morphism $\varphi\colon B \longrightarrow V$.*

B is the parameter space for "choices of blowup starting with n distinct points $q_1, \ldots, q_n$, with the blowup over q_j following the j-th given discrete data". For $b \in B$, $\varphi(b) = (q_1, q_2, \ldots, q_n)$ is the ordered set of points where we first blow up.

- *A quasi projective variety $\mathcal{Y}$ with morphism $\Pi\colon \mathcal{Y} \longrightarrow B$.*

Setting $\psi := pr_1 \circ \Pi$, the morphism $\psi\colon \mathcal{Y} \longrightarrow B$ is the universal family of blown up surfaces. For a point $b \in B$, we denote by $\mathcal{Y}_b := \psi^{-1}(b)$ the fibre of $\mathcal{Y}$ over b, and $\pi_b\colon \mathcal{Y}_b \longrightarrow X$ the restriction of Π over b.

- *For each $j = 1, \ldots, n$, and $i = 1, \ldots, s_j$, a closed subscheme $\mathcal{E}_{i,j} \subset \mathcal{Y}$.*

The $\mathcal{E}_{i,j}$ are the relative families of exceptional divisors. The varieties and maps satisfy the following.

(a) *ψ is a smooth proper morphism of relative dimension 2.*
(b) *Via ψ, each $\mathcal{E}_{i,j}$ is a $\mathbb{P}^1$-bundle over B.*
(c) *φ is a smooth surjective morphism, and B is smooth and irreducible.*
(d) *For each $b \in B$, the morphism $\pi_b\colon \mathcal{Y}_b \longrightarrow X$ is a blow up of X starting at the points $q_1, \ldots, q_n$ given by $\varphi(b) = (q_1, \ldots, q_n)$, with further blowups over each q_j following the pattern described by j-th set of given discrete data.*

(e) *Conversely, given a birational map $\pi\colon Y \longrightarrow X$ which is obtained by blowing up X at points $q_1,\dots, q_n$, and then continuing over each q_j in the pattern described by the j-th set of discrete data, there exists $b \in B$ such that $\mathcal{Y}_b = Y$ and $\pi_b = \pi$ as objects and morphisms over X.*

(f) *If X is defined over k, then B, $\mathcal{Y}$, the $\mathcal{E}_{i,j}$, the morphisms Π, ψ, φ, and the inclusion morphisms $\mathcal{E}_{i,j} \hookrightarrow \mathcal{Y}$ are defined over k.*

(g) *If, in addition, the k-points of X are Zariski dense, then the k-points of B and V are Zariski dense, and the map $B(k) \longrightarrow V(k)$ induced by φ is surjective.*

Proof. For each j, $j = 1,\dots, n$, applying Proposition 8.2 to the j-th set of discrete data we obtain: a parameter space B_j; a map $\varphi_j\colon B_j \longrightarrow X$; a family $\psi_j\colon \mathcal{Y}_j \longrightarrow B_j$; and a universal blowdown map $\Pi_j\colon \mathcal{Y}_j \longrightarrow B_j \times X$. We define B as the fibre product of $B_1 \times \cdots \times B_n$ and V over X^n:

$$\begin{tikzcd} B \arrow[r, hook] \arrow[d, "\varphi"] \arrow[dr, phantom, "\square"] & B_1 \times B_2 \times \cdots \times B_n \arrow[d, "\varphi_1\times\cdots\times\varphi_n"] \\ V \arrow[r, hook] & X^n \end{tikzcd} \tag{8.3.1}$$

We label by φ the map $B \longrightarrow V$ in the fibre product above, and note for use below that since V is an open subset of X^n, B is an open subset of $B_1 \times\cdots\times B_n$. For a point $b \in B$, we identify it with its image in $B_1 \times\cdots\times B_n$, writing $b = (\overline{q}_1, \dots, \overline{q}_n)$. Thus points of B parameterize instructions $\overline{q}_1,\dots, \overline{q}_n$ for blowing up X at $q_j = \varphi_j(\overline{q}_j)$, $j = 1, \dots, n$ (with $q_i \neq q_j$ if $i \neq j$) and continuing to blow up over each q_j as specified by the j-th set of discrete data.

In order to construct the universal family $\mathcal{Y}$ over B, we take advantage of the fact that if $Y_1 \longrightarrow X$ and $Y_2 \longrightarrow X$ are two blowups over different points of X, then the fibre product $Y_1 \times_X Y_2$ is the simultaneous blow up of X at both points. For each j, $j = 1,\dots, n$, let $\mathcal{Y}_j^B$ denote the base change of $\mathcal{Y}_j$ to B, pulling back via

$$B \hookrightarrow B_1 \times \cdots \times B_n \xrightarrow{pr_j} B_j,$$

where pr_j again denotes projection onto the j-th factor. By pulling back the morphism Π_j, we get morphisms $\Pi_j^B\colon \mathcal{Y}_j^B \longrightarrow B \times X$ for each j. We set $\mathcal{Y}$ to be the fibre product of the $\mathcal{Y}_j^B$ over $B \times X$:

$$\mathcal{Y} := \mathcal{Y}_1^B \times_{B\times X} \mathcal{Y}_2^B \times_{B\times X} \cdots \times_{B\times X} \mathcal{Y}_n^B. \tag{8.3.2}$$

