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On the Distribution of the Area Enclosed by a Random Walk on $\mathbf{Z}^2$

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Let Γ_{2n} be the set of paths with $2n$ steps of unit length in $\mathbf{Z}^2$, which begin and end at $(0, 0)$. For $\gamma \in \Gamma_{2n}$, let $area(\gamma) \in \mathbf{Z}$ denote the oriented area enclosed by γ. We show that for every positive even integer k, there exists a rational function R_k with integer coefficients, such that:

$$\frac{1}{|\Gamma_{2n}|} \sum_{\gamma \in \Gamma_{2n}} [area(\gamma)]^k = R_k(n), \qquad n > 2k.$$

We calculate explicitly the degree and leading coefficient of R_k. We show how as a consequence of this (and by also using the enumeration of up-down permutations, and the exponential formula for cycles of permutations) one can derive the asymptotic distribution of the area enclosed by a random path in Γ_{2n}. The formula for the asymptotic distribution can be stated as follows: for $\alpha < \beta$ in $\mathbf{R}$

$$\lim_{n \to \infty} \frac{|\{\gamma \in \Gamma_{2n} \mid n\alpha < area(\gamma) < n\beta\}|}{|\Gamma_{2n}|} = \frac{1}{2}(\tanh(\pi\beta) - \tanh(\pi\alpha)).$$

Since the appropriately re-normalized random walks with n steps on $\mathbf{Z}^2$ converge, as $n \to \infty$, to the 2-dimensional Brownian motion, this argument can be viewed as a combinatorial approach to the formula of Lévy for the area enclosed by a random 2-dimensional Brownian path. © 1998 Academic Press

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1. INTRODUCTION

Let γ be a path of length m with unit steps in $\mathbf{Z}^2$, i.e. an $(m+1)$-tuple of the form:

$$\gamma = (a_0, a_1, ..., a_m), \tag{1.1}$$

with $a_i = (a_i', a_i'') \in \mathbf{Z}^2$ for $0 \leqslant i \leqslant m$, and such that $|a_i' - a_{i-1}'| + |a_i'' - a_{i-1}''| = 1$ for every $1 \leqslant i \leqslant m$. The path γ is said to be closed if $a_m = a_0$; if this happens then the length of γ is necessarily even, $m = 2n$.

For every positive integer n, we make the notation

$$\Gamma_{2n} = \{\gamma \text{ as in (1.1)} \mid m = 2n,\ a_0 = a_{2n} = (0, 0)\}. \tag{1.2}$$

The cardinality of Γ_{2n} is $\binom{2n}{n}^2$ (see e.g. [10] VIII.5.2).

Now, every $\gamma \in \Gamma_{2n}$ encloses a certain (oriented) area; by Green's theorem this equals:

$$area(\gamma) = \int_\gamma x\, dy = -\int_\gamma y\, dx, \tag{1.3}$$

where in the path integral of (1.3), $\gamma = (a_0, a_1, ..., a_{2n})$ is identified with the continuous function $\gamma\colon [0, 1] \to \mathbf{R}^2$ which has $\gamma(i/2n) = a_i$, $0 \leqslant i \leqslant 2n$, and is linear on each interval $[(i-1)/2n, i/2n]$, $1 \leqslant i \leqslant 2n$. Combinatorial formulas for $area(\gamma)$ can also be given, if one takes into account that in fact $area(\gamma) \in \mathbf{Z}$ is nothing but the number of lattice squares enclosed by γ, where each square is counted with the multiplicity given by the winding number of γ around it. The combinatorial formulation of (1.3) which we use is presented in Proposition 2.5 below.

Our first goal in the present paper is to point out that:

1.1. THEOREM. *Let k be an even positive integer. There exists a rational function with integer coefficients, R_k, such that*

$$\binom{2n}{n}^{-2} \sum_{\gamma \in \Gamma_{2n}} [area(\gamma)]^k = R_k(n) \tag{1.4}$$

for every $n > 2k$.

The assertion of Theorem 1.1 also holds for k odd, but in a trivial way, since in that case the left-hand side of (1.4) is zero. (Indeed, if $-\gamma$ denotes the path obtained by traversing $\gamma \in \Gamma_{2n}$ in reversed direction, then $area(-\gamma) = -area(\gamma)$; thus for k odd, the terms $[area(\gamma)]^k$ and

$[area(-\gamma)]^k$ in the summation of (1.4) will cancel each other.) For k even, we have for instance

$$R_2(X)=\frac{X^2(X-1)}{6(2X-1)}, \tag{1.5}$$

$$R_4(X)=\frac{X^3(X-1)(7X^2-18X+13)}{60(2X-1)(2X-3)}. \tag{1.6}$$

For an arbitrary even k, the proof of Theorem 1.1 provides an explicit formula for R_k, involving a family of polynomials $Q_{\pi,\rho}$, with π, ρ partitions of the set $\{1, ..., k\}$ –see the Equations (2.10), (2.11) in Theorem 2.9 below. This formula is not too simple (and in fact it is not clear if R_k could be at all brought to a simple form, for arbitrary k); however, if we are concerned with the asymptotics of $R_k(n)$ as $n\to\infty$, then we have:

1.2. THEOREM 1. *For every $k\geqslant 2$, even, the degree of R_k (i.e., the difference of the degrees of the numerator and denominator of R_k) is equal to k.*

2. *For $k\geqslant 2$, even, let λ_k denote the leading coefficient of R_k (i.e., the ratio of the leading coefficients of the numerator and denominator of R_k). Let us also put $\lambda_0=1$ and $\lambda_k=0$ for $k\geqslant 1$, odd. Then the exponential power series of the λ_k's has radius of convergence 2π, and satisfies*

$$\sum_{k=0}^{\infty}\frac{\lambda_k}{k!}(2z)^k=\frac{z}{\sin z}\qquad |z|<\pi. \tag{1.7}$$

These results can be used for deriving the asymptotic distribution of the area enclosed by a random path in Γ_{2n}. Let us view Γ_{2n} as a probability space, where each $\gamma\in\Gamma_{2n}$ has a probability of $\binom{2n}{n}^{-2}$; and let us view the area enclosed by a path in Γ_{2n} as a random variable, *area*: $\Gamma_{2n}\to\mathbf{Z}$. We denote the distribution of this random variable by μ_n,

$$\mu_n\stackrel{\text{def}}{=}\binom{2n}{n}^{-2}\sum_{\gamma\in\Gamma_{2n}}\delta_{area(\gamma)}=\binom{2n}{n}^{-2}\sum_{k\in\mathbf{Z}}A_{2n,k}\delta_k, \tag{1.8}$$

where for every $k\in\mathbf{Z}$ we have denoted

$$A_{2n,k}\stackrel{\text{def}}{=}|\{\gamma\in\Gamma_{2n}\mid area(\gamma)=k\}|, \tag{1.9}$$

and where "δ_k" stands for the Dirac measure at k. For every $k\geqslant 0$, the moment of order k of μ_n is

$$\int_{-\infty}^{\infty}t^k\,d\mu_n(t)=\binom{2n}{n}^{-2}\sum_{\gamma\in\Gamma_{2n}}[area(\gamma)]^k; \tag{1.10}$$

this equals 0 when k is odd, and equals (by Theorem 1.1) $R_k(n)$ when k is even and $n>2k$. But then it becomes clear that the numbers $(\lambda_k)_{k=0}^{\infty}$ considered in Theorem 1.2.2 can be also expressed as

$$\lambda_k=\lim_{n\to\infty}\frac{1}{n^k}\int_{-\infty}^{\infty}t^k\,d\mu_n(t),\qquad k\geqslant 0. \tag{1.11}$$

Equation (1.11) and a basic application of the method of moments (as described for instance in [5], Section 8) give us that the sequence of distributions $(\mu_n)_{n=1}^{\infty}$ becomes weakly convergent as soon as the μ_n's are normalized by their variances. The method of moments also tells how the limit of the normalized μ_n's can be calculated explicitly—modulo normalization constants, one only needs to replace z by iz and then do an inverse Fourier transform in the right-hand side of (1.7). Stated directly in terms of areas enclosed by random walks, the conclusion of this calculation is the following:

1.3 COROLLARY. *For every $\alpha<\beta$ in $\mathbf{R}$,*

$$\lim_{n\to\infty}\binom{2n}{n}^{-2}|\{\gamma\in\Gamma_{2n}\mid n\alpha<area(\gamma)<n\beta\}|=\frac{\tanh(\pi\beta)-\tanh(\pi\alpha)}{2}. \tag{1.12}$$

The asymptotic formula (1.12) was known in the theory of Brownian motion. Indeed, the appropriately re-normalized random walks with n steps on $\mathbf{Z}^2$ approach, as $n\to\infty$, the 2-dimensional Brownian motion (see e.g. [2] Sections 9 and 10); therefore Corollary 1.3 can also be obtained as an application of the formula found by P. Lévy ([7]) for the distribution of the area enclosed by a planar Brownian motion. Conversely, the above derivation of the asymptotics (1.12) can be viewed as a combinatorial approach to the Lévy's area formula, where an extra piece of structure is put into evidence at the level of the approximating random walks. For a discussion of the analytical aspects of Lévy's formula see e.g. [6], and the references mentioned there.

While circulating a preliminary version of this paper, we heard of yet another approach to the asymptotics (1.12), very recently found by Bellissard et al. [4], who exploit the connection between areas enclosed by random walks and the spectral properties of a certain 2-dimensional Schrödinger operator called the Harper operator. (For a presentation of combinatorial aspects of the theory of the Harper operator, see also Section 2 of [3].)

Besides the Introduction, the paper has four more sections. The proof of Theorem 1.1 is presented in Section 2, and the one of Theorem 1.2 is split between the Sections 3 and 4. The derivation of the asymptotics (1.12) is briefly presented in the final Section 5.

2. THE PROOF OF THEOREM 1.1

Our starting point here is a combinatorial formula for the area enclosed by a path $\gamma \in \Gamma_{2n}$. The idea is that Equation (1.3) for $area(\gamma)$ takes a particularly nice form when γ is replaced by a certain pair (ξ, η) of its one-dimensional projections. We will describe the correspondence $\gamma \mapsto (\xi, \eta)$ in Proposition 2.3, after introducing the appropriate notation.

2.1. *Notation.* Let n be a positive integer. Let

$$X_{2n} \stackrel{\text{def}}{=} \left\{ \xi = (\xi_1, ..., \xi_{2n}) \in \{-1, 1\}^{2n} \;\middle|\; \sum_{i=1}^{2n} \xi_i = 0 \right\}. \tag{2.1}$$

An element $\xi = (\xi_1, ..., \xi_{2n}) \in X_{2n}$ can be viewed as a "closed one-dimensional path" of length $2n$ since it can be identified with the path $(b_0, b_1, ..., b_{2n})$ in $\mathbf{Z}$ having $b_i = \xi_1 + \cdots + \xi_i$, $0 \leqslant i \leqslant 2n$. Thus X_{2n} is a one-dimensional counterpart of Γ_{2n}.

2.2. DEFINITION. Let n be a positive integer, and let $\gamma = (a_0, a_1, ..., a_{2n})$ be in Γ_{2n}, where $a_i = (a_i', a_i'') \in \mathbf{Z}^2$ for $0 \leqslant i \leqslant 2n$. Recall that $a_0 = a_{2n} = (0, 0)$ and that $|a_i' - a_{i-1}'| + |a_i'' - a_{i-1}''| = 1$ for every $1 \leqslant i \leqslant 2n$.

1. We denote: $H(\gamma) = \{1 \leqslant i \leqslant 2n \mid a_{i-1}'' = a_i''\}$ (the set of horizontal steps of γ), and $V(\gamma) = \{1 \leqslant i \leqslant 2n \mid a_{i-1}' = a_i'\}$ (the set of vertical steps of γ). $\{1, ..., 2n\}$ is the disjoint union of $H(\gamma)$ and $V(\gamma)$.

2. The $(2n)$-tuple ξ which has its ith component ξ_i equal to $(a_i' - a_{i-1}') + (a_i'' - a_{i-1}'') \in \{-1, 1\}$, $1 \leqslant i \leqslant 2n$, is in X_{2n} and will be denoted by $\xi(\gamma)$. Similarly, the $(2n)$-tuple η which has its ith component equal to $(a_i' - a_{i-1}') - (a_i'' - a_{i-1}'')$, $1 \leqslant i \leqslant 2n$, is in X_{2n} and will be denoted by $\eta(\gamma)$. From a geometric point of view $\xi(\gamma)$ and $\eta(\gamma)$ are the projections of γ onto the lines $y = x$ and $y = -x$ respectively.

