Crossings of set-partitions and addition of graded-independent random variables

James A. Mingo*
Department of Mathematics and Statistics, Queen's University
Kingston, K7L 3N6, CANADA
james.mingo@queensu.ca

Alexandru Nica†
Department of Mathematics
University of Michigan
Ann Arbor, MI, 48109, U.S.A.
andu@math.lsa.umich.edu

Abstract

We consider a q-deformation, introduced in [7], of the cumulants associated with a linear functional on polynomials; combinatorially, the deformation is defined using crossing numbers of set-partitions. The paper is concerned with the case $q = -1$. We show that if the linear functional $\mu : \mathbf{C}[X] \to \mathbf{C}$ is symmetric (i.e. $\mu(X^n) = 0$ for n odd), then the exponential generating function of the even (-1)-cumulants of μ is equal to $\log \left(\int e^{t^2 z} d\mu(t) \right)$, (as a formal power series in z). We discuss the connection of this fact with the form of independence for (non-commutative) random variables which corresponds to the operation of $\mathbf{Z}_2$-graded tensor product.

1 Introduction and statement of the result

The idea that counting crossings of set-partitions might play a role in non-commutative probabilistic considerations appeared in [7], in the context of a q-deformation of the R-transform of Voiculescu. The R-transform plays a central role in the "harmonic analysis" of free random variables, and can be approached in two ways: (a) by the original method of Voiculescu [13], which consists in modeling probability distributions via a construction with Toeplitz operators; (b) by the method of Speicher [10], which uses the combinatorics of non-crossing partitions and the so-called "free (or non-crossing) cumulants" associated to a probability distribution. For the q-deformation in [7], one starts with the approach of Voiculescu, and replaces (in the Toeplitz construction of [13]) the unilateral shift S by

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a weighted shift S_q satisfying the relation $S_q^* S_q = qS_qS_q^* + I$. Then the R_q-transform so obtained turns out to also have a nice combinatorial description in terms of *q-cumulants*. The latter notion is defined by formula (1.1) (which, modulo certain renormalizations with q-factorials, is the same as Equation (1.10) of [7]).

1.1 Definition Let $\mu : \mathbf{C}[X] \to \mathbf{C}$ be a linear functional with $\mu(1) = 1$, and let q be a parameter. We define the *q-cumulants* of μ to be the sequence $(\alpha_n)_{n=1}^{\infty}$, defined implicitly by the sequence of equations:

$$\begin{cases} \mu(X) &= \alpha_1 \\ \mu(X^2) &= \alpha_2 + \alpha_1^2 \\ \mu(X^3) &= \alpha_3 + 3\alpha_2\alpha_1 + \alpha_1^3 \\ \mu(X^4) &= \alpha_4 + 4\alpha_3\alpha_1 + (2+q)\alpha_2^2 + 6\alpha_2\alpha_1^2 + \alpha_1^4 \\ &\vdots \\ \mu(X^n) &= \displaystyle\sum_{\substack{\pi\in\mathcal{P}([n]) \\ \pi=\{B_1,\ldots,B_r\}}} q^{c_o(\pi)}\alpha_{|B_1|}\cdots\alpha_{|B_r|} \\ &\vdots \end{cases} \tag{1.1}$$

where $\mathcal{P}([n])$ is the set of all partitions of $[n] \overset{def}{=} \{1, \ldots, n\}$, $|B_i|$ denotes the number of elements of the set B_i, and $c_o(\pi)$ is the number of *left-reduced crossings* of the partition $\pi \in \mathcal{P}([n])$ (see Definitions 2.1 below).

Note that (1.1) is a lower triangular system of (non-linear) equations, hence it always has a unique solution.

For $q = 1$, the α_n's determined by (1.1) are what are classically called the cumulants, or semi-invariants of μ (see e.g. [9], Section II.12.8), while for $q = 0$ they are the non-crossing (or free) cumulants of Speicher [10]. The importance of the sequence of q-cumulants in these two cases stems from their property of linearizing the addition of *commuting independent* and of *freely independent* random variables, respectively. The notions of independence considered here are formulated in the framework of algebras with specified states (as in [6] or [14], for instance).

1.2 Definitions

1° We shall call $(\mathcal{A}, \varphi)$ a *non-commutative probability space* if $\mathcal{A}$ is a unital complex algebra and $\varphi : \mathcal{A} \to \mathbf{C}$ is a linear functional with $\varphi(1) = 1$. If $\mathcal{A}$ is a C*-algebra and φ is positive, then $(\mathcal{A}, \varphi)$ is called a *C^*-probability space*. Given $(\mathcal{A}, \varphi)$ and an element $a \in \mathcal{A}$, the functional $\mu_a : \mathbf{C}[X] \to \mathbf{C}$ defined by: $\mu_a(f) = \varphi(f(a))$, $f \in \mathbf{C}[X]$, is called *the distribution of* a (with respect to φ).

2^{o} The unital subalgebras $\mathcal{A}_1, \mathcal{A}_2 \subseteq \mathcal{A}$ are said to be *commuting independent* in $(\mathcal{A}, \varphi)$ if they commute ($a_1 a_2 = a_2 a_1$ for every $a_1 \in \mathcal{A}_1$ and $a_2 \in \mathcal{A}_2$), and if moreover: $\varphi(a_1 a_2) = \varphi(a_1)\varphi(a_2)$, for every $a_1 \in \mathcal{A}_1$ and $a_2 \in \mathcal{A}_2$.

3^{o} The unital subalgebras $\mathcal{A}_1, \mathcal{A}_2 \subseteq \mathcal{A}$ are called *freely independent* in $(\mathcal{A}, \varphi)$ if for every word whose elements ("letters") alternate between $\mathcal{A}_1$ and $\mathcal{A}_2$, $a_1 \in \mathcal{A}_{i_1}$, $a_2 \in \mathcal{A}_{i_2}, \ldots, a_k \in \mathcal{A}_{i_k}$, with $i_1 \neq i_2$, $i_2 \neq i_3, \ldots, i_{k-1} \neq i_k$, we have the implication: $\varphi(a_1) = \varphi(a_2) = \cdots = \varphi(a_k) = 0 \Rightarrow \varphi(a_1 a_2 \cdots a_k) = 0$.

The linearizing property of cumulants for $q = 1$ and $q = 0$ can be stated thusly.

1.3 Linearization for q = 1 (see e.g. [9], Section II.12.8, or [7], the comment in Remark 1.5) Let $(\alpha_n)_{n=1}^{\infty}$ be the sequence of 1-cumulants (semi-invariants) of the functional $\mu : \mathbf{C}[X] \to \mathbf{C}$, and let us denote by $R_1(\mu)$ the formal power series:

$$[R_1(\mu)](z) = \sum_{n=1}^{\infty} \frac{\alpha_n}{n!} z^n \qquad (\alpha_n\text{'s for } q = 1). \tag{1.2}$$

Then

$$[R_1(\mu)](z) = \log \left(\sum_{n=0}^{\infty} \frac{\mu(X^n)}{n!} z^n \right), \tag{1.3}$$

where by the logarithm we mean the inverse of the map exp, viewed formally as a bijection from power series without constant term onto power series with constant term equal to 1. The relation between (1.3) and (1.1) is that if we exponentiate (1.3), and take the coefficient of z^n on both sides, then the n-th equation in (1.1) is obtained.

A consequence of (1.3) is that whenever $\mathcal{A}_1, \mathcal{A}_2 \subseteq \mathcal{A}$ are commuting independent subalgebras in some $(\mathcal{A}, \varphi)$, we have:

$$R_1(\mu_{a_1+a_2}) = R_1(\mu_{a_1}) + R_1(\mu_{a_2}) \tag{1.4}$$

for every $a_1 \in \mathcal{A}_1$, $a_2 \in \mathcal{A}_2$, (with the distributions $\mu_{a_1}, \mu_{a_2}, \mu_{a_1+a_2}$ defined as in $1.2.1^{\text{o}}$). Indeed, if the Fourier transform $[\mathcal{F}(\mu)](z)$ of μ is formally defined as the power series obtained by taking the Taylor expansion of the exponential in $\int e^{itz} d\mu(t)$, then $R_1(\mu)$ is just $\log[\mathcal{F}(\mu)](-iz)$. For a_1 and a_2 commuting independent, we have that $\mu_{a_1+a_2}$ is the convolution of μ_{a_1} and μ_{a_2}, and (1.4) follows from the multiplicativity of the Fourier transform.

