

5.1 Inclusion–Exclusion Principle

One identity with alternating signs is especially famous.

Theorem 5.1.1 (Inclusion-exclusion)**.** *For any subsets $\mathcal{A}_1, \mathcal{A}_2, \dots, \mathcal{A}_n$ of a fixed superset $\mathcal{X}$, the number of elements in $\mathcal{X}$ that are contained in none of $\mathcal{A}_j$ is*

$$\left| \mathcal{X} \setminus \bigcup_{j=1}^{n} \mathcal{A}_j \right| = \sum_{J \subseteq [n]} (-1)^{|J|} \left| \bigcap_{j \in J} \mathcal{A}_j \right| = \sum_{J \subseteq [n]} (-1)^{|J|} \, |\mathcal{A}_J| \, ,$$

where $\mathcal{A}_J := \bigcap_{j \in J} \mathcal{A}_j$ for all subsets $J \subseteq [n]$.

The empty intersection is the fixed superset: $\mathcal{A}_\varnothing = \mathcal{X}$.

Involutive proof. For any $x \in \mathcal{X}$, set $P_x := \{ j \in [n] \mid x \in \mathcal{A}_j \}$. Consider the set of all pairs (x, J) where $x \in \mathcal{X}$ and $J \subseteq P_x$.

Positive block: All pairs (x, J) such that $|J|$ is even.

Negative block: All pairs (x, J) such that $|J|$ is odd.

Involution: Given a pair (x, J), set $i_x := \max P_x$ and consider the map $(x, J) \mapsto (x, J \ominus \{i_x\})$. Since $(J \ominus \{i_x\}) \ominus \{i_x\} = J$, this operation defines an involution. As $J \ominus \{i_x\}$ has either one more or one less element than J, this involution is sign-reversing. However, this map is undefined when $P_x = \varnothing$. In this case, the set J must empty, so $|J| = 0$ is even. We declare that the pairs (x, J) such that $P_x = \varnothing$ are fixed points.

Since the fixed-point set is $\left\{ x \in \mathcal{X} \;\middle|\; x \notin \bigcup_{j=1}^{n} \mathcal{A}_j \right\}$, we see that

$$\begin{aligned} \sum_{J \subseteq [n]} (-1)^{|J|} \, |\mathcal{A}_J| &= \sum_{|J| \subseteq [n]} \sum_{x \in \mathcal{A}_J} (-1)^{|J|} = \sum_{(x,J) \in \mathcal{X} \times 2^{[n]}} (-1)^{|J|} \\ &= \sum_{(x,\varnothing) \in \mathcal{X} \times 2^{[n]}} (-1)^0 = |\mathcal{X} \setminus (\mathcal{A}_1 \cup \mathcal{A}_2 \cup \cdots \cup \mathcal{A}_n)| \, . \end{aligned}$$ □

When $n = 2$ or $n = 3$, the inclusion-exclusion principle gives

$$|\mathcal{X} \setminus (\mathcal{A}_1 \cup \mathcal{A}_2)| = |\mathcal{X}| - |\mathcal{A}_1| - |\mathcal{A}_2| + |\mathcal{A}_1 \cap \mathcal{A}_2| \, ,$$
$$|\mathcal{X} \setminus (\mathcal{A}_1 \cup \mathcal{A}_2 \cup \mathcal{A}_3)| = |\mathcal{X}| - |\mathcal{A}_1| - |\mathcal{A}_2| - |\mathcal{A}_3| + |\mathcal{A}_1 \cap \mathcal{A}_2| + |\mathcal{A}_1 \cap \mathcal{A}_3| + |\mathcal{A}_2 \cap \mathcal{A}_2| - |\mathcal{A}_1 \cap \mathcal{A}_2 \cap \mathcal{A}_3| \, .$$

Problem 5.1.2. A school has 100 students, 50 students studying French, 40 students studying English, 30 students studying Chinese, 15 students studying any pair of languages, and 5 students studying all three. How many students are not studying any of these languages?