2.3. PROPOSITION. *For every positive integer $n \geqslant 1$, the map*

$$\Gamma_{2n} \ni \gamma \mapsto (\xi(\gamma), \eta(\gamma)) \in X_{2n} \times X_{2n} \tag{2.2}$$

is bijective.

Proof. Fix for the moment a pair of elements $\xi = (\xi_1, ..., \xi_{2n})$ and $\eta = (\eta_1, ..., \eta_{2n})$ in X_{2n}. We consider

$$H = \{1 \leqslant i \leqslant 2n \mid \xi_i = \eta_i\}, \qquad V = \{1 \leqslant i \leqslant 2n \mid \xi_i = -\eta_i\}, \tag{2.3}$$

and note that

$$\sum_{i \in H} \xi_i = \frac{1}{2}\left(\sum_{i=1}^{2n} \xi_i + \sum_{i=1}^{2n} \eta_i \right) = 0, \qquad \sum_{i \in V} \xi_i = \frac{1}{2}\left(\sum_{i=1}^{2n} \xi_i - \sum_{i=1}^{2n} \eta_i \right) = 0. \tag{2.4}$$

If we put $a_0 = (0, 0)$, and for $1 \leqslant i \leqslant 2n$

$$a_i = \begin{cases} a_{i-1} + (\xi_i, 0) & \text{if } i \in H \\ a_{i-1} + (0, \xi_i) & \text{if } i \in V, \end{cases}$$

then (2.4) implies that $a_{2n} = (0, 0)$; therefore $\gamma = (a_0, a_1, ..., a_{2n})$ is in Γ_{2n}.

The construction described in the preceding paragraph defines a map $X_{2n} \times X_{2n} \ni (\xi, \eta) \mapsto \gamma \in \Gamma_{2n}$, which is easily verified to be an inverse for the map in (2.2). Q.E.D.

2.4. *Remark.* For $n \geqslant 1$ and $\xi, \eta \in X_{2n}$ it will be convenient to use the notations $H(\xi, \eta) = \{1 \leqslant i \leqslant 2n \mid \xi_i = \eta_i\}$, and $V(\xi, \eta) = \{1 \leqslant i \leqslant 2n \mid \xi_i = -\eta_i\}$. The sets $H(\xi, \eta)$ and $V(\xi, \eta)$ are disjoint with union $\{1, ..., 2n\}$, and, as is clear from (2.3), we have compatibility with the notations introduced in Definition 2.2.1 –in the sense that

$$H(\xi(\gamma), \eta(\gamma)) = H(\gamma), \qquad V(\xi(\gamma), \eta(\gamma)) = V(\gamma)$$

for every $\gamma \in \Gamma_{2n}$.

2.5. PROPOSITION. *Let n be a positive integer, and let γ be a path in Γ_{2n}. With $\xi(\gamma) = (\xi_1, ..., \xi_{2n})$, as in Definition 2.2.2, we have that*

$$area(\gamma) = - \sum_{\substack{1 \leqslant i < j \leqslant 2n \\ i \in V(\gamma),\, j \in H(\gamma)}} \xi_i \xi_j. \tag{2.5}$$

Proof. We identify γ with a continuous function $\gamma: [0, 1] \to \mathbf{R}^2$, in the way mentioned following Equation (1.3). Denoting by γ_j the restriction of the function γ to the interval $[(j-1)/2n, j/2n]$, $1 \leqslant j \leqslant 2n$, we can write

$$\int_\gamma y\, dx = \sum_{j=1}^{2n} \int_{\gamma_j} y\, dx = \sum_{j \in H(\gamma)} \int_{\gamma_j} y\, dx + \sum_{j \in V(\gamma)} \int_{\gamma_j} y\, dx. \tag{2.6}$$

For a $j \in V(\gamma)$ we have that the coordinate x is constant along γ_j, and therefore $\int_{\gamma_j} y\, dx = 0$. On the other hand for a $j \in H(\gamma)$ we have that along γ_j: the coordinate y is constantly equal to $\sum_{i \in V(\gamma),\, i < j} \xi_i$, while the variation in the coordinate x is equal to ξ_j; this leads to the formula

$$\int_{\gamma_j} y\, dx = \left(\sum_{\substack{i < j \\ i \in V(\gamma)}} \xi_i \right) \xi_j. \tag{2.7}$$

By placing (2.7) into (2.6), we obtain

$$\int_\gamma y\,dx = \sum_{j\in H(\gamma)} \left(\sum_{\substack{i<j \\ i\in V(\gamma)}} \xi_i\xi_j \right) = \sum_{\substack{1\leqslant i<j\leqslant 2n \\ i\in V(\gamma),\ j\in H(\gamma)}} \xi_i\xi_j.$$

Since $area(\gamma) = -\int_\gamma y\,dx$, we thus arrive at Equation (2.5). Q.E.D.

2.6. COROLLARY. *For every even positive integer k and for every positive integer n we have that*

$$\sum_{\gamma\in\Gamma_{2n}} [area(\gamma)]^k = \sum_{\xi,\eta\in X_{2n}} \sum_{\substack{1\leqslant i_1<j_1\leqslant 2n,\ \ldots \\ \ldots,\ 1\leqslant i_k<j_k\leqslant 2n \\ i_1,\ldots,i_k\in V(\xi,\eta) \\ j_1,\ldots,j_k\in H(\xi,\eta)}} \xi_{i_1}\cdots\xi_{i_k}\xi_{j_1}\cdots\xi_{j_k}. \tag{2.8}$$

Proof. One only needs to raise (2.5) to the power k, then do a summation over $\gamma\in\Gamma_{2n}$, and finally replace this summation (over Γ_{2n}) by one over $X_{2n}\times X_{2n}$, with the change of variable $\gamma\leftrightarrow(\xi,\eta)$ indicated by Proposition 2.3. Q.E.D.

Theorem 1.1 will be obtained by transforming the double-summation on the right-hand side of (2.8). In order to state how this is done (Theorem 2.9 below), we first need more notation.

2.7. *Notation.* Let k be a positive integer.

1. We will use the short-hand notation $[k]=\{1,\ldots,k\}$. If $\pi=\{B_1,\ldots,B_p\}$ is a partition of $[k]$ (i.e. $B_1,\ldots,B_p$ are non-void, pairwise disjoint sets such that $B_1\cup\cdots\cup B_p=[k]$), then $B_1,\ldots,B_p$ are called the *blocks* of π; their number, p, will be denoted by $|\pi|$. The set of all partitions of $[k]$ will be denoted by $\mathscr{P}([k])$.

2. Let π be a partition in $\mathscr{P}([k])$. A subset $A\subseteq[k]$ will be said to be *π-saturated* if it is a union of blocks of π. If A is π-saturated, then the number of blocks of π which are contained in A will be denoted by $|A/\pi|$. Note that if $A\subseteq[k]$ is π-saturated, then so is the complement $A^c=[k]\backslash A$, and we have $|A/\pi|+|A^c/\pi|=|\pi|$.

3. Let $\pi=\{B_1,\ldots,B_p\}$ and $\rho=\{C_1,\ldots,C_q\}$ be partitions in $\mathscr{P}([k])$. A *(π,ρ)-restricted permutation* will be a permutation (i.e. re-writing in some order) of $p+q$ objects –call them $X_1,\ldots,X_p,Y_1,\ldots,Y_q$ –such that X_s comes before Y_t in the permutation whenever the blocks B_s (of π) and C_t (of ρ) have a non-void intersection, for $1\leqslant s\leqslant p$ and $1\leqslant t\leqslant q$. The number of (π,ρ)-restricted permutations will be denoted by $RP(\pi,\rho)$.

To give a concrete example: if $k=6$, $\pi=\{(1,2),(3,4),(5,6)\}$, $\rho=\{(1),(2,3),(4,5),(6)\}$, then by letting X_1, X_2, X_3 correspond to the

blocks of π (in the order written above) and by letting Y_1, Y_2, Y_3, Y_4 correspond to the blocks of ρ (in the order written above) we have that $X_1Y_1X_3Y_4X_2Y_3Y_2$ is a (π,ρ)-restricted permutation. One can check by direct counting that for these particular π and ρ we have $RP(\pi,\rho)=61$ (see also Remark 4.10 below).

For $\pi=\{B_1,\ldots,B_p\}$ and $\rho=\{C_1,\ldots,C_q\}$ as in Notation 2.7.3, one can also think about (π,ρ)-restricted permutations in the following way. Let $\mathscr{R}$ be the rank-1 poset with elements corresponding to the blocks of π and ρ, such that every block of π has rank 0, every block of ρ has rank 1, and such that the block B_s of π precedes the block C_t of ρ if and only if $B_s\cap C_t\neq\varnothing$. Then the (π,ρ)-restricted permutations can be identified with the linear extensions of $\mathscr{R}$ (as defined for instance in [11], Section 3.5).

2.8. *Notation.* Another (standard) notation which we will repeatedly use is the one for falling factorials. Recall that for $x\in\mathbf{R}$ and d a non-negative integer one writes

$$(x)_d = \begin{cases} x(x-1)\cdots(x-d+1) & \text{if } d>0 \\ 1 & \text{if } d=0. \end{cases} \tag{2.9}$$

We will also use Equation (2.9) with x replaced by an indeterminate X, and where $(X)_d$ will stand for the corresponding polynomial of degree d in X.

We will prove the following statement, which clearly implies Theorem 1.1.

2.9. THEOREM. *Let k be an even positive integer. For every $n>2k$ we have that*

$$\binom{2n}{n}^{-2} \sum_{\gamma\in\Gamma_{2n}} [area(\gamma)]^k = \sum_{\pi,\,\rho\in\mathscr{P}([k])} \frac{RP(\pi,\rho)}{(|\pi|+|\rho|)!}\cdot\frac{Q_{\pi,\rho}(n)}{(2n)_{|\pi|+|\rho|}}, \tag{2.10}$$

where, for $\pi,\ \rho\in\mathscr{P}([k])$, $Q_{\pi,\rho}(X)$ is the polynomial

$$\sum_{\substack{A,\,B\subseteq[k] \\ A\ \pi\text{-saturated} \\ B\ \rho\text{-saturated}}} (-1)^{|A|+|B|}(X)_{|A/\pi|+|B/\rho|}(X)_{|A^c/\pi|+|B/\rho|} \times (X)_{|A/\pi|+|B^c/\rho|}(X)_{|A^c/\pi|+|B^c/\rho|}. \tag{2.11}$$

Proof. We fix the positive integers k and n (k even, $n>2k$) about which we want to show that (2.10) holds.

Given two functions $i, j:[k]\to[2n]$, let us agree to write "$i\prec j$" if $i([k])\cap j([k])=\varnothing$ and if the inequality $i(p)<j(p)$ holds for every

$p \in [k]$. With this notation, Equation (2.8) of Corollary 2.6 can be written in the form:

$$\sum_{\gamma \in \Gamma_{2n}} [area(\gamma)]^k = \sum_{(\xi, \eta; i, j) \in \mathscr{I}_{n,k}} \xi_{i(1)} \cdots \xi_{i(k)} \xi_{j(1)} \cdots \xi_{j(k)}, \tag{2.12}$$

where the index set $\mathscr{I}_{n,k}$ appearing on the right-hand side is:

$$\mathscr{I}_{n,k} = \left\{ (\xi, \eta; i, j) \;\middle|\; \begin{array}{c} \xi, \eta \in X_{2n},\ i, j : [k] \to [2n], \\ i \prec j,\ i([k]) \subseteq V(\xi, \eta),\ j([k]) \subseteq H(\xi, \eta) \end{array} \right\} \tag{2.13}$$

(and where the notations X_{2n}, $H(\xi, \eta)$, $V(\xi, \eta)$ are as used earlier in this section—see, e.g., Remark 2.4).

