Classification of $a(2)$-finite Coxeter groups

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(Joint work with Richard Green)
I. Fully commutative elements in Coxeter groups

II. Kazhdan–Lusztig theory and the $a$-function

III. Classification of $a(2)$-finite Coxeter groups
1. Fully commutative elements in Coxeter groups
Coxeter Groups

- **Coxeter groups** are groups with special presentations; the presentations are encoded by weighted graphs.

\[ W = \langle S \mid R \rangle, \quad S = \{ s, t, u \}, \]

relations: \( s^2 = t^2 = u^2 = 1 \),

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We call the relations \( (sts \cdots = tst \cdots) \) braid relations.

If \( m(s, t) = 2 \), we also call \( (st = ts) \) a commutation relation.
Elements in Coxeter groups are represented by words, but not uniquely. For example, if $m_{s,u} = 2$ and $m_{t,u} = 3$, then

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Reduced words

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- Of the words expressing an element $w$, the ones of minimal length are called the reduced words of $w$. 
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$$ustut = sutut = suutu = stu.$$  

Of the words expressing an element $w$, the ones of minimal length are called the reduced words of $w$.

**Theorem (Matsumoto-Tits)**

*Every two reduced words of an element are connected via a finite sequence of braid relations.*
We define an element $w \in W$ to be \textit{fully commutative} if all its reduced words are connected via only commutation relations.
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By the Matsumoto-Tits Theorem, \( w \) is fully commutative if and only if no reduced word of \( w \) contains a contiguous subword \( sts \ldots \) of length \( m(s, t) \) where \( s, t \in S \) and \( m(s, t) \geq 3 \).
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**Example**

If \( m_{s,u} = 2 \) and \( m_{t,u} = 3 \), then \( ustu = sutu \) is not fully commutative.
Fully commutative elements have close connections with

- symmetric functions and Schubert polynomials
  (Billey–Haiman, Billey–Jockusch–Stanley, Fomin–Kirillov, etc)

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- Temperley–Lieb algebras
  (Fan, Green, Green–Losonczy)

- Kazhdan–Lusztig polynomials and cells
  (Green, Green–Losonczy, Shi)
Heaps of words

Definition

Let \( w = s_1 \cdots s_q \in S^* \). The heap of \( w \) is the labeled poset

\[
H(w) = (\{1, \cdots, q\}, \{s_1, \cdots, s_q\}, \leq)
\]

where

- each \( i \) is labeled by \( s_i \);
- “\( \leq \)” is obtained via the transitive closure of “\( \prec \)”, with

\[
i \prec j \quad \text{iff} \quad i < j \quad \text{and} \quad m_{s_i, s_j} \neq 2.
\]
Lattice embedding of heaps

Heaps can be naturally represented in the lattice $S \times \mathbb{N}$.

**Example**

In

\[
\begin{align*}
abcdb &\overset{b,d}{=} abcbd & \overset{b,c}{=} acbcd,
\end{align*}
\]

Note that commuting generators fall “independently”.

\[
\begin{array}{ccc}
\begin{array}{c}
5b \\
\text{3c} \\
\text{2b} \\
\text{1a}
\end{array} & \cong & \begin{array}{c}
4b \\
\text{3c} \\
\text{2b} \\
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5d \\
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\]
Heaps of fully-commutative elements

Proposition (Stembridge, 1996)

Let \( w \in W \). Then the heaps of the reduced words of \( w \) are all isomorphic as labeled posets if and only if \( w \) is fully commutative.

- This means we may now define the heap of a fully commutative element \( w \) to be that of any reduced word of \( w \).
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**Proposition (Stembridge, 1996)**

A word $w$ is the reduced word of a fully commutative element if and only if in $H(w)$:

- No column contains two points connected by an edge.
- For any $s, t \in S$ such that $m_{s,t} \geq 3$, there is no convex chain of edges connecting a sequence $s, t, \cdots$ of $m(s, t)$ points.

- This provides a visual criterion for full commutativity.
Suppose $s, t \in S$ and $m(s, t) \geq 3$. Let $I = \{s, t\}$. Using coset decompositions, one may define generalized star operations with respect to $I$ on elements of $W$. If $w$ is fully commutative and $w'_{st}$ is a reduced word of $w$, then $w \ast = w'_{s}$. More generally, star operations always amounts to addition or removal of an extremal letter. On heaps, star operations also always amounts to addition or removal of an extremal node.
Generalized star operations

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- On heaps, star operations also always amount to addition or removal of an extremal node.
II. Kazhdan–Lusztig theory and the a-function
Each Coxeter system \((W, S)\) gives rise to a Hecke algebra \(H\). The algebra \(H\) is an associative algebra over \(\mathcal{A} := \mathbb{Z}[v, v^{-1}]\) and has an \(\mathcal{A}\)-basis \(\{c_w : w \in W\}\) called the Kazhdan–Lusztig basis.
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**Proposition (Lusztig, 1985)**

Write

\[
c_x c_y = \sum_{z \in W} h_{x,y,z} c_z, \quad \forall x, y \in W.
\]

*For each \(z \in W\), there is a unique integer \(a(z) \geq 0\) such that*

- \(h_{x,y,z} \in v^{a(z)}\mathbb{Z}[v^{-1}]\) for all \(x, y \in W\);
- \(h_{x,y,z} \notin v^{a(z)-1}\mathbb{Z}[v^{-1}]\) for some \(x, y \in W\).
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The function \(a : W \rightarrow \mathbb{N}\) is closely related to the construction of the asymptotic Hecke algebra and the representation theory of \(H\).
Call a Coxeter group $\mathfrak{a}(n)$-finite if it contains finitely many elements of $\mathfrak{a}$-value $n$. Let $w \in W$. 

Lusztig showed:

\[ a(w) = 0 \iff w = 1; \]
\[ a(w) = 1 \iff w \text{ has a unique non-empty reduced word}; \]

An irreducible Coxeter group is $\mathfrak{a}(1)$-finite $\iff$ its Coxeter diagram is a tree and has at most one edge of weight larger than 3.

Question: Is there a similar classification of all $\mathfrak{a}(2)$-finite Coxeter groups in terms of Coxeter diagrams?
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### Computation of \(a\)-values

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**Proposition (Green–X, 2017)**

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\text{Let } w \in W. \text{ If } a(w) = 2, \text{ then } w \text{ is fully commutative.}
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- We will mostly compute/verify \(a\)-values in two indirect ways:
  1. by using a heap characterization of \(a\);
  2. by considering Kazhdan–Lusztig cells.
The heap characterization of $a$

Theorem (Shi, 2005)

Let $W$ be a Weyl group or an affine Weyl group. Let $w \in W$ be fully commutative. Let $\mathcal{AC}$ be the set of all antichains in $H(w)$ and $n(w) = \max(|A| : A \in \mathcal{AC})$. Then $a(w) = n(w)$. 

Two remarkable features of this result: $a$ is difficult to compute, but $n$ is easy. Exact $m$-values do not affect $n$, but why not $a$?

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The result $a(w) = n(w)$ also holds if $W$ is star-reducible.
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- Star operations are known to preserve cell membership, so one way to show \( x \sim y \) is to relate them by star operations.
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- Another method of checking $x \sim y$ involves checking $\mu(x, y) \neq 0$, where $\mu(x, y)$ is a coefficient in the Kazhdan–Lusztig polynomial $p_{x,y}$. 
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- Another method of checking $x \sim y$ involves checking $\mu(x, y) \neq 0$, where $\mu(x, y)$ is a coefficient in the Kazhdan–Lusztig polynomial $p_{x, y}$. The most technical part of our classification involves computing $\mu$-coefficients by a recursion of the form

$$\mu(x^*, y) + \mu(x^*, y) = \mu(x, y^*) + \mu(x, y^*)$$

.$$
III. Classification of \( a(2) \)-finite Coxeter groups
An irreducible Coxeter group is a(2)-finite if and only if its Coxeter diagram is complete or one of the following graphs.

\begin{align*}
A_n & \quad \begin{array}{c}
\text{-----} \\
(n \geq 1)
\end{array} \\
B_n & \quad \begin{array}{c}
\text{-----} \\
(n \geq 2)
\end{array} \\
\tilde{C}_n & \quad \begin{array}{c}
\text{-----} \\
(n \geq 4)
\end{array} \\
E_{q,r} & \quad \begin{array}{c}
\text{-----} \\
(q, r \geq 1)
\end{array} \\
F_n & \quad \begin{array}{c}
\text{-----} \\
(n \geq 4)
\end{array} \\
H_n & \quad \begin{array}{c}
\text{-----} \\
(n \geq 3)
\end{array} \\
I_2(m) & \quad \begin{array}{c}
\text{-----} \\
(5 \leq m \leq \infty)
\end{array}
\end{align*}
The proof of the “if” direction is short, but relies on Stembridge’s classification of $fc$-finite groups and a result of Ernst on the Temperley–Lieb algebra of type $\tilde{C}$.
The proof strategy

- The proof of the “if” direction is short, but relies on Stembridge’s classification of $fc$-finite groups and a result of Ernst on the Temperley–Lieb algebra of type $\tilde{C}$.

- To prove the “only if” direction, we show that a Coxeter group would be $a(2)$-infinite if its Coxeter diagram contains certain configurations.
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- To prove the “only if” direction, we show that a Coxeter group would be \( a(2)\)-infinite if its Coxeter diagram contains certain configurations. We do so by constructing infinitely many “witnesses” of \( a\)-value 2 for each “forbidden configuration”.
The proof strategy

- The proof of the “if” direction is short, but relies on Stembridge’s classification of fc-finite groups and a result of Ernst on the Temperley–Lieb algebra of type \( \tilde{C} \).

- To prove the “only if” direction, we show that a Coxeter group would be \( a(2) \)-infinite if its Coxeter diagram contains certain configurations. We do so by constructing infinitely many “witnesses” of \( a \)-value 2 for each “forbidden configuration”.

  A graph theoretical argument then shows that to avoid these configurations, the Coxeter diagram has to be in our list.
Forbidden configurations

- $m_1 \geq 5$
- $m_2 \geq 4$
- $m \geq 6$
- $m \geq 4$
- $5$
- $4$
- $4$
- $4$

Classification of a(2)-finite Coxeter groups
Example

Forbidden configuration:

Witnesses:

(type $\tilde{D}$)
Example

Forbidden configuration:

(star-reducible)
Example

Forbidden configuration:

\[ m_1 \quad v_0 \quad v_1 \quad v_2 \quad v_n \quad v_{n+1} \quad m_2 \]

\[(m_1 \geq 5, \ m_2 \geq 4)\]
Fully commutative elements

The $a$-function

Classification of $a(2)$-finite Coxeter groups

Example

Forbidden configuration:

Witnesses:
A remark on Shi’s result

Our witnesses actually all have $n$-value 2, so our proof could be significantly simplified if Shi’s $a(w) = n(w)$ result is true for general Coxeter groups. It would be interesting to know if this is the case.
Thank you!