Solution. By the inclusion-exclusion principle, the number of students at the school not studying any of the three languages is $100 - 50 - 40 - 30 + 15 + 15 + 15 - 5 = 20$. □

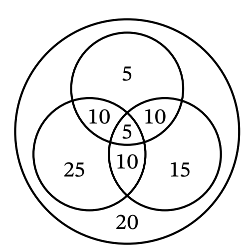

Figure 5.1: Venn diagram of language students

Problem 5.1.3. Count the permutations σ of the set [9] such that $\sigma(1) \geqslant 2$ and $\sigma(9) \leqslant 7$.

Solution. The inclusion-exclusion principle gives

$$\underbrace{9!}_{\text{all permutations}} - \underbrace{(2)(8!)}_{\sigma(1) \in \{1,2\}} - \underbrace{(8!)(2)}_{\sigma(9) \in \{8,9\}} + \underbrace{(2)(7!)(2)}_{\text{both conditions}} = 221760 \, .$$ □

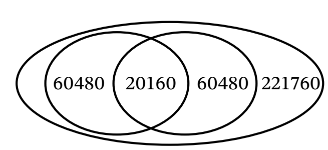

Figure 5.2: Venn diagram of permutations

To complement these illustrations of the inclusion–exclusion principle, we describe two of the more celebrated applications.

Problem 5.1.4 (Derangements). How many permutations σ of the set $[n]$ have no fixed points, that is $\sigma(j) \neq j$ for all $j \in [n]$? A permutation in which no element appears in its canonical position is called a ***derangement***. Thus, this question is equivalent to counting all derangements.

Proof. For any nonnegative integer n, let D_n be the number of derangements of the set $[n]$. Consider the subset $\mathcal{A}_j$ consist of all permutations of the set $[n]$ with $\sigma(j) = j$. For all subsets $J \subseteq [n]$, it follows that the set $\mathcal{A}_J$ consists of permutations that fix the elements in J, so we have $|\mathcal{A}_J| = (n - |J|)!$. The inclusion-exclusion principle and the definition of the binomial coefficient imply that

$$D_n = \sum_{J \subseteq [n]} (-1)^{|J|} \, |\mathcal{A}_J| = \sum_{k=0}^{n} (-1)^k \binom{n}{k} (n-k)! = n! \sum_{k=0}^{n} \frac{(-1)^k}{k!} \,. \; \square$$

The first few numbers D_n are $1, 0, 1, 2, 9, 44, 265, 1854, 14833, \ldots$

Since $e^{-1} = \sum_{k \in \mathbb{N}} (-1)^k / k!$, we see that $n!/e$ is a good approximation to D_n. In fact, one can show that D_n is the nearest integer to $n!/e$.

Problem 5.1.5 (Surjections). Let n and k be nonnegative integers. Count the surjections from the set $[n]$ onto the set $[k]$?

Solution. For all $j \in [k]$, let $\mathcal{A}_j$ consist of maps from the set $[n]$ to the set $[k]$ such that j is not in the image. For all subsets $J \subseteq [k]$, it follows that $\mathcal{A}_J$ consists of the maps from $[n]$ to $[k]$ that miss the elements in J, so we have $|\mathcal{A}_J| = (k - |J|)^n$. The inclusion-exclusion principle implies that the number of surjections is

$$\sum_{J \subseteq [k]} (-1)^{|J|} \, |\mathcal{A}_J| = \sum_{j=0}^{k} (-1)^j \binom{k}{j} (k-j)^n \,. \qquad \square$$

Remark 5.1.6. Although there are no surjections when $n < k$, it is not obvious that the expression $\sum_{j=0}^{k} (-1)^j \binom{k}{j} (k-j)^n$ vanishes under this hypothesis.

Corollary 5.1.7. *For any nonnegative integers n and k, we have*

$$k! \left\{ {n \atop k} \right\} = \sum_{j \in \mathbb{Z}} (-1)^j \binom{k}{j} (k-j)^n \,.$$

Double-counting proof. How many surjective maps from the set $[n]$ to the set $[k]$ are there?

Answer 1: A map $f : [n] \to [k]$ is determine by its k preimages $f^{-1}(i) := \{ j \in [n] \mid f(j) = i \}$ for all $i \in [k]$. The map f is surjective if all k of the preimages are nonempty. The definition of the Stirling subset numbers implies that there are $\left\{ {n \atop k} \right\}$ ways to partition the set $[n]$ into k blocks. There are $k!$ to order these blocks. Thus, the number of surjective maps is $k! \left\{ {n \atop k} \right\}$.

