Solutions 2

P2.1. Consider the monomial ideals $I := \langle \boldsymbol{x}^{\boldsymbol{u}_1}, \boldsymbol{x}^{\boldsymbol{u}_2}, \dots, \boldsymbol{x}^{\boldsymbol{u}_\ell} \rangle$ and $J := \langle \boldsymbol{x}^{\boldsymbol{v}_1}, \boldsymbol{x}^{\boldsymbol{v}_2}, \dots, \boldsymbol{x}^{\boldsymbol{v}_m} \rangle$ in the polynomial ring $S := \mathbb{K}[x_1, x_2, \dots, x_n]$.

i. For any monomial $\boldsymbol{x}^{\boldsymbol{w}}$ in S, prove that the ideal $(J : \boldsymbol{x}^{\boldsymbol{w}}) := \{ f \in S \mid f\, \boldsymbol{x}^{\boldsymbol{w}} \in J \}$ is generated by the monomials $\boldsymbol{x}^{\boldsymbol{v}_j} / \gcd(\boldsymbol{x}^{\boldsymbol{v}_j}, \boldsymbol{x}^{\boldsymbol{w}})$ for all $1 \leqslant j \leqslant m$.

ii. Prove that intersection $J \cap I$ is generated by monomials $\operatorname{lcm}(\boldsymbol{x}^{\boldsymbol{v}_j}, \boldsymbol{x}^{\boldsymbol{u}_i})$ for all $1 \leqslant j \leqslant m$ and all $1 \leqslant i \leqslant \ell$.

Solution.

i. Since the monomial $\boldsymbol{x}^{\boldsymbol{w}}\, \boldsymbol{x}^{\boldsymbol{v}_j} / \gcd(\boldsymbol{x}^{\boldsymbol{v}_j}, x^{\boldsymbol{w}})$ is clearly divisible by $\boldsymbol{x}^{\boldsymbol{v}_j}$, we have

$$\left\langle \frac{\boldsymbol{x}^{\boldsymbol{v}_j}}{\gcd(\boldsymbol{x}^{\boldsymbol{v}_j}, \boldsymbol{x}^{\boldsymbol{w}})} \;\middle|\; 1 \leqslant j \leqslant m \right\rangle \subseteq (J : \boldsymbol{x}^{\boldsymbol{w}})$$

On the other hand, given $f \in (J : \boldsymbol{x}^{\boldsymbol{w}})$, we have $f\, \boldsymbol{x}^{\boldsymbol{w}} \in J$ and each term in the product $f\, \boldsymbol{x}^{\boldsymbol{w}}$ is a multiply of $\boldsymbol{x}^{\boldsymbol{v}_j}$ for some $1 \leqslant j \leqslant m$. Unique factorization implies that each term in f is a multiply of $\boldsymbol{x}^{\boldsymbol{v}_j} / \gcd(\boldsymbol{x}^{\boldsymbol{v}_j}, \boldsymbol{x}^{\boldsymbol{w}})$ for some $1 \leqslant j \leqslant m$. Thus, we deduce that

$$\left\langle \frac{\boldsymbol{x}^{\boldsymbol{v}_j}}{\gcd(\boldsymbol{x}^{\boldsymbol{v}_j}, \boldsymbol{x}^{\boldsymbol{w}})} \;\middle|\; 1 \leqslant j \leqslant m \right\rangle \supseteq (J : \boldsymbol{x}^{\boldsymbol{w}})\,.$$

ii. Since the monomial $\operatorname{lcm}(\boldsymbol{x}^{\boldsymbol{v}_j}, \boldsymbol{x}^{\boldsymbol{u}_i})$ is divisible by both $\boldsymbol{x}^{\boldsymbol{v}_j}$ and $\boldsymbol{x}^{\boldsymbol{u}_i}$, it lies in $J \cap I$. Conversely, suppose $f \in J \cap I$. Because $f \in J$, each term in f is a multiply of $\boldsymbol{x}^{\boldsymbol{v}_j}$ for some $1 \leqslant j \leqslant m$. Similarly, we have $f \in I$ and each term in f is a multiply of $\boldsymbol{x}^{\boldsymbol{u}_i}$ for some $1 \leqslant i \leqslant \ell$. Hence, the definition of the least common multiple implies that each term in f is a multiply of $\operatorname{lcm}(\boldsymbol{x}^{\boldsymbol{v}_j}, \boldsymbol{x}^{\boldsymbol{u}_i})$ for some $1 \leqslant j \leqslant m$ and some $1 \leqslant i \leqslant \ell$. It follows that $\langle \operatorname{lcm}(\boldsymbol{x}^{\boldsymbol{v}_j}, \boldsymbol{x}^{\boldsymbol{u}_i}) \mid 1 \leqslant j \leqslant m, 1 \leqslant i \leqslant l \rangle = J \cap I$. □