Thus, the fibre of $\mathcal{Y}$ over a point $(\overline{q}_1, \dots, \overline{q}_n) \in B$ is

$$\mathcal{Y}_{(\overline{q}_1,\dots,\overline{q}_n)} = \mathcal{Y}_{1,\overline{q}_1} \times_X \mathcal{Y}_{2,\overline{q}_2} \times_X \cdots \times_X \mathcal{Y}_{n,\overline{q}_n}. \tag{8.3.3}$$

Since the points $q_1, \dots, q_n$ are distinct, this fibre product is the simultaneous blowup of X over $q_1, \dots, q_n$, with further blowups over each q_i as specified by $\overline{q}_i$.

We define $\Pi\colon \mathcal{Y} \longrightarrow B \times X$ to be the natural map from $\mathcal{Y}$ to $B \times X$ induced by (8.3.2) and set $\psi\colon \mathcal{Y} \longrightarrow B$ to be the composition of Π and projection onto the first factor. Finally, we define the exceptional divisors $\mathcal{E}_{i,j}$ by first pulling back each such divisor in $\mathcal{Y}_j$ to $\mathcal{Y}_j^B$ to produce a divisor $\mathcal{E}_{i,j}^B$ in $\mathcal{Y}_j^B$, and then pulling back to $\mathcal{Y}$.

The properties (a)–(g) follow readily from the corresponding properties in Proposition 8.2. Specifically:

(a) By Proposition 8.2(h) each of the maps $\Pi_j\colon \mathcal{Y}_j \longrightarrow B_j \times X$ is proper, and therefore each of the morphisms $\Pi_j^B\colon \mathcal{Y}_j^B \longrightarrow B \times X$, obtained as the base change of Π_j to B, is also proper. From the definition (8.3.2) of $\mathcal{Y}$ as a fibre product, we conclude that $\Pi\colon \mathcal{Y} \longrightarrow B \times X$ is proper. Since X is projective, $pr_1\colon B \times X \longrightarrow B$ is proper, and therefore $\psi = pr_1 \circ \Pi$ is proper.

We next show that ψ is smooth of relative dimension 2, which may be checked locally on $\mathcal{Y}$. Let $\varphi_j^B\colon B \longrightarrow X$ be the pullback φ_j to B (so $\varphi_j^B = pr_j \circ \varphi$). By Proposition 8.2(h) each of the Π_j is an isomorphism over the complement of $\Gamma_{\varphi_j} \subset B_j \times X$, and, pulling back, we conclude that each of the maps Π_j^B is an isomorphism over the complement of $\Gamma_{\varphi_j^B} \subset B \times X$, with Γ again denoting the graph of a morphism.

For a point $b \in B$, by the definition of V ("$q_i \neq q_j$") we have that $\varphi_i^B(b) \neq \varphi_j^B(b)$ if $i \neq j$, i.e., each of the graphs $\Gamma_{\varphi_j^B}$, $j = 1, \dots, n$, is disjoint. Thus, a point $z \in B \times X$ lies on at most one, and possibly none, of the $\Gamma_{\varphi_j^B}$.

Given $y \in \mathcal{Y}$, set $z = \Pi(y) \in B \times X$. By the previous remark, there exists j so that for all $i \neq j$, $z \notin \Gamma_{\varphi_i^B}$ (and possibly even when $i = j$). Thus, over a neighbourhood U of z, each of the maps $\Pi_i^B\colon \mathcal{Y}_i^B \longrightarrow B \times X$ is an isomorphism when $i \neq j$. By the definition (8.3.2) of $\mathcal{Y}$ as a fibre product, we conclude that $(\mathcal{Y}, \Pi)$ is isomorphic to $(\mathcal{Y}_j^B, \Pi_j^B)$ as objects and morphisms over $U \subset B \times X$. Via this isomorphism, $\psi = pr_1 \circ \Pi$ is equal to $pr_1 \circ \Pi_j^B$ near y.

Let $\psi_j^B \colon \mathcal{Y}_j^B \longrightarrow B$ be the base change of $\psi_j \colon \mathcal{Y}_j \longrightarrow B_j$ from B_j to B. The base change diagram

$$\begin{tikzcd}
\mathcal{Y}_j^B \arrow[r] \arrow[d, "\Pi_j^B"] \arrow[dd, bend right=60, "\psi_j^B"'] \arrow[dr, phantom, "\square"] & \mathcal{Y}_j \arrow[d, "\Pi_j"] \arrow[dd, bend left=60, "\psi_j"] \\
B \times X \arrow[r] \arrow[d, "pr_1"] \arrow[dr, phantom, "\square"] & B_j \times X \arrow[d, "pr_1"] \\
B \arrow[r] & B_j
\end{tikzcd}$$

shows that $pr_1 \circ \Pi_j^B = \psi_j^B$. By Proposition 8.2(a) ψ_j is smooth of relative dimension 2, and hence the same is true for ψ_j^B. By the local equality $\psi = pr_1 \circ \Pi_j^B$ above, we conclude that ψ is smooth of relative dimension 2 near y. Since $y \in \mathcal{Y}$ was arbitrary, this finishes the proof of (a).