Answer 2: In the previous problem, we establish that there are $\sum_{j=0}^{k} (-1)^j \binom{k}{j} (k-j)^n$ surjective maps. $\square$

5.2 Inverting Infinite Matrices

A large subfamily of combinatorial identies may be recast in terms of inverting infinite matrices.

Proposition 5.2.1. *For all nonnegative integers m and n, we have*

$$\sum_{k \in \mathbb{Z}} (-1)^k \binom{n}{k} \binom{k}{m} = (-1)^n \, \delta_{n,m} \,.$$

The ***Kronecker delta*** function is defined to be

$$\delta_{j,k} := \begin{cases} 1 & \text{if } j = k, \\ 0 & \text{if } j \neq k. \end{cases}$$

In other words, it gives the entries of the identity matrix $I = [\delta_{j,k}]$.

Involutive proof. Consider the set of all nested pairs $(\mathcal{A}, \mathcal{B})$ such that $\mathcal{B} \subseteq \mathcal{A} \subseteq [n]$ and $|\mathcal{B}| = m$.

Positive block: A nested pair $(\mathcal{A}, \mathcal{B})$ is positive when $|\mathcal{A}|$ is even. The definition of the binomial coefficients implies that there are $\sum_{k \in \mathbb{Z}} \binom{n}{2k} \binom{2k}{m}$ positive pairs.

Negative block: A nested pair $(\mathcal{A}, \mathcal{B})$ is negative when $|\mathcal{A}|$ is odd. The definition of the binomial coefficients implies that there are $\sum_{k \in \mathbb{Z}} \binom{n}{2k+1} \binom{2k+1}{m}$ negative pairs.

Involution: To define a sign-reversing involution, we deal with three separate cases.

$n < m$: Under this assumption, the positive and negative blocks are both empty.

$n = m$: When m is even, the positive block contains one pair $([n], [n])$ and the negative block is empty. When m is odd, the positive part is empty and the negative part contains one pair. Either way, the unique sign-reversing involution has a one fixed point whose sign is $(-1)^n$.

$n > m$: For any nested pair $(\mathcal{A}, \mathcal{B})$, let a be the largest element of the set $[n]$ that is not in $\mathcal{B}$; such an element must exist because $n > m$. We map $(\mathcal{A}, \mathcal{B})$ to $(\mathcal{A} \ominus \{a\}, \mathcal{B})$. Since $|\mathcal{A}|$ and $|\mathcal{A} \ominus \{a\}|$ have opposite parity, we have constructed a sign-reversing involution with no fixed points.

Combining the three cases, we see that

$$\begin{aligned} \sum_{k \in \mathbb{Z}} \binom{n}{2k} \binom{2k}{m} - \sum_{k \in \mathbb{Z}} \binom{n}{2k+1} \binom{2k+1}{m} &= \sum_{k \in \mathbb{Z}} (-1)^k \binom{n}{k} \binom{k}{m} \\ &= (-1)^n \, \delta_{n,m} \,. \end{aligned}$$

□

Up to appropriate signs, the matrix of binomial coefficients is its own inverse;

$$I = \begin{bmatrix} \binom{0}{0} & \binom{0}{1} & \cdots & \binom{0}{n} \\ \binom{1}{0} & \binom{1}{1} & \cdots & \binom{2}{n} \\ \vdots & \vdots & \ddots & \vdots \\ \binom{n}{0} & \binom{n}{1} & \cdots & \binom{n}{n} \end{bmatrix} \begin{bmatrix} (-1)^{0+0}\binom{0}{0} & (-1)^{0+1}\binom{0}{1} & \cdots & (-1)^{0+n}\binom{0}{n} \\ (-1)^{1+0}\binom{1}{0} & (-1)^{1+1}\binom{1}{1} & \cdots & (-1)^{1+n}\binom{1}{n} \\ \vdots & \vdots & \ddots & \vdots \\ (-1)^{n+0}\binom{n}{0} & (-1)^{n+1}\binom{n}{1} & \cdots & (-1)^{n+n}\binom{n}{n} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & \cdots \\ 1 & 1 & 0 & 0 & 0 & \cdots \\ 1 & 2 & 1 & 0 & 0 & \cdots \\ 1 & 3 & 3 & 1 & 0 & \cdots \\ 1 & 4 & 6 & 4 & 1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & \cdots \\ -1 & 1 & 0 & 0 & 0 & \cdots \\ 1 & -2 & 1 & 0 & 0 & \cdots \\ -1 & 3 & -3 & 1 & 0 & \cdots \\ 1 & -4 & 6 & -4 & 1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}.$$

