Math 221 Queen's University, Department of Mathematics

Vector Calculus, tutorial 2

September 2013

1. Volume in the first octant bounded by cylinder $z = 16 - x^2$ and the plane $y = 5$.

Draw a diagram, and compute the volume.

This parabolic cylinder is parallel to the y-axis. In the first octant it lies over a rectanglular region

$$\mathbf{R} = \{(x, y) \mid 0 \le x \le 4, \quad 0 \le y \le 5\}$$

The crossections perpendicular to the x-axis are rectangles, and

$$\begin{aligned}
\text{Volume} &= \int\int_{\mathbf{R}} z dA \\
&= \int_0^4 A(x) dx \\
&= \int_0^4 5(16 - x^2) dx \\
&= 80x \Big|_0^4 - \frac{5}{3} x^3 \Big|_0^4 \\
&= 320 - \frac{320}{3} \\
&= \frac{640}{3}
\end{aligned}$$

If we crossection perpendicular to the y-axis we get parabolic curves bounding the crossection

$$\text{Volume} = \int\int_{\mathbf{R}} z dA$$

$$
\begin{aligned}
&= \int_0^5 A(y)dy \\
&= \int_0^5 \int_0^4 (16 - x^2)dxdy \\
&= 80x \Big|_0^4 - \frac{5}{3}x^3 \Big|_0^4 \\
&= 320 - \frac{320}{3}
\end{aligned}
$$

2. The volume bounded by the planes

$$z = 0, z = x, x + y = 2, y = x.$$

These four planes bound a finite region in $\mathbb{R}^3$. Sketch the planes, and determine the volume by triple integral.

To see this notice that the planes y=x, and x+y+2 are vertical planes and of course z=0 is horizontal. The extra plane z=x bounds the region. To see this notice that if $x < 0$ then $z < 0$ on the plane $z = x$. Since this cant happen we notice next that therefore $y > 0$ since otherwise $y = x < 0$ on the plane $y = x$. Therefore the region lies in the first octant , below the plane $z = x$ and above the region $\mathbf{D}$ in the horizontal plane bounded by $x = 0, y = x, x + y = 2$. This is a isosceles triangular shaped region, with the symmetric vertex at $(1, 1)$. We can crossection this region perpendicular to the y-axis.

$$\text{Volume} = \int\int_{\mathbf{D}} zdA$$

$$
\begin{aligned}
&= \int_0^1 A(y)dy \\
&= \int_0^1 \int_y^{2-y} x dx dy \\
&= \int_0^1 \frac{1}{2} x^2 \Big|_y^{2-y} dy \\
&= \int_0^1 \frac{1}{2} \left(4 - 4y + y^2 - y^2\right) dy \\
&= \frac{1}{2} \left(4y - 2y^2\right) \Big|_0^1 \\
&= 1
\end{aligned}
$$

3. Volume bounded by the cylinders $x^2 + y^2 = r^2$ and $y^2 + z^2 = r^2$.

This is a challenging question. Try to compute the volume by looking at the region **D** in the x-y plane bounded by the circle $x^2 + y^2 = r^2$.

The cylinder $x^2 + y^2 = r^2$ intersects the horizontal plane $z = 0$ in a circle of radius r, centered at the origin. Call the region interior to this circle **D**. The secret to unlocking this problem is to crossection this domain perpendicular to the y-axis. The crossections perpendicular to the x-axis give more complicated formulas.

Look at a crossection in the region **D** which is perpendicular to the y-axis. The cylinder $y^2 + z^2 = r^2$ intersects this vertical crossection in a square of sidelength $2\sqrt{r^2 - y^2}$. To see this notice that the x-variable is bounded on this crossection by $-\sqrt{r^2 - y^2} \leq x \leq \sqrt{r^2 - y^2}$. The vertical crossection above this line segment is parallel to the cylinder $y^2 + z^2 = r^2$, and on this crossection the height of this cylinder above the horizonal plane is $z = \pm\sqrt{r^2 - y^2}$. The difference of the z values is $2\sqrt{r^2 - y^2}$. This shows that the crossection perpendicular to the y-axis is a square of sidelength $2\sqrt{r^2 - y_2}$ and area $4(r^2 - y^2)$. Denote this area by $A(y)$.

Next we will integrate $A(y)$ to get the volume of the intersecting cylinders which is the same as integrating twice the height of the upper half of the horizontal cylinder as a double integral over the region **D**

$$\begin{aligned}
\text{Volume} &= \int\int_{\mathbf{D}} 2z dA \\
&= \int_{-r}^{+r} A(y) dy \\
&= \int_{-r}^{+r} 4(r^2 - y^2) dy \\
&= 4r^2 y \Big|_{-r}^{+r} - \frac{4}{3} y^3 \Big|_{-r}^{+r} \\
&= 8r^3 - \frac{8}{3} r^3 \\
&= \frac{16}{3} r^3
\end{aligned}$$