2.3 Multichoose Identities

Inspired by our binomial identities, we pursue analogues for the multichoose coefficients. Curiously, this also leads to new results about binomial coefficients themselves.

Proposition 2.3.1 (Absorption)**.** *For any integer k, we have*

$$k\left(\!\!\binom{x}{k}\!\!\right) = x\left(\!\!\binom{x+1}{k-1}\!\!\right).$$

Double-counting proof. Since both sides vanish when $k \leqslant 0$, we may assume that k is a positive integer. Once again, it suffices to verify the identity when $x = n$ is sufficiently large integer. How many k-tuples $(\lambda_1, \lambda_2, \dots, \lambda_k) \in \mathbb{N}^k$ satisfying $n \geqslant \lambda_1 \geqslant \lambda_2 \geqslant \cdots \geqslant \lambda_k \geqslant 1$ and having one entry underlined are there?

Answer 1: Definition 2.2.1 implies that there are $\left(\!\binom{n}{k}\!\right)$ ways to create the non-increasing sequence. Since there are k ways to underline an entry, the total number k-tuples having the desired form is $k\left(\!\binom{n}{k}\!\right)$.

Answer 2: First determine the integer to be underlined. There are n such choices. Suppose the underlined number is r. Next, create a positive $(k-1)$-tuple with entries between 1 and $n+1$. There are $\left(\!\binom{n+1}{k-1}\!\right)$ such sequences. Any r's among the chosen entries go to the left of the underlined r. Any $n+1$'s among the chosen entries are converted to r's and repositioned to the right of the underlined r. Thus, there are $n\left(\!\binom{n+1}{k-1}\!\right)$ sequences having the required form. □

Remark 2.3.2. The absorption identities for multichoose and binomial coefficients are different:

$$k\binom{n+k-1}{k} = n\binom{n+k-1}{k-1} \quad \text{versus} \quad k\binom{n}{k} = n\binom{n-1}{k-1}.$$

While not a perfect match for the upper sum identity [2.0.8], we do have a lower sum identity for multichoose coefficients.

Proposition 2.3.3 (Parallel Sum)**.** *For any integer k, we have*

$$\left(\!\!\binom{x+1}{k}\!\!\right) = \sum_{j=0}^{k} \left(\!\!\binom{x}{j}\!\!\right).$$

The sum of the first k entries in the n-th row of Table 2.9 equals the $(n+1,k)$-entry.

Double-counting proof. Since both sides vanish when $k < 0$, we may assume that k is a nonnegative integer. It is enough to prove this identity when $x = n$ is a sufficiently large integer. How many ways can we allocate k votes to $n+1$ candidates?

Answer 1: Definition 2.2.1 implies that there are $\left(\!\binom{n+1}{k}\!\right)$ allocations.

Answer 2: Focus on the number of votes allocated to everyone other than candidate $n+1$. The first n candidates receive a total of j votes for some $0 \leqslant j \leqslant k$, so there are $\left(\!\binom{n}{j}\!\right)$ ways to allocate these votes. The last candidate receives all of the remaining $k-j$ votes. Thus, there are $\sum_{j=0}^{k}\left(\!\binom{n}{j}\!\right)$ allocations. □

Remark 2.3.4. Proposition 2.2.2, together with this parallel sum identity for multichoose coefficients, give a parallel sum identity for binomial coefficients: for any integer k, we have

$$\binom{x+k}{k} = \sum_{j=0}^{k} \binom{x+j-1}{j}.$$

The multichoose analogue for the binomial theorem [2.1.6] requires formal power series, which will discuss more carefully later in the text.

Theorem 2.3.5. *For any nonnegative integer n, we have*

$$\left(\sum_{j\in\mathbb{N}} x^j\right)^n = \sum_{k\in\mathbb{N}} \left(\!\!\binom{n}{k}\!\!\right) x^k .$$

Proof. Expanding the product gives

$$\prod_{i=1}^{n} \left(\sum_{j\in\mathbb{N}} x_i^j\right) = \sum_{M} \prod_{x_i \in S} x_i^{\nu(i)}$$

where the righthand summands are indexed by the multisets M over $[n]$ or equivalently the functions $\nu\colon [n] \to \mathbb{N}$ satisfying $\sum_{j=1}^{n} \nu(j) < \infty$. Setting $x_i := x$ for all $1 \leqslant j \leqslant n$, we obtain

$$\left(\sum_{j\in\mathbb{N}} x^j\right)^n = \sum_{M} x^{\nu(1)+\nu(2)+\cdots+\nu(n)} = \sum_{M} x^{|M|} = \sum_{k\in\mathbb{N}} \left(\!\!\binom{n}{k}\!\!\right) x^k . \qquad \square$$

There is a legitimate multichoose counterpart to the upper sum identity [2.0.8].