Corollary 5.2.2 (Binomial inversion). *For any two sequences (u_n) and (v_n), the following conditions are equivalent.*

a. *For any nonnegative integer n, we have $u_n = \sum_{k \in \mathbb{Z}} \binom{n}{k} v_k$.*

b. *For any nonnegative integer n, we have $v_n = \sum_{k \in \mathbb{Z}} (-1)^{n-k} \binom{n}{k} u_k$.*

Proof. The inverse of the matrix whose (n,k)-entry is $\binom{n}{k}$ is the matrix whose (n,k)-entry is $(-1)^{n-k}\binom{n}{k}$. □

This provides an alternative approach to Corollary 5.1.7.

Problem 5.2.3. For all nonnegative integers m and n, prove that

$$\left\{ {n \atop m} \right\} = \frac{1}{m!} \sum_{k \in \mathbb{Z}} (-1)^{n-k} \binom{m}{k} k^n \,.$$

Solution. The power conversion identity [3.0.6] asserts that

$$m^n = \sum_{k\in\mathbb{Z}} \left\{ {n \atop k} \right\} m^{\underline{k}} = \sum_{k\in\mathbb{Z}} \left\{ {n \atop k} \right\} k! \binom{m}{k} = \sum_{k\in\mathbb{Z}} \binom{m}{k} \left(k! \left\{ {n \atop k} \right\} \right),$$

so binomial inversion gives $m! \left\{ {n \atop m} \right\} = \sum_{k\in\mathbb{Z}} (-1)^{n-k} \binom{m}{j} k^n$. □

As we have come to expect, there is a Stirling analogue of our inverse for the binomial matrix.

Proposition 5.2.4. *For all nonnegative integers m and n, we have*

$$\sum_{k\in\mathbb{Z}} (-1)^k \left[{n \atop k} \right] \left\{ {k \atop m} \right\} = (-1)^n \, \delta_{m,n} \,.$$

Involutive solution. Consider all groupings in which n individuals are seated around k nonempty circular tables that are then placed into m indistinguishable nonempty rooms.

Positive block: All acceptable groupings with an even number of tables. The definitions of the Stirling numbers implies that there are $\sum_{k\in\mathbb{Z}} \left[{n \atop 2k} \right] \left\{ {2k \atop m} \right\}$ such groupings.

Negative block: All acceptable groupings with an odd number of tables. The definitions of the Stirling numbers implies that there are $\sum_{k\in\mathbb{Z}} \left[{n \atop 2k+1} \right] \left\{ {2k+1 \atop m} \right\}$ such groupings.

Involution: To define a sign-reversing involution on the set of all acceptable groupings, we deal with three separate cases.

$m > n$: When there are more rooms than individual, there are no acceptable groupings.

$m = n$: There is only one grouping—each room contains one table and each table has one person seated at it. It follows that the unique sign-reversing involution has one fixed point whose sign equals $(-1)^n$.

$m < n$: Given an grouping, let a be the largest numbered individual who is not in a room by themselves and let b the next largest numbered individual in the same room. As with cycle decompositions, we adopt the convention of tables lists in increasing order by the largest numbered individual seated at a table. If a and b are not at the same table, then we merge these to tables by inserting all the people on the second table (starting with b and proceeding counterclockwise) to the right a on the first table. If a and b are at the same table, then split the table into two: the people starting with a and working counterclockwise up to, but not including, b are on one table and everyone else is on the other (with relative orders are preserved). By construction, this operation is a sign-reversing involution.