Given a function $i : [k] \to [2n]$, let us denote by $ker(i)$ the partition of $[k]$ into level sets of i (i.e., $1 \leqslant p < q \leqslant k$ are in the same block of $ker(i)$ if and only if $i(p) = i(q)$). We consider the map

$$\Phi : \mathscr{I}_{n,k} \to \left\{ (\pi, \rho; A, B) \;\middle|\; \begin{array}{c} \pi, \rho \in \mathscr{P}([k]),\ A, B \subseteq [k], \\ A\ \pi\text{-saturated},\ B\ \rho\text{-saturated} \end{array} \right\} \tag{2.14}$$

which to an element $(\xi, \eta; i, j) \in \mathscr{I}_{n,k}$ associates the $(\pi, \rho; A, B)$ determined as follows

$$\begin{cases} \pi = ker(i),\ A = \{1 \leqslant p \leqslant k \mid \xi_{i(p)} = -1\}, \\ \rho = ker(j),\ B = \{1 \leqslant q \leqslant k \mid \xi_{j(q)} = -1\}. \end{cases} \tag{2.15}$$

It is immediate that in (2.15) the set A is π-saturated and the set B is ρ-saturated, so that Φ does indeed take values in the codomain stated in (2.14).

If $(\xi, \eta; i, j) \in \mathscr{I}_{n,k}$ and $(\pi, \rho; A, B)$ are related as in (2.15), it is clear that

$$\xi_{i(1)} \cdots \xi_{i(k)} = (-1)^{|A|}, \qquad \xi_{j(1)} \cdots \xi_{j(k)} = (-1)^{|B|}.$$

This means in particular that the term indexed by $(\xi, \eta; i, j)$ in the summation of (2.12) depends only on $(\pi, \rho; A, B)$. As a consequence, the named summation can be pulled forward by using the map Φ; when this is done, (2.12) becomes

$$\sum_{\gamma \in \Gamma_{2n}} [area(\gamma)]^k = \sum_{\pi, \rho \in \mathscr{P}([k])} \sum_{\substack{A, B \subseteq [k] \\ A\ \pi\text{-saturated} \\ B\ \rho\text{-saturated}}} (-1)^{|A| + |B|} \cdot |\Phi^{-1}(\pi, \rho; A, B)|. \tag{2.16}$$

Our next task is to calculate $|\Phi^{-1}(\pi, \rho; A, B)|$ for some arbitrary (but fixed) choice of $\pi, \rho \in \mathscr{P}([k])$ and of $A, B \subseteq [k]$, such that A is π-saturated and B is ρ-saturated. If $(\xi, \eta; i, j) \in \Phi^{-1}(\pi, \rho; A, B)$, then in particular we

must have (from (2.13) and (2.15)) that $i \prec j$, $ker(i) = \pi$, $ker(j) = \rho$; so let us start by pointing out that

$$\begin{aligned} &|\{(i, j) \mid i, j : [k] \to [2n],\ i \prec j,\ ker(i) = \pi,\ ker(j) = \rho\}| \\ &\quad = 2n \binom{2n}{|\pi| + |\rho|} \cdot RP(\pi, \rho) \end{aligned} \tag{2.17}$$

(with $RP(\pi, \rho)$ as defined in Notation 2.7.3). Indeed, if i, j are as in (2.17), then the sets $i([k])$ and $j([k])$ are disjoint (from $i \prec j$) and have $|i([k])| = |\pi|$, $|j([k])| = |\rho|$ (due to the kernel conditions), hence necessarily $|i([k]) \cup j([k])| = |\pi| + |\rho|$. As a consequence, the left-hand side of (2.17) splits as

$$\begin{aligned} \sum_{\substack{M \subseteq [2n] \\ |M| = |\pi| + |\rho|}} |\{(i, j) \mid i, j : [k] \to [2n], \quad i \prec j, \quad ker(i) = \pi, \\ ker(j) = \rho,\ i([k]) \cup j([k]) = M\}|. \end{aligned} \tag{2.18}$$

But it is immediately seen that for every choice of $M \subseteq [2n]$ with $|M| = |\pi| + |\rho|$, counting the couples (i, j) with the properties listed in (2.18) is the same thing as counting the (π, ρ)-restricted permutations. Thus all the terms in (2.18) are equal to $RP(\pi, \rho)$, and (2.17) follows.

For the π, ρ, A, B which were fixed in the preceding paragraph, let us now also pick a pair of functions $i, j : [k] \to [2n]$ which satisfy the conditions in (2.17), and let us count in how many ways can we choose $\xi, \eta \in X_{2n}$ such that $(\xi, \eta; i, j) \in \Phi^{-1}(\pi, \rho; A, B)$. In this context, where all of π, ρ, A, B, i, j are fixed, we have that for arbitrary $\xi, \eta \in X_{2n}$ the following equivalence holds:

$$(\xi, \eta; i, j) \in \Phi^{-1}(\pi, \rho; A, B) \Leftrightarrow \begin{cases} \xi_{i(p)} = -1, & \eta_{i(p)} = +1 & \text{for} & p \in A \\ \xi_{i(p)} = +1, & \eta_{i(p)} = -1 & \text{for} & p \in A^c \\ \xi_{j(q)} = -1, & \eta_{j(q)} = -1 & \text{for} & q \in B \\ \xi_{j(q)} = +1, & \eta_{j(q)} = +1 & \text{for} & q \in B^c. \end{cases} \tag{2.19}$$

The verification of (2.19) is straightforward; we only mention here that in the proof of "$\Leftarrow$" one also needs to check that $(\xi, \eta; i, j) \in \mathscr{I}_{n,k}$, i.e. that we have $i([k]) \subseteq V(\xi, \eta)$, $j([k]) \subseteq H(\xi, \eta)$ –however, these properties are built in the equations on the right-hand side of (2.19). The merit of (2.19) is that it shows clearly what our counting problem is about: we have certain entries of ξ and certain entries of η the values of which are prescribed, and what we need to do is fill in the remaining entries of ξ and η, independently, such that each of ξ and η arrives to have n entries of 1 and n entries of -1. A direct count shows that ξ has $|A/\pi| + |B/\rho|$ entries prescribed to be -1, and $|A^c/\pi| + |B^c/\rho|$ entries prescribed to be $+1$; while η has

$|A^c/\pi| + |B/\rho|$ entries prescribed to be -1, and $|A/\pi| + |B^c/\rho|$ entries prescribed to be $+1$. Note that all these numbers of prescribed entries are smaller than n, e.g. $|A/\pi| + |B/\rho| \leqslant |\pi| + |\rho| \leqslant 2k < n$. It is an immediate verification that the number of ways of completing a vector $\zeta \in X_{2n}$ where $s < n$ entries of ζ are prescribed to be -1 and $t < n$ entries of ζ are prescribed to be $+1$ is $\binom{2n-s-t}{n-s}$. We hence obtain that the number of possible choices for a pair (ξ, η) satisfying the conditions in (2.19) is

$$\binom{2n - |\pi| - |\rho|}{n - |A/\pi| - |B/\rho|} \cdot \binom{2n - |\pi| - |\rho|}{n - |A^c/\pi| - |B/\rho|}, \tag{2.20}$$

where the first factor in (2.20) counts the choices for ξ, and the second one counts the choices for η (we also took into account, for instance in the first factor, that $(|A/\pi| + |B/\rho|) + (|A^c/\pi| + |B^c/\rho|) = |\pi| + |\rho|$).

The conclusion of the preceding two paragraphs (obtained by putting (2.17) and (2.20) together) is that for every $\pi, \rho \in \mathscr{P}([k])$ and $A, B \subseteq [k]$ such that A is π-saturated and B is ρ-saturated:

$$|\Phi^{-1}(\pi, \rho; A, B)| = \binom{2n}{|\pi| + |\rho|} \cdot RP(\pi, \rho) \binom{(2n - |\pi| - |\rho|}{n - |A/\pi| - |B/\rho|} \cdot \binom{2n - |\pi| - |\rho|}{n - |A^c/\pi| - |B/\rho|}. \tag{2.21}$$

If we plug (2.21) into the right-hand side of (2.16), then divide by $\binom{2n}{n}^2$ and do the appropriate cancellations in the occurring factorials, we obtain

$$\binom{2n}{n}^{-2} \sum_{\gamma \in \Gamma_{2n}} [area(\gamma)]^k = \sum_{\pi, \rho \in \mathscr{P}([k])} \sum_{\substack{A, B \subseteq [k] \\ A\ \pi\text{-saturated} \\ B\ \rho\text{-saturated}}} (-1)^{|A| + |B|} \cdot \frac{RP(\pi, \rho)}{(|\pi| + |\rho|)!} \times \frac{(n)_{|A/\pi| + |B/\rho|} (n)_{|A^c/\pi| + |B/\rho|} (n)_{|A/\pi| + |B^c/\rho|} (n)_{|A^c/\pi| + |B^c/\rho|}}{(2n)_{|\pi| + |\rho|}},$$

out of which (2.10), (2.11) immediately follow. Q.E.D.

3. THE DEGREE AND DOMINANT COEFFICIENT OF $Q_{\pi, \rho}(X)$

Let k be an even positive integer, let π, ρ be in $\mathscr{P}([k])$, and consider the polynomial $Q_{\pi, \rho}(X)$ defined by Equation (2.11) in Theorem 2.9. A straightforward calculation shows that every term in the sum in (2.11) is

a polynomial of degree $2(|\pi| + |\rho|)$. However, the cancellations between terms generally make the degree of $Q_{\pi, \rho}$ strictly smaller than that.

3.1. *Notation.* For k a positive integer and $\pi \in \mathscr{P}([k])$ we will denote by $|\pi|_o$ (respectively $|\pi|_e$) the number of blocks of π which have odd (respectively even) cardinality. Note that if k is even, then $|\pi|_o$ is an even number, for every $\pi \in \mathscr{P}([k])$.

3.2. THEOREM. *Let k be an even positive integer, and let π, ρ be in $\mathscr{P}([k])$. The degree of $Q_{\pi, \rho}(X)$ is*

$$2(|\pi|_e + |\rho|_e) + \tfrac{3}{2}(|\pi|_o + |\rho|_o), \tag{3.1}$$

and its coefficient of highest degree is

$$(-1)^{(|\pi|_o + |\rho|_o)/2} \cdot 2^{(|\pi| + |\rho|)} \cdot (|\pi|_o - 1)!! \cdot (|\rho|_o - 1)!!; \tag{3.2}$$

in (3.2) *we used the customary notation $m!! = 1 \cdot 3 \cdot 5 \cdots m$ for $m \geqslant 1$, odd, with the convention that $(-1)!! = 1$.*

By comparing (3.1) with $2(|\pi| + |\rho|) = 2(|\pi_e| + |\rho_e|) + 2(|\pi_o| + |\rho_o|)$, we thus see that in (2.11) the cancellations decrease the degree by $(|\pi_o| + |\rho_o|)/2$. (This will be essential for the considerations in Section 4 below.)

Our goal in the present section is to prove Theorem 3.2. We start by noticing that both the definition of $Q_{\pi, \rho}$ and the conclusion of Theorem 3.2 are more conveniently formulated in terms of the "direct sum" of π and ρ, as in the following statement which involves only one partition instead of two.

3.3. THEOREM. *Let k be an even positive integer, let σ be in $\mathscr{P}([2k])$, and let $S \subseteq [2k]$ be a σ-saturated set (in the sense of Notation 2.7.2) for which $|S| = k$. Consider the involution Θ on the family of σ-saturated subsets of $[2k]$, defined by the formula*

$$\Theta(D) = (D \cap S) \cup (D^c \cap S^c) \tag{3.3}$$

(for $D \subseteq [2k]$, σ-saturated, and with $D^c = [2k] \backslash D$, $S^c = [2k] \backslash S$). Let Q be the polynomial

$$Q(X) = \sum_{\substack{D \subseteq [2k] \\ \sigma\text{-saturated}}} (-1)^{|D|} (X)_{|D/\sigma|} (X)_{|D^c/\sigma|} (X)_{|\Theta(D)/\sigma|} (X)_{|\Theta(D)^c/\sigma|}, \tag{3.4}$$

where $|D/\sigma|$, $|D^c/\sigma|$, ... *are also as in Notation* 2.7.2. *Then*

$$degree(Q) = 2|\sigma|_e + \tfrac{3}{2}|\sigma|_o\,, \tag{3.5}$$

and the coefficient of highest degree in Q *is equal to*

$$(-1)^{|\sigma|_o/2} \cdot 2^{|\sigma|} \cdot (|S/\sigma|_o - 1)!! \cdot (|S^c/\sigma|_o - 1)!!, \tag{3.6}$$

where $|S/\sigma|_o$ (*respectively* $|S^c/\sigma|_o$) *denotes the number of of blocks of* σ *which are contained in* S (*respectively* S^c) *and have odd cardinality.*

Theorem 3.2 is the particular case of Theorem 3.3 where we take $\sigma \in \mathscr{P}([2k])$ to be obtained by putting together a copy of π partitioning $[k] \subseteq [2k]$ with a copy of ρ partitioning $\{k+1, ..., 2k\} \subseteq [2k]$, and where we take $S = [k]$. Indeed, it is easily verified that in this case the polynomial Q of (3.4) reduces to $Q_{\pi,\rho}$, while on the other hand (3.5) and (3.6) reduce to (3.1) and (3.2).