Proposition 2.3.6 (Upper sum)**.** *For all nonnegative integers n and k excluding the degenerate case $(n,k) = (0,0)$, we have*

$$\left(\!\!\binom{n}{k}\!\!\right) = \sum_{j=1}^{n} \left(\!\!\binom{j}{k-1}\!\!\right).$$

Double-counting proof. How many k-tuples $(\lambda_1, \lambda_2, \dots, \lambda_k) \in \mathbb{N}^k$ satisfying $n \geqslant \lambda_1 \geqslant \lambda_2 \geqslant \cdots \geqslant \lambda_k \geqslant 1$ are there?

Answer 1: Definition 2.2.1 implies that there are $\left(\!\binom{n}{k}\!\right)$ k-tuples having the desired form.

Answer 2: Focus on the largest entry λ_1. For all $1 \leqslant j \leqslant n$, the number of non-increasing integer vectors in $\mathbb{N}^k$ with $\lambda_1 = j$ equals the number of $(k-1)$-tuples $(\lambda_2, \lambda_3, \dots, \lambda_k) \in \mathbb{N}^{k-1}$ satisfying $j \geqslant \lambda_2 \geqslant \lambda_3 \geqslant \cdots \geqslant \lambda_k \geqslant 1$. Since Definition 2.2.1 implies that there are $\left(\!\binom{j}{k-1}\!\right)$ such vectors, there is a total of $\sum_{j=1}^{n} \left(\!\binom{j}{k-1}\!\right)$ non-increasing positive integer k-tuples. $\square$

Algebraic proof. Combining Proposition 2.2.2, the symmetry identity for binomial coefficients [2.0.5], and the parallel sum identity for binomial coefficients [2.3.4] gives

$$\sum_{j=1}^{n} \left(\!\!\binom{j}{k-1}\!\!\right) = \sum_{j=1}^{n} \binom{j+k-2}{k-1} = \sum_{j=0}^{n-1} \binom{j+k-1}{j}$$
$$= \binom{k+n-1}{n-1} = \binom{n+k-1}{k} = \left(\!\!\binom{n}{k}\!\!\right). \qquad \square$$

We end this subsection with an identity intertwining binomial and multichoose coefficients.

Problem 2.3.7. For any nonnegative integer n and any integer k, demonstrate that

$$\left(\!\!\binom{n}{k}\!\!\right) = \sum_{j=0}^{n} \binom{n}{j} \left(\!\!\binom{j}{k-j}\!\!\right).$$

Double-counting solution. How many ways can we allocate k votes to n candidates?

Answer 1: Definition 2.2.1 implies that there are $\left(\!\binom{n}{k}\!\right)$ allocations.

Answer 2: Focus on the candidates that receive at least one vote. When j candidates, for some $0 \leqslant j \leqslant n$, receive at least one vote, there are $\binom{n}{j}$ ways to select them. Since each of these candidate receives a vote, there are $\left(\!\binom{j}{k-j}\!\right)$ ways to allocate the remaining $k - j$ votes. In total, there are $\sum_{j=0}^{n} \binom{n}{j} \left(\!\binom{j}{k-j}\!\right)$ allocations. $\square$

Algebraic proof. Combining Proposition 2.2.2 and the Vandermonde identity for binomial coefficients [2.1.5] give

$$\sum_{j=0}^{n} \binom{n}{j} \left(\!\!\binom{j}{k-j}\!\!\right) = \sum_{j=0}^{n} \binom{n}{j} \binom{k-1}{k-j} = \binom{n+k-1}{k} = \left(\!\!\binom{n}{k}\!\!\right). \qquad \square$$

3

Stirling Objects

We study a few more famous sequences. These numbers resemble binomial coefficients in several ways: they depend on a pair of integers (one being nonnegative), satisfy similar addition formulas, and manifest as the coefficients in natural families of polynomials. As introduced by Jovan Karamata and promoted by Donald Knuth, our notation underscores these analogies. Our approach also stresses double-counting.

3.0 Set Partitions

We start be some vocabulary to describe the subdivisions of a set.