1.4 Linearization for $\mathbf{q = 0}$ Let $(\alpha_n)_{n=1}^{\infty}$ be the sequence of 0-cumulants of the functional $\mu : \mathbf{C}[X] \to \mathbf{C}$, and $R(\mu)$ be the formal power series:

$$[R(\mu)](z) \;=\; \sum_{n=1}^{\infty} \alpha_n z^n \qquad\qquad (\alpha_n\text{'s for } q = 0); \tag{1.5}$$

this is Speicher's equivalent definition of the R-transform of Voiculescu. Note that in contrast to Equation (1.2), $R(\mu)$ is the ordinary (rather than exponential) generating series for the corresponding sequence of cumulants. Then

$$\Big(\frac{1 + [R(\mu)](z)}{z}\Big)^{<-1>} (\zeta) \;=\; \sum_{n=0}^{\infty} \frac{\mu(X^n)}{\zeta^{n+1}}, \tag{1.6}$$

where "$< -1 >$" denotes formal inversion under composition for power series (see Theorem 2.9 in [13], also the Theorem in [10], Section 3). If $\mathcal{A}_1, \mathcal{A}_2 \subseteq \mathcal{A}$ are freely independent subalgebras in some $(\mathcal{A}, \varphi)$, then for every $a_1 \in \mathcal{A}_1$, $a_2 \in \mathcal{A}_2$, we have that:

$$R(\mu_{a_1+a_2}) \;=\; R(\mu_{a_1}) + R(\mu_{a_2}); \tag{1.7}$$

for the proof of (1.7), see Corollary 2.2 of [13], also Section 4 of [10].

Now, in connection to the considerations of 1.3, 1.4, it is natural to raise the following question:

Are there other values of $q \neq 0, 1$, for which there is some form of independence for non-commutative random variables, and for which analogues of the formulas (1.4) and (1.7) hold?

The interest in this problem was aroused by the work in [2] (see also [3]), which suggested that a theory of q-independence might be developed around the so-called *q-commutation relations*; the deformation of the R-transform discussed in [7] is in fact based on the 1-dimensional instance of these relations.

On the other hand, however, very tangible evidence was recently produced in [5] and [11] which showed that for $q \neq 0, 1$, one cannot expect to have some form of q-independence defined in terms of moments of non-commutative random variables, and some q-cumulants (defined by (1.1) or some other formula) which would linearize the addition of q-independent random variables, in an arbitrary non-commutative probability space $(\mathcal{A}, \varphi)$.

In this paper we consider the case $q = -1$, and note that an interesting form of independence exists when $(\mathcal{A}, \varphi)$ is $\mathbf{Z}_2$-graded. This leaves open the possibility that (without ignoring the results of van Leeuwen-Maassen and Speicher in [5], [11]), a notion of q-independence may exist when the non-commutative probability space has some additional structure.

1.5 Definition A non-commutative probability space is $\mathbf{Z}_2$-graded if, together with $\mathcal{A}$ and φ of 1.2.1°, one is also given an automorphism ($*$-automorphism in the C^*-case) of order two, $\gamma : \mathcal{A} \to \mathcal{A}$, such that $\varphi \circ \gamma = \varphi$. Recall that an element $a \in \mathcal{A}$ is said to be homogeneous if $\gamma(a) = (-1)^{\partial a} a$, where $\partial a \in \{0, 1\}$ is called the degree of a. A unital subalgebra $\mathcal{A}_1 \subseteq \mathcal{A}$ is called homogeneous if it is invariant under γ. Two homogeneous unital subalgebras $\mathcal{A}_1, \mathcal{A}_2 \subseteq \mathcal{A}$ will be called *graded independent* if they gradedly commute, i.e. $a_1 a_2 = (-1)^{\partial a_1 \partial a_2} a_2 a_1$ for all homogeneous $a_1 \in \mathcal{A}_1$ and $a_2 \in \mathcal{A}_2$, and if $\varphi(a_1 a_2) = \varphi(a_1)\varphi(a_2)$ for all $a_1 \in \mathcal{A}_1$ and $a_2 \in \mathcal{A}_2$.

Although we couldn't find it treated in the literature, graded independence is most likely part of the folklore of quantum probability. The corresponding "product operation" (in the same way as the tensor product and the free product correspond to commuting and free independence, respectively) is the graded tensor product of algebras, as described for instance in [1], Section 14, (see also Sections 3.3–3.4 below).

The connection with cumulants is thus.

1.6 Definition Let $\mu : \mathbf{C}[X] \to \mathbf{C}$ be a functional such that $\mu(1) = 1$ and $\mu(X^n) = 0$ for odd n, and let $(\alpha_n)_{n=1}^{\infty}$ be the sequence of (-1)-cumulants of μ. Then $\alpha_n = 0$ for odd n. We define the R_{-1}-transform of μ to be the formal power series

$$[R_{-1}(\mu)](z) \;=\; \sum_{n=1}^{\infty} \frac{\alpha_{2n}}{n!} z^{2n} \qquad (\alpha_n\text{'s for } q = -1). \tag{1.8}$$

1.7 Theorem (the case q = –1)

1° For μ as in 1.6, we have the equality:

$$[R_{-1}(\mu)](z) \;=\; \log\left(\sum_{n=0}^{\infty} \frac{\mu(X^{2n})}{n!} z^{2n}\right). \tag{1.9}$$

2° If $(\mathcal{A}, \gamma, \varphi)$ is a graded non-commutative probability space, and if $\mathcal{A}_1, \mathcal{A}_2 \subseteq \mathcal{A}$ are graded independent subalgebras (in the sense of Definition 1.5), then for homogeneous elements $a_1 \in \mathcal{A}_1$, $a_2 \in \mathcal{A}_2$, with $\partial a_1 = \partial a_2 = 1$, we have:

$$R_{-1}(\mu_{a_1+a_2}) \;=\; R_{-1}(\mu_{a_1}) + R_{-1}(\mu_{a_2}). \tag{1.10}$$

Note that the distributions appearing in Eqn. (1.10) have vanishing moments of odd orders (and hence their R_{-1}-transforms are defined). Indeed, if $a \in \mathcal{A}$ has $\partial a = 1$, then a^n also has $\partial(a^n) = 1$ for odd n, and the condition $\varphi \circ \gamma = \varphi$ applied to a^n yields $\varphi(a^n) = 0$, i.e. $\mu_a(X^n) = 0$ for odd n. The reason for considering in 1.7.2° only the case when $\partial a_1 = \partial a_2 = 1$ is that when either $\partial a_1 = 0$ or $\partial a_2 = 0$, we have just commuting independence.

Note also that if μ is in fact a probability measure with compact support then, by 1.7.1°

$$[R_{-1}(\mu)](z) = \log\left(\int_{-\infty}^{\infty} e^{t^2 z^2} d\mu(t)\right) \tag{1.11}$$

is a complex analytic function in a neighbourhood of 0.

By comparing Equations (1.9) and (1.3) we obtain that.

1.8 Corollary Let $(\mathcal{A}, \varphi)$ be a non-commutative probability space and let $a \in \mathcal{A}$ be such that $\varphi(a^n) = 0$ for odd n. Then

$$[R_{-1}(\mu_a)](z) = [R_1(\mu_{a^2})](z^2). \tag{1.12}$$

In 2.7 below we shall in fact prove (1.12) first and from this deduce (1.9). We should also point out that (1.9) and (1.12) are essentially statements about the combinatorics of crossings of set-partitions, and can be given an elegant proof via an involution on the set of 2-divisible partitions of $\{1, 2, \ldots, 2n\}$.

The rest of the paper is divided into two sections. In Section 2 we present the considerations on crossings of set-partitions which lead to the proof of Theorem 1.7. In Section 3, without attempting to be comprehensive, we survey a few facts about graded independence, and its connections with the R_{-1}-transform.

2 Proof of the result

We start by recording, in 2.1–2.3, a few definitions and basic facts about crossings of set-partitions.

2.1 Definitions 1° If $\pi = \{B_1, \ldots, B_r\}$ is a partition of $\{1, \ldots, n\}$ (i.e., $B_1, \ldots, B_r$ are non-void, pairwise disjoint sets, such that $B_1 \cup \cdots \cup B_r = \{1, \ldots, n\}$), then the equivalence relation on $\{1, \ldots, n\}$ with equivalence classes $B_1, \ldots, B_r$ will be denoted by $\stackrel{\pi}{\sim}$. The sets $B_1, \ldots, B_r$ are called the *blocks* of π. The number of elements in the block B_k, $1 \le k \le r$, will be denoted by $|B_k|$. For the sake of brevity, we will use in what follows the notation $[n] \stackrel{def}{=} \{1, \dots, n\}$.

2^{o} For every integer $n \geq 1$, the set of all partitions of $[n]$ will be denoted by $\mathcal{P}([n])$.

3^{o} For $\pi \in \mathcal{P}([n])$, we shall call a *crossing* of π a 4-tuple (i, i', j, j') such that $1 \leq i < i' < j < j' \leq n$, and such that $i \stackrel{\pi}{\sim} j \stackrel{\pi}{\not\sim} i' \stackrel{\pi}{\sim} j'$. A crossing (i, i', j, j') of π will be called *left-reduced* if i and i' are minimal in the blocks of π containing them (i.e: there exists no $1 \leq i_o < i$ such that $i_o \stackrel{\pi}{\sim} i$, and similarly for i'). The total number of crossings and the number of left-reduced crossings of the partition π will be denoted by $c(\pi)$ and $c_o(\pi)$, respectively ($0 \leq c_o(\pi) \leq c(\pi)$).