It is therefore sufficient to prove Theorem 3.3. From this point and until the end of the section we fix k, σ, S, Q as in the statement of Theorem 3.3, and we will prove that (3.5) and (3.6) hold for these fixed k, σ, S, Q.

3.4. Definition (Matching-joiner graphs). Let $\mathscr{B}$ denote the set of blocks of σ; in some sense $\mathscr{B}$ and σ are the same thing, but it is convenient to make a notational distinction between them. We will work with graphs that have $\mathscr{B}$ as set of vertices, and thus often refer to a $B \in \mathscr{B}$ as "vertex" —although B is at the same time a subset of $[2k]$. We will also use the notation $\mathscr{B}_o$ (respectively $\mathscr{B}_e$) for the subset of $\mathscr{B}$ consisting of the blocks of σ which have odd (respectively even) cardinality. We have of course $|\mathscr{B}| = |\sigma|$, and $|\mathscr{B}_o| = |\sigma|_o$, $|\mathscr{B}_e| = |\sigma|_e$.

Recall that a graph with vertex set $\mathscr{B}$ and edge set $\mathscr{M}$ is called a (*partial*) *matching* of $\mathscr{B}$ if there are no two distinct edges in $\mathscr{M}$ which have a vertex in common. The graphical objects of interest for what follows are those of the form:

$$\mathscr{G} = (\mathscr{B}; \mathscr{M}_1, \mathscr{M}_2), \tag{3.7}$$

such that for each $i = 1, 2$ the graph with vertex set $\mathscr{B}$ and edge set $\mathscr{M}_i$ is a matching. We will call the $\mathscr{G}$'s as in (3.7) by the (ad-hoc) name of *matching-joiner graphs*, or *m.j.-graphs* for short.

Let $\mathscr{G} = (\mathscr{B}; \mathscr{M}_1, \mathscr{M}_2)$ be an m.j.-graph. We will say that two vertices $B, B' \in \mathscr{B}$ are in the same *connected component* of $\mathscr{G}$ if one can go from B to B' by traveling along edges from $\mathscr{M}_1 \cup \mathscr{M}_2$. It is easily seen that $\mathscr{B}$ is partitioned into connected components of $\mathscr{G}$, and moreover that each of these components is either reduced to a point or it is a (possibly open, possibly closed) path consisting of edges alternating between $\mathscr{M}_1$ and $\mathscr{M}_2$.

3.5. Definition (Compatibility index for an m.j.-graph). Let $\mathscr{G} = (\mathscr{B}; \mathscr{M}_1, \mathscr{M}_2)$ be an m.j.-graph. A σ-saturated set $D \subseteq [2k]$ will be said to be *compatible* with $\mathscr{G}$ when it has the following two properties: (a) if two vertices $B, B' \in \mathscr{B}$ are connected by an edge in $\mathscr{M}_1$, then exactly one of B, B' (when viewed as subset of $[2k]$) is contained in D; and: (b) if two vertices $B, B' \in \mathscr{B}$ are connected by an edge in $\mathscr{M}_2$, then exactly one of B, B' is contained in $\Theta(D)$, the σ-saturated set defined by (3.3).

We define a number $cd(\mathscr{G})$, which we will call the *compatibility index* of $\mathscr{G}$, by the formula

$$cd(\mathscr{G}) = \sum_D (-1)^{|D|}, \tag{3.8}$$

with D running in the family of all the σ-saturated sets $D \subseteq [2k]$ which are compatible with $\mathscr{G}$; if there are no such sets, we take by convention $cd(\mathscr{G}) = 0$.

3.6. Proposition. *The polynomial* Q *defined by Equation* (3.4) *also has the formula*

$$Q(X) = \sum_{\substack{\mathscr{G} = (\mathscr{B}; \mathscr{M}_1, \mathscr{M}_2) \\ \text{m.j.-graph}}} cd(\mathscr{G})(X)_{|\sigma| - |\mathscr{M}_1|}\,(X)_{|\sigma| - |\mathscr{M}_2|}\,. \tag{3.9}$$

Proof. We will show that for every integer $x > |\sigma|$ we have the equality of integers:

$$Q(x) = \sum_{\substack{\mathscr{G} = (\mathscr{B}; \mathscr{M}_1, \mathscr{M}_2) \\ \text{m.j.-graph}}} cd(\mathscr{G})(x)_{|\sigma| - |\mathscr{M}_1|}\,(x)_{|\sigma| - |\mathscr{M}_2|}\,; \tag{3.10}$$

this will imply the equality of polynomials (3.9) (since x in (3.10) can take infinitely many values).

So, let us fix $x > |\sigma|$. We denote by $\mathscr{J}$ the set of all triples $(D; u, v)$ where $D \subseteq [2k]$ is a σ-saturated set and $u, v: \mathscr{B} \to [x]$ are functions such that: (a) u is injective on $\{B \in \mathscr{B} \mid B \subseteq D\}$ and on $\{B \in \mathscr{B} \mid B \subseteq D^c\}$, and: (b) v is injective on $\{B \in \mathscr{B} \mid B \subseteq \Theta(D)\}$ and on $\{B \in \mathscr{B} \mid B \subseteq \Theta(D)^c\}$. We claim that

$$Q(x) = \sum_{(D; u, v) \in \mathscr{J}} (-1)^{|D|}. \tag{3.11}$$

Indeed, the right-hand side of (3.11) can be evaluated as

$$\sum_{\substack{D\subseteq[2k]\\ \sigma\text{-saturated}}}\left(\sum_{\substack{u,\,v:\,\mathscr{B}\to[x]\\ \text{such that}\\ (D;\,u,\,v)\in\mathscr{J}}}(-1)^{|D|}\right)$$
$$=\sum_{\substack{D\subseteq[2k]\\ \sigma\text{-saturated}}}(-1)^{|D|}\cdot|\{(u,v)\mid u,v:\mathscr{B}\to[x],(D;u,v)\in\mathscr{J}\}|$$
$$=\sum_{\substack{D\subseteq[2k]\\ \sigma\text{-saturated}}}(-1)^{|D|}\,(x)_{|D/\sigma|}\,(x)_{|D^c/\sigma|}\,(x)_{|\Theta(D)/\sigma|}\,(x)_{|\Theta(D)^c/\sigma|}\,, \tag{3.12}$$

where at the last equal sign we took into account that constructing a pair (u, v) such that $(D; u, v)\in\mathscr{J}$ is tantamount to constructing, independently of each other, four injective functions from $\{B\in\mathscr{B}\mid B\subseteq D\}$, $\{B\in\mathscr{B}\mid B\subseteq D^c\}$, $\{B\in\mathscr{B}\mid B\subseteq\Theta(D)\}$ and $\{B\in\mathscr{B}\mid B\subseteq\Theta(D)^c\}$ into $[x]$ (and that there are $(x)_{|D/\sigma|}$ injective functions $\{B\in\mathscr{B}\mid B\subseteq D\}\to[x]$, there are $(x)_{|D^c/\sigma|}$ injective functions $\{B\in\mathscr{B}\mid B\subseteq D^c\}\to[x]$, etc). The quantity in (3.12) is $Q(x)$, thus (3.11) is proved.

But now, the right-hand side of (3.11) can also be evaluated in a different way:

$$\sum_{(D;\,u,\,v)\in\mathscr{J}}(-1)^{|D|}=\sum_{u,\,v:\,\mathscr{B}\to[x]}\left(\sum_{\substack{D\subseteq[2k],\ \sigma\text{-saturated}\\ \text{such that }(D;\,u,\,v)\in\mathscr{J}}}(-1)^{|D|}\right). \tag{3.13}$$

Let us say about a function $u\colon\mathscr{B}\to[x]$ that it is "at most 2-to-1" if $|\{B\in\mathscr{B}\mid u(B)=y\}|\leqslant 2$ for every $y\in[x]$. If u is at most 2-to-1, then we can associate with it a matching of $\mathscr{B}$, with edge set $\mathscr{M}_u$ which is obtained by drawing an edge between two vertices $B\neq B'$ in $\mathscr{B}$ if and only if $u(B)=u(B')$.

The first summation on the right-hand side of (3.13) can be restricted to the situation when both u and v are at most 2-to-1. Indeed, if for instance u is not at most 2-to-1, then for no σ-saturated $D\subseteq[2k]$ can we have that u is injective on both $\{B\in\mathscr{B}\mid B\subseteq D\}$ and $\{B\in\mathscr{B}\mid B\subseteq D^c\}$—thus there is no D such that $(D; u, v)\in\mathscr{J}$.

Let us consider on the other hand two maps $u, v\colon\mathscr{B}\to[x]$ which are at most 2-to-1. Then the two matchings associated to u and v can be considered together in a matching-joiner graph $\mathscr{G}_{u,v}\overset{\text{def}}{=}(\mathscr{B};\mathscr{M}_u,\mathscr{M}_v)$. It is easy to see that the second sum on the right-hand side of (3.13) gives nothing but the compatibility index $cd(\mathscr{G}_{u,v})$. Thus (3.13) can be continued with:

$$=\sum_{\substack{u,\,v:\,\mathscr{B}\to[x]\\ \text{at most 2-to-1}}}cd(\mathscr{G}_{u,v})$$
$$=\sum_{\substack{\mathscr{G}=(\mathscr{B};\,\mathscr{M}_1,\,\mathscr{M}_2)\\ \text{m.j.-graph}}}cd(\mathscr{G})\cdot|\{(u,v)|\ u,v\text{ at most 2-to-1},\ \mathscr{G}_{u,v}=\mathscr{G}\}|$$
$$=\sum_{\substack{\mathscr{G}=(\mathscr{B};\,\mathscr{M}_1,\,\mathscr{M}_2)\\ \text{m.j.-graph}}}cd(\mathscr{G})\cdot|\{u\colon\mathscr{B}\to[x]|\ u\text{ at most 2-to-1},\ \mathscr{M}_u=\mathscr{M}_1\}|$$
$$\cdot|\{v\colon\mathscr{B}\to[x]|\ v\text{ at most 2-to-1},\ \mathscr{M}_v=\mathscr{M}_2\}|. \tag{3.14}$$

Finally, if $\mathscr{M}_1$ is the edge set of a matching of $\mathscr{B}$, then the number of functions $u\colon\mathscr{B}\to[x]$ which are at most 2-to-1 and have $\mathscr{M}_u=\mathscr{M}_1$ is clearly equal to the number of injective functions from the set of connected components of the matching, into $[x]$. There are $|\sigma|-|\mathscr{M}_1|$ such connected components, (counting $|\mathscr{M}_1|$ edges and $|\mathscr{B}|-2|\mathscr{M}_1|=|\sigma|-2|\mathscr{M}_1|$ isolated vertices), therefore the number of functions u as above turns out to be $(x)_{|\sigma|-|\mathscr{M}_1|}$. Similarly, the number of functions $v\colon\mathscr{B}\to[x]$ in (3.14) which are at most 2-to-1 and have $\mathscr{M}_v=\mathscr{M}_2$ is found to be equal to $(x)_{|\sigma|-|\mathscr{M}_2|}$. Thus (3.14) coincides with the right-hand side of (3.10). Q.E.D.

3.7. LEMMA. *Let $\mathscr{G}=(\mathscr{B};\mathscr{M}_1,\mathscr{M}_2)$ be an m.j.-graph, and assume that there exists a connected component $\mathscr{C}\subseteq\mathscr{B}$ of $\mathscr{G}$ such that the subset $\bigcup_{B\in\mathscr{C}}B$ of $[2k]$ has odd cardinality. Then $cd(\mathscr{G})=0$.*

Proof. We assume that the family of σ-saturated sets $D\subseteq[2k]$ which are compatible with $\mathscr{G}$ is non-void, otherwise there is nothing to prove. We define an involution on this family of compatible sets by setting $D\mapsto D'$, with

$$D'=\left(\left(\bigcup_{B\in\mathscr{C}}B\right)^c\cap D\right)\cup\left(\left(\bigcup_{B\in\mathscr{C}}B\right)\setminus D\right). \tag{3.15}$$

We leave to the reader the straightforward verifications that D' of (3.15) is still a σ-saturated and $\mathscr{G}$-compatible set, that $D\mapsto D'$ is an involution (i.e. the same rule (3.15) would send D' back into D), and that $(-1)^{|D|}+(-1)^{|D'|}=0$. These facts have as consequence that

$$cd(\mathscr{G})=\sum_{\text{D compatible with }\mathscr{G}}(-1)^{|D|}=0,$$

since the terms of the latter sum can be grouped in pairs such that they cancel each other. Q.E.D.