The analysis of our sign-reversing involution yields

$$\sum_{k \in \mathbb{Z}} \begin{bmatrix} n \\ 2k \end{bmatrix} \begin{Bmatrix} 2k \\ m \end{Bmatrix} - \sum_{k \in \mathbb{Z}} \begin{bmatrix} n \\ 2k+1 \end{bmatrix} \begin{Bmatrix} 2k+1 \\ m \end{Bmatrix} = \sum_{k \in \mathbb{Z}} (-1)^k \begin{bmatrix} n \\ k \end{bmatrix} \begin{Bmatrix} k \\ m \end{Bmatrix} = (-1)^n \, \delta_{m,n} \, . \qquad \square$$

Up to appropriate signs, the matrix of Stirling subset numbers is the inverse of the matrix of Stirling cycle numbers;

$$I = \begin{bmatrix} \left[\begin{smallmatrix} 0 \\ 0 \end{smallmatrix}\right] & \left[\begin{smallmatrix} 0 \\ 1 \end{smallmatrix}\right] & \cdots & \left[\begin{smallmatrix} 0 \\ n \end{smallmatrix}\right] \\ \left[\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right] & \left[\begin{smallmatrix} 1 \\ 1 \end{smallmatrix}\right] & \cdots & \left[\begin{smallmatrix} 1 \\ n \end{smallmatrix}\right] \\ \vdots & \vdots & \ddots & \vdots \\ \left[\begin{smallmatrix} n \\ 0 \end{smallmatrix}\right] & \left[\begin{smallmatrix} n \\ 1 \end{smallmatrix}\right] & \cdots & \left[\begin{smallmatrix} n \\ n \end{smallmatrix}\right] \end{bmatrix} \begin{bmatrix} (-1)^{0+0} \left\{\begin{smallmatrix} 0 \\ 0 \end{smallmatrix}\right\} & (-1)^{0+1} \left\{\begin{smallmatrix} 0 \\ 1 \end{smallmatrix}\right\} & \cdots & (-1)^{0+n} \left\{\begin{smallmatrix} 0 \\ n \end{smallmatrix}\right\} \\ (-1)^{1+0} \left\{\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right\} & (-1)^{1+1} \left\{\begin{smallmatrix} 1 \\ 1 \end{smallmatrix}\right\} & \cdots & (-1)^{1+n} \left\{\begin{smallmatrix} 1 \\ n \end{smallmatrix}\right\} \\ \vdots & \vdots & \ddots & \vdots \\ (-1)^{n+0} \left\{\begin{smallmatrix} n \\ 0 \end{smallmatrix}\right\} & (-1)^{n+1} \left\{\begin{smallmatrix} n \\ 1 \end{smallmatrix}\right\} & \cdots & (-1)^{n+n} \left\{\begin{smallmatrix} n \\ n \end{smallmatrix}\right\} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & \cdots \\ 0 & 1 & 0 & 0 & 0 & \cdots \\ 0 & 1 & 1 & 0 & 0 & \cdots \\ 0 & 2 & 3 & 1 & 0 & \cdots \\ 0 & 6 & 11 & 6 & 1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & \cdots \\ 0 & 1 & 0 & 0 & 0 & \cdots \\ 0 & -1 & 1 & 0 & 0 & \cdots \\ 0 & 1 & -3 & 1 & 0 & \cdots \\ 0 & -1 & 7 & -6 & 1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix} .$$

Exercises

Problem 5.2.5. For all nonnegative integers m and n, show that

$$\sum_{k \in \mathbb{Z}} (-1)^k \binom{n}{k} \left(\!\!\binom{k}{m}\!\!\right) = (-1)^n \left(\!\!\binom{n}{m-n}\!\!\right).$$

6
Power Series

Notation 6.0.1. Throughout, the ring R denotes a commutative domain of characteristic zero. In particular, the ring $\mathbb{Z}$ is contained in R and the product of two elements in R equals zero if and only if one of the elements is zero. The ring $\mathbb{Z}$ of integers, the field $\mathbb{Q}$ of rational numbers, the field $\mathbb{C}$ of complex numbers, and the field $\mathbb{C}(y)$ of rational functions over $\mathbb{C}$ are examples of such a ring.

