Problems 4

Due: Friday, 22 October 2021 before 17:00 EDT

Students registered in MATH 402 should submit solutions to 4 of the following problems. Students in MATH 802 should submit solutions to all 5.

1. For any nonnegative integer n, the ***Bell number*** ϖ_n counts all the partitions of the set $[n]$. Prove each of the following identities via a double-counting argument.

(i) For any nonnegative integer n, demonstrate that $\varpi_n = \sum_{k \in \mathbb{Z}} \left\{ {n \atop k} \right\}$.

(ii) For any nonnegative integer n, demonstrate that $\varpi_{n+1} = \sum_{j \in \mathbb{Z}} \binom{n}{j} \varpi_j$.

2. For any nonnegative integer n, prove the following variants of the binomial theorem.

(i) $(x+y)^{\overline{n}} = \sum_{k \in \mathbb{Z}} \binom{n}{k} x^{\overline{k}} \, y^{\overline{n-k}}$

(ii) $(x+y)^{\underline{n}} = \sum_{k \in \mathbb{Z}} \binom{n}{k} x^{\underline{k}} \, y^{\underline{n-k}}$

3. Prove each of the following identities via a double-counting argument.

(i) For all nonnegative integer m and n, establish that $\left\{ {n+1 \atop m+1} \right\} = \sum_{k=0}^{n} \left\{ {k \atop m} \right\} (m+1)^{n-k}$.

(ii) For all nonnegative integer m and n, establish that $\left[{n+1 \atop m+1} \right] = \sum_{k=0}^{n} \left[{k \atop m} \right] n^{\underline{n-k}}$.

4. Prove the following identities via a double-counting argument.

(i) For all nonnegative integer m and n, show that $\left\{ {m+n+1 \atop m} \right\} = \sum_{k=0}^{m} k \left\{ {n+k \atop k} \right\}$.

(ii) For all nonnegative integer m and n, show that $\left[{m+n+1 \atop m} \right] = \sum_{k=0}^{m} (n+k) \left[{n+k \atop k} \right]$.

5. For any nonnegative integer n, a ***Stirling permutation*** is a permutation of the multiset $M_n := \{1^2, 2^2, \dots, n^2\}$ such that, for each element j in the permutation, all elements between the two copies of j are larger than j. The 15 Stirling permutations of M_3 are

1 1 2 2 3 3,	1 1 2 3 3 2,	1 1 3 3 2 2,	1 3 3 1 2 2,	3 3 1 1 2 2,
1 2 2 1 3 3,	1 2 2 3 3 1,	1 2 3 3 2 1,	1 3 3 2 2 1,	3 3 1 2 2 1,
2 2 1 1 3 3,	2 2 1 3 3 1,	2 2 3 3 1 1,	2 3 3 2 1 1,	3 3 2 2 1 1.

For any nonnegative integer n and any integer k, the ***Eulerian number of the second kind***, denoted $\left\langle\!\left\langle {n \atop k} \right\rangle\!\right\rangle$, counts the number of Stirling permutations of the multiset M_n that have k ascents. For instance, we have $\left\langle\!\left\langle {3 \atop 0} \right\rangle\!\right\rangle = 1$, $\left\langle\!\left\langle {3 \atop 1} \right\rangle\!\right\rangle = 8$, and $\left\langle\!\left\langle {3 \atop 2} \right\rangle\!\right\rangle = 6$.

(i) For any nonnegative integer n, provide an inductive proof that the number of Stirling permutations of M_{n+1} is $(2n+1)!!$.

(ii) For any nonnegative integer n and any integer k, prove via double-counting the additive identity for Eulerian number of the second kind:

$$\left\langle\!\!\left\langle {n+1 \atop k+1} \right\rangle\!\!\right\rangle = (2n-k) \left\langle\!\!\left\langle {n \atop k} \right\rangle\!\!\right\rangle + (k+2) \left\langle\!\!\left\langle {n \atop k+1} \right\rangle\!\!\right\rangle .$$

(iii) For all $0 \leqslant n, k \leqslant 7$, compute the matrix whose (n,k)-entry is $\left\langle\!\left\langle {n \atop k} \right\rangle\!\right\rangle$.