4^{o} It is easily verified that for $\pi \in \mathcal{P}([n])$ we have $c(\pi) = 0 \Leftrightarrow c_o(\pi) = 0$. A partition π with $c(\pi) = c_o(\pi) = 0$ is said to be *non-crossing.*

2.2 Remark A picture of the number of crossings of a given partition $\pi \in \mathcal{P}([n])$ can be obtained by marking n points $P_1, \dots, P_n$ (in counter-clockwise order, say) around a circle, and then by drawing the line segment P_iP_j whenever $i \stackrel{\pi}{\sim} j$; then $c(\pi)$ is the number of pairs of intersecting chords that were drawn in this way. In order to put into evidence the smaller number $c_o(\pi)$, we consider again the points $P_1, \dots, P_n$ on the circle, but only draw, for each block B of π, the bouquet of chords $\{P_{\min(B)}\ P_j \mid j \in B,\ j \neq \min(B)\}$ (and then count the number of intersections). For example, the situation of a partition with only two blocks is illustrated in the Figure 1.

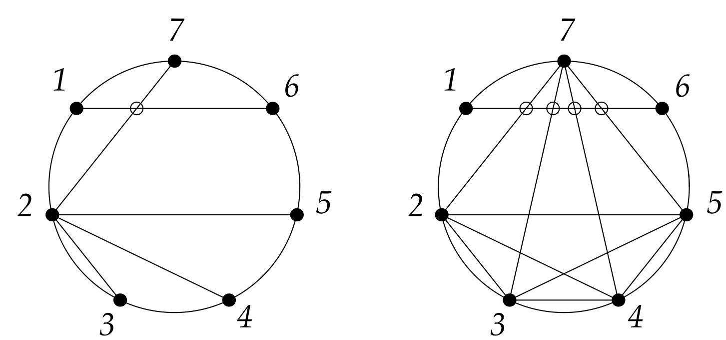

Figure 1: $\pi = \{\{1, 6\}, \{2, 3, 4, 5, 7\}\}$ has $c(\pi) = 4,\ c_o(\pi) = 1$.

From the pictorial image for $c(\pi)$, it is clear that this quantity is preserved when either a rotation or the order-reversing permutation is applied to the partition π. More precisely, let us consider the natural action of the symmetric group $\mathcal{S}_n$ on $\mathcal{P}([n])$ (where for $\tau \in \mathcal{S}_n$ and $\pi = \{B_1, \dots, B_r\} \in \mathcal{P}([n])$ we have $\tau \cdot \pi \stackrel{def}{=} \{\tau(B_1), \dots, \tau(B_r)\}$), let ρ be the cycle

$\rho(i) \equiv i+1 \bmod(n)$, and let σ be the order-reversing permutation $\sigma(i) \equiv n+1-i$. Then $c(\rho \cdot \pi) = c(\pi)$ and $c(\sigma \cdot \pi) = c(\pi)$ for every $\pi \in \mathcal{P}([n])$, due to the fact that the circular pictures for $c(\rho \cdot \pi)$ and $c(\sigma \cdot \pi)$ are obtained from the one for $c(\pi)$ via some rigid motions of the plane. As is easily seen by looking at examples, these invariance properties are not shared by the number of left-reduced crossings $c_o(\pi)$.

2.3 Remark For $\emptyset \neq A, B \subseteq \mathbf{N} \setminus \{0\}$ finite sets, such that $A \cap B = \emptyset$ and $\min(A) < \min(B)$, let us make the notations:

$$
\begin{aligned}
c_o(A,B) &= \mathrm{card}\{(a,b) \mid a \in A,\ b \in B,\ \min(B) < a < b\}; \\
c(A,B) &= \mathrm{card}\{(a,b,a',b') \mid a,a' \in A,\ b,b' \in B,\ a < b < a' < b'\} \\
&\quad + \mathrm{card}\{(b,a,b',a') \mid a,a' \in A,\ b,b' \in B,\ b < a < b' < a'\}.
\end{aligned}
\tag{2.1}
$$

Then, if $\pi = \{B_1, \ldots, B_r\} \in \mathcal{P}([n])$ has the blocks listed such that $\min(B_1) < \min(B_2) < \cdots < \min(B_r)$, we have:

$$
\left\{
\begin{aligned}
c_o(\pi) &= \textstyle\sum_{1 \le i < j \le r} c_o(B_i, B_j), \\
c(\pi) &= \textstyle\sum_{1 \le i < j \le r} c(B_i, B_j).
\end{aligned}
\right.
\tag{2.2}
$$

This is, clearly, just a reformulation of the definitions of $c_o(\pi)$ and $c(\pi)$ in 2.1.3°.

Now, the Equation (1.1) used in the definition of q-cumulants involved the number of left-reduced crossings $c_o(\pi)$ (rather than the total number of crossings $c(\pi)$). As mentioned in the Introduction, the reason for preferring to use c_o (in [7], and consequently also in Definition 1.1 above) was that the q-cumulants so obtained correspond to a natural q-deformation of the shift in Voiculescu's equivalent picture of the R-transform.

But on the other hand, as the reader may have already noticed, for both the cases $q = 0$ and $q = 1$ (discussed in 1.4, 1.3), making distinction between $c_o(\pi)$ and $c(\pi)$ in Eqn.(1.1) is of no importance. Indeed, this is trivial for $q = 1$; while for $q = 0$ we have (in view of 2.1.4°) that with either $c_o(\pi)$ or $c(\pi)$ as exponent for q, Eqn.(1.1) reduces to

$$
\mu(X^n) = \sum_{\substack{\pi \in NC(n) \\ \pi = \{B_1, \ldots, B_r\}}} \alpha_{|B_1|} \cdots \alpha_{|B_r|} \qquad (\alpha_n\text{'s for } q = 0),
\tag{2.3}
$$

where $NC(n)$ denotes the set of non-crossing partitions of $\{1, \ldots, n\}$.

Let us now turn to the case $q = -1$, discussed in Theorem 1.7. If the functional μ has $\mu(X^n) = 0$ for n odd, then Equation (1.1) becomes:

$$\mu(X^{2n}) \;=\; \sum_{\substack{\pi \in \mathcal{P}([2n]) \\ \pi = \{B_1, \ldots, B_r\} \\ |B_1|, \ldots, |B_r| \textit{ even}}} (-1)^{c_o(\pi)} \alpha_{|B_1|} \cdots \alpha_{|B_r|} \qquad (\alpha_n\text{'s for } q = -1). \tag{2.4}$$

One would expect at first glance that here indeed, choosing to work with $c_o(\pi)$ rather than with $c(\pi)$ makes a difference; and it is puzzling that (again) this is not really the case, as is shown by the next proposition.

2.4 Proposition If $\pi = \{B_1, \ldots, B_r\} \in \mathcal{P}([2n])$ is such that $|B_1|, \ldots, |B_r|$ are all even, then $c_o(\pi)$ and $c(\pi)$ have the same parity.

Proof By Remark 2.3, it suffices to show that if $\emptyset \neq A, B \subseteq \mathbf{N} \setminus \{0\}$ are finite sets, having: $A \cap B = \emptyset$, $\min(A) < \min(B)$, and $|A|, |B|$ both even, then $c_o(A, B)$ and $c(A, B)$ have the same parity.

For the rest of the proof we fix A B with the above stated properties. Let us write explicitly $A = \{a_1, a_2, \ldots, a_{2m}\}$ (with $a_1 < a_2 < \cdots < a_{2m}$), and let us make the notations:

$$\beta_i \;=\; \mathrm{card}\{b \in B \mid a_i < b < a_{i+1}\}, \quad 1 \leq i \leq 2m-1, \qquad \beta_{2m} \;=\; \mathrm{card}\{b \in B \mid b > a_{2m}\};$$

we also set, for convenience, $\beta_0 = 0 = \mathrm{card}\{b \in B \mid b < a_1\}$. Each of the numbers $\beta_0, \beta_1, \ldots, \beta_{2m}$ is a non-negative integer, and $\sum_{i=0}^{2m} \beta_i = |B|$ is even. We denote by s, $1 \leq s \leq 2m$, the smallest index with the property that $\beta_s \neq 0$. We shall actually assume that $s < 2m$, otherwise it is clear that $c_o(A, B) = c(A, B) = 0$, and there isn't anything left to prove.