3.8. LEMMA. *Let $\mathscr{G}=(\mathscr{B};\mathscr{M}_1,\mathscr{M}_2)$ be an m.j.-graph, and assume that for every connected component $\mathscr{C}\subseteq\mathscr{B}$ of $\mathscr{G}$, the subset $\bigcup_{B\in\mathscr{C}} B$ of $[2k]$ has even cardinality. Then*

$$|\mathscr{M}_1|+|\mathscr{M}_2|\geqslant\tfrac{1}{2}|\sigma|_o. \tag{3.16}$$

Equality holds in (3.16) *if and only if the sets $\mathscr{B}_1$, $\mathscr{B}_2$ of vertices covered by the edges in $\mathscr{M}_1$, $\mathscr{M}_2$ (respectively) have $\mathscr{B}_1\cap\mathscr{B}_2=\varnothing$ and $\mathscr{B}_1\cup\mathscr{B}_2=\mathscr{B}_o$ (the subset of $\mathscr{B}$ consisting of the blocks of σ which have odd cardinality).*

Proof. The meaning of the hypothesis on $\mathscr{G}$ is that every connected component of $\mathscr{G}$ contains an even number of vertices from $\mathscr{B}_o$. By deleting an edge from each cyclic component of $\mathscr{G}$ (which only helps in (3.16)) we may assume that all the connected components of $\mathscr{G}$ are open paths. If a component $\mathscr{C}$ of $\mathscr{G}$ contains $2p\neq 0$ vertices from $\mathscr{B}_o$—say that we call them $B_1, B_2, \ldots, B_{2p}$, in the order they are reached while traveling along $\mathscr{C}$—then $\mathscr{C}$ must have at least p edges (in fact at least $2p-1\geqslant p$ edges, because there is at least one edge between B_1 and B_2, at least one between B_2 and B_3, etc). By summing this inequality over all the connected components of $\mathscr{G}$ we obtain (3.16). The conditions under which equality can hold in (3.16) are also easily determined in this way. Q.E.D.

3.9. LEMMA. *Let $\mathscr{G}=(\mathscr{B};\mathscr{M}_1,\mathscr{M}_2)$ be an m.j.-graph for which equality holds in* (3.16). *Let h denote the number of edges in $\mathscr{M}_2$ which connect a vertex B such that $B\subseteq S$ with a vertex B' such that $B'\subseteq S^c$ (where $S\subseteq[2k]$ is the σ-saturated set in the statement of Theorem* 3.3). *Then*

$$cd(\mathscr{G})=2^{|\sigma|_e+|\sigma|_o/2}\cdot(-1)^{|\sigma|_o/2}\cdot(-1)^h. \tag{3.17}$$

Proof. According to Lemma 3.8, every $B\in\mathscr{B}_e$ is an isolated vertex of $\mathscr{G}$, while on the other hand the vertices from $\mathscr{B}_o$ are grouped by the edges in $\mathscr{M}_1\cup\mathscr{M}_2$ into $|\mathscr{B}_o|/2=|\sigma|_o/2$ pairs.

If we want to construct a σ-saturated set $D\subseteq[2k]$ which is compatible with $\mathscr{G}$ (in the sense of Definition 3.5), then:

— For every $B\in\mathscr{B}_e$ we have a choice between two possibilities —either including B into D, or leaving B outside D (i.e. with $B\cap D=\varnothing$).

— For every two vertices $B, B'\in\mathscr{B}_o$ which are connected by an edge in $\mathscr{M}_1$, we know that exactly one of B, B' has to be included in D, while the other has to be left outside D. So we also have here a choice between two possibilities—which of the B, B' is contained in D and which is not.

— For every two vertices $B, B'\in\mathscr{B}_o$ which are connected by an edge in $\mathscr{M}_2$, we know that one of B, B' has to be contained in $\Theta(D)=(D\cap S)\cup(D^c\cap S^c)$, while the other has to be contained in $\Theta(D)^c$. If B, B' are both contained in S or both contained in S^c, this comes again to saying that exactly one of B, B' must be included in D, and we have a choice between the same two possibilities as in the preceding paragraph. On the other hand, if one of B, B' is contained in S while the other is contained in S^c, then by looking at the relation between D and $\Theta(D)$ we find that what is needed is still a choice between two possibilities: either we include both B and B' into D, or we leave them both out.

The above discussion shows clearly that there are $2^{|\mathscr{B}_e|}\cdot 2^{|\mathscr{B}_o|/2}$ (i.e. $2^{|\sigma|_e}\cdot 2^{|\sigma|_o/2}$) possibilities of constructing a σ-saturated set D which is compatible with $\mathscr{G}$. Moreover, a direct inspection shows that in all the possible ways of constructing D, the cardinality $|D|$ has the same parity as $(|\mathscr{B}_o|/2)-h$. Thus the sum defining $cd(\mathscr{G})$ in Equation (3.8) has in this case $2^{|\sigma|_e}\cdot 2^{|\sigma|_o/2}$ terms, all equal to $(-1)^{(|\sigma|_o/2)-h}$, and (3.17) follows. Q.E.D.

3.10. *The proof of Theorem* 3.3. Consider the expression found for Q in Equation (3.9) of Proposition 3.6. The summation in (3.9) can of course be restricted to the m.j.-graphs $\mathscr{G}$ with $cd(\mathscr{G})\neq 0$; but from Lemmas 3.7 and 3.8 it follows that for any such graph $\mathscr{G}=(\mathscr{B};\mathscr{M}_1,\mathscr{M}_2)$:

$$\begin{aligned}\operatorname{degree}(cd(\mathscr{G})(X)_{|\sigma|-|\mathscr{M}_1|}(X)_{|\sigma|-|\mathscr{M}_2|}) &= 2|\sigma|-(|\mathscr{M}_1|+|\mathscr{M}_2|)\\ &\overset{(3.16)}{\leqslant} 2|\sigma|-\tfrac{1}{2}|\sigma|_o=2|\sigma|_e+\tfrac{3}{2}|\sigma|_o.\end{aligned}$$

This makes clear that the degree of Q cannot exceed $2|\sigma|_e+\frac{3}{2}|\sigma|_o$. In order to verify that degree(Q) is in fact equal to this number, and at the same time calculate the dominant coefficient of Q, it remains that we add together the dominant coefficients of the polynomials $cd(\mathscr{G})(X)_{|\sigma|-|\mathscr{M}_1|}(X)_{|\sigma|-|\mathscr{M}_2|}$ for all the m.j.-graphs $\mathscr{G}=(\mathscr{B};\mathscr{M}_1,\mathscr{M}_2)$ for which equality is reached in (3.16).

Let us call "complete odd matching" a matching of $\mathscr{B}$ which has $|\sigma_o|/2$ edges, covering the $|\sigma|_o$ vertices in $\mathscr{B}_o$ (but which leaves every $B\in\mathscr{B}_e$ as an isolated vertex). Lemma 3.8 tells us how to parameterize the family of m.j.-graphs $\mathscr{G}$ for which (3.16) holds with equality: we pick an arbitrary complete odd matching of $\mathscr{B}$, with edge set $\mathscr{M}$; then we pick an arbitrary subset $\mathscr{M}_2\subseteq\mathscr{M}$, and we set $\mathscr{G}=(\mathscr{B};\mathscr{M}\backslash\mathscr{M}_2,\mathscr{M}_2)$. Thus the coefficient of the term of degree $2|\sigma|_e+\frac{3}{2}|\sigma|_o$ in Q is equal to:

$$\sum_{\substack{\mathscr{M}\text{ an edge set of a}\\ \text{complete odd matching}}}\ \sum_{\mathscr{M}_2\subseteq\mathscr{M}} cd(\mathscr{B};\mathscr{M}\backslash\mathscr{M}_2,\mathscr{M}_2). \tag{3.18}$$

For a complete odd matching with edge set $\mathscr{M}$, let us denote by $\mathscr{T}_{\mathscr{M}}\subseteq\mathscr{M}$ the set of edges of the matching which have endpoints B, B' such that one

of B, B' is contained in S while the other is contained in S^c. Lemma 3.9 tells us that for $\mathcal{M}$ and $\mathcal{M}_2$ as in (3.18):

$$cd(\mathcal{B}; \mathcal{M}\backslash\mathcal{M}_2, \mathcal{M}_2) = 2^{|\sigma|_e + |\sigma|_o/2} \cdot (-1)^{|\sigma|_o/2} \cdot (-1)^{|\mathcal{M}_2 \cap \mathcal{T}_\mathcal{M}|};$$

thus the sum in (3.18) equals

$$2^{|\sigma|_e + |\sigma|_o/2} \cdot (-1)^{|\sigma|_o/2} \sum_{\substack{\mathcal{M} \text{ an edge set of a} \\ \text{complete odd matching}}} \left(\sum_{\mathcal{M}_2 \subseteq \mathcal{M}} (-1)^{|\mathcal{M}_2 \cap \mathcal{T}_\mathcal{M}|} \right). \quad (3.19)$$

Now, let us fix for the moment a complete odd matching of $\mathcal{B}$, with edge set $\mathcal{M}$. If we write an arbitrary $\mathcal{M}_2 \subseteq \mathcal{M}$ as the disjoint union of an arbitrary set of $\mathcal{M}\backslash\mathcal{T}_\mathcal{M}$ and an arbitrary subset of $\mathcal{T}_\mathcal{M}$, we obtain

$$\begin{aligned} \sum_{\mathcal{M}_2 \subseteq \mathcal{M}} (-1)^{|\mathcal{M}_2 \cap \mathcal{T}_\mathcal{M}|} &= 2^{|\mathcal{M}\backslash\mathcal{T}_\mathcal{M}|} \cdot \sum_{\mathcal{N} \subseteq \mathcal{T}_\mathcal{M}} (-1)^{|\mathcal{N}|} \\ &= 2^{|\mathcal{M}\backslash\mathcal{T}_\mathcal{M}|} \cdot (1-1)^{|\mathcal{T}_\mathcal{M}|} \\ &= \begin{cases} 0 & \text{if} \quad \mathcal{T}_\mathcal{M} \neq \varnothing \\ 2^{|\mathcal{M}|} (= 2^{|\sigma|_o/2}) & \text{if} \quad \mathcal{T}_\mathcal{M} = \varnothing. \end{cases} \end{aligned} \quad (3.20)$$

On the other hand, it is clear that a generic complete odd matching with edge set $\mathcal{M}$ such that $\mathcal{T}_\mathcal{M} = \varnothing$ is obtained by constructing, independently of each other, two complete matchings for the sets of vertices $\{B \in \mathcal{B}_o \mid B \subseteq S\}$ and $\{B \in \mathcal{B}_o \mid B \subseteq S^c\}$. The latter two sets have $|S/\sigma|_o$ and $|S^c/\sigma|_o$ elements, respectively; and since (as is easily seen by induction, for instance) the number of ways of grouping a given set of $2p$ objects into p pairs is $(2p-1)!!$, it follows that the number of complete odd matchings having an edge set $\mathcal{M}$ with $\mathcal{T}_\mathcal{M} = \varnothing$ is

$$(|S/\sigma|_o - 1)!! \cdot (|S^c/\sigma|_o - 1)!! \quad (3.21)$$

When (3.20) and (3.21) are replaced in (3.19), Equation (3.6) is obtained. Q.E.D.

4. THE PROOF OF THEOREM 1.2

4.1. Definition . Let k be an even positive integer. We will denote by $\mathcal{P}_2([k])$ (respectively by $\mathcal{P}_{1\text{-}2}([k])$) the set of partitions $\pi \in \mathcal{P}([k])$ such that every block of π has exactly (respectively not more than) two elements.

4.2. *Remarks.* 1. The set $\mathcal{P}([k])$ of partitions of $[k]$ bears a natural partial order, by refinement: $\pi \leqslant \rho$ if and only if every block of ρ is a union of blocks of π. The poset $(\mathcal{P}([k]), \leqslant)$ is a lattice (see e.g. [11], Example 3.10.4). We will use the customary notation $\pi \vee \rho$ for the least common upper bound, in this lattice, of $\pi, \rho \in \mathcal{P}([k])$.