(b) By Proposition 8.2(b), each of the subschemes produced in the case $n = 1$ is a $\mathbb{P}^1$-bundle over the base B_j. The $\mathcal{E}_{i,j}$ are obtained by pulling these families back to B, and are therefore also $\mathbb{P}^1$-bundles over B.

(c) By Proposition 8.2(c), each φ_j is smooth and surjective; hence the product map $\varphi_1 \times \cdots \times \varphi_n$ is smooth and surjective. Thus, φ, being the base change of this map, is also smooth and surjective. Again by Proposition 8.2(c), each of the B_i is smooth and irreducible, and hence the product $B_1 \times \cdots \times B_n$ is smooth and irreducible. Since B is an open subset of this product, B is also smooth and irreducible.

(d) For a point $b = (\overline{q}_1, \ldots, \overline{q}^n) \in B$, by (8.3.3) we have that $\mathcal{Y}_b$ is the fibre product of $\mathcal{Y}_{1,\overline{q}_1}$ through $\mathcal{Y}_{n,\overline{q}_n}$ over X. By Proposition 8.2(d), each of the $\mathcal{Y}_{j,\overline{q}_j}$ is obtained from X by blowing up at $\varphi_j(\overline{q}_j) = q_j$, and then continuing following the pattern of the j-th set of discrete data. Since the $\varphi_j(\overline{q}_j)$, $j = 1, \ldots, n$, are disjoint, the fibre product $\mathcal{Y}_b$ is the surface obtained by performing all of these blowups simultaneously.

(e) Given such a surface Y, by Proposition 8.2(e) for each $j = 1, \ldots, n$ there is a point $\overline{q}_j$ in B_j such that $\mathcal{Y}_{j,\overline{q}_j}$ is the blowup of X starting at q_j and continuing in the same way as Y was blown up over q_j. Setting $b = (\overline{q}_1, \ldots, \overline{q}_n)$ it follows from (8.3.3) that $Y = \mathcal{Y}_b$ and $\pi_b = \pi$.

(f) By Proposition 8.2(f), if X is defined over k, then so are all the B_j, and therefore so is the product $B_1 \times \cdots \times B_n$. The open set $V \subset X^n$ is defined over k, and therefore so is B (being given by the fibre product (8.3.1)). By Proposition 8.2(f) again, each of the $\mathcal{Y}_j$ is defined over k, as are the maps to $B_j \times X$. Pulling these back to B, it follows that the $\mathcal{Y}_j^B$ and maps $\mathcal{Y}_j^B \longrightarrow B \times X$ are also defined over k, and therefore $\mathcal{Y}$ and Π, being defined by the fibre product (8.3.2), are also defined over k. Finally, invoking Proposition 8.2(f) one more time, each of the families of exceptional divisors in each $\mathcal{Y}_j$ is defined over k, and therefore their pullbacks, the $\mathcal{E}_{i,j}$, are also defined over k.

(g) If the k-points of X are Zariski dense, then by Proposition 8.2(g), the k-points of each B_j are Zariski dense, and therefore the k-points of B (being an open subset of $B_1 \times \cdots \times B_n$) are also Zariski dense. Furthermore, again by Proposition 8.2(g), each of the maps $B_j(k) \longrightarrow X(k)$ is surjective, so that the map $B_1(k) \times B_2(k) \times \cdots \times B_n(k) \longrightarrow X^n(k)$ is surjective. By the definition of B as the fibre product (8.3.1), it follows that $B(k) \longrightarrow V(k)$ is surjective.

This completes the proof of Proposition 8.3. $\square$

We record two further results we will need before proceeding to the proof of Theorem 8.1.

Lemma 8.4. *Let Y be a smooth projective surface, $g\colon Y \longrightarrow \mathbb{P}^2$ a birational morphism, and set $L = g^*\mathcal{O}_{\mathbb{P}^2}(1)$. Then $c_1(L)^2 = 1$, and $H^i(Y, L) = H^i(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(1))$ for all $i \geq 0$.*

Proof. Since g is birational, $\deg(g) = 1$, and therefore $c_1(L)^2 = c_1(\mathcal{O}_{\mathbb{P}^2}(1))^2 = 1$. By the projection formula we have

$$R^i g_* L = R^i g_*(g^*\mathcal{O}_{\mathbb{P}^2}(1)) = (R^i g_* \mathcal{O}_Y) \otimes_{\mathcal{O}_{\mathbb{P}^2}} \mathcal{O}_{\mathbb{P}^2}(1)$$

for all $i \geq 0$. On the other hand, since g is a proper birational map between smooth varieties, we also have $g_*\mathcal{O}_Y = \mathcal{O}_{\mathbb{P}^2}$ and $R^i g_* \mathcal{O}_{\mathbb{P}^2} = 0$ for $i \geq 1$. Together these give $g_* L = \mathcal{O}_{\mathbb{P}^2}(1)$ and $R^i g_* L = 0$ for $i \geq 1$. The Leray spectral sequence for computing $H^{\bullet}(Y, L)$ thus degenerates immediately, giving $H^i(Y, L) = H^i(\mathbb{P}^2, g_* L) = H^i(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(1))$ for all $i \geq 0$. $\square$

