Solutions 02

P2.1. The *Fibonacci numbers* are defined by $F_0 := 0$, $F_1 := 1$, and $F_n := F_{n-1}+F_{n-2}$ for all $n \geqslant 2$. The first few terms are $0, 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, 233, 377, 610, 987, 1597, \dots$. For any nonnegative integer n, verify that $F_{n+1}^2 = F_{n+2}\,F_n + (-1)^n$.

Solution. We proceed by induction on n. When $n = 0$ and $n = 1$, we have

$$\begin{aligned} F_1^2 - F_2\,F_0 - (-1)^0 &= (1)^2 - (1)(0) - 1 = 0 && \text{and} \\ F_2^2 - F_3\,F_1 - (-1)^1 &= (1)^2 - (2)(1) + 1 = 0\,, \end{aligned}$$

so the base cases hold. Assume that $F_{n+1}^2 - F_{n+2}F_n - (-1)^n = 0$. Using the defining recurrence relation twice and the induction hypothesis gives

$$\begin{aligned} F_{n+2}^2 - F_{n+3}\,F_{n+1} - (-1)^{n+1} &= F_{n+2}^2 - (F_{n+2} + F_{n+1})F_{n+1} + (-1)^n \\ &= F_{n+2}^2 - F_{n+2}\,F_{n+1} - F_{n+1}\,F_{n+1} + (-1)^n \\ &= F_{n+2}(F_{n+2} - F_{n+1}) - F_{n+1}^2 + (-1)^n \\ &= F_{n+2}\,F_n - F_{n+1}^2 + (-1)^n \\ &= -\big(F_{n+1}^2 - F_{n+2}\,F_n\big) + (-1)^n = -(-1)^n + (-1)^n = 0 \end{aligned}$$

which completes the induction. □

P2.2. Establish the cancellation law for addition: for any nonnegative integers k, m, and n, show that the equation $m + k = n + k$ implies that $m = n$.

Solution. Fix nonnegative integers m and n. Consider the subset

$$\mathcal{X} := \big\{k \in \mathbb{N} \mid m + k = n + k \text{ implies that } m = n\big\}.$$

The definition of addition gives $m + 0 = m$ and $n + 0 = n$. Hence, the equation $m + 0 = n + 0$ implies that $m = m + 0 = n + 0 = n$, so $0 \in \mathcal{X}$. Suppose that $m + \mathrm{S}(k) = n + \mathrm{S}(k)$ for some nonnegative integer k. The definition of addition gives $\mathrm{S}(m + k) = m + \mathrm{S}(k) = n + \mathrm{S}(k) = \mathrm{S}(n + k)$. Since the Peano Condition (C1) asserts that the successor function S is injective, we deduce that $m + k = n + k$. When $k \in \mathcal{X}$, it follows that $m = n$, so $\mathrm{S}(k) \in \mathcal{X}$. Thus, the principle of induction yields $\mathcal{X} = \mathbb{N}$. □

P2.3. Use the well-ordering of the nonnegative integers to prove that any nonempty subset of the nonnegative integers that is bounded above has a unique greatest element.

Solution. Let $\mathcal{X}$ be a nonempty subset of nonnegative integers that is bounded above. Consider the associated subset $\mathcal{Y} \subset \mathbb{N}$ consisting of all upper bounds for $\mathcal{X}$;

$$\mathcal{Y} := \big\{n \in \mathbb{N} \mid x \leqslant n \text{ for all } x \in \mathcal{X}\big\}.$$

The set $\mathcal{Y}$ is nonempty because $\mathcal{X}$ is bounded above. As $\mathcal{X} \neq \varnothing$, we see that $0 \notin \mathcal{Y}$. By the Well-Ordering Principle, the subset $\mathcal{Y}$ has a unique least element m.

Suppose that m is not in $\mathcal{X}$. Since $m \neq 0$, there exists a nonnegative integer n such that $\mathrm{S}(n) = m$. We claim that n would also be an upper bound for $\mathcal{X}$. Indeed, for any $x \in \mathcal{X}$, we would have $x < m$ because $x \neq m$. Hence, there would exist a nonzero k in $\mathbb{N}$ such that $m = x + k$. If $k = 1$, then we would have $n + 1 = \mathrm{S}(n) = m = x + 1$, so

$n = x$. If $k > 1$, then there would exist a nonzero j in $\mathbb{N}$ such that $k = 1 + j$. It would follow that $n + 1 = \mathrm{S}(n) = m = x + k = x + j + 1$, so $n = x + j$ and $x < n$. In either case, we would have $x \leqslant n$, so n would be an upper bound for $\mathcal{X}$ and $n \in \mathcal{Y}$. However, the inequality $n < m$ would contradict the fact that m is the least element in $\mathcal{Y}$. Thus, we deduce that m is in $\mathcal{X}$.

Since the least upper bound m for $\mathcal{X}$ belongs to $\mathcal{X}$, it is the greatest element in $\mathcal{X}$. □