Note that if k is even and if $\mathcal{P}_2([k])$, $\mathcal{P}_{1\text{-}2}([k])$ are as in Definition 4.1, then:

$$\mathcal{P}_{1\text{-}2}([k]) = \{\pi \in \mathcal{P}([k]) \mid \text{there exists } \pi' \in \mathcal{P}_2([k]) \text{such that } \pi \leqslant \pi'\};$$

i.e., $\mathcal{P}_{1\text{-}2}([k])$ is the order ideal generated by $\mathcal{P}_2([k])$.

2. Assume that k is even, let $\mathcal{S}_k$ denote the group of all permutations of $[k]$, and let $\mathcal{S}_k^{(even)} \subseteq \mathcal{S}$ consist of the permutations which have only cycles of even length. One can find a bijection between $\mathcal{P}_2([k]) \times \mathcal{P}_2([k])$ and $\mathcal{S}_k^{(even)}$, with the property that if $(\pi, \rho) \in \mathcal{P}_2([k]) \times \mathcal{P}_2([k])$ corresponds to $\tau \in \mathcal{S}_k^{(even)}$, then the cycles of τ coincide with the blocks of $\pi \vee \rho$. In the direction from τ to (π, ρ), such a bijection can be described as follows: for each cycle of τ we consider its unique writing in the form $i_1 \to i_2 \to \cdots \to i_{2p-1} \to i_{2p} \to i_1$ which has $i_1 < i_j$ for $2 \leqslant j \leqslant 2p$, and we set (i_1, i_2), (i_3, i_4), ..., (i_{2p-1}, i_{2p}) to be blocks of π, and (i_2, i_3), ..., (i_{2p-2}, i_{2p-1}), (i_{2p}, i_1) to be blocks of ρ.

4.3. Proposition. *Let k be an even positive integer, and let R_k be the rational function of Theorem* 1.1 (*i.e., which is given by the Equations* (2.10), (2.11) *in Theorem* 2.9). *Then the degree of R_k is less than or equal to k; and the coefficient of order k in the expansion of R_k has the formula*

$$\lambda_k = \sum_{\pi, \rho \in \mathcal{P}_{1\text{-}2}([k])} \frac{(-1)^{|\pi| + |\rho|} RP(\pi, \rho)}{(|\pi| + |\rho|)!} \cdot (|\pi|_o - 1)!! \, (|\rho|_o - 1)!!, \quad (4.1)$$

with $\mathcal{P}_{1\text{-}2}([k])$ as above, and where $RP(\pi, \rho)$ is as introduced in Notation 2.7; *the double factorial notation is the same as in Theorem* 3.2.

Proof. For every $\pi, \rho \in \mathcal{P}([k])$,

$$\begin{aligned} \text{degree}(Q_{\pi, \rho}) &= 2(|\pi|_e + |\rho|_e) + \tfrac{3}{2}(|\pi|_o + |\rho|_o) \text{ by Theorem 3.2} \\ &= |\pi| + |\rho| + \tfrac{1}{2}(2|\pi|_e + |\pi|_o) + \tfrac{1}{2}(2|\rho|_e + |\rho|_o). \end{aligned} \quad (4.2)$$

We have $k \geqslant 2|\pi|_e + |\pi|_o$, with equality if and only if $\pi \in \mathcal{P}_{1\text{-}2}([k])$ –this is just because every block B of π satisfies: $|B|$ even $\Rightarrow |B| \geqslant 2$, and $|B|$ odd $\Rightarrow |B| \geqslant 1$. A similar inequality holds for ρ, hence we can continue (4.2) with

$$\leqslant |\pi| + |\rho| + \tfrac{1}{2}k + \tfrac{1}{2}k = |\pi| + |\rho| + k; \quad (4.3)$$

equality holds in (04.3.2) if and only if π, ρ are both in $\mathscr{P}_{1\text{-}2}([k])$. Thus

$$\text{degree}\ \frac{Q_{\pi,\rho}}{(2n)_{|\pi|+|\rho|}} \leqslant k,$$

with equality only when $\pi, \rho \in \mathscr{P}_{1\text{-}2}([k])$. This and (2.10) prove that $\text{degree}(R_k) \leqslant k$. Moreover, if we use the formula found for the leading coefficient of $Q_{\pi,\rho}$ in Theorem 3.2, we obtain that the coefficient of order k in R_k is

$$\lambda_k = \sum_{\pi,\,\rho \in \mathscr{P}_{1\text{-}2}([k])} \frac{RP(\pi,\rho)}{(|\pi|+|\rho|)!} \cdot 2^{-(|\pi|+|\rho|)} \cdot ((-1)^{(|\pi|_o+|\rho|_o)/2} \cdot 2^{(|\pi|+|\rho|)} \cdot (|\pi|_o-1)!!\,(|\rho|_o-1)!!). \tag{4.4}$$

We finally observe that for every $\pi, \rho \in \mathscr{P}_{1\text{-}2}([k])$ we have $(|\pi|_o+|\rho|_o)/2 = |\pi|+|\rho|-k$ (this is easily obtained from the equalities $2|\pi|_e+|\pi|_o = k = 2|\rho|_e+|\rho|_o$, also used in (4.3)). Thus $(|\pi|_o+|\rho|_o)/2$ can be replaced with $|\pi|+|\rho|$ as exponent of -1 in (4.4), and in this way (4.1) is obtained. Q.E.D.

We next put into evidence a more convenient expression for the λ_k's found in Proposition 4.3, where they appear as summations over groups of permutations.

4.4. *Notations.* Let k be an even positive integer.

1. For every $\pi, \rho \in \mathscr{P}([k])$ we denote

$$\varphi(\pi,\rho) \overset{\text{def}}{=} \frac{(-1)^{|\pi|+|\rho|} RP(\pi,\rho)}{(|\pi|+|\rho|)!}; \tag{4.5}$$

if $\pi, \rho \in \mathscr{P}_{1\text{-}2}([k])$, this is the first factor in the general term of the summation in (4.1).

2. For every $\pi', \rho' \in \mathscr{P}_2([k])$ we denote

$$\psi(\pi',\rho') \overset{\text{def}}{=} \sum_{\pi \leqslant \pi',\ \rho \leqslant \rho'} \varphi(\pi,\rho). \tag{4.6}$$

3. If τ is a permutation of $[k]$ having only cycles of even length, and if $(\pi', \rho') \in \mathscr{P}_2([k]) \times \mathscr{P}_2([k])$ corresponds to τ by the bijection described in Remark 4.2.2, then we denote:

$$\tilde{\psi}(\tau) \overset{\text{def}}{=} \psi(\pi',\rho'). \tag{4.7}$$

We extend the function $\tilde{\psi}$ to the set $\mathscr{S}_k$ of all permutations of $[k]$, by setting $\tilde{\psi}(\tau) = 0$ for every $\tau \in \mathscr{S}_k$ which has at least one cycle of odd length.

4.5 Proposition. *In the context of Proposition* 4.3, *and with the notations introduced in* 4.4, *we have*

$$\lambda_k = \sum_{\tau \in \mathscr{S}_k} \tilde{\psi}(\tau). \tag{4.8}$$

Proof. If $\pi \in \mathscr{P}_{1\text{-}2}([k])$, then constructing a partition $\pi' \in \mathscr{P}_2([k])$ such that $\pi' \geqslant \pi$ amounts to choosing a way of grouping into pairs the $|\pi|_o$ one-element blocks of π; this can be done in $(|\pi|_o-1)!!$ ways.

But then we can write

$$\begin{aligned}
\lambda_k &= \sum_{\pi,\,\rho \in \mathscr{P}_{1\text{-}2}([k])} \varphi(\pi,\rho) \cdot (|\pi|_o-1)!! \cdot (|\rho|_o-1)!! \\
&= \sum_{\pi,\,\rho \in \mathscr{P}_{1\text{-}2}([k])} \ \sum_{\substack{\pi',\,\rho' \in \mathscr{P}_2([k]) \\ \pi' \geqslant \pi,\ \rho' \geqslant \rho}} \varphi(\pi,\rho) \\
&= \sum_{\pi',\,\rho' \in \mathscr{P}_2([k])} \ \sum_{\pi \leqslant \pi',\ \rho \leqslant \rho'} \varphi(\pi,\rho) \\
&= \sum_{\pi',\,\rho' \in \mathscr{P}_2([k])} \psi(\pi',\rho') = \sum_{\tau \in \mathscr{S}_k} \tilde{\psi}(\tau).
\end{aligned}$$

Q.E.D.

The next step of the argument is to point out that the function on permutations $\tilde{\psi}$ defined in 4.4.3 has a useful "multiplicative" property.

4.6. *Notation.* If k is an even positive integer, we denote by β_k the value of $\tilde{\psi}$ at the cyclic permutation $1 \to 2 \to \cdots \to k \to 1$ of $[k]$. We complete the sequence $\beta_2, \beta_4, \beta_6, \ldots$ obtained in this way to a sequence $(\beta_k)_{k=1}^{\infty}$, by setting $\beta_k = 0$ for k odd.

4.7. Proposition. *Let k be an even positive integer, let τ be in $\mathscr{S}_k$, and for every $1 \leqslant i \leqslant k$ let $c_i(\tau)$ denote the number of cycles of length i of τ. Then*

$$\tilde{\psi}(\tau) = \beta_1^{c_1(\tau)} \cdots \beta_k^{c_k(\tau)}. \tag{4.9}$$

Proof. For the duration of this proof, it is convenient to extend the various notations considered in 4.4 to the situation where we deal with partitions and permutations of an arbitrary finite set S (rather than just having $S = [k]$ for some k).

It is first of all obvious that, back in the Notations 2.7, the hypothesis of π, ρ being partitions of the special set $[k]$ was not essential; the number $RP(\pi,\rho)$ can be defined, in exactly the same way, for π, ρ partitions of an

arbitrary finite set S. It is also immediate that if $f: S_1 \to S_2$ is a bijection between finite sets, and if the partitions π_1, ρ_1 of S_1 are transported by f into the partitions π_2, ρ_2 of S_2, then $RP(\pi_1, \rho_1) = RP(\pi_2, \rho_2)$.

But then the number $\varphi(\pi, \rho)$ can be defined whenever π, ρ are partitions of some finite set S, by the same formula as in (4.5). Moreover, if S has even cardinality and if $\pi', \rho' \in \mathscr{P}_2(S)$ (i.e. π', ρ' are partitions of S into blocks of exactly two elements), then the same formula as in (4.6) can be used to define the number $\psi(\pi', \rho')$. It is clear that $\varphi(\pi, \rho)$ and $\psi(\pi', \rho')$ also have the property of invariance under transportations with bijections, which was observed for $RP(\pi, \rho)$ at the end of the preceding paragraph.

Now, let S be a finite set, let τ be a permutation of S, and let us define the number $\tilde{\psi}(\tau)$. If τ has at least one cycle of odd length (which in particular is always the case when $|S|$ is odd), then we take $\tilde{\psi}(\tau) = 0$. If τ has only cycles of even length, then $|S|$ must be even, and there exist $\pi', \rho' \in \mathscr{P}_2(S)$ such that the blocks of $\pi' \vee \rho'$ coincide with the cycles of τ (π', ρ' with this property can be constructed in a way similar to the one described in Remark 4.2.2); we define:

$$\tilde{\psi}(\tau) \stackrel{\text{def}}{=} \psi(\pi', \rho'). \tag{4.10}$$

This definition makes sense; indeed, if $\pi'', \rho'' \in \mathscr{P}_2(S)$ have the same property as π', ρ', then $\pi'' \vee \rho'' = \pi' \vee \rho'$, and it is easy to construct a bijection $f: S \to S$ which sends π' into π'' and ρ' into ρ''; we therefore have $\psi(\pi', \rho') = \psi(\pi'', \rho'')$, by the invariance of ψ under transportation with bijections.

From Equation (4.10) it is clear that $\tilde{\psi}$ itself is "invariant under transportation with bijections", which in this case means that $\tilde{\psi}(\tau)$ depends solely on the cycle structure of the permutation τ.

We will show the following: let S be a finite set, let τ be a permutation of S, and let S_1, S_2 be non-void disjoint subsets of S which are invariant under τ and have $S_1 \cup S_2 = S$. Then denoting by τ_1, τ_2 the permutations of S_1, S_2 which are induced by τ, we have

$$\tilde{\psi}(\tau) = \tilde{\psi}(\tau_1)\, \tilde{\psi}(\tau_2). \tag{4.11}$$

It is immediate that our goal in this proof, Equation (4.9), follows from an iteration of (4.11).

The case of (4.11) when τ has at least one cycle of odd length is trivial: one of τ_1, τ_2 must also have cycles of odd length, and both sides of (4.11) are zero. We will therefore assume that τ has only cycles of even length; the same must then be true for τ_1, τ_2 (and the cardinalities of S, S_1, S_2 are all even).

Let $\pi', \rho' \in \mathscr{P}_2(S)$ be such that the blocks of $\pi' \vee \rho'$ coincide with the cycles of τ. Then for $i = 1, 2$ we have that: S_i is both π'-saturated and ρ'-saturated; $\pi'_{|S_i}, \rho'_{|S_i} \in \mathscr{P}_2(S_i)$; and the blocks of $(\pi'_{|S_i}) \vee (\rho'_{|S_i})$ coincide with the cycles of τ_i. By taking (4.10) into account, we obtain that (4.11) reduces to:

$$\psi(\pi', \rho') = \psi(\pi'_{|S_1}, \rho'_{|S_1})\, \psi(\pi'_{|S_2}, \rho'_{|S_2}),$$

i.e., to

$$\sum_{\pi \leqslant \pi',\ \rho \leqslant \rho'} \varphi(\pi, \rho) = \left(\sum_{\pi_1 \leqslant \pi'|S_1,\ \rho_1 \leqslant \rho'|S_1} \varphi(\pi_1, \rho_1) \right) \cdot \left(\sum_{\pi_2 \leqslant \pi'|S_2,\ \rho_2 \leqslant \rho'|S_2} \varphi(\pi_2, \rho_2) \right). \tag{4.12}$$

Since a generic $\pi \leqslant \pi'$ (respectively $\rho \leqslant \rho'$) is assembled from $\pi_1 = \pi_{|S_1} \leqslant \pi'_{|S_1}$ and $\pi_2 = \pi_{|S_2} \leqslant \pi'_{|S_2}$ (respectively $\rho_1 = \rho_{|S_1} \leqslant \rho'_{|S_1}$ and $\rho_2 = \rho_{|S_2} \leqslant \rho'_{|S_2}$), (4.12) will in turn follow if we prove the equality

$$\varphi(\pi, \rho) = \varphi(\pi_{|S_1}, \rho_{|S_1})\, \varphi(\pi_{|S_2}, \rho_{|S_2}), \tag{4.13}$$

holding for partitions π, ρ of S such that S_i is both π-saturated and ρ-saturated, $i = 1, 2$. After substituting $\varphi(\pi, \rho)$, $\varphi(\pi_{|S_1}, \rho_{|S_1})$, $\varphi(\pi_{|S_2}, \rho_{|S_2})$ by what was their definition in (4.5), and after a simplification of signs, Equation (4.13) gets the form

$$RP(\pi, \rho) = \binom{|\pi| + |\rho|}{|\pi_{|S_1}| + |\rho_{|S_1}|} \cdot RP(\pi_{|S_1}, \rho_{|S_1}) \cdot RP(\pi_{|S_2}, \rho_{|S_2}). \tag{4.14}$$

Finally, (4.14) is obtained via a direct counting argument. Indeed, a generic (π, ρ)-restricted permutation counted by $RP(\pi, \rho)$ can be created in two steps:

(a) We first decide on which $|\pi_{|S_1}| + |\rho_{|S_1}|$ positions (out of the possible $|\pi| + |\rho|$) we want to place the blocks of π and ρ which are contained in S_1. The number of ways of doing this is counted by the binomial coefficient in (4.14).

(b) We then arrange the blocks of π and ρ which are contained in S_1 in the $|\pi_{|S_1}| + |\rho_{|S_1}|$ positions selected for them, and the blocks of π and ρ which are contained in S_2 within the remaining $|\pi_{|S_2}| + |\rho_{|S_2}|$ positions. These arrangements can be made in $RP(\pi_{|S_1}, \rho_{|S_1})$ and respectively $RP(\pi_{|S_2}, \rho_{|S_2})$ ways, and combined with the step (a) above this leads precisely to Equation (4.14). Q.E.D.

4.8. COROLLARY. *Let λ_k be as defined in Proposition* 4.3, *for $k \geqslant 2$, even; and let us also set $\lambda_0 = 1$ and $\lambda_k = 0$ for k odd. Then, as formal power series, we have*

$$\sum_{k=0}^{\infty} \frac{\lambda_k}{k!} z^k = \exp\left(\sum_{k=1}^{\infty} \frac{\beta_k}{k} z^k \right), \tag{4.15}$$

where the β_k's are as defined in Notation 4.6.

Proof. In view of (4.8) and (4.9), we can write

$$\lambda_k = \sum_{\tau \in \mathscr{S}_k} \beta_1^{c_1(\tau)} \cdots \beta_k^{c_k(\tau)}, \tag{4.16}$$

for every $k \geqslant 2$, even. Equation (4.16) also holds for k odd, when both its sides are zero (the right-hand side vanishes in this case because every $\tau \in \mathscr{S}_k$ must have at least one cycle of odd length). Hence we have

$$\sum_{k=0}^{\infty} \frac{\lambda_k}{k!} z^k = 1 + \sum_{k=1}^{\infty} \frac{z^k}{k!} \sum_{\tau \in \mathscr{S}_k} \beta_1^{c_1(\tau)} \beta_2^{c_2(\tau)} \cdots \beta_k^{c_k(\tau)}. \tag{4.17}$$

But the well-known exponential formula for cycles of permutations (see for instance [9] Section 4.2) says precisely that the right-hand side of (4.17) is equal to $\exp(\sum_{k=1}^{\infty} (\beta_k/k) z^k)$. Q.E.D.

4.9. *Remark.* Let us now fix an even positive integer k, and look for a convenient expression for the number β_k defined in Notation 4.6. By also taking into account the Notations 4.4, we see that:

$$\beta_k = \psi(\pi'_k, \rho'_k) = \sum_{\pi \leqslant \pi'_k,\ \rho \leqslant \rho'_k} \varphi(\pi, \rho), \tag{4.18}$$

where

$$\pi'_k = \{(1,2), (3,4), ..., (k-1, k)\}, \quad \rho'_k = \{(2,3), ..., (k-2, k-1), (k, 1)\}. \tag{4.19}$$

There are $2^{k/2}$ partitions $\pi \leqslant \pi'_k$ (since every such π is completely determined by the subset of the $k/2$ blocks of π'_k which are also blocks of π); similarly, there are $2^{k/2}$ partitions $\rho \leqslant \rho'_k$. Thus the sum on the right-hand side of (4.18) has 2^k terms. We will single out and give a special name to the term of the sum corresponding to $\pi = \pi'_k$, $\rho = \rho'_k$:

$$\alpha_k^{(1)} \stackrel{\text{def}}{=} \varphi(\pi'_k, \rho'_k). \tag{4.20}$$

Let us on the other hand consider, for every $j \geqslant 1$, the number:

$$\alpha_j^{(2)} \stackrel{\text{def}}{=} \begin{cases} \varphi(\{(1,2), (3,4), ..., (j-1, j)\}, \{(1), (2,3), ..., (j-2, j-1), (j)\}) \\ \quad \text{if} \quad j \text{ is even} \\ \varphi(\{(1,2), ..., (j-2, j-1), (j)\}, \{(1), (2,3), ..., (j-1, j)\}) \\ \quad \text{if} \quad j \text{ is odd.} \end{cases} \tag{4.21}$$

If we pick some partitions $\pi \leqslant \pi'_k$, $\rho \leqslant \rho'_k$ such that $(\pi, \rho) \neq (\pi'_k, \rho'_k)$, and if we form the join $\pi \vee \rho = \{B_1, ..., B_p\}$, then for every $1 \leqslant i \leqslant p$ the pair of partitions $(\pi_{|B_i}, \rho_{|B_i})$ will be of the form considered in (4.21). An argument identical to the one which proved Equation (4.13) in the proof of Proposition 4.7 (but now applied to a decomposition into p subsets, instead of 2 subsets) gives:

$$\varphi(\pi, \rho) = \prod_{i=1}^{p} \varphi(\pi_{|B_i}, \rho_{|B_i}) = \alpha_{|B_1|}^{(2)} \cdots \alpha_{|B_p|}^{(2)}.$$

We thus obtain the formula:

$$\beta_k = \alpha_k^{(1)} + \sum_{\substack{\pi \leqslant \pi'_k,\ \rho \leqslant \rho'_k \\ (\pi, \rho) \neq (\pi'_k, \rho'_k) \\ \pi \vee \rho = \{B_1, ..., B_p\}}} \alpha_{|B_1|}^{(2)} \cdots \alpha_{|B_p|}^{(2)}. \tag{4.22}$$

This formula can be further simplified, if we notice that the partition $\pi \vee \rho$ appearing in (4.22) is always an "interval partition". Or in other words: assuming the blocks $B_1, ..., B_p$ to be written in increasing order of their minimal elements, each of $B_2, ..., B_p$ is a subinterval of $[k]$, and B_1 either is a subinterval too, or it is of the form $\{1, ...l'\} \cup \{l'', ..., k\}$ for some $1 \leqslant l' < l'' \leqslant k$. For this reason the partition $\pi \vee \rho = \{B_1, ..., B_p\}$—and π, ρ themselves as a consequence—are completely determined if we know the cardinalities $|B_1|, ..., |B_p|$ and the value of $\max(B_1) \in [k]$. A moment's reflection shows that this observation leads to the following re-writing of (4.22): if $\mathscr{P}_{\text{ord}}(k)$ denotes the set of compositions (i.e. ordered partitions) of the number k,

$$\mathscr{P}_{\text{ord}}(k) = \{(l_1, ..., l_p) \mid l_1, ..., l_p \geqslant 1,\ l_1 + \cdots + l_p = k\}, \tag{4.23}$$

then

$$\beta_k = \alpha_k^{(1)} + \sum_{(l_1, ..., l_p) \in \mathscr{P}_{\text{ord}}(k)} l_1 \cdot \alpha_{l_1}^{(2)} \cdots \alpha_{l_p}^{(2)}. \tag{4.24}$$

Our next goal is to find a closed formula for the generating function of the β_k's. To this end, we will find closed forms for the generating functions

of the sequences $(\alpha_k^{(1)})_{k\geqslant 2,\, even}$ and $(\alpha_j^{(2)})_{j=1}^{\infty}$ introduced in Remark 4.9, and then we will make use of Equation (4.24).