Lemma 8.5. *Let $\psi_U\colon \mathcal{Y}_U \longrightarrow U$ be a flat proper morphism and $\mathcal{L}_U$ a line bundle on $\mathcal{Y}_U$. For a point $b \in U$ we use $\mathcal{Y}_b$ for the fibre $\mathcal{Y}_b := \psi_U^{-1}(b)$ and $\mathcal{L}_b$ for the restriction of $\mathcal{L}_U$ to $\mathcal{Y}_b$. We assume that there is an integer m so that $h^0(\mathcal{Y}_b, \mathcal{L}_b) = m$ for all $b \in U$, and that there is a $b_0 \in U$ such that $H^0(\mathcal{Y}_{b_0}, \mathcal{L}_{b_0})$ is base point free on $\mathcal{Y}_{b_0}$.*

Then there is a nonempty open subset $U' \subseteq U$ such that $H^0(\mathcal{Y}_b, \mathcal{L}_b)$ is base point free on $\mathcal{Y}_b$ for all $b \in U'$.

Proof. On each $\mathcal{Y}_b$ we have the natural evaluation homomorphism $H^0(\mathcal{Y}_b, L_b) \otimes \mathcal{O}_{\mathcal{Y}_b} \longrightarrow \mathcal{L}_b$, and $\mathcal{L}_b$ is basepoint free if and only if this map is surjective. This evaluation homomorphism globalizes to a map of bundles on $\mathcal{Y}_U$.

Since $h^0(\mathcal{Y}_b, \mathcal{L}_b) = m$ for all $b \in U$, by the theorem on cohomology and base change ([EGA III_2, Corollaire 7.9.10], [Ha, Theorem 12.9], or [Mu, Corollary

2, p. 50]) $\psi_{U*}(\mathcal{L}_U)$ is a vector bundle of rank m on U. The natural adjunction morphism $\psi_U^*\psi_{U*}(\mathcal{L}_U) \longrightarrow \mathcal{L}_U$ restricts on each fibre to be the evaluation morphism above (this again uses the theorem on cohomology and base change). Let $\mathcal{Q}$ be the cokernel of this map, so that we have the exact sequence of sheaves on $\mathcal{Y}_U$:

$$\psi_U^*\psi_{U*}(\mathcal{L}_U) \longrightarrow \mathcal{L}_U \longrightarrow \mathcal{Q} \longrightarrow 0.$$

By right exactness of the tensor product, restricting the sequence above to $\mathcal{Y}_b$ gives the exact sequence

$$H^0(\mathcal{Y}_b, \mathcal{L}_b) \otimes \mathcal{O}_{\mathcal{Y}_b} \longrightarrow \mathcal{L}_b \longrightarrow \mathcal{Q}|_{\mathcal{Y}_b} \longrightarrow 0.$$

Therefore $\mathcal{L}_b$ is basepoint free on $\mathcal{Y}_b$ if and only if $\mathcal{Q}|_{\mathcal{Y}_b} = 0$. Let $W \subseteq \mathcal{Y}_U$ be the support of $\mathcal{Q}$, and $Z = \psi_U(W)$. Then W is closed in $\mathcal{Y}_U$ since $\mathcal{Q}$ is a coherent sheaf, and Z is closed in U since ψ_U is proper. Thus $\mathcal{L}_b$ is basepoint free if and only if $b \in U' := U - Z$, and U' is nonempty since $b_0 \in U'$. □

Proof of Theorem 8.1. We start by proving (a).

Let $p_1, \ldots, p_n$ be the points of indeterminacy of f. By blowing up at each p_j, and then possibly further blowing up at points of the exceptional divisors over these points (and on points of further exceptional divisors) we obtain a birational morphism $\pi\colon Y \longrightarrow X$ resolving f. Let $g\colon Y \longrightarrow \mathbb{P}^2$ be the resulting birational morphism to $\mathbb{P}^2$, and set $L = g^*\mathcal{O}_{\mathbb{P}^2}(1)$. By Lemma 8.4 we have $c_1(L)^2 = 1$, $h^0(Y, L) = 3$, and $h^1(Y, L) = h^2(Y, L) = 0$.

For $j = 1, \ldots, n$, let s_j be the number of blowups over p_j (including the first blow up at p_j), and let $E_{i,j}$, $i = 1, \ldots, s_j$ be the exceptional divisors over p_j, listed in the order they appear when blowing up. The Picard group of Y is the direct sum $\pi^* \operatorname{Pic}(X) \bigoplus \left(\oplus_{j=1}^n \oplus_{i=1}^{s_j} \mathbb{Z}[E_{i,j}]\right)$, and therefore there are unique integers $r_{i,j}$ and a unique line bundle M on X such that $L = (\pi^*M)(\sum_{j=1}^n \sum_{i=1}^{s_j} r_{i,j} E_{i,j})$.