Hence our picture of A and B is as shown below.

$$\mathsf{a}_1\mathsf{a}_2\cdots\mathsf{a}_s\,\underbrace{\overset{\overset{\min(B)}{\downarrow}}{\boxed{\mathsf{b}}}\;\cdots\mathsf{b}}_{\beta_s}\,\mathsf{a}_{s+1}\cdots\mathsf{a}_{2m-1}\underbrace{\mathsf{b}\cdots\mathsf{b}}_{\beta_{2m-1}}\mathsf{a}_{2m}\underbrace{\mathsf{b}\cdots\mathsf{b}}_{\beta_{2m}}$$

A direct calculation made on this picture gives us the equations:

$$c_o(A, B) \;=\; \sum_{i=s+1}^{2m} (\beta_i + \cdots + \beta_{2m}), \tag{2.5}$$

$$c(A, B) \;=\; \sum_{1 \leq i < j \leq 2m} (\beta_i + \cdots + \beta_{j-1})(\beta_j + \cdots + \beta_{2m}) \tag{2.6}$$

$$+\sum_{1\le i<j\le 2m}(\beta_0+\cdots+\beta_{i-1})(\beta_i+\cdots+\beta_{j-1}).$$

Indeed: in (2.5), $\beta_i+\cdots+\beta_{2m}$ counts, for each $s+1\le i\le 2m$, the "left-reduced crossings" of the form $(\min(A),\ \min(B),\ a_i,\ b)$, $b\in B$. In (2.6), the (i,j) term of the first sum counts the crossings of the type (a_i,b,a_j,b'), $b,b'\in B$, and the (i,j) term of the second sum counts the crossings of the type (b,a_i,b',a_j), $b,b'\in B$.

We will show that both $c_o(A,B)$ and $c(A,B)$ have the same parity as $\beta_1+\beta_3+\cdots+\beta_{2m-1}$.

For $c_o(A,B)$: $\beta_i+\cdots+\beta_{2m}$ has the same parity as $\beta_0+\cdots+\beta_{i-1}$, and moreover $\beta_0+\cdots+\beta_{i-1}=0$ for $1\le i\le s$, hence:

$$c_o(A,B)\ \overset{mod\ 2}{=}\ \sum_{i=s+1}^{2m}(\beta_0+\cdots+\beta_{i-1})\ =\ \sum_{i=1}^{2m}(\beta_0+\cdots+\beta_{i-1})$$

$$=\sum_{k=0}^{2m}(2m-k)\beta_k\ \overset{mod\ 2}{=}\ \beta_1+\beta_3+\cdots+\beta_{2m-1}.$$

For $c(A,B)$, we write:

$$c(A,B)\ =\ \sum_{1\le i<j\le 2m}(\beta_i+\cdots+\beta_{j-1})(|B|-(\beta_i+\cdots+\beta_{j-1}))$$

$$\overset{mod\ 2}{=}\ \sum_{1\le i<j\le 2m}(\beta_i+\cdots+\beta_{j-1})$$

(because $(\beta_0+\cdots+\beta_{i-1})+(\beta_j+\cdots+\beta_{2m})=(|B|-(\beta_i+\cdots+\beta_{j-1}))$, then by taking into account that $|B|$ is even, and that $(\beta_i+\cdots+\beta_{j-1})^2\ \overset{mod\ 2}{=}\ (\beta_i+\cdots+\beta_{j-1})$);

$$=\ \sum_{k=1}^{2m-1}\ \mathrm{card}\{(i,j)\mid 1\le i\le k<j\le 2m\}\cdot\beta_k$$

$$=\ \sum_{k=1}^{2m-1}k(2m-k)\beta_k\ \overset{mod\ 2}{=}\ \beta_1+\beta_3+\cdots+\beta_{2m-1}. \quad \textbf{QED}$$

We next introduce the involution on the set of 2-divisible partitions of $\{1,\ldots,2n\}$ which will be used in the proof of Theorem 1.7.

2.5 Definition and Proposition Let n be a positive integer, and let $\mathcal{P}_{\mathrm{even}}([2n])$ denote the set of all partitions $\pi=\{B_1,\ldots,B_r\}$ of $\{1,\ldots,2n\}$ such that the block-sizes $|B_1|,\ldots,|B_r|$ are even numbers. We define a map χ from $\mathcal{P}_{\mathrm{even}}([2n])$ to itself, which acts on a given partition $\pi\in\mathcal{P}_{\mathrm{even}}([2n])$ as follows:

- if $2i-1\overset{\pi}{\sim}2i$ for every $1\le i\le n$, then $\chi(\pi)=\pi$;

- otherwise, we choose the smallest $1 \leq i \leq n$ such that $2i-1 \overset{\pi}{\not\sim} 2i$, and we put $\chi(\pi) = \tau \cdot \pi$, with τ the permutation of $\{1, \ldots, 2n\}$ transposing $2i-1$ and $2i$ (i.e., $\tau(2i-1) = 2i$, $\tau(2i) = 2i-1$, and $\tau(j) = j$ for $j \neq 2i-1, 2i$), and where the notation "$\tau \cdot \pi$" is used in the same way as in Remark 2.2.

Then the map χ has the following properties.

1° $\chi \circ \chi$ is the identity on $\mathcal{P}_{\text{even}}([2n])$.

2° The set of fixed points of χ is in natural bijection with $\mathcal{P}([n])$.

3° Every partition fixed by χ has an even number of crossings.

4° If $\pi \in \mathcal{P}_{\text{even}}([2n])$ is not fixed by χ, then the crossing numbers $c_o(\pi)$ and $c_o(\chi(\pi))$ are of different parities.

Proof 1° If $\pi \in \mathcal{P}_{\text{even}}([2n])$ does not have $2i-1 \overset{\pi}{\sim} 2i$ for all $1 \leq i \leq n$, then $\chi(\pi)$ is obtained from π by swapping two elements of different blocks of π; this implies in particular that $\chi(\pi)$ is still in $\mathcal{P}_{\text{even}}([2n])$ (so that considering $\chi(\chi(\pi))$ makes indeed sense). It is moreover immediate that applying χ to $\chi(\pi)$ consists in swapping again the same two elements, and, consequently, that $\chi(\chi(\pi)) = \pi$.

2° The set of fixed points of χ is $\{\pi \in \mathcal{P}_{\text{even}}([2n]) \mid 2i-1 \overset{\pi}{\sim} 2i \text{ for every } 1 \leq i \leq n\}$. If to a partition π in this set we associate the partition ρ of $\{1, \ldots, n\}$ determined by: $i \overset{\rho}{\sim} j \overset{def}{\Leftrightarrow} 2i \overset{\pi}{\sim} 2j$, for $1 \leq i, j \leq n$, then we obtain a bijection $(\pi \mapsto \rho)$ onto $\mathcal{P}([n])$, as desired.

3° Let $\pi \in \mathcal{P}_{\text{even}}([2n])$ be such that $\chi(\pi) = \pi$, i.e. such that $2i-1 \overset{\pi}{\sim} 2i$ for every $1 \leq i \leq n$. Then, clearly, the rule $(i, j, i', j') \to (i, j, i'+1, j')$ establishes a bijection between the sets $\{(i, j, i', j') \text{ left-reduced crossing of } \pi \mid i' \text{ is odd}\}$ and $\{(i, j, i', j') \text{ left-reduced crossing of } \pi \mid i' \text{ is even}\}$. Since $c_o(\pi)$ is the sum of the cardinalities of these two sets, it follows that $c_o(\pi)$ is even.

4° We first note that it suffices to check $c_o(\pi) \overset{mod\ 2}{\neq} c_o(\chi(\pi))$ for $\pi \in \mathcal{P}_{\text{even}}([2n])$ such that $1 \overset{\pi}{\not\sim} 2$. Indeed, by combining the facts observed in Remark 2.2 and Proposition 2.4, one sees that applying a cyclic permutation to a partition $\pi \in \mathcal{P}_{\text{even}}([2n])$ preserves the parity of $c_o(\pi)$. But then, the verification that $c_o(\pi) \overset{mod\ 2}{\neq} c_o(\chi(\pi))$ for an arbitrary $\pi \in \mathcal{P}_{\text{even}}([2n])$ with the property that $\{1 \leq i \leq n \mid 2i-1 \overset{\pi}{\not\sim} 2i\} \neq \emptyset$ is reduced to the particular case mentioned above (when $\{1 \leq i \leq n \mid 2i-1 \overset{\pi}{\not\sim} 2i\} \ni 1$) by performing the appropriate cyclic permutation.

Consequently, we shall fix, for the rest of the proof, a permutation $\pi \in \mathcal{P}_{\text{even}}([2n])$ such that $1 \overset{\pi}{\not\sim} 2$, and we shall prove that $c_o(\pi) \overset{mod\ 2}{\neq} c_o(\chi(\pi))$. If the blocks of π are $B_1, \ldots, B_r$, listed in the increasing order of their minimal elements, then our assumption implies that

$r \geq 2$, $\min(B_1) = 1$, $\min(B_2) = 2$. Moreover, we have that $\chi(\pi) = \{B_2', B_1', B_3, \ldots, B_r\}$, where $B_1' = (B_1 \setminus \{1\}) \cup \{2\}$, $B_2' = (B_2 \setminus \{2\}) \cup \{1\}$, and where the blocks of $\chi(\pi)$ are also listed in the increasing order of their minimal elements. It is straightforward that $c_o(B_1, B_i)$ $= c_o(B_1', B_i)$ and $c_o(B_2, B_i) = c_o(B_2', B_i)$ for every $3 \leq i \leq r$, hence by invoking the first formula (2.2) of 2.3 we get:

$$
\begin{aligned}
c_o(\pi) + c_o(\chi(\pi)) \;&=\; c_o(B_1, B_2) + c_o(B_2', B_1') \\
+\, 2 &\left(\sum_{i=3}^{r} c_o(B_1, B_i) + \sum_{i=3}^{r} c_o(B_2, B_i) + \sum_{3 \leq i < j \leq r} c_o(B_i, B_j) \right).
\end{aligned}
$$

Our stated goal is equivalent to showing that $c_o(\pi) + c_o(\chi(\pi))$ is odd, and is therefore reduced to only showing that $c_o(B_1, B_2) + c_o(B_2', B_1')$ is odd.