4.10. *Remark.* The first terms in the sequence $(\alpha_j^{(2)})_{j=1}^{\infty}$ are $\alpha_1^{(2)}=1/2$, $\alpha_2^{(2)}=-1/3$, $\alpha_3^{(2)}=5/24$, $\alpha_4^{(2)}=-2/15$, $\alpha_5^{(2)}=61/720$, ... This sequence is easy to track due to the fortunate coincidence that the number of restricted permutations considered in (4.21) counts (modulo taking an inverse in the symmetric group) the permutations τ of $[j+1]$ which satisfy the alternating inequalities $\tau(1)>\tau(2)<\tau(3)>\cdots$ The enumeration of the permutations with the latter property is well-known (see e.g. [11], Section 3.16). In fact, by classical work of D. André [1] we have the identities

$$\begin{aligned} 1+\alpha_1^{(2)}z^2+\alpha_3^{(2)}z^4+\alpha_5^{(2)}z^6+\cdots &= \sec z,\\ z-\alpha_2^{(2)}z^3-\alpha_4^{(2)}z^5-\alpha_6^{(2)}z^7-\cdots &= \tan z. \end{aligned} \tag{4.25}$$

In order to determine the generating function for the numbers $\alpha_k^{(1)}$, we observe a connection between $\alpha_k^{(1)}$ and $\alpha_{k-2}^{(2)}$. Due to symmetry, $(k/2)^{-1}RP(\pi_k', \rho_k')$ counts the (π_k', ρ_k')-restricted permutations which start with a prescribed block of π_k' –the block $(k-1, k)$, for instance. But it is an easy exercise to see that the (π_k', ρ_k')-restricted permutations satisfying this extra condition are in bijection with the permutations counted by the RP-number implicitly involved in (4.21), for $j=k-2$. Let us denote (for notational convenience) the latter RP-number by just RP_{k-2}. We have:

$$\alpha_k^{(1)}=\varphi(\pi_k', \rho_k')=\frac{(-1)^k\, RP(\pi_k', \rho_k')}{k!},$$

and similarly

$$\alpha_{k-2}^{(2)}=\frac{(-1)^{k-1}\, RP_{k-2}}{(k-1)!};$$

so the relation $(k/2)^{-1}\, RP(\pi_k', \rho_k')=RP_{k-2}$ gets converted into:

$$\alpha_k^{(1)}=-\tfrac{1}{2}\alpha_{k-2}^{(2)}, \qquad \text{for} \quad k\geqslant 4,\ even. \tag{4.26}$$

As a consequence of (4.25), (4.26) and the obvious fact that $\alpha_2^{(1)}=1/2$, we obtain

$$\sum_{k\geqslant 2,\ even} \alpha_k^{(1)} z^k=\tfrac{1}{2}z\tan z. \tag{4.27}$$

In order to convert Equation (4.24) into a formula at the level of generating functions, we will use the following:

4.11. Lemma. *Let $(\alpha_k)_{k=1}^{\infty}$ be a sequence of real numbers, and for every $k\geqslant 1$ let us define*

$$a_k=\sum_{(l_1,\ldots,l_p)\in\mathscr{P}_{\mathrm{ord}}(k)} l_1\cdot\alpha_{l_1}\cdots\alpha_{l_p}, \tag{4.28}$$

where $\mathscr{P}_{\mathrm{ord}}(k)$ is as in (4.23). *Consider the generating functions $f(z)=\sum_{k=1}^{\infty}\alpha_k z^k$ and $g(z)=\sum_{k=1}^{\infty}a_k z^k$. Then*

$$g(z)=zf'(z)(1-f(z))^{-1}. \tag{4.29}$$

Proof. By taking into account that $(1-f(z))^{-1}=\sum_{k=0}^{\infty}f(z)^k$ and that $zf'(z)=\sum_{l=1}^{\infty}l\,\alpha_l z^l$, it is immediately verified that the coefficient of order k of $zf'(z)(1-f(z))^{-1}$ has the form indicated on the right hand side of (4.28). Q.E.D.

The Proof of Theorem 1.2. We will now put together the calculations made throughout the present section, in order to obtain Equation (1.7) of Theorem 1.2.

Let $(\alpha_j^{(2)})_{j=1}^{\infty}$ be as defined in (4.21) of Remark 4.9, and let us set, for every $k\geqslant 1$,

$$a_k^{(2)}=\sum_{(l_1,\ldots,l_p)\in\mathscr{P}_{\mathrm{ord}}(k)} l_1\cdot\alpha_{l_1}^{(2)}\cdots\alpha_{l_p}^{(2)}. \tag{4.30}$$

Then Equation (4.24) in Remark 4.9 becomes

$$\beta_k=\alpha_k^{(1)}+a_k^{(2)}, \qquad \text{for} \quad k\geqslant 2,\ even,$$

so that

$$\sum_{k=1}^{\infty}\beta_k z^k=\sum_{k\geqslant 2,\ even}\beta_k z^k=\sum_{k\geqslant 2,\ even}\alpha_k^{(1)}z^k+\sum_{k\geqslant 2,\ even}a_k^{(2)}z^k. \tag{4.31}$$

The series $\sum_{k\geqslant 2,\ even}\alpha_k^{(1)}z^k$ is given in closed form in (4.27). In order to obtain a closed form for $\sum_{k\geqslant 2,\ even}a_k^{(2)}z^k$, we will use Lemma 4.11. More precisely, let us set $f(z)=\sum_{k=1}^{\infty}\alpha_k^{(2)}z^k$ and $g(z)=\sum_{k=1}^{\infty}a_k^{(2)}z^k$. Then $f(z)=((\sec z-1)-(\tan z-z))/z$ by (4.25), so that Lemma 4.11 gives

$$\begin{aligned} g(z)=zf'(z)(1-f(z))^{-1}&=\frac{z\sin z-\cos z+\cos^2 z-z+\sin z\cos z}{\cos z(\cos z+\sin z-1)}\\ &=\frac{u(z)+v(z)}{2\cos z\cdot\sin(z/2)(\cos(z/2)-\sin(z/2))}, \end{aligned}$$

where we denoted $u(z)=z\sin z-\cos z+\cos^2 z$ (even expression in z) and $v(z)=-z+\sin z\cos z$ (odd expression in z). The closed form for

$\sum_{k\geqslant 2,\, even} a_k^{(2)} z^k$ is obtained by separating the even part of the latter expression. It is convenient to first amplify with $\cos z/2 + \sin z/2$, and thus get

$$\sum_{k\geqslant 2,\, even} a_k^{(2)} z^k = \frac{u(z)\sin z/2 + v(z)\cos z/2}{2\cos z \cdot \sin z/2 \cdot \cos z}.$$

We then substitute $u(z), v(z)$ by their expressions, and observe that some simplifications occur (namely, that $z\cdot\sin z\cdot\sin(z/2) - z\cdot\cos(z/2) = -z\cdot\cos(z/2)\cdot\cos z$, and that $-\cos z\cdot\sin(z/2) + \sin z\cdot\cos z\cdot\cos(z/2) = \cos^2 z\cdot\sin(z/2)$); the result of the calculation arrives in this way to be

$$\sum_{k\geqslant 2,\, even} a_k^{(2)} z^k = 1 - \frac{z\cos z/2}{2\cos z\cdot\sin z/2}. \tag{4.32}$$

We substitute (4.27) and (4.32) in (4.31), and we obtain (after some more simplifications)

$$\sum_{k=1}^{\infty} \beta_k z^k = 1 - \frac{z}{2}\cot(z/2). \tag{4.33}$$

In view of Corollary 4.8, Equation (1.7) which we want to establish comes to

$$\exp\left(\sum_{k=1}^{\infty} \frac{\beta_k}{k} z^k\right) = \frac{z/2}{\sin(z/2)}. \tag{4.34}$$

Both sides of (4.34) are formal power series with constant coefficient equal to 1, so it will suffice to prove the equality obtained from (4.34) by taking a formal logarithm and differentiating. But applying these operations in (4.34) leads to:

$$\frac{1}{z}\left(\sum_{k=1}^{\infty} \beta_k\, z^k\right) = \left(\frac{z/2}{\sin(z/2)}\right)' \cdot \frac{\sin(z/2)}{z/2},$$

which is easily verified when $\sum_{k=1}^{\infty} \beta_k z^k$ is replaced from (4.33).

This concludes the proof of Theorem 1.2. We note that in Proposition 4.3 it was only stated that degree$(R_k) \leqslant k$ (instead of "$=k$"), because from the form given to λ_k at that stage it was not obvious that $\lambda_k \neq 0$. However, this is easily checked now that (1.7) is established. Q.E.D.

5. THE PROOF OF COROLLARY 1.3

For the reader's convenience, we make an outline of how the asymptotics (1.12) can be obtained, as a consequence of Theorem 1.2. As mentioned in the Introduction, this is a straightforward application of the method of moments.

5.1. *The Proof of Corollary* 1.3. For every $n \geqslant 1$, let μ_n be the probability distribution defined by Equation (1.8), and let σ_n^2 denote the variance of μ_n:

$$\sigma_n^2 = \int_{-\infty}^{\infty} t^2\, d\mu_n(t) = \binom{2n}{n}^{-2} \sum_{\gamma\in\Gamma_{2n}} [area(\gamma)]^2.$$

As implied by the Equations (1.4) and (1.5), for large n we have $\sigma_n \sim n/\sqrt{12}$.

Let us also consider, for every $n \geqslant 1$, the measure ν_n which is obtained when we normalize μ_n by its variance. (This means, in other words, that ν_n is obtained by replacing δ_k by δ_{k/σ_n} in the right-most summation of (1.8)). The moments of order k of μ_n and ν_n are then connected by the formula:

$$\int_{-\infty}^{\infty} t^k\, d\nu_n(t) = \sigma_n^{-k} \int_{-\infty}^{\infty} t^k\, d\mu_n(t),\ k \geqslant 0. \tag{5.1}$$

If we take into account that (for k fixed and $n\to\infty$) $\sigma_n^{-k} \sim 12^{k/2} n^{-k}$, then from Equations (5.1) and (1.11) we obtain that

$$\lim_{n\to\infty} \int_{-\infty}^{\infty} t^k\, d\nu_n(t) = 12^{k/2}\lambda_k, \qquad k \geqslant 0. \tag{5.2}$$

The method of moments says that: if the limits in (5.2) exist, and if in addition the exponential generating series of these limits has positive radius of convergence, then the sequence of probability distributions $(\nu_n)_{n=1}^{\infty}$ is weakly convergent. And moreover (under the two hypotheses stated above), the weak limit ν of the ν_n's is uniquely determined by the fact that it has finite moments of all orders, which equal

$$\int_{-\infty}^{\infty} t^k\, d\nu(t) = \lim_{n\to\infty} \int_{-\infty}^{\infty} t^k\, d\nu_n(t), \qquad k \geqslant 0.$$

(For a presentation of these facts, see e.g. [5], Sect. 8.12.)

But now, the exponential generating series of the limits in (5.2) is, by the Theorem 1.2

$$\sum_{k=0}^{\infty} \frac{\lambda_k 12^{k/2}}{k!} z^k = \sum_{k=0}^{\infty} \frac{\lambda_k}{k!} (z\sqrt{12})^k = \frac{z\sqrt{3}}{\sin(z\sqrt{3})}, \tag{5.3}$$

with positive radius of convergence $\pi/\sqrt{3}$. Thus the considerations of the preceding paragraph do indeed apply; moreover the right-hand side of

(5.3) yields a closed form for the exponential moment-generating series of the limit distribution $\nu = \lim_{n\to\infty} \nu_n$. As is well-known, we can obtain back ν from this moment-generating series, by doing the substitution $z \mapsto iz$, followed by an inverse Fourier transform. The inverse Fourier transform needed here can be found for instance in [8], Formula 7.21; we get that ν is absolutely continuous, with

$$\frac{d\nu(t)}{dt} = \frac{\pi}{4\sqrt{3}} \operatorname{sech}^2\left(\frac{\pi t}{2\sqrt{3}}\right). \tag{5.4}$$

The distribution function of ν is then found to be $F(t) = \frac{1}{2}(1 + \tanh(\pi t/\sqrt{12}))$. Since F is continuous everywhere on $\mathbf{R}$, one of the equivalent characterizations of a weak limit of distributions (see e.g. [5], Sect. 8.1) says that

$$\lim_{n\to\infty} \nu_n(I) = \nu(I) = F(b) - F(a), \tag{5.5}$$

for every bounded interval $I \subseteq \mathbf{R}$ having $\inf(I) = a$, $\sup(I) = b$. The asymptotic formula (1.12) is easily obtained from (5.5), if we observe that

$$\binom{2n}{n}^{-2} |\{\gamma \in \Gamma_{2n} \mid n\alpha < area(\gamma) < n\beta\}| = \nu_n\left(\left(\frac{n}{\sigma_n}\alpha, \frac{n}{\sigma_n}\beta\right)\right),$$

and if we use the fact that $n/\sigma_n \to \sqrt{12}$ for $n \to \infty$. Q.E.D.

5.2. *Remark*. As explained in the Introduction, the asymptotics (1.12) can be obtained by using Lévy's formula for the area enclosed by a random 2-dimensional Brownian path (and the connection of the random walks on $\mathbf{Z}^2$ with the 2-dimensional Brownian motion). The above derivation of (1.12) can be in fact viewed as a combinatorial approach to Lévy's formula, where an extra piece of information—the existence of rational expressions giving the moments—is put into evidence at the level of random walks. We believe that this is by no means the only example of a combinatorial piece of structure which exists at the level of the random walks, and is reflected in some way (in the limit) in the framework of the Brownian motion. To give an example, it is tempting to conjecture that the inequality

$$|\{\gamma \in \Gamma_{2n} \mid area(\gamma) = k\}| \geqslant |\{\gamma \in \Gamma_{2n} \mid area(\gamma) = k+1\}| \tag{5.6}$$

holds for every $n \geqslant 1$ and $k \geqslant 0$. The "limit as $n \to \infty$" of (5.6) is just the obvious fact that the density function in (5.4) is decreasing on $[0, \infty)$. The inequality (5.6) is challenging, due to its simplicity, and is strongly supported by numerical evidence (e.g., for $n \leqslant 36$), but we don't know a proof at this moment.

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