Let $\mathcal{Y}$, B, Π, ψ, and $\mathcal{E}_{i,j}$ ($j = 1, \ldots, n$, $i = 1, \ldots, s_j$) be the parameter spaces, maps, and divisors obtained by applying Proposition 8.3, with the j-th discrete data being that which describes the pattern of blowups over p_j used to resolve f above. By Proposition 8.3(e) there is a point $b_0 \in B$ such that $\mathcal{Y}_{b_0} = Y$ and $\pi_{b_0} = \pi$.

On $\mathcal{Y}$ we set $\mathcal{L} := (\Pi^* pr_2^* M)(\sum_{j=1}^n \sum_{i=1}^{s_j} r_{i,j}\mathcal{E}_{i,j})$ where $pr_2\colon B \times X \longrightarrow X$ is the second projection. For a point $b \in B$ we denote by $E_{i,j,b}$ and $\mathcal{L}_b$ the restrictions of $\mathcal{E}_{i,j}$ and $\mathcal{L}$ respectively to $\mathcal{Y}_b$. Thus $\mathcal{L}_b$ is the line bundle $(\pi_b^*M)(\sum_{j=1}^n \sum_{i=1}^{s_j} E_{i,j,b})$ on $\mathcal{Y}_b$, and for $b_0 \in B$ we have $\mathcal{L}_{b_0} = L$.

In this way we have constructed a smooth family of blowups of X, with a flat family of line bundles which specialize to Y and L. By flatness, for each $b \in B$ we have $c_1(\mathcal{L}_b)^2 = c_1(L)^2 = 1$ and $\chi(\mathcal{Y}_b, L_b) = \chi(Y, L) = 3$.

Since $h^1(\mathcal{Y}_{b_0}, \mathcal{L}_{b_0}) = h^1(Y, L) = 0$, and $h^2(\mathcal{Y}_{b_0}, \mathcal{L}_{b_0}) = h^2(Y, L) = 0$, by semicontinuity of the dimension of the cohomology groups on fibres ([Mu, Corollary, p. 50] or [Ha, Theorem 12.8]) there is a nonempty open set $U \subset B$ containing b_0 such that $h^1(\mathcal{Y}_b, L_b) = h^2(\mathcal{Y}_b, \mathcal{L}_b) = 0$ for all $b \in U$. Thus, for $b \in U$ we have $h^0(\mathcal{Y}_b, \mathcal{L}_b) = \chi(\mathcal{Y}_b, \mathcal{L}_b) = 3$.

Set $\mathcal{Y}_U = \psi^{-1}(U)$, with ψ_U denoting the restriction of ψ to $\mathcal{Y}_U$, and let $\mathcal{L}_U := \mathcal{L}|_{\mathcal{Y}_U}$. By Lemma 8.5 there is a nonempty open set $U' \subseteq U$ such that for each $b \in U'$ the three dimensional space of sections $H^0(\mathcal{Y}_b, \mathcal{L}_b)$ is basepoint free on $\mathcal{Y}_b$. This space of sections therefore induces a morphism $g_b \colon \mathcal{Y}_b \longrightarrow \mathbb{P}^2$, necessarily birational since $c_1(\mathcal{L}_b)^2 = 1$.

By Proposition 8.3(c) $\varphi\colon B \longrightarrow V$ is surjective and B irreducible, and hence the nonempty open set $U' \subseteq B$ dominates V. Therefore for a general $(q_1, \ldots, q_n) \in V \subset X^n$ there is a point $b \in U'$ with $\varphi(b) = (q_1, \ldots, q_n)$, and hence a birational map $f_I := g_b \circ \pi_b^{-1}$, $f_I \colon X \dashrightarrow \mathbb{P}^2$, with indeterminacy locus $I = \{q_1, \ldots, q_n\}$. This proves (a).

We now turn to the proof of (c), and the k-rationality of the constructions. If X is defined over k, then by Proposition 8.3(f), $\mathcal{Y}$, B, the morphisms Π and ψ, and the exceptional divisors $\mathcal{E}_{i,j}$ are also all defined over k. The only ingredient in the argument which is not automatically defined over k is the line bundle M. But if the initial birational map $f\colon X \dashrightarrow \mathbb{P}^2$ is defined over k, then so is the resolution $Y \longrightarrow \mathbb{P}^2$, and one concludes that M is then defined over k.[2] Therefore the line bundle $\mathcal{L}$ on $\mathcal{Y}$ used in the argument for (a) is also defined over k.

Proceeding with the argument, we arrive at a nonempty open set[3] $U' \subset B$ such that for all $b \in U'$ the resulting $\mathcal{Y}_b$ has a birational morphism to $\mathbb{P}^2$, and thus that X has a birational map to $\mathbb{P}^2$ with indeterminacy locus $(q_1, \ldots, q_n) := \varphi(b) \in V$.

If $b \in U'(k)$ then the surface $\mathcal{Y}_b$, and, from above, the line bundle $\mathcal{L}_b$ are defined over k, and thus so are the resulting birational morphism $\mathcal{Y}_b \longrightarrow \mathbb{P}^2$, and birational map $X \dashrightarrow \mathbb{P}^2$.

To prove (c), it is therefore sufficient to show that the image of $U'(k)$ in V is Zariski-dense. By Proposition 8.3(g) we have that $B(k)$ is Zariski-dense in B. Since U' is a nonempty open subset of B, we conclude that the $U'(k)$ is Zariski-dense in B, and since $\varphi\colon B \longrightarrow V$ is a surjective map between irreducible varieties, that $\varphi(U'(k))$ is Zariski-dense in V.

[2]The individual divisors $E_{i,j}$ on Y need not be defined over k, but if p_j and $p_{j'}$ in X are in the same $\mathrm{Gal}(\overline{k}/k)$ orbit, then $r_{i,j'} = r_{i,j}$ for all $i = 1, \ldots, s_j = s_{j'}$, and so $M = \pi_* L(-\sum_{j=1}^{n} \sum_{i=1}^{s_j} r_{i,j} E_{i,j})$ is defined over k.

[3]Since $\mathcal{L}$ is defined over k, the set U' is also defined over k, although this does not matter for the argument.

This finishes the proof of Theorem 8.1. □

If $Z \subset X(k)$ is a finite set of points, it is possible to improve the conclusion of Theorem 8.1(d) to obtain a birational map $f\colon X \dashrightarrow \mathbb{P}^2$ so that f is defined at each point of Z, and f^{-1} is defined at each point of $f(Z)$. For instance, see Lemma 3.2, the result of combining Theorem 8.1 and Proposition 8.6.

Proposition 8.6. *Let Y be a smooth projective surface defined over k, $Z \subset Y(k)$ a finite set of points, and let $f\colon Y \longrightarrow \mathbb{P}^2$ be a birational morphism defined over k. Then there exists a blowup $\pi\colon \widetilde{Y} \longrightarrow Y$ at finitely many points of $Y(k)$ disjoint from Z, and a birational morphism $g\colon \widetilde{Y} \longrightarrow \mathbb{P}^2$ also defined over k, such that for each point $z \in Z$, g^{-1} is defined at $g(z)$. (Here, via π, we are identifying $Z \subset Y$ with the corresponding subset of $\widetilde{Y}$.) The points blown up at are general, and may be taken to avoid any proper closed subset $W \subset Y$.*

To explain the idea of the argument, suppose that $f\colon Y \longrightarrow \mathbb{P}^2$ is the blowup of $\mathbb{P}^2$ at a single (k-rational) point with exceptional divisor E, and that $Z = \{z\}$, with $z \in E$. We want to find a different morphism to $\mathbb{P}^2$ where E is not contracted. To do this let $\widetilde{Y}$ be the blow up of Y at two further k-rational points q_1 and q_2 of Y such that $f(z)$, $f(q_1)$, and $f(q_2)$ are not on the same line in $\mathbb{P}^2$. Thus $\widetilde{Y}$ is the blowup of $\mathbb{P}^2$ at three points. The associated Cremona transformation, blowing down the proper transforms of the lines connecting each of the three pairs of those points, sends E to a line in $\mathbb{P}^2$, and thus has the property we are looking for.

In general (i.e., returning to the general setup of the proposition), for each $z \in Z$, $f^{-1}(f(z))$ may be a tree of $\mathbb{P}^1$'s, and we will have to successively apply basic quadratic Cremona transformations to get at the component of the tree containing z, and do this for each $z \in Z$.

Proof. The argument is inductive, and we first define an invariant $N(f, Z)$ to keep track of the steps. Let $h\colon S \longrightarrow S'$ be a birational morphism of smooth projective surfaces, and s a point of S. If h is an isomorphism near s then we set $\ell(h, s) = 0$. Otherwise, let $T := h^{-1}(h(s))$ be the tree of h-exceptional divisors through s. This tree has a distinguished component E_1, the exceptional divisor from first blowup of S' at $h(s)$ when resolving h^{-1}. Since T is a tree, for any component E' of T there is a unique sequence of components from E_1 to E'. If s lies on a single component E' of T, we set $\ell(h, s)$ to be the number of components in the path from E_1 to E'. If s lies on two components of T, we do the same, choosing for E' the component closest to E_1, i.e., the component which gives the shorter path. Thus, for example, $\ell(h, s) = 1$ if and only if $s \in E_1$. Finally we set

$$N(f, Z) := \sum_{z \in Z} \ell(f, z). \tag{8.6.1}$$

Therefore $N(f, Z) \geq 0$, and $N(f, Z) = 0$ if and only if f is an isomorphism near each point of Z, i.e., if and only if f^{-1} is defined at each point of $f(z)$, $z \in Z$.

For use below, we record the following formula. Suppose that f factors as $f = \gamma \circ \psi$ with $\psi\colon Y \longrightarrow V$ a map to a smooth surface. The tree of f-exceptional curves through z is obtained from the tree of γ-exceptional curves through $\psi(z)$ by adding the tree of ψ-exceptional curves through z, as well as possibly adding other ψ-exceptional trees lying over points different from $\psi(z)$. The computation of ℓ only depends on the path from the distinguished component to the component containing z, and we conclude that for any $z \in Y$ we have

$$\ell(f, z) = \ell(\psi, z) + \ell(\gamma, \psi(z)). \tag{8.6.2}$$

We now establish the inductive step. Assuming that $N(f, Z) > 0$ we will construct a map $\pi\colon Y_1 \longrightarrow Y$, the blowup of Y at two general k-rational points of Y disjoint from Z, and a birational morphism $f_1\colon Y_1 \longrightarrow \mathbb{P}^2$ such that $\ell(f_1, z) \leq \ell(f, z)$ for each point $z \in Z$, and such that for at least one of the points the inequality is strict. (Here via π we are again considering Z as a subset of Y_1.) Thus, $N(f_1, Z) < N(f, Z)$, and iterating this procedure we arrive at a morphism g with $N(g, Z) = 0$. By the remark after (8.6.1), such a g satisfies the conditions of the morphism to be constructed, proving the proposition.

