Problem Set #17

Due: Thursday, 9 February 2012

1. If we think of an electron as a particle, the function $P(r) := 1 - (2r^2 + 2r + 1)e^{-2r}$ is the cumulative distribution function of the distance r of the electron in a hydrogen atom from the center of the atom. (The distance is measured in Bohr radii; 1 Bohr radius $= 5.29 \times 10^{-11}$ m. Niels Bohr (1885-1962) was a Danish physicist.) For example, $P(1) = 1 - 5e^{-2} \approx 0.32$ means that the electron is within 1 Bohr radius from the center of the atom 32% of the time.

(a) Find a formula for the density function of this distribution. Sketch the density function and the cumulative distribution function.

(b) Find the median distance and the mean distance. Near what value of r is an electron most likely to be found?

2. Suppose that $a < b$. The purpose of this problem is to show that if f is a quadratic polynomial, then we have

$$\int_a^b f(x)\,dx = \frac{b-a}{3}\left(\frac{f(a)}{2} + 2f\left(\tfrac{a+b}{2}\right) + \frac{f(b)}{2}\right).$$

(a) Show that this equation holds for $f_0(x) = 1$, $f_1(x) = x$ and $f_2(x) = x^2$.

(b) Show that the equation holds for any quadratic polynomial $f(x) = Ax^2 + Bx + C$.

3. Consider the following method for approximating $\int_a^b f(x)\,dx$. Partition the interval $[a,b]$ into n equal subintervals. On each subinterval approximate the function f by a quadratic polynomial that agrees with f at both endpoints and at the midpoint of the subinterval.

(a) Explain why the integral of f on the subinterval $[x_{i-1}, x_i]$ is approximately equal to the expression

$$\frac{x_i - x_{i-1}}{3}\left[\frac{f(x_{i-1})}{2} + 2f\left(\frac{x_{i-1}+x_i}{2}\right) + \frac{f(x_i)}{2}\right].$$

(b) Show that if we add up these approximations, we get Simpson's rule:

$$\int_a^b f(x)\,dx \approx \tfrac{2}{3}\mathrm{MID}(n) + \tfrac{1}{3}\mathrm{TRAP}(n).$$