6.0 Algebra of Formal Power Series

Loosely speaking, a formal power series is a polynomial that may have infinitely many terms. The collection of all such objects forms a new ring.

Definition 6.0.2. For any commutative domain R, the ***algebra of formal power series*** in the variable x over the ring R is denoted by $R[[x]]$. As a set, $R[[x]]$ consists of all power series $f := \sum_{j \in \mathbb{N}} a_j \, x^j$ where $a_j \in R$ for all j. Each element $f \in R[[x]]$ is identified with an infinite sequence $(a_0, a_1, a_2, \dots)$ of elements in R. For example, we have

$$\sum_{j \in \mathbb{N}} j! \, x^j \in \mathbb{Z}[[x]] \qquad \text{and} \qquad \sum_{j \in \mathbb{N}} 2^{-j} \, x^j \in \mathbb{Q}[[x]] \, .$$

Addition and multiplication of formal power series are defined termwise by

$$\Big(\sum_{j \in \mathbb{N}} a_j \, x^j \Big) + \Big(\sum_{j \in \mathbb{N}} b_j \, x^j \Big) = \sum_{j \in \mathbb{N}} (a_j + b_j) \, x^j \, .$$

$$\Big(\sum_{j \in \mathbb{N}} a_j \, x^j \Big) \Big(\sum_{j \in \mathbb{N}} b_j \, x^j \Big) = \sum_{j \in \mathbb{N}} \left(\sum_{k=0}^{j} a_k b_{j-k} \right) x^j \, .$$

With these operations, the additive identity (or zero element) is $0 := \sum_{j \in \mathbb{N}} 0 \, x^j$ and multiplicative identity is $1 := 1 + \sum_{j > 0} 0 \, x^j$. The ring R embeds into $R[[x]]$ by sending $a \in R$ to $a \, 1 = a + \sum_{j>0} 0 \, x^j$.

Theorem 6.0.3. *For any commutative domain R, the set $R[[x]]$ of formal power series forms a commutative unital associative R-algebra.*

Proof. For addition in $R[[x]]$, commutativity, associativity, the existence of an additive identity, and the existence of an additive inverse are inherited from the corresponding properties in the coefficient ring R. For multiplication in $R[[x]]$, commutativity, associativity, and the existence of a multiplicative inverse depend on distributivity in R as well as the corresponding property in R. For any $f := \sum_{j \in \mathbb{N}} a_j\, x^j$, $g := \sum_{j \in \mathbb{N}} b_j\, x^j$, and $h := \sum_{j \in \mathbb{N}} c_j\, x^j$, we have

$$f g = \sum_{j \in \mathbb{N}} \left(\sum_{k=0}^{j} a_k\, b_{k-j} \right) x^j = \sum_{j \in \mathbb{N}} \left(\sum_{k=0}^{j} b_k\, a_{j-k} \right) x^j = g f$$

$$\begin{aligned}(f g) h &= \left[\sum_{j \in \mathbb{N}} \left(\sum_{\ell=0}^{j} a_\ell\, b_{j-\ell} \right) x^j \right] \left[\sum_{j \in \mathbb{N}} c_j\, x^j \right] = \sum_{j \in \mathbb{N}} \left[\sum_{k=0}^{j} \left(\sum_{\ell=0}^{k} a_\ell\, b_{k-\ell} \right) c_{j-k} \right] x^j \\ &= \sum_{j \in \mathbb{N}} \left[\sum_{\ell=0}^{j} \left(\sum_{k=\ell}^{j} a_\ell\, b_{k-\ell} c_{j-k} \right) \right] x^j = \sum_{j \in \mathbb{N}} \left[\sum_{\ell=0}^{j} \left(\sum_{\ell=0}^{j-\ell} a_\ell\, b_k c_{j-k-\ell} \right) \right] x^j \\ &= \sum_{j \in \mathbb{N}} \left[\sum_{\ell=0}^{j} a_\ell \left(\sum_{\ell=0}^{j-\ell} b_k c_{j-k-\ell} \right) \right] x^j = \left[\sum_{j \in \mathbb{N}} a_j\, x^j \right] \left[\sum_{j \in \mathbb{N}} \left(\sum_{k=0}^{j} b_k\, c_{j-k} \right) x^j \right] = f(gh)\,.\end{aligned}$$

Distributivity in $R[[x]]$ just rests on corresponding property in R. Finally, the algebra $R[[x]]$ is a R-module because the canonical embedding $R \hookrightarrow R[[x]]$ is compatible with both addition and multiplication. □

A formal power series need not have a largest degree term, but the degree of the smallest nonzero term is a useful invariant.

