Solutions 03

P3.1. Define a binary relation on the set $\mathbb{R}$ of real numbers: for any real numbers x and y, we write $x \sim y$ if there exists an integer k such that $x - y = 2k\pi$.

i. Verify that this is an equivalence relation.

ii. Describe a system of distinct representatives (also known as a transversal) for this equivalence relation.

iii. Is addition well-defined on the quotient set $\mathbb{R}/\!\sim$?

iv. Is multiplication well-defined on the quotient set $\mathbb{R}/\!\sim$?

Solution.

i. Let x, y, and z be real numbers.

(*reflexivity*) For any real number x, we have $x - x = 0 = 2(0)\pi$, so $x \sim x$.

(*symmetry*) Suppose that $x \sim y$. By definition, there exist an integer k such that $x - y = 2k\pi$. It follows that $y - x = 2(-k)\pi$, so $y \sim x$.

(*transitivity*) Suppose that $x \sim y$ and $y \sim z$. There are integers k and ℓ such that $x - y = 2k\pi$ and $y - z = 2\ell\pi$. It follows that

$$x - z = (x - y) + (y - z) = 2k\pi + 2\ell\pi = 2(k + \ell)\pi,$$

so $x \sim z$.

Since the relation is reflexive, symmetric, and transitive, we conclude that it is an equivalence relation.

ii. We claim that the real numbers x satisfying the inequalities $0 \leqslant x < 2\pi$ form a system of distinct representatives for the equivalence relation.

We first show that every equivalence class is represented by a real number in this half-open interval $[0, 2\pi)$. By definition, the floor function $x \mapsto \lfloor x \rfloor$ sends a real number x to the greatest integer less than or equal to x;

$$\lfloor x \rfloor := \max\{m \in \mathbb{Z} \mid m \leqslant x\}.$$

Given a real number y, set $k := \lfloor y/(2\pi) \rfloor$. It follows that $k \in \mathbb{Z}$ and

$$k \leqslant \frac{y}{2\pi} < k + 1 \quad \Leftrightarrow \quad 2k\pi \leqslant y < 2(k+1)\pi \quad \Leftrightarrow \quad 0 \leqslant y - 2k\pi < 2\pi.$$

Since $y \sim (y - 2k\pi)$, the real number $y - 2k\pi$ represents the equivalence class of y and belongs to the half-open interval $[0, 2\pi)$.

It remains to demonstrate that no two equivalence classes for real numbers in the half-open interval $[0, 2\pi)$ coincide. Consider real numbers x and y such that $0 \leqslant x < 2\pi$, $0 \leqslant y < 2\pi$, and $[x] = [y]$. The inequalities give $0 \leqslant |x - y| < 2\pi$. The equality implies that there is an integer k such that $|x - y| = 2k\pi$. It follows that $0 \leqslant 2k\pi = |x - y| < 2\pi$, so $k = 0$. We conclude that $|x - y| = 0$ and $x = y$.

iii. We claim that addition of real numbers is independent on the choosen representatives in the equivalence classes. Let w, x, y, and z be real numbers such that $w \sim x$ and $y \sim z$. By definition, there exists integers j and k such that $w - x = 2j\pi$ and $y - z = 2k\pi$. It follows that

$$(w + y) - (x + z) = (w - x) + (y - z) = 2j\pi + 2k\pi = 2(j + k)\pi.$$

Since $w + y \sim x + z$, addition on the quotient $\mathbb{R}/\!\sim$ is well-defined.

iv. We demonstrate that multiplication of real numbers depends on the choice of representatives in the equivalence classes. For instance, we have $0 \sim 2\pi$, but

$$(0)\left(\frac{\pi}{8}\right) = 0 \qquad \text{and} \qquad 0 < (2\pi)\left(\frac{\pi}{8}\right) = \frac{\pi^2}{4} = \left(\frac{\pi}{4}\right)(\pi) < \pi$$

From part ii, we see that 0 and $\pi^2/4$ are not equivalent. Thus, multiplication on the quotient $\mathbb{R}/\!\sim$ is *not* well-defined. □

Remark. The quotient $\mathbb{R}/\!\sim$ may be identified with the unit circle $\{z \in \mathbb{C} \mid |z| = 1\}$. Since the trigonometric functions are independent of the choice of representative, the equivalence class $\theta \in \mathbb{R}/\!\sim$ corresponds to $\cos(\theta) + \sqrt{-1}\,\sin(\theta) \in \mathbb{C}$.

P3.2. The *absolute value* function $|\bullet| : \mathbb{Z} \to \mathbb{N}$ is defined, for any integer m, by

$$|m| := \begin{cases} m & \text{if } m \geqslant 0, \\ -m & \text{if } m < 0. \end{cases}$$

i. Let n be a nonnegative integer. For any integer m, prove that $-n \leqslant m \leqslant n$ if and only if $|m| \leqslant n$.

ii. For any integers m and n, show that $\big||n| - |m|\big| \leqslant |n + m| \leqslant |n| + |m|$.

Solution.

i. We consider two cases.

(*nonnegative*) Suppose that $m \geqslant 0$. It follows that $|m| = m$. For any nonnegative integer n, we have $m \leqslant n$ if and only if $|m| \leqslant n$. Since $-n \leqslant 0$ and $m \geqslant 0$, we also have $-n \leqslant m$.

(*negative*) Suppose that $m < 0$. It follows that $|m| = -m$. For any nonnegative integer n, we have $-n \leqslant m$ if and only if $-m \leqslant n$ which is equivalent to $|m| \leqslant n$. Since $n \geqslant 0$ and $m < 0$, we also have $m \leqslant n$.

ii. We start by proving the second inequality. By part i, we have $-|n| \leqslant n \leqslant |n|$ and $-|m| \leqslant m \leqslant |m|$. Addition gives $-(|n| + |m|) \leqslant n + m \leqslant |n| + |m|$. Using part i again, we obtain $|n + m| \leqslant |n| + |m|$.

The first inequality is a special case of the second. Specifically, we have $|m| = |(n + m) + (-n)| \leqslant |m + n| + |n|$, which yields $-|n + m| \leqslant |n| - |m|$. We also have $|n| = |(n + m) + (-m)| \leqslant |n + m| + |m|$, which gives $|n| - |m| \leqslant |n + m|$. Hence, part i establishes that $\big||n| - |m|\big| \leqslant |n + m|$. □

P3.3. Let k, m, and n be integers. Verify that $\gcd\big(k, \gcd(m, n)\big) = \gcd\big(\gcd(k, m), n\big)$.

Solution. Let $d := \gcd\big(k, \gcd(m, n)\big)$ and $e := \gcd\big(\gcd(k, m), n\big)$. We first claim that d divides e. The definition of d implies that d divides k and d divides $\gcd(m, n)$, so d divides k, m, and n. It follows that d divides $\gcd(k, m)$ and d divides n, whence d divides $e := \gcd\big(\gcd(k, m), n\big)$.

We next claim that e divides d. The definition of e implies that e divides $\gcd(k, m)$ and e divides n, so e divides k, m, and n. It follows that e divides k and e divides $\gcd(m, n)$, whence e divides $d := \gcd\big(k, \gcd(m, n)\big)$.

Finally, because d divides e and e divides d, there exists integers i and j such that $e = id$ and $d = je$. We deduce that $e = (ij)e$ and $e(1 - ij) = 0$, so either $e = 0$ or $i = j = \pm 1$. Since d and e are nonnegative integers, we conclude that $d = e$. □