Let us make the notations $C_1 = B_1 \setminus \{1\} = B_1' \setminus \{2\}$, $C_2 = B_2 \setminus \{2\} = B_2' \setminus \{1\}$. Then, by the very definition of c_o for a pair of sets, in Eqn.(2.1), we have:

$$
\left\{
\begin{array}{rcl}
c_o(B_1, B_2) & = & \mathrm{card}\{(c_1, c_2) \mid c_1 \in C_1,\ c_2 \in C_2,\ c_1 < c_2\}, \\[2ex]
c_o(B_2', B_1') & = & \mathrm{card}\{(c_2, c_1) \mid c_2 \in C_2,\ c_1 \in C_1, c_2 < c_1\}.
\end{array}
\right. \tag{2.7}
$$

From the fact that

$$
\{(c_1, c_2) \mid c_1 \in C_1,\ c_2 \in C_2,\ c_1 < c_2\} \,\cup\, \{(c_1, c_2) \mid c_1 \in C_1,\ c_2 \in C_2,\ c_1 > c_2\} \;=\; C_1 \times C_2,
$$

we obtain, by taking cardinalities (and by using (2.7), where the couples in the second Eqn.(2.7) are switched), that $c_o(B_1, B_2) + c_o(B_2', B_1') = |C_1| \times |C_2|$. The latter number is indeed odd, because $|C_1| = |B_1| - 1$, $|C_2| = |B_2| - 1$, and $|B_1|, |B_2|$ are even. **QED**

Now, in order to show how Proposition 2.5 is applied to prove Theorem 1.7, it is convenient that, besides partitions of the set $[n] = \{1, \ldots, n\}$, we also consider partitions of the *number* n, i.e. decreasing sequences of integers $\lambda = (\lambda_1 \geq \cdots \geq \lambda_m > 0)$ having $\sum_{i=1}^{m} \lambda_i = n$. We shall denote the set of all partitions of the number n by $\mathcal{P}_n$ (as opposed to $\mathcal{P}([n])$, which stands for all the partitions of $[n] = \{1, \ldots, n\}$). Every set-partition $\pi = \{B_1, \ldots, B_r\} \in \mathcal{P}([n])$ determines a partition $\lambda \in \mathcal{P}_n$ which is obtained by writing the block-sizes $|B_1|, \ldots, |B_r|$ in decreasing order; this partition λ will be called *the type of* π, and will be denoted by *type*(π).

Let us look again at the sum on the right-hand side of Equation (1.1), and let us remark that in its general term, indexed by $\pi \in \mathcal{P}([n])$, only the factor $q^{c_o(\pi)}$ really depends on π

— while the product $\alpha_{|B_1|}\cdots\alpha_{|B_r|}$ actually depends only on *type*(π). In other words, (1.1) has the alternative writing:

$$\mu(X^n) \;=\; \sum_{\substack{\lambda\in\mathcal{P}_n \\ \lambda=(\lambda_1\geq\cdots\geq\lambda_m>0)}} \Bigg(\sum_{\substack{\pi\in\mathcal{P}([n]) \\ type(\pi)=\lambda}} q^{c_o(\pi)} \Bigg) \; \alpha_{\lambda_1}\cdots\alpha_{\lambda_m}, \tag{2.8}$$

which can be advantageous if we have some information about the 'interior' summation $\sum q^{c_o(\pi)}$. For the case needed in Theorem 1.7.1°, this summation is handled by using the following consequence of Proposition 2.5.

2.6 Corollary Let n be a positive integer, and consider a partition $\lambda = (\lambda_1 \geq \cdots \geq \lambda_m > 0) \in \mathcal{P}_n$. We denote by 2λ the partition $(2\lambda_1 \geq \cdots \geq 2\lambda_m > 0) \in \mathcal{P}_{2n}$. Then:

$$\sum_{\substack{\pi\in\mathcal{P}_{\mathrm{even}}([2n]) \\ type(\pi)=2\lambda}} (-1)^{c_o(\pi)} \;=\; \sum_{\substack{\pi\in\mathcal{P}([n]) \\ type(\pi)=\lambda}} 1. \tag{2.9}$$

Proof It is immediate that the involutive bijection χ on $\mathcal{P}_{\mathrm{even}}([2n])$ discussed in 2.5 preserves the type of a set-partition; hence, it induces an involutive bijection of $\{\pi \in \mathcal{P}_{\mathrm{even}}([2n]) \mid type(\pi) = 2\lambda\}$, which will also be denoted by χ.

We have that

$$\sum_{\substack{\pi\in\mathcal{P}_{\mathrm{even}}([2n]) \\ type(\pi)=2\lambda \\ \chi(\pi)\neq\pi}} (-1)^{c_o(\pi)} \;=\; 0. \tag{2.10}$$

Indeed, the terms of the sum in (2.10) can be grouped into pairs, indexed by couples of partitions (π_1, π_2) such that $\pi_1 \neq \pi_2$ and $\chi(\pi_1) = \pi_2$, $\chi(\pi_2) = \pi_1$; and the terms in each pair cancel each other, as implied by 2.5.4°.

From (2.10) it follows that the left-hand side of (2.9) is equal to:

$$\sum_{\substack{\pi\in\mathcal{P}_{\mathrm{even}}([2n]) \\ type(\pi)=2\lambda \\ \chi(\pi)=\pi}} (-1)^{c_o(\pi)}. \tag{2.11}$$

Finally, let us note that the natural bijection $\{\pi \in \mathcal{P}_{\mathrm{even}}([2n]) \mid \chi(\pi) = \pi\} \to \mathcal{P}([n])$ remarked in the proof of 2.5.2° sends $\{\pi \in \mathcal{P}_{\mathrm{even}}([2n]) \mid \chi(\pi) = \pi,\; type(\pi) = 2\lambda\}$ into $\{\rho \in \mathcal{P}([n]) \mid type(\rho) = \lambda\}$. By using this restricted bijection, and by taking into account that all the terms in (2.11) are - as implied by 2.5.3° - equal to 1, we see that (2.11) is also equal to the right-hand side of (2.9). **QED**

Note: the right-hand side of (2.9) is presented as a sum of 1's for the convenience of the proofs (both of Proposition 2.6, and in Section 2.7 below). The quantity there is, of course, just card$\{\pi \in \mathcal{P}_n \mid type(\pi) = \lambda\}$; it is well-known, moreover, that we can write this cardinality as $n!/((1!)^{k_1} \cdots (n!)^{k_n} \cdot k_1! \cdots k_n!)$, where $k_1, \ldots, k_n$ are the number of parts of λ that are equal to $1, \ldots, n$, respectively (see e.g. [4], Theorem 5.1.B).

2.7 Proof of Theorem 1.7 It will be more convenient to prove the statement of Corollary 1.8; for this reason, we start by showing how, conversely, the assertion of 1.8 implies those of Theorem 1.7.

1° Let $\mu : \mathbf{C}[X] \to \mathbf{C}$ be a linear functional with $\mu(1) = 1$ and $\mu(X^n) = 0$ for n odd. Then by taking $\mathcal{A} = \mathbf{C}[X]$, $\varphi = \mu$, $a = X$, we have that $\mu = \mu_a$ in $(\mathcal{A}, \varphi)$. But then (assuming Corollary 1.8 to be true), we obtain:

$$\begin{aligned} [R_{-1}(\mu)](z) &= [R_{-1}(\mu_a)](z) \overset{(1.12)}{=} [R_1(\mu_{a^2})](z^2) \\ \overset{(1.3)}{=} \log\left(\sum_{n=0}^{\infty} \frac{\mu_{a^2}(X^n)}{n!}(z^2)^n\right) &= \log\left(\sum_{n=0}^{\infty} \frac{\mu(X^{2n})}{n!} z^{2n}\right), \end{aligned}$$

with the last equality in the sequence holding because $\mu_{a^2}(X^n) = \varphi(a^{2n}) = \mu(X^{2n})$, $n \geq 0$.