P2.2. Demonstrate that the following properties uniquely determine the monomial orders $>_{\text{lex}}$ and $>_{\text{grevlex}}$ among all monomial orders $>$ on the polynomial ring $S := \mathbb{K}[x_1, x_2, \dots, x_n]$ satisfying $x_1 > x_2 > \cdots > x_n$.

i. For any polynomial f in S such that $\operatorname{LT}_{\text{lex}}(f) \in \mathbb{K}[x_i, x_{i+1}, \dots, x_n]$ for some $1 \leqslant i \leqslant n$, we have $f \in \mathbb{K}[x_i, x_{i+1}, \dots, x_n]$.

ii. The monomial order $>_{\text{grevlex}}$ refines the partial order given by total degree and, for any homogeneous $f \in S$ such that $\operatorname{LT}_{\text{grevlex}}(f) \in \langle x_i, x_{i+1}, \dots, x_n \rangle$ for some $1 \leqslant i \leqslant n$, we have $f \in \langle x_i, x_{i+1}, \dots, x_n \rangle$.

Solution.

i. By definition, we have $\boldsymbol{x}^{\boldsymbol{u}} >_{\text{lex}} \boldsymbol{x}^{\boldsymbol{v}}$ if and only if there is an index $i \in \{1, 2, \dots, n\}$ such that $u_1 = v_1$, $u_2 = v_2$, …, $u_{i-1} = v_{i-1}$, and $u_i > v_i$. Set $\boldsymbol{x}^{\boldsymbol{u}} := \operatorname{LM}_{\text{lex}}(f)$ and let $\boldsymbol{x}^{\boldsymbol{v}}$ be any other monomial appearing in a polynomial f. The relation $\boldsymbol{x}^{\boldsymbol{u}} \in \mathbb{K}[x_i, x_{i+1}, \dots, x_n]$ implies that $u_1 = \cdots = u_{i-1} = 0$. Since $\boldsymbol{x}^{\boldsymbol{u}} >_{\text{lex}} \boldsymbol{x}^{\boldsymbol{v}}$, it follows that $v_1 = \cdots = v_{i-1} = 0$ and $\boldsymbol{x}^{\boldsymbol{v}} \in \mathbb{K}[x_i, x_{i+1}, \dots, x_n]$.

Conversely, suppose that $>$ is a monomial order on S such that the relation $\operatorname{LT}_{>}(f) \in \mathbb{K}[x_i, x_{i+1}, \dots, x_n]$ for some $1 \leqslant i \leqslant n$ implies that $f \in \mathbb{K}[x_i, x_{i+1}, \dots, x_n]$.

CC BY NC ND

Consider monomials $\boldsymbol{x}^{\boldsymbol{u}}$ and $\boldsymbol{x}^{\boldsymbol{v}}$ in S such that $\boldsymbol{x}^{\boldsymbol{u}} > \boldsymbol{x}^{\boldsymbol{v}}$. By setting $\boldsymbol{x}^{\boldsymbol{w}} := \gcd(\boldsymbol{x}^{\boldsymbol{u}}, \boldsymbol{x}^{\boldsymbol{v}})$, we have $\boldsymbol{x}^{\boldsymbol{u}} = \boldsymbol{x}^{\boldsymbol{w}}\, \boldsymbol{x}^{\boldsymbol{u}'}$ and $\boldsymbol{x}^{\boldsymbol{v}} = \boldsymbol{x}^{\boldsymbol{w}}\, \boldsymbol{x}^{\boldsymbol{v}'}$ where $\min(u'_j, v'_j) = 0$ for all $1 \leqslant j \leqslant n$. Since $>$ is a monomial order, it follows that $\boldsymbol{x}^{\boldsymbol{u}'} > \boldsymbol{x}^{\boldsymbol{v}'}$. Let i be the largest integer such that $u'_1 = u'_2 = \cdots = