JOURNAL OF COMBINATORIAL THEORY, Series A **73**, 168–169 (1996)
Article No. 0010

Note

There Exist Caps Which Block All Spaces of Fixed Codimension in $\mathbb{PG}(n, 2)$*

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Communicated by the Managing Editors

Received November 29, 1994

Recall that a *cap* in $\mathbb{PG}(n, 2)$ is simply a set of points with no three collinear. A cap which intersected all codimension 2 subspaces would yield an interesting example of a 2-block: 2-blocks have been much studied in the literature.

An interesting folklore conjecture, which received considerable attention, had it that in fact no cap is a 2-block. Although this conjecture can be shown to be true for small dimensions, we show that is far from being the case in general.

* This note developed from conversations at the Workshop on Modular Incidence Matrices held at the Fields Institute, University of Waterloo, August, 1994. We thank the organizers for organizing an excellent workshop.

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0097-3165/96 $12.00

PROPOSITION. *For any $k \geqslant 0$ and sufficiently large n there exists a cap in $\mathbb{PG}(n, 2)$ which intersects all subspaces of dimension $n-k$ of $\mathbb{PG}(n, 2)$.*

Proof. Let V be a vector space of dimension $n+1$ over the binary field $\mathbb{F}_2$. Choose a basis $\{e_0, e_1, ..., e_n\}$ of V. Let C be a subset of the projective space $\mathbb{P}(V)$ such that C contains only points of the form $\langle e_i + e_j \rangle$ where $i \neq j$. Then C can be regarded as a graph Γ with vertex set $\{0, 1, ..., n\}$ and where the pair (i, j) is an edge whenever $\langle e_i + e_j \rangle \in C$. Note that C is a cap if and only if if Γ is triangle free.

Suppose W is a subspace of $\mathbb{P}(V)$ disjoint from C and of codimension k. Then W is the intersection of k hyperplanes, $W = H_1 \cap H_2 \cap \cdots \cap H_k$. Colour the vertex i of Γ by the set $\{ j \mid \langle e_i \rangle \in H_j \}$, so that Γ is coloured with (at most) 2^k colours. If (i, j) is an edge of Γ then the point $\langle e_i + e_j \rangle$ of C is not in W. Therefore, some hyperplane, H_t say, does not contain $\langle e_i + e_j \rangle$. It follows that not both of the points $\langle e_i \rangle, \langle e_j \rangle$ are in H_t. However, since every line intersects every hyperplane, we conclude that exactly one of the two points $\langle e_i \rangle$, $\langle e_j \rangle$ lies in H_t. But this means that the vertices i and j have different colours. Therefore we have a proper colouring of Γ.

Now it is well-known, and has been proved by many people independently (probably first by W.T. Tutte [T]), that there exist triangle-free graphs with arbitrarily high chromatic number. Taking Γ to be a triangle-free graph of chromatic number langer than 2^k will produce a cap that meets all projective $(n-k)$-spaces. ∎

This work is related to question on binary linear codes and families of subsets which are symmetric-difference free. These connections will be developed elsewhere.

REFERENCE

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