Problem Set #23

Due: Thursday, 29 March 2012

1. Let $J(x) := \sum_{k=0}^{\infty} \frac{(-1)^k}{(k!)^2} \left(\frac{x}{2}\right)^{2k}$. The function J is called a ***Bessel function of order zero***.

(a) What is the radius of convergence of this power series?

(b) Show that J satisfies the differential equation $x^2 J''(x) + xJ'(x) + x^2 J(x) = 0$.

2. **(a)** The ***error function*** is defined by $\operatorname{erf}(x) := \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2}\, dt$. Find its Taylor expansion.

(b) By looking at the Taylor series, decide with of the following functions

$$\ln(1+y^2) \qquad \sin(y^2) \qquad 1-\cos(y)$$

is largest and which is smallest for values of y near 0.

3. The Fibonacci sequence $(a_n)_{n=0}^{\infty}$ is defined by $a_0 = 1$, $a_1 = 1$ and $a_{n+1} := a_n + a_{n-1}$ for all $n \geqslant 1$.

(a) Show that $\frac{a_{n+1}}{a_n} \leqslant 2$ for $n \geqslant 0$.

(b) Let $f(x) := \sum_{n=0}^{\infty} a_n x^n$. Show that $f(x)$ converges if $|x| < \frac{1}{2}$.

(c) If $|x| < \frac{1}{2}$ then prove that $f(x) = \frac{1}{1-x-x^2}$.

(d) Use the partial fraction decomposition for $\frac{1}{1-x-x^2}$ and the geometric series to obtain another power series expression for f.

(e) By comparing coefficients, show that $a_n = \frac{1}{\sqrt{5}} \left(\left(\frac{1+\sqrt{5}}{2}\right)^{n+1} - \left(\frac{1-\sqrt{5}}{2}\right)^{n+1} \right)$.