2° Let $(\mathcal{A}, \gamma, \varphi)$ be a graded non-commutative probability space, let $\mathcal{A}_1, \mathcal{A}_2 \subseteq \mathcal{A}$ be gradedly independent subalgebras, and let $a_1 \in \mathcal{A}_1$, $a_2 \in \mathcal{A}_2$ be such that $\partial a_1 = \partial a_2 = 1$. Then $\partial(a_1 + a_2) = 1$, hence

$$[R_{-1}(\mu_{a_1+a_2})](z) \overset{(1.12)}{=} [R_1(\mu_{(a_1+a_2)^2})](z^2) = [R_1(\mu_{a_1^2+a_2^2})](z^2) \tag{2.12}$$

(where $(a_1 + a_2)^2 = a_1^2 + a_2^2$ because a_1, a_2 anti-commute). But the subalgebras of $\mathcal{A}$ generated by a_1^2 and a_2^2 are, obviously, commuting independent, hence (2.12) can be continued with:

$$\overset{(1.4)}{=} [R_1(\mu_{a_1^2})](z^2) + [R_1(\mu_{a_2^2})](z^2),$$

and another use of Corollary 1.8 shows the latter quantity to be equal to $[R_{-1}(\mu_{a_1}) + R_{-1}(\mu_{a_2})](z)$, as desired.

Hence, from now on we will concentrate on proving the statement of Corollary 1.8; i.e., we will fix a non-commutative probability space $(\mathcal{A}, \varphi)$ and an element $a \in \mathcal{A}$ such that $\varphi(a^n) = 0$ for odd n, and we will prove that (1.12) holds.

Let $(\alpha_n)_{n=1}^{\infty}$ denote the sequence of (-1)-cumulants of μ_a (but where we know that $\alpha_n = 0$ for n odd), and let $(\beta_n)_{n=1}^{\infty}$ denote the sequence of 1-cumulants of μ_{a^2}. Recall that the formulas for $R_{-1}(\mu_a)$, $R_1(\mu_{a^2})$ are:

$$[R_{-1}(\mu_a)](z) \;=\; \sum_{n=1}^{\infty} \frac{\alpha_{2n}}{n!} z^{2n}, \qquad [R_1(\mu_{a^2})](z) \;=\; \sum_{n=1}^{\infty} \frac{\beta_n}{n!} z^n,$$

which makes clear that proving (1.12) is tantamount to proving:

$$\beta_n \;=\; \alpha_{2n}, \quad \text{for every } n \geq 1. \tag{2.13}$$

We shall verify (2.13) by induction on n. For $n = 1$: Equation (1.1) written for $\mu = \mu_{a^2}$, $n = 1$, $q = 1$, reduces to $\mu_{a^2}(X) = \beta_1$, while Equation (2.4) written for $\mu = \mu_a$ and $n = 1$ takes the form $\mu_a(X^2) = \alpha_2$. But $\mu_{a^2}(X) = \varphi(a^2) = \mu_a(X^2)$; hence we have $\beta_1 = \alpha_2$.

It remains to verify the induction step, $(1, 2, \ldots, n-1) \Rightarrow \; n$, for $n \geq 2$. We write Eqn.(1.1) for $\mu = \mu_{a^2}$, $q = 1$, and Eqn.(2.4) for $\mu = \mu_a$:

$$\mu_{a^2}(X^n) \;=\; \sum_{\substack{\pi \in \mathcal{P}([n]) \\ \pi = \{B_1, \ldots, B_r\}}} \beta_{|B_1|} \cdots \beta_{|B_r|}; \tag{2.14}$$

$$\mu_a(X^{2n}) \;=\; \sum_{\substack{\pi \in \mathcal{P}_{\mathrm{even}}([2n]) \\ \pi = \{A_1, \ldots, A_r\}}} (-1)^{c_o(\pi)} \, \alpha_{|A_1|} \cdots \alpha_{|A_r|}. \tag{2.15}$$

The left-hand sides of (2.14) and (2.15) are both equal to $\varphi(a^{2n})$, so we can also equate the right-hand sides of the two equations. When doing this, and also performing the transformation indicated in (2.8), we obtain:

$$\sum_{\substack{\lambda \in \mathcal{P}_n \\ \lambda = (\lambda_1 \geq \cdots \geq \lambda_m > 0)}} \Big(\sum_{\substack{\pi \in \mathcal{P}([n]) \\ \mathit{type}(\pi) = \lambda}} 1 \Big) \, \beta_{\lambda_1} \cdots \beta_{\lambda_m} \tag{2.16}$$

$$= \sum_{\substack{\lambda \in \mathcal{P}_n \\ \lambda = (\lambda_1 \geq \cdots \geq \lambda_m > 0)}} \Big(\sum_{\substack{\pi \in \mathcal{P}_{\mathrm{even}}([2n]) \\ \mathit{type}(\pi) = 2\lambda}} (-1)^{c_o(\pi)} \Big) \;\; \alpha_{2\lambda_1} \cdots \alpha_{2\lambda_m}$$

(where on the right-hand side of (2.16) we used the obvious fact that every partition into even parts of $2n$ is of the form 2λ for some $\lambda \in \mathcal{P}_n$).

But now, by taking Corollary 2.6 into account, we can re-write (2.16) as:

$$\sum_{\substack{\lambda \in \mathcal{P}_n \\ \lambda = (\lambda_1 \geq \cdots \geq \lambda_m > 0)}} \mathrm{card}\{\pi \in \mathcal{P}([n]) \mid \mathit{type}(\pi) = \lambda\} \cdot (\beta_{\lambda_1} \cdots \beta_{\lambda_m} - \alpha_{2\lambda_1} \cdots \alpha_{2\lambda_m}) \;=\; 0. \tag{2.17}$$

For every $\lambda = (\lambda_1 \geq \cdots \geq \lambda_m > 0)$ which has at least two parts ($m \geq 2$), the equality $\beta_{\lambda_1} \cdots \beta_{\lambda_m} - \alpha_{2\lambda_1} \cdots \alpha_{2\lambda_m} = 0$ is guaranteed by the induction hypothesis, because we must have $\lambda_1, \ldots, \lambda_m < n$. So in the sum of (2.17) we can actually delete all the terms with the exception of the one corresponding to the partition with one part $(n) \in \mathcal{P}_n$. Since, obviously, $\mathrm{card}\{\pi \in \mathcal{P}([n]) \mid \mathit{type}(\pi) = (n)\} = 1$, (2.17) comes thus to $\beta_n - \alpha_{2n} = 0$, which is what we wanted. **QED**

3 Considerations on graded independence

Without attempting to give an organized treatment of graded independence, we survey here, for the reader's convenience, a few facts which support the legitimacy of this notion.

Two examples of well-known C^*-algebras, where graded independence arises naturally, are the following.

3.1 Examples

1^{o} Clifford algebras. Let V be a Euclidean space (i.e., a finite dimensional real Hilbert space), and let $\mathcal{C}(V)$ be the Clifford C^*-algebra of V. This means that we have a linear embedding, λ, of V into the space of selfadjoint elements of $\mathcal{C}(V)$, such that:

$$\lambda(v)\lambda(w) + \lambda(w)\lambda(v) \ = \ 2\langle v \mid w\rangle I, \quad v, w \in V; \tag{3.1}$$

and moreover, $\mathcal{C}(V)$ and λ are universal with respect to the property stated in (3.1) (see e.g. [8], Chapter 1). Due to universality, every orthogonal transformation of V extends to an automorphism of $\mathcal{C}(V)$; in particular, the map $v \to -v$ on V extends to a grading automorphism $\gamma \in \mathrm{Aut}(\mathcal{C}(V))$. If $\dim_{\mathbf{R}}(V) = n$, then the (complex) linear dimension of $\mathcal{C}(V)$ is 2^n. More precisely, if $v_1, \ldots, v_n$ is an orthonormal basis of V, and if for $A = \{i_1 < \cdots < i_k\} \subseteq \{1, \ldots, n\}$ we put $v_A \stackrel{def}{=} v_{i_1} \cdots v_{i_k} \in \mathcal{C}(V)$ (with the convention $v_A = I$ for $A = \emptyset$), then the v_A's, $A \subseteq \{1, \ldots, n\}$, form a linear basis of $\mathcal{C}(V)$. There exists a unique trace-state φ on $\mathcal{C}(V)$ which is invariant under the grading automorphism γ; if a basis of V is chosen, as above, then φ is determined by the formula:

$$\varphi(v_A) \ = \ \begin{cases} 1, & \text{if } A = \emptyset \\ 0, & \text{if } A \neq \emptyset, \end{cases} \qquad A \subseteq \{1, \ldots, n\}$$

(see [8], Theorem 1.1.7).

If V_1 is a linear subspace of V, then the C^*-subalgebra of $\mathcal{C}(V)$ generated by $\{\lambda(v) \mid v \in V_1\}$ is naturally identified with $\mathcal{C}(V_1)$. The example of graded-independence appearing in this framework is the following: if $V_1, V_2 \subseteq V$ are linear subspaces such that $V_1 \perp V_2$, then $\mathcal{C}(V_1)$ and $\mathcal{C}(V_2)$ are graded-independent in $(\mathcal{C}(V), \gamma, \varphi)$.