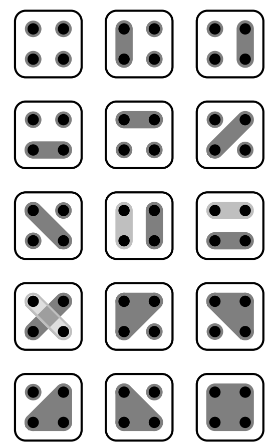

Figure 3.1: The 15 partitions of the set [4]

Definition 3.0.1. A ***partition*** of a finite set $\mathcal{A}$ is a collection $\{\mathcal{B}_1, \mathcal{B}_2, \dots, \mathcal{B}_\ell\}$ of subsets of $\mathcal{A}$ such that
- $\mathcal{B}_j \neq \varnothing$ for all $1 \leqslant j \leqslant \ell$,
- $\mathcal{B}_j \cap \mathcal{B}_k = \varnothing$ for all $j \neq k$, and
- $\mathcal{B}_1 \cup \mathcal{B}_2 \cup \cdots \cup \mathcal{B}_\ell = \mathcal{A}$.

The subset $\mathcal{B}_j$ is a ***block*** of the partition.

In other words, a partition of $\mathcal{A}$ is a collection of nonempty disjoint subsets of $\mathcal{A}$ such that every element in $\mathcal{A}$ is in exactly one of the subsets.

Definition 3.0.2. For all nonnegative integers n and all integers k, the ***Stirling subset number*** $\left\{ {n \atop k} \right\}$ is the number of partitions of the set $[n]$ into k blocks. We read $\left\{ {n \atop k} \right\}$ as "n subset k" and the curly braces remind us of the 'subsets'.

Historically, these are known as the Stirling number of the second kind. Using Figure 3.1, we see that $\left\{ {4 \atop 4} \right\} = 1$, $\left\{ {4 \atop 3} \right\} = 6$, $\left\{ {4 \atop 2} \right\} = 7$, $\left\{ {4 \atop 1} \right\} = 1$, and $\left\{ {4 \atop 0} \right\} = 0$.

Some special values are easy to determine.
- For all $n < k$ and all $k < 0$, we have $\left\{ {n \atop k} \right\} = 0$ because there are no partitions of $[n]$ with k blocks.
- For all nonnegative integers n, we have $\left\{ {n \atop n} \right\} = 1$ because there is a unique partiton $\{1\} \cup \{2\} \cup \{3\} \cup \cdots \cup \{n\}$ of $[n]$ with n blocks. When $n = 0$, there is a unique way to partition the empty set into zero nonempty blocks.
- For all positive integers n, we have $\left\{ {n \atop 1} \right\} = 1$ because $\{1, 2, \dots, n\}$ is the unique partition of $[n]$ with 1 block.
- For all positive integer k, we have $\left\{ {0 \atop k} \right\} = 0$ because a block is nonempty.
- For all positive integers n, we have $\left\{ {n \atop 0} \right\} = 0$ because a nonempty set needs at least one block.

- For all positive integers n, we have $\genfrac\{\}{0pt}{}{n}{2} = 2^{n-1} - 1$ because the block not containing n is a nonempty subset of $\{1, 2, \dots, n-1\}$.
- For all positive integers n, we have $\genfrac\{\}{0pt}{}{n}{n-1} = \binom{n}{2}$ because a partition of the set $[n]$ into $n-1$ blocks must consist of $n-2$ singletons and one doubleton.

Just like binomial coefficients, the Stirling subset numbers satisfy a concise two-term recurrence.

Proposition 3.0.3 (Addition). *For all nonnegative integers n and all integer k, we have*

$$\genfrac\{\}{0pt}{}{n+1}{k+1} = \genfrac\{\}{0pt}{}{n}{k} + (k+1)\genfrac\{\}{0pt}{}{n}{k+1}.$$

Double-Counting Proof. How many partitions of $[n+1]$ have $k+1$ blocks?

Answer 1: The definition of the Stirling subset numbers implies that there are $\genfrac\{\}{0pt}{}{n+1}{k+1}$ partitions of the set $[n+1]$ into $k+1$ blocks.

Answer 2: Focus on whether the singleton $\{n+1\}$ is a block in the partition. When it is, the remaining elements lie in the set $[n]$ and can be partitioned into k blocks in $\genfrac\{\}{0pt}{}{n}{k}$ ways. When it is not, we partition the set $[n]$ into $k+1$ blocks and then insert the element $n+1$ into one of the blocks, which can be done in $(k+1)\genfrac\{\}{0pt}{}{n}{k+1}$ ways. Thus, we have a total of $\genfrac\{\}{0pt}{}{n}{k} + (k+1)\genfrac\{\}{0pt}{}{n}{k+1}$ partitions of $[n+1]$ into $k+1$ blocks. □

n	$\genfrac\{\}{0pt}{}{n}{0}$	$\genfrac\{\}{0pt}{}{n}{1}$	$\genfrac\{\}{0pt}{}{n}{2}$	$\genfrac\{\}{0pt}{}{n}{3}$	$\genfrac\{\}{0pt}{}{n}{4}$	$\genfrac\{\}{0pt}{}{n}{5}$	$\genfrac\{\}{0pt}{}{n}{6}$
0	1	0	0	0	0	0	0
1	0	1	0	0	0	0	0
2	0	1	1	0	0	0	0
3	0	1	3	1	0	0	0
4	0	1	7	6	1	0	0
5	0	1	15	25	10	1	0
6	0	1	31	90	65	15	1

Figure 3.2: Matrix of Stirling subset numbers

Echoing our development of the multichoose coefficients, the version of the binomial theorem [2.1.6] for Stirling subset numbers uses formal power series.

Theorem 3.0.4. *For all nonnegative integers m, we have*

$$\frac{x^m}{(1-x)(1-2x)(1-3x)\cdots(1-mx)} = \prod_{j=1}^{m} \frac{x}{1-jx} = \sum_{n\in\mathbb{N}} \genfrac\{\}{0pt}{}{n}{m} x^n .$$

Inductive proof. When $m = 0$, we have $1 = \sum_{n\in\mathbb{N}} \genfrac\{\}{0pt}{}{n}{0} x^n$, so the base case holds. For all positive integers m, assume that

$$\sum_{n\in\mathbb{N}} \genfrac\{\}{0pt}{}{n}{m-1} x^n = \prod_{j=1}^{m-1} \frac{x}{1-mx} .$$

The addition formula and the induction hypothesis give

$$\begin{aligned} \sum_{n\in\mathbb{N}} \genfrac\{\}{0pt}{}{n}{m} x^n &= \sum_{n\in\mathbb{N}} \left(m\genfrac\{\}{0pt}{}{n-1}{m} + \genfrac\{\}{0pt}{}{n-1}{m-1} \right) x^n \\ &= mx\left(\sum_{n\in\mathbb{N}} \genfrac\{\}{0pt}{}{n}{m} x^n \right) + x\left(\sum_{n\in\mathbb{N}} \genfrac\{\}{0pt}{}{n}{m-1} x^n \right) \end{aligned}$$

which implies that

$$(1-mx)\left(\sum_{n\in\mathbb{N}} \genfrac\{\}{0pt}{}{n}{m} x^n \right) = x\left(\prod_{j=1}^{m-1} \frac{x}{1-mx} \right),$$

so we conclude that $\prod_{j=1}^{m} \frac{x}{1-mx} = \sum_{n\in\mathbb{N}} \genfrac\{\}{0pt}{}{n}{m} x^n$. □

To recognize the Stirling subset numbers as entries in a change of basis matrix, we first introduce two bases for the rational vector space of univariate polynomials.

Notation 3.0.5. For all nonnegative integers k, the ***falling factorial power*** and ***rising factorial powers*** are

$$n^{\underline{k}} := n(n-1)\cdots(n-k+1) = \prod_{j=1}^{k}(n-j+1)$$
$$n^{\overline{k}} := n(n+1)\cdots(n+k-1) = \prod_{j=1}^{k}(n+j-1)\,.$$

Both binomial and multichoose coefficients can be neatly expressed via falling and rising factorial powers;

$$\binom{x}{k} = \frac{x^{\underline{k}}}{k!}\,,$$
$$\left(\!\binom{x}{k}\!\right) = \frac{x^{\overline{k}}}{k!}\,.$$

When read aloud, the quantity $x^{\underline{k}}$ is "n to the k falling" and the quantity $x^{\overline{k}}$ is "n to the k rising". Observe that $n^{\underline{0}} = n^{\overline{0}} = 1$ and $n! = n^{\underline{n}} = 1^{\overline{n}}$.

The Stirling subset numbers are the coefficients for falling factorial powers that yield ordinary powers.

Proposition 3.0.6 (Power conversion). *For any nonnegative integers n, we have*

$$x^n = \sum_{k\in\mathbb{Z}} \left\{ {n \atop k} \right\} x^{\underline{k}}\,.$$

Double-counting proof. Since a nonzero polynomial in $\mathbb{Q}[x]$ has at most finitely many zeros, it suffices to prove this identity when $x = m$ is a nonnegative integer. How many ways can n students be assigned to m different classrooms where some classrooms are allowed to be empty?

Answer 1: Each student is assigned to one of m classrooms, so there are m^n assignments.

Answer 2: Focus on the number k of nonempty classrooms. The definition of the Stirling subset numbers implies that there are $\left\{ {n \atop k} \right\}$ ways to partition the students into k blocks. The k blocks of students can be assigned to classrooms in

$$m^{\underline{k}} = (m)(m-1)\cdots(m-k+1)$$

ways, because no two blocks can be assigned to the same classroom. Therefore, there are $\sum_{k\in\mathbb{Z}} \left\{ {n \atop k} \right\} x^{\underline{k}}$ assignments. □

Inductive proof. When $n = 0$, we have $x^0 = 1 = \left\{ {0 \atop 0} \right\} x^{\underline{0}}$, so the base case holds. For all positive integers n, assume that

$$x^{n-1} = \sum_{k\in\mathbb{Z}} \left\{ {n-1 \atop k} \right\} x^{\underline{k}}\,.$$

Since $x^{\underline{k+1}} = x^{\underline{k}}(x-k)$, we see that $(x)(x^{\underline{k}}) = x^{\underline{k+1}} + k\,x^{\underline{k}}$ for all nonnegative integers k. The induction hypothesis and addition formula give

$$\begin{aligned} x^{\underline{n}} &= (x)(x^{\underline{n-1}}) = x\left(\sum_{k\in\mathbb{Z}} \left\{ {n-1 \atop k} \right\} x^{\underline{k}}\right) \\ &= \sum_{k\in\mathbb{Z}} \left\{ {n-1 \atop k} \right\} (x)(x^{\underline{k}}) = \sum_{k\in\mathbb{Z}} \left\{ {n-1 \atop k} \right\} \left(x^{\underline{k+1}} + k x^{\underline{k}}\right) \\ &= \sum_{k\in\mathbb{Z}} \left\{ {n-1 \atop k-1} \right\} x^{\underline{k}} + \sum_{k\in\mathbb{Z}} k \left\{ {n-1 \atop k} \right\} x^{\underline{k}} \\ &= \sum_{k\in\mathbb{Z}} \left(\left\{ {n-1 \atop k-1} \right\} + k \left\{ {n-1 \atop k} \right\} \right) x^{\underline{k}} = \sum_{k\in\mathbb{Z}} \left\{ {n \atop k} \right\} x^{\underline{k}} . \end{aligned}$$ □

Exercises

Problem 3.0.7. For all $n \in \mathbb{N}$, the ***Bell number*** ϖ_n counts all the partitions of the set $[n]$. Prove each of the following identities via a double-counting argument.

(i) For all $n \in \mathbb{N}$, show that $\varpi_n = \sum_{k\in\mathbb{Z}} \left\{ {n \atop k} \right\}$.
(ii) For all $n \in \mathbb{N}$, show that $\varpi_{n+1} = \sum_{j\in\mathbb{Z}} \binom{n}{j} \varpi_j$.

Problem 3.0.8. For all $n \in \mathbb{N}$, prove the following variants of the Binomial Theorem.

(i) $(x+y)^{\overline{n}} = \sum_{k\in\mathbb{Z}} \binom{n}{k} x^{\overline{k}}\, y^{\overline{n-k}}$,
(ii) $(x+y)^{\underline{n}} = \sum_{k\in\mathbb{Z}} \binom{n}{k} x^{\underline{k}}\, y^{\underline{n-k}}$.

Problem 3.0.9. For all $n \in \mathbb{N}$ and all $k \in \mathbb{Z}$, the ***Lah number*** $\left\lfloor {n \atop k} \right\rfloor$ is the number of partitions of the set $[n]$ into k nonempty lists (the elements in each block are ordered, but the blocks are unordered). Prove each of the following identities via a double-counting argument.

(i) For all positive $n, k \in \mathbb{N}$, verify that $\left\lfloor {n \atop k} \right\rfloor k! = n! \binom{n-1}{k-1}$.
(ii) For all $n, k \in \mathbb{N}$, verify that $\left\lfloor {n+1 \atop k+1} \right\rfloor = \left\lfloor {n \atop k} \right\rfloor + (n+k+1) \left\lfloor {n \atop k+1} \right\rfloor$.
(iii) For all $0 \leqslant n, k \leqslant 5$, compute the matrix whose (n,k)-entry is $\left\lfloor {n \atop k} \right\rfloor$.

$$\begin{array}{ll} (1,2) \sqcup (3) \sqcup (4), & (2,1) \sqcup (3) \sqcup (4), \\ (1,3) \sqcup (2) \sqcup (4), & (3,1) \sqcup (2) \sqcup (4), \\ (1,4) \sqcup (3) \sqcup (4), & (4,1) \sqcup (3) \sqcup (4), \\ (1) \sqcup (2,3) \sqcup (4), & (1) \sqcup (3,2) \sqcup (4), \\ (1) \sqcup (2,4) \sqcup (3), & (1) \sqcup (4,2) \sqcup (3), \\ (1) \sqcup (2) \sqcup (3,4), & (1) \sqcup (2) \sqcup (4,3). \end{array}$$

The set $[4]$ can be partitioned into 3 nonempty lists in $12 = \left\lfloor {4 \atop 3} \right\rfloor$ ways.

3.1 Cycles in Permutations

For our next variant on the binomial coefficients, we scrutinize subdivisions of permutations. To explicate the appropriate notion of a subdivision, we start with technical observation.

Lemma 3.1.1. *For some nonnegative integer n, let σ be a permutation of the set $[n]$. For any $i \in [n]$, there exists $m \in [n]$ such that $\sigma^m(i) = i$.*

Proof. Fix $i \in [n]$ and consider $\sigma(i), \sigma^2(i), \dots, \sigma^n(i)$. If none of these images were equal to i, then the pigeonhole principle [0.0.2] would imply that two of these images are equal: $\sigma^j(i) = \sigma^k(i)$ where $1 \leqslant j < k \leqslant n$. Applying the inverse function σ^{-1} to both sides of this equation j-times, we would obtain $i = \sigma^{k-j}(i)$ which is a contradiction. Thus, there exists $m \in [n]$ such that $\sigma^m(i) = i$. □

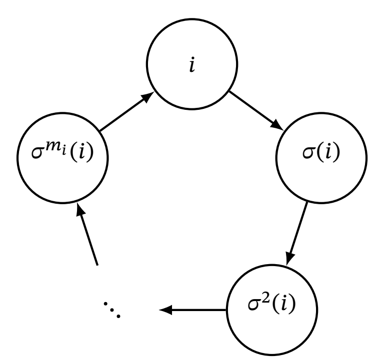

Figure 3.3: Direct graph of a cycle

Given a permutation σ of the set $[n]$ and an element $i \in [n]$, let m_i be the smallest positive integer such that $\sigma^{m_i}(i) = i$. The sequence $\big(i \;\; \sigma(i) \;\; \sigma^2(i) \;\; \sigma^3(i) \;\; \cdots \;\; \sigma^{m_i-1}(i)\big)$ is a ***cycle*** of length m_i. Using the directed graph interpretation for a permutation, a cycle is just a strongly connected component (meaning there is a directed path between any two vertices).

(1)(2)(3)(4)	(2 1)(3)(4)
(2)(3 1)(4)	(2)(3)(4 1)
(1)(3 2)(4)	(1)(3)(4 2)
(1)(2)(4 3)	(2 1)(4 2)
(3 1)(4 2)	(3 2)(4 1)
(1)(4 2 3)	(1)(4 3 2)
(2)(4 1 3)	(2)(4 3 1)
(3)(4 1 2)	(3)(4 2 1)
(3 1 2)(4)	(3 2 1)(4)
(4 1 2 3)	(4 1 3 2)
(4 2 1 3)	(4 2 3 1)
(4 3 1 2)	(4 3 2 1)

Figure 3.4: Cycle decompositions for the 24 permutations of [4]

Theorem 3.1.2. *For any nonnegative integer n, every permutation of the finite set $[n]$ may be expressed as a disjoint union of cycles.*

Proof. Lemma 3.1.1 shows that every element in the set $[n]$ belongs to some cycle. By design, distinct cycles are disjoint. □

To eliminate the ambiguity in a cycle decomposition, we almost always begin a cycle with its largest element and list the cycles in increasing order of their first element.

Definition 3.1.3. For any nonnegative integer n and any integer k, the ***Stirling cycle number*** $\left[{n \atop k}\right]$ is the number of permutations of the set $[n]$ with exactly k cycles. We read $\left[{n \atop k}\right]$ as "n cycle k".

Historically, these are known as the Stirling number of the first kind. From Table 3.4, we see that $\left[{4 \atop 4}\right] = 1$, $\left[{4 \atop 3}\right] = 6$, $\left[{4 \atop 2}\right] = 11$, $\left[{4 \atop 1}\right] = 6$, and $\left[{4 \atop 0}\right] = 0$.

Alternatively, the number $\left[{n \atop k}\right]$ counts the ways for n people to sit around k identical circular tables where there are no empty tables. Some special values are easy to determine.

- For all $k < 0$ and all $n < k$, we have $\left[{n \atop k}\right] = 0$ because number of cycles is nonnegative but cannot exceed the number of elements.
- For all nonnegative integers n, we have $\left[{n \atop n}\right] = 1$ because the unique decomposition with n cycles is $(1)(2)(3)\cdots(n)$. When $n = 0$, the unique permutation of the empty set has no cycles.
- For all positive integers n, we have $\left[{n \atop 1}\right] = (n-1)!$ because any permutation of the set $[n]$ with one cycle has the form $(n \;\; a_1 \;\; a_2 \;\; \cdots \;\; a_{n-1})$ where $a_1 \;\; a_2 \;\; \cdots \;\; a_{n-1}$ is the one-line notation for a permutation of the set $[n-1]$.
- For all positive integers k, we have $\left[{0 \atop k}\right] = 0$ because the unique permutation of the empty set $\varnothing$ has no cycles.
- For all positive integers n, we have $\left[{n \atop 0}\right] = 0$ because any permutation of a nonempty set must have a cycle.
- For all positive integers n, we have $\left[{n \atop n-1}\right] = \binom{n}{2}$ because any permutation of the set $[n]$ with $n-1$ cycles must consists of $n-1$ cycles of length 1 and one cycle of length 2.

n	$\left[{n \atop 0}\right]$	$\left[{n \atop 1}\right]$	$\left[{n \atop 2}\right]$	$\left[{n \atop 3}\right]$	$\left[{n \atop 4}\right]$	$\left[{n \atop 5}\right]$	$\left[{n \atop 6}\right]$
0	1	0	0	0	0	0	0
1	0	1	0	0	0	0	0
2	0	1	1	0	0	0	0
3	0	2	3	1	0	0	0
4	0	6	11	6	1	0	0
5	0	24	50	35	10	1	0
6	0	120	274	225	85	15	1

Figure 3.5: Matrix of Stirling cycle numbers

The Stirling cycle numbers satisfy a two-term recurrence, which we view as a perturbation of the addition formula [2.0.6] for binomial coefficients.

Proposition 3.1.4 (Addition)**.** *For all nonnegative integers n and all integers k, we have*

$$\left[{n+1 \atop k+1}\right] = \left[{n \atop k}\right] + n\left[{n \atop k+1}\right].$$

Double-counting proof. How many of the permutation of the set $[n+1]$ have $k+1$ cycles?

Answer 1: The definition of the Stirling cycle numbers implies that there are $\left[{n+1 \atop k+1}\right]$ permutations of $[n+1]$ with $k+1$ cycles.

Answer 2: Focus on the element $n+1$. When $(n+1)$ appears in the cycle decomposition, the remaining n elements may be placed into k cycles in $\left[{n \atop k}\right]$ ways. When $n+1$ belongs to a cycle of length greater than 1, we first place the elements in the set $[n]$ into $k+1$ cycles and then insert the element $n+1$ to the immediate right of some element. This two-step process can be done in $n\left[{n \atop k+1}\right]$ ways. Therefore, the total number of permutations with k cycles is $\left[{n \atop k}\right] + n\left[{n \atop k+1}\right]$. □

For a fixed numerator, summing of all possible denominators in the Stirling cycle numbers has a clear meaning; compare with Problem 2.0.7.

Problem 3.1.5. For all nonnegative integers n, we have

$$n! = \sum_{k \in \mathbb{Z}} \left[{n \atop k}\right].$$

The sum of the entries in the n-th row of Table 3.5 is $n!$.

Double-counting solution. How many ways can n people be seated around n indistinguishable circular tables?

Answer 1: The first person sits down at a table. As the tables are indistinguishable, this can be done in 1 way. The second person has 2 choices: either sit to the right of the first person (which is equivalent to sitting to the left) or start a new table. Regardless of 2's decision, the third person has three choices: either sit on the immediate the right of the first person, sit on the immediate right of the second person, or start a new table. In general, the k-th person will have k choices: sit to the immediate right of the j-th person for all $1 \leqslant j \leqslant k-1$, or start a new table. We conclude that there are $n!$ possible arrangements.