The diagram below, whose pieces we will fill in as we go along, is useful for keeping track of the elements of the argument; the subscripts on the two copies of $\mathbb{P}^2$ are used to distinguish between them.

$$\begin{tikzcd}
Y_1 \arrow[r, "\pi"] \arrow[d, "\psi_1"'] \arrow[dd, bend right=60, "f_1"'] \arrow[dr, phantom, "\square"] & Y \arrow[d, "\psi_0"] \arrow[dd, bend left=60, "f"] \\
V_1 \arrow[r, "\alpha"] \arrow[d, "\gamma_1"] & V_0 \arrow[d, "\gamma_0"] \\
\mathbb{P}^2_1 \arrow[r, dashed, "\delta"] & \mathbb{P}^2_0
\end{tikzcd} \tag{8.6.3}$$

We are assuming that $N(f, Z) > 0$, and so there is a $z_0 \in Z$ with $\ell(f, z_0) \geq 1$. Fix one such z_0, and let $\gamma_0\colon V_0 \longrightarrow \mathbb{P}^2_0$ be the blowup of $\mathbb{P}^2_0$ at $f(z_0)$, with exceptional divisor E_1. By the structure theorem for birational maps between smooth surfaces (e.g., [B, Théorème II.11]) f factors as $f = \gamma_0 \circ \psi_0$ for a map $\psi_0\colon Y \longrightarrow V_0$.

We split Z into two subsets, setting

$$Z' := \{z \in Z \mid f(z) = f(z_0)\} \qquad \Big(= \{z \in Z \mid \psi_0(z) \in E_1\}\Big),$$

and setting Z'' to be the complement of Z' in Z.

For $u, v \in Y(k)$ let L_{uv} be the line in $\mathbb{P}^2_0$ passing through $f(u)$ and $f(v)$ (assuming $f(u) \neq f(v)$). We choose q_1 and q_2 in $Y(k)$ general enough so that the following conditions are satisfied:

(i) q_1 and q_2 do not lie on any of the exceptional curves of f (and thus do not lie on any of the exceptional curves of ψ_0).
(ii) $f(z_0)$, $f(q_1)$, and $f(q_2)$ do not lie on the same line in $\mathbb{P}^2_0$.
(iii) None of the lines $L_{z_0q_1}$, $L_{z_0q_2}$ or $L_{q_1q_2}$ pass through $f(z)$, for any $z \in Z''$.
(iv) The proper transforms of $L_{z_0q_1}$ and $L_{z_0q_2}$ in V_0 do not meet E_1 at any of the points $\psi_0(z)$, for any $z \in Z'$.

For $z \in Z$, if $\ell(f, z) \neq 0$ then (i) implies that q_1, q_2 must be different from z. If $\ell(f, z) = 0$ then $z \in Z''$, and (iii) implies that q_1, q_2 are also different from z. Thus q_1, q_2 are disjoint from Z. As stated in the proposition, and for use inductively, we may also fix a proper closed subset $W \subset Y$ and require

(v) q_1 and q_2 are not in W.

When applying the argument inductively, W should at least contain the exceptional divisors of previous applications of the inductive step, so that the end $\widetilde{Y}$ resulting from the process is the blowup of the original Y at finitely many points (as opposed to being an iterated blowup, blowing up at points of exceptional divisors).

We set $\pi\colon Y_1 \longrightarrow Y$ to be the blowup of Y at q_1 and q_2, and $\alpha\colon V_1 \longrightarrow V_0$ to be the blowup of V_0 at $\psi_0(q_1)$ and $\psi_0(q_2)$. Since ψ_0 is a local isomorphism at q_1 and q_2 (by (i)), Y_1 is also the fibre product $Y_1 = Y \times_{V_0} V_1$, and we set $\psi_1\colon Y_1 \longrightarrow V_1$ to be the induced morphism.

The variety V_1 is the blowup of $\mathbb{P}^2_0$ at three distinct points, with $\gamma_0 \circ \alpha$ the blowdown map. We let $\gamma_1\colon V_1 \longrightarrow \mathbb{P}^2_1$ be the other blowdown map associated with this configuration, blowing down the proper transforms of $L_{z_0q_1}$, L_{z_0,q_2}, and $L_{q_1q_2}$ in V_1, and define $f_1 := \gamma_1 \circ \psi_1$.

Setting $\delta := \gamma_0 \circ \alpha \circ \gamma_1^{-1}$, the resulting birational map $\delta\colon \mathbb{P}^2_1 \dashrightarrow \mathbb{P}^2_0$ is a standard quadratic Cremona transformation and makes the bottom square of (8.6.3) commutative, where defined.