2^{o} The rational rotation algebra $\mathcal{A}_{1/2}$, i.e. the universal C^*-algebra generated by two unitaries u, v such that $uv = -vu$. Due to universality, the prescription $u \to -u$, $v \to -v$ extends to a grading automorphism $\gamma \in \mathrm{Aut}(\mathcal{A}_{1/2})$. The family $\{u^m v^n \mid m, n \in \mathbf{Z}\}$ is a linear basis for a dense $*$-subalgebra of $\mathcal{A}_{1/2}$, and the formula

$$\varphi(u^m v^n) \ = \ \begin{cases} 1, & \text{if } m = n = 0 \\ 0, & \text{otherwise,} \end{cases} \qquad m, n \in \mathbf{Z},$$

determines a trace-state on $\mathcal{A}_{1/2}$. In this framework, the C^*-subalgebras generated by u and by v in $\mathcal{A}_{1/2}$ (each of them isomorphic to the algebra $C(S^1)$ of continuous functions on the circle) are gradedly-independent in $(\mathcal{A}_{1/2}, \gamma, \varphi)$.

3.2 Remark As mentioned following to Definition 1.5, a general recipe for producing examples of graded independence is provided by the graded tensor product operation, $\widehat{\otimes}$. Both the examples in 3.1 could be introduced in this way: in 3.1.1^{o} we have (in the notations used there) that $V_1 \perp V_2 \Rightarrow \mathcal{C}(V_1) \widehat{\otimes} \mathcal{C}(V_2) \simeq \mathcal{C}(V_1 \oplus V_2) \subseteq \mathcal{C}(V)$; while in 3.1.$2^{\mathrm{o}}$ we have $\mathcal{A}_{1/2} \simeq C(S^1) \widehat{\otimes} C(S^1)$, where the grading considered on $C(S^1)$ is $[\gamma(f)](z) = f(-z)$, $f \in C(S^1), z \in S^1$, and the state on $C(S^1)$ is the integral against the Haar measure. For the reader's convenience, we shall briefly review (in 3.3–3.4 below) the construction of the graded tensor product, adapted to the framework of Definition 1.5, i.e. to the situation when the graded algebras to be tensored are equipped with specified homogeneous linear functionals.

We first consider the purely algebraic framework.

3.3 Algebraic graded tensor product Let $(\mathcal{A}_1, \gamma_1, \varphi_1)$ and $(\mathcal{A}_2, \gamma_2, \varphi_2)$ be graded non-commutative probability spaces. Their graded tensor product $(\mathcal{A}_1 \widehat{\odot} \mathcal{A}_2, \gamma, \varphi)$ is defined as follows: as a vector space, $\mathcal{A}_1 \widehat{\odot} \mathcal{A}_2$ is the usual tensor product of the vector spaces $\mathcal{A}_1$ and $\mathcal{A}_2$. The multiplication on $\mathcal{A}_1 \widehat{\odot} \mathcal{A}_2$ is determined by the rule:

$$(a_1 \otimes a_2)(a_1' \otimes a_2') \ = \ (-1)^{\partial a_2 \partial a_1'} (a_1 a_1' \otimes a_2 a_2'), \quad a_1, a_1' \in \mathcal{A}_1, a_2, a_2' \in \mathcal{A}_2, \text{ homogeneous,} \tag{3.2}$$

while the grading automorphism $\gamma : \mathcal{A}_1 \widehat{\odot} \mathcal{A}_2 \to \mathcal{A}_1 \widehat{\odot} \mathcal{A}_2$ and the functional $\varphi : \mathcal{A}_1 \widehat{\odot} \mathcal{A}_2 \to \mathbf{C}$

are determined by the formulas:

$$\gamma(a_1 \otimes a_2) \;=\; \gamma_1(a_1) \otimes \gamma_2(a_2), \;\; \varphi(a_1 \otimes a_2) \;=\; \varphi_1(a_1)\varphi_2(a_2), \quad a_1 \in \mathcal{A}_1, a_2 \in \mathcal{A}_2 \quad (3.3)$$

(i.e., we just have $\gamma = \gamma_1 \otimes \gamma_2$, $\varphi = \varphi_1 \otimes \varphi_2$).

3.4 C^*-algebra graded tensor product We now pass to the situation when $(\mathcal{A}_1, \gamma_1, \varphi_1)$ and $(\mathcal{A}_2, \gamma_2, \varphi_2)$ are C^*-probability spaces. For the sake of brevity, we shall only discuss the case when the states φ_1 and φ_2 are faithful, i.e. they do not have non-zero positive elements in their kernels. We start by considering the algebraic graded tensor product $(\mathcal{A}_1 \widehat{\odot} \mathcal{A}_2, \gamma, \varphi)$ defined in 3.3. The existence of $*$-operations on $\mathcal{A}_1$ and $\mathcal{A}_2$ gives us a $*$-operation on $\mathcal{A}_1 \widehat{\odot} \mathcal{A}_2$, determined by the formula:

$$(a_1 \otimes a_2)^* \;=\; (-1)^{\partial a_1 \partial a_2}(a_1^* \otimes a_2^*), \quad a_1 \in \mathcal{A}_1, a_2 \in \mathcal{A}_2, \text{ homogeneous.} \qquad (3.4)$$

We next observe that, when $\mathcal{A}_1 \widehat{\odot} \mathcal{A}_2$ is viewed as a $*$-algebra, the following hold:

(A) $\varphi : \mathcal{A}_1 \widehat{\odot} \mathcal{A}_2 \to \mathbf{C}$ is positive definite (i.e., $\varphi(x^*x) \geq 0$ for every $x \in \mathcal{A}_1 \widehat{\odot} \mathcal{A}_2$);

(B) φ is faithful (i.e., if $x \in \mathcal{A}_1 \widehat{\odot} \mathcal{A}_2$ has $\varphi(x^*x) = 0$, then $x = 0$);

(C) for every $x \in \mathcal{A}_1 \widehat{\odot} \mathcal{A}_2$, the positive element x^*x can be majorized by a constant (where "$x^*x \leq C$" means by definition that $C - x^*x$ can be written as $\sum_{i=1}^n x_i^* x_i$ for some x_i's in $\mathcal{A}_1 \widehat{\odot} \mathcal{A}_2$).

The assertions (A),(B),(C) have elementary proofs, very similar to the ones of the corresponding statements in the theory of the spatial (ungraded) tensor products of C^*-algebras. (See for instance [12], Chapter IV: Theorem 4.9 for the analogues of (A) and (B), Lemma 4.4 (ii) for the analogue of (C).) The close analogy is due to the fact that when $x \in \mathcal{A}_1 \widehat{\odot} \mathcal{A}_2$ is written as $\sum_{i=1}^n a_i \otimes b_i$, for some homogeneous $a_1, \ldots, a_n \in \mathcal{A}_1$, $b_1, \ldots, b_n \in \mathcal{A}_2$, the use of (3.2) and (3.4) and the fact that φ_1 and φ_2 vanish on elements of odd degree gives us:

$$\varphi(x^*x) \;=\; \sum_{i,j=1}^n \varphi_1(a_i^* a_j)\varphi_2(b_i^* b_j). \qquad (3.5)$$

The above properties (A),(B),(C) ensure that the GNS representation $\pi : \mathcal{A}_1 \widehat{\odot} \mathcal{A}_2 \to \mathcal{L}(\mathcal{H})$ of the functional φ can be constructed. (Thus, $\mathcal{H}$ is the completion of $\mathcal{A}_1 \widehat{\odot} \mathcal{A}_2$ under the sesquilinear form $\langle x \mid y \rangle \overset{def}{=} \varphi(y^*x)$; and π acts on $\mathcal{H}$ by left translation operators – which are bounded, due to (C).) Moreover, (B) implies that π is one-to-one. We define the C^*-graded tensor product of $(\mathcal{A}_1, \gamma_1, \varphi_1)$ and $(\mathcal{A}_2, \gamma_2, \varphi_2)$ to be:

$$\mathcal{A}_1 \widehat{\otimes} \mathcal{A}_2 \overset{def}{=} \text{ the norm closure of } \pi(\mathcal{A}_1 \widehat{\odot} \mathcal{A}_2) \subseteq \mathcal{L}(\mathcal{H}), \qquad (3.6)$$

endowed with the unique grading $\widetilde{\gamma}$ and homogeneous state $\widetilde{\varphi}$ which extend continuously the grading γ and the state φ on $\mathcal{A}_1 \widehat{\odot} \mathcal{A}_2$. An alternative definition of $\widetilde{\gamma}$ is as the automorphism of $\mathcal{A}_1 \widehat{\otimes} \mathcal{A}_2$ implemented by the unitary $U \in \mathcal{L}(\mathcal{H})$, which works by the rule $Ux = \gamma(x)$, $x \in \mathcal{A}_1 \widehat{\odot} \mathcal{A}_2 \subseteq \mathcal{H}$. Also, $\widetilde{\varphi}$ can be alternatively defined as the vector-state on $\mathcal{A}_1 \widehat{\otimes} \mathcal{A}_2$ determined by the canonical cyclic vector $I \otimes I \in \mathcal{A}_1 \widehat{\odot} \mathcal{A}_2 \subseteq \mathcal{H}$.