Answer 2: Focus on the number of nonempty tables. When there are k nonempty tables, the definition of Stirling cycle numbers implies that there are $\left[{n \atop k}\right]$ seatings, so the total number of seatings is $\sum_{k \in \mathbb{Z}} \left[{n \atop k}\right]$. □

Unlike Stirling subset numbers, the version of the binomial theorem [2.1.6] for Stirling cycle numbers uses only polynomials.

Theorem 3.1.6 (Power conversion)**.** *For all nonnegative integers n, we have*

$$x^{\overline{n}} = \sum_{k \in \mathbb{Z}} \left[{n \atop k}\right] x^k .$$

Counting proof. Consider expanding the product

$$x^{\overline{n}} = \underbrace{x(x+1)(x+2)\cdots\big(x+(n-1)\big)}_{n \text{ factors}} .$$

Every monomial in the expansion is the product of n factors where each factor is either x or an element of $\{0, 1, \dots, n-1\}$. How many different ways can one create the monomial x^k? Each such monomial arises from choosing x from k factors and distinct elements from the set $\{0, 1, \dots, n-1\}$ from the complementary $n-k$ factors. Hence, it suffices to prove that

$$\begin{bmatrix} n \\ k \end{bmatrix} = \sum_{0 \leqslant i_1 < i_2 < \cdots < i_{n-k} \leqslant n-1} i_1 \, i_2 \, \cdots \, i_{n-k} \,,$$

using a double-counting argument. How permutations of the set $[n]$ have k cycles?

Answer 1: The definition of the Stirling cycle numbers implies that there are $\left[\begin{smallmatrix} n \\ k \end{smallmatrix}\right]$ permutations of the set $[n]$ with k cycles.

Answer 2: Focus on the smallest element in each cycle. Let $\mathcal{A}$ be the set of smallest elements and consider the complementary set $\{i_1+1, i_2+1, \dots, i_{n-k}+1\} := [n] \setminus \mathcal{A}$ where

$$0 \leqslant i_1 < i_2 < \cdots < i_{n-k} \leqslant n-1 \,.$$

For all $1 \leqslant j \leqslant n-k$, the nonnegative integer i_j equals the sum of $j-1$ and the number of elements in $\mathcal{A}$ than are less that $i_j + 1$. The following algorithm constructs all of the permutations having $\mathcal{A}$ as its set of smallest elements:

place each element of $\mathcal{A}$ in its own cycle of length 1;
for j from 1 to $n-k$
insert $i_j + 1$ to the right of any element
in a cycle starting with number less than $i_j + 1$.

At the j-th step of this loop, there are i_j places to insert $i_j + 1$, so the number of permutations having $\mathcal{A}$ as its set of smallest elements equals the product $i_1 \, i_2 \, \cdots i_{n-k}$. Therefore, the total number of permutations of the set $[n]$ having k cycles is

$$\sum_{0 \leqslant i_1 < i_2 < \cdots < i_{n-k} \leqslant n-1} i_1 \, i_2 \, \cdots \, i_{n-k} \,. \qquad \square$$

Inductive proof. When $n = 0$, we have $x^{\overline{0}} = 1 = \left[\begin{smallmatrix} 0 \\ 0 \end{smallmatrix}\right] x^0$, so the base case holds. For all positive integers n, assume that

$$x^{\overline{n-1}} = \sum_{k \in \mathbb{Z}} \begin{bmatrix} n-1 \\ k \end{bmatrix} x^k \,.$$

This induction hypothesis and the additive identity 3.1.4 give

$$\begin{aligned} x^{\overline{n}} &= (x+n-1)(x^{\overline{n-1}}) = \big(x + (n-1)\big) \left(\sum_{k \in \mathbb{Z}} \begin{bmatrix} n-1 \\ k \end{bmatrix} x^k \right) \\ &= \sum_{k \in \mathbb{Z}} \begin{bmatrix} n-1 \\ k-1 \end{bmatrix} x^k + \sum_{k \in \mathbb{Z}} (n-1) \begin{bmatrix} n-1 \\ k-1 \end{bmatrix} x^k \\ &= \sum_{k \in \mathbb{Z}} \left(\begin{bmatrix} n-1 \\ k-1 \end{bmatrix} + (n-1) \begin{bmatrix} n-1 \\ k-1 \end{bmatrix} \right) x^k = \sum_{k \in \mathbb{Z}} \begin{bmatrix} n \\ k \end{bmatrix} x^k \,. \qquad \square \end{aligned}$$