Definition 6.0.4. The ***order*** of a nonzero formal power series $f := \sum_{j \in \mathbb{N}} a_j\, x^j$ is $\operatorname{ord}(f) := \min\{k \in \mathbb{N} \mid a_k \neq 0\}$.

Proposition 6.0.5. *Let $f, g \in R[[x]]$ be two nonzero power series. When $f + g \neq 0$, we have $\operatorname{ord}(f+g) \geqslant \min\big(\operatorname{ord}(f), \operatorname{ord}(g)\big)$. We also have $\operatorname{ord}(fg) = \operatorname{ord}(f) + \operatorname{ord}(g)$, so the algebra $R[[x]]$ is a domain.*

Proof. Consider the two formal power series $f := \sum_{j \in \mathbb{N}} a_j\, x^j$ and $g := \sum_{j \in \mathbb{N}} b_j\, x^j$ such that $m := \operatorname{ord}(f)$ and $n := \operatorname{ord}(g)$. It follows that $a_j = 0 = b_j$ for all $j < \min(m, n)$. Since $a_j + b_j = 0$ for all $j < \min(m, n)$, we see that $\operatorname{ord}(f + g) \geqslant \min(m, n)$. The definition of the product of formal power series implies that the coefficient of x^j in fg equals $\sum_{k=0}^{j} a_k\, b_{j-k}$. It follows that either $k < m$ and $a_k = 0$ or $j - k < n$ and $b_{j-k} = 0$, so the coefficient of x^j in fg vanishes for all $j < m + n$. Furthermore, the coefficient of x^{m+n} is $\sum_{k=0}^{m+n} a_k\, b_{m+n-k} = a_m\, b_n$. Since R is a domain, $a_m \neq 0$, and $b_n \neq 0$, we deduce that $a_m\, b_n \neq 0$ and $\operatorname{ord}(fg) = m + n$. □

Remark 6.0.6. Unlike with polynomials, a formal power series does not determine a function on the coefficient ring R. Except for evaluation at $x = 0$, the map which sends the variable x to an element of the coefficient ring R is typically not a well-defined.

Remark 6.0.7. Although not relevant for our combinatorial applications, there are several equivalent ways to view $R[[x]]$ as a topological algebra.

- We may give $R[[x]] \cong R^{\mathbb{N}}$ the product topology where each copy of R is given the discrete topology.

- We may give $R[[x]]$ the $\langle x \rangle$-adic topology. The ideals $\langle x \rangle^j$, for all $j \in \mathbb{N}$, form the basis of open neighbourhoods of 0.
- We may give $R[[x]]$ the metric topology where the distance between distinct elements $\sum_{j \in \mathbb{N}} a_j \, x^j$ and $\sum_{j \in \mathbb{N}} b_j \, x^j$ is 2^{-k} where $k := \min(k \in \mathbb{N} \mid a_k \neq b_k)$. As a metric space, $R[[x]]$ is complete.

With this topological structure, an infinite sum converges if and only if the sequence of its terms converges to 0.

Exercises

Problem 6.0.8. Let R be a commutative domain of characteristic zero. Consider two formal power series

$$f(x) := \sum_{j \in \mathbb{N}} a_j \, x^j \in R[[x]] \qquad \text{and} \qquad g(x) := \sum_{j \in \mathbb{N}} b_j \, x^j \in R[[x]] \, .$$

If $f(x) \in R[x]$ or $b_0 = 0$, then show that the composition

$$f\big(g(x)\big) := \sum_{j \in \mathbb{N}} a_j \, \big(g(x)\big)^j$$

is a well-defined element of $R[[x]]$.