Set $Z_1 := \pi^{-1}(Z)$. Since q_1 and q_2 are disjoint from Z, π identifies Z_1 with Z, but we use this notation to emphasize that the maps now start from Y_1. The decomposition $Z = Z' \sqcup Z''$ into disjoint subsets induces a decomposition

$Z_1 = Z_1' \sqcup Z_1''$. Since Z_1' is nonempty (Z' contains z_0), to see that f_1 satisfies $N(f_1, Z_1) < N(f, Z)$, it suffices to verify

(a) $\ell(f_1, z) = \ell(f, \pi(z))$ for each $z \in Z_1''$, and
(b) $\ell(f_1, z) = \ell(f, \pi(z)) - 1$ for each $z \in Z_1'$.

Let $U_0 \subset \mathbb{P}_0^2$ be the complement of $L_{z_0q_1}$, $L_{z_0q_2}$, and $L_{q_1q_2}$. Let E_{q_1} and E_{q_2} be the exceptional divisors of α over $\psi_0(q_1)$ and $\psi_0(q_2)$ respectively, set $F_1 := \alpha^{-1}(E_1)$, and let U_1 be the complement of the three lines $\gamma_1(E_{q_1})$, $\gamma_1(E_{q_2})$, and $\gamma_1(F_1)$ in $\mathbb{P}_1^2$.

Both γ_0 and α are isomorphisms over U_0, and similarly γ_1 is an isomorphism over U_1. Moreover, δ induces an isomorphism $U_1 \xrightarrow{\sim} U_0$ (i.e., δ and γ_1 are also isomorphisms over U_0). Thus, over U_0, the bottom square of (8.6.3) is the diagram of isomorphisms

$$\begin{tikzcd}
\gamma_1^{-1}(U_1) \arrow[r, "\alpha", "\sim"'] \arrow[d, "\gamma_1", "\wr"'] & \gamma_0^{-1}(U_0) \arrow[d, "\gamma_0", "\wr"'] \\
U_1 \arrow[r, "\delta", "\sim"'] & U_0
\end{tikzcd}$$

Since the top of (8.6.3) is a fibre square, we conclude that π induces an isomorphism

$$f_1^{-1}(U_1) \xrightarrow{\sim} f^{-1}(U_0) \tag{8.6.4}$$

of varieties over U_0. For any z in Z_1'', condition (iii) shows that $f(\pi(z))$ is in U_0, and thus also that $f_1(z)$ is in U_1. From the definition, $\ell(f_1, z)$ can be computed on the inverse image of any neighbourhood of $f_1(z)$, and similarly $\ell(f, \pi(z))$ can be computed on the inverse image of any neighbourhood of $f(\pi(z))$. By (8.6.4) we may assume these neighbourhoods are the same, with the isomorphism taking f_1 to f. It follows that for $z \in Z_1''$ we have $\ell(f_1, z) = \ell(f, \pi(z))$, proving (a).

Let $U \subset V_0$ be the complement of q_1 and q_2, so that α is an isomorphism over U, and hence by the top fibre square in (8.6.3), π induces an isomorphism

$$\psi_1^{-1}(\alpha^{-1}(U)) \xrightarrow{\sim} \psi_0^{-1}(U) \tag{8.6.5}$$

of varieties over U. By (ii), q_1 and q_2 do not lie on E_1, and so $E_1 \subset U$. For a point $z \in Z_1'$, by definition $\psi_0(\pi(z)) \in E_1$, and in particular $\psi_0(\pi(z)) \in U$ Applying the reasoning at the end of (a), with (8.6.5) in place of (8.6.4) we conclude that $\ell(\psi_1, z) = \ell(\psi_0, \pi(z))$.

Since $\psi_0(\pi(z))$ lies on E_1, the unique curve contracted by γ_0, we have $\ell(\gamma_0, \psi_0(\pi(z))) = 1$. By (iv) the exceptional divisors of γ_1 (the proper transforms of $L_{z_0q_1}$, L_{z_0q2}, and $L_{q_1q_2}$) do not meet $F_1 = \alpha^{-1}(E_1)$ at $\psi_1(z) =$

$\alpha^{-1}(\psi_0(\pi(z)))$, and so $\ell(\gamma_1, \psi_1(z)) = 0$. Combining these and using (8.6.2) we obtain

$$\begin{aligned} \ell(f_1, z) &\overset{(8.6.2)}{=} \ell(\psi_1, z) + \ell(\gamma_1, \psi_1(z)) = \ell(\psi_1, z) = \ell(\psi_0, \pi(z)) \\ &\overset{(8.6.2)}{=} \ell(f, \pi(z)) - \ell(\gamma_0, \psi_0(\pi(z))) = \ell(f, \pi(z)) - 1, \end{aligned}$$

proving (b).

It is clear from the k-rational nature of the construction that if f is defined over k, then so is f_1, and therefore so is the morphism g produced as the result of iterating the inductive steps. □

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Department of Pure Mathematics, University of Waterloo, Waterloo, Ontario N2L 3G1, Canada

Email address: dmckinnon@uwaterloo.ca

Department of Mathematics and Statistics, Queens University, Kingston, Ontario, Canada

Email address: mike.roth@queensu.ca