Finally, let us note that the formulas:

$$J_1(a_1) \ = \ \pi(a_1 \otimes I), \quad a_1 \in \mathcal{A}_1, \qquad J_2(a_2) \ = \ \pi(I \otimes a_2), \quad a_2 \in \mathcal{A}_2, \tag{3.7}$$

define one-to-one unital $*$-homomorphisms from $\mathcal{A}_1, \mathcal{A}_2$ into $\mathcal{A}_1 \widehat{\otimes} \mathcal{A}_2$, such that the gradings and the specified states on these C^*-algebras are preserved. It is a straightforward verification that $J_1(\mathcal{A}_1)$ and $J_2(\mathcal{A}_2)$ are graded-independent in $(\mathcal{A}_1 \widehat{\otimes} \mathcal{A}_2, \widetilde{\gamma}, \widetilde{\varphi})$. Thus the graded tensor product produces isomorphic copies of the given spaces $(\mathcal{A}_1, \gamma_1, \varphi_1)$ and $(\mathcal{A}_2, \gamma_2, \varphi_2)$, which sit in a graded-independent position.

3.5 Remark Graded independence can be considered (and the graded tensor product construction can be carried over) when more than two algebras are involved; the homogeneous unital subalgebras $\mathcal{A}_1, \ldots, \mathcal{A}_n \subseteq \mathcal{A}$ are graded-independent in $(\mathcal{A}, \gamma, \varphi)$ if $\mathcal{A}_i$ gradedly commutes with $\mathcal{A}_j$ for every $i \neq j$, and if we have $\varphi(a_1 \cdots a_n) = \varphi(a_1) \cdots \varphi(a_n)$ for every $a_1 \in \mathcal{A}_1, \ldots, a_n \in \mathcal{A}_n$. Moreover, it is straightforward to verify that we have good "associativity" properties, in the sense that, for instance, if $\mathcal{A}_1, \mathcal{A}_2, \mathcal{A}_3$ are graded-independent in $(\mathcal{A}, \gamma, \varphi)$, and if $\mathcal{A}'$ denotes the algebra generated by $\mathcal{A}_1 \cup \mathcal{A}_2$, then $\mathcal{A}'$ and $\mathcal{A}_3$ are graded-independent. The reason for mentioning this aspect is that, as shown in [11], associativity arguments of this nature give a major obstruction for the existence of "independence theories" in an an arbitrary non-commutative probability space $(\mathcal{A}, \varphi)$.

3.6 Remark A "central limit" type statement that can be made in the framework of graded independence is the following. Let $(\mathcal{A}, \gamma, \varphi)$ be a graded non-commutative probability space, and let $(a_n)_{n=1}^{\infty}$ be a sequence of homogeneous elements of degree 1 in $\mathcal{A}$, which are graded-independent (in the sense that, for every $N \geq 1$, so are the unital subalgebras $\mathcal{A}_1, \ldots, \mathcal{A}_N \subseteq \mathcal{A}$ generated by $a_1, \ldots, a_N$, respectively). We assume, for simplicity, that all the a_n's have the same distribution μ with respect to φ; note that we necessarily have $\mu(X^n) = 0$ for n odd (see the comment following Theorem 1.7). Consider the sums $s_N = (a_1 + \cdots + a_N)/\sqrt{N}$, $N \geq 1$. Then the distributions μ_{s_N} converge moment-wise to the atomic measure $\frac{1}{2}(\delta_{-\sigma} + \delta_{\sigma})$, where $\pm\sigma$ are the square roots of $\mu(X^2)$.

The above assertion appears in fact as a particular case of a result of von Waldenfels ([15], Theorem 1, where one sets $s = 2$ and $|I| = 1$). We would like to point out here that an alternative proof can be obtained by reproducing the proof of the classical central limit theorem via characteristic functions, but where the characteristic function is replaced by the R_{-1}-transform. Indeed, by Theorem 1.7.1$^{\text{o}}$ and a homogeneity argument which comes directly from Eqn.(1.1), we have (in the notations of the preceding paragraph) that:

$$[R_{-1}(\mu_{s_N})](z) \;=\; \sum_{i=1}^{N} [R_{-1}(\mu_{a_i/\sqrt{N}})](z) \;=\; N[R_{-1}(\mu)](z/\sqrt{N}), \quad N \geq 1. \tag{3.8}$$

So if the sequence of even (-1)-cumulants of μ is $(\alpha_{2n})_{n=1}^{\infty}$, then (3.8) gives:

$$[R_{-1}(\mu_{s_N})](z) \;=\; \sum_{n=1}^{\infty} \frac{\alpha_{2n}}{n!N^{n-1}} z^{2n};$$

hence for $N \to \infty$, $R_{-1}(\mu_{s_N})$ converges coefficient-wise to $\alpha_2 z^2$. But then from (1.1) it clearly follows that μ_{s_N} itself converges moment-wise, for $N \to \infty$, to the inverse R_{-1} - transform of $\alpha_2 z^2$; and the latter functional is found, by using 1.7.1$^{\text{o}}$, to be $\frac{1}{2}(\delta_{-\sigma} + \delta_{\sigma})$, with $\sigma^2 = \alpha_2 = \mu(X^2)$.

An interesting remark is that the cited Theorem 1 of [15] does actually involve parities of crossing numbers for certain even partitions – namely those which only have blocks of two elements. More precisely: if $\pi = \{\{b_1', b_1''\}, \ldots, \{b_k', b_k''\}\}$ is such a partition of $\{1, 2, \ldots, 2k\}$, then the statement of Theorem 1 in [15] (for $s = 2$) uses the signature of the permutation $\begin{pmatrix} 1 & 2 & \cdots & 2k-1 & 2k \\ b_1' & b_1'' & \cdots & b_k' & b_k'' \end{pmatrix}$; and it can be verified that this signature is equal to $(-1)^{c_o(\pi)}$. The multivariable framework of the results in [15] suggests that a theory of the multi-dimensional R_{-1}-transform could also be developed (but we did not insist on pursuing this line).

3.7 Remark In the context of Example 3.1.2$^{\text{o}}$, it is amusing to see what Equation (1.10) of Theorem 1.7 gives about the distribution of the element (also called "the Harper operator") $h = u + u^{-1} + v + v^{-1} \in \mathcal{A}_{1/2}$. Since $u + u^{-1}$ and $v + v^{-1}$ are graded-independent and of degree 1, we have:

$$R_{-1}(\mu_h) \;=\; R_{-1}(\mu_{u+u^{-1}}) + R_{-1}(\mu_{v+v^{-1}}) \;=\; 2R_{-1}(\mu_{u+u^{-1}}). \tag{3.9}$$

By exponentiating (3.9), and by using (1.9) and the fact that

$$\mu_{u+u^{-1}}(X^{2n}) \;=\; \int_{S^1} (z + z^{-1})^{2n} \, dz \;=\; \int_0^1 (2\cos(2\pi t))^{2n} \, dt \;=\; \binom{2n}{n}, \quad n \geq 0,$$

we thus obtain:

$$\sum_{n=0}^{\infty} \frac{\mu_h(X^{2n})}{n!} z^{2n} = \left(\sum_{n=0}^{\infty} \frac{1}{n!} \left(\begin{array}{c} 2n \\ n \end{array} \right) z^{2n} \right)^2 . \tag{3.10}$$

Since $\mu_h(X^{2n}) = \mu_{h^2}(X^n)$, $n \geq 0$, the left-hand side of (3.10) is $[\mathcal{F}(\mu_{h^2})](-iz^2)$ (with $\mathcal{F}$ denoting Fourier transform, in the same sense as in Section 1.3). The right-hand side of (3.10) can be also summed, by involving the Bessel function $J_0(z) = \sum_{n=0}^{\infty} \frac{1}{(n!)^2} \left(\frac{iz}{2}\right)^{2n}$; and in this way we get:

$$[\mathcal{F}(\mu_{h^2})](z) = e^{4iz} J_0(2z)^2. \tag{3.11}$$

(Of course, (3.10) and (3.11) could have been reached directly, without using R_{-1}.) By inverting the Fourier transform in (3.11) and by passing from h^2 to h (in the straightforward way), it follows that μ_h is absolutely continuous with respect to Lebesgue measure with density $\dfrac{d\mu}{dt}(t) = |t| f(t^2 - 4)$ for $-2\sqrt{2} \leq t \leq 2\sqrt{2}$, where f denotes the convolution of the function $\dfrac{1}{\pi\sqrt{4 - t^2}}$ (on $[-2, 2]$) with itself.

It would be interesting to determine the spectral distribution of $h = u + u^{-1} + v + v^{-1}$ for more general rotation algebras (i.e. when $uv = e^{2\pi i p/q} vu$ for some $p, q \in \mathbf{N} \setminus \{0\}$, instead of just $uv = -vu$), but this seems to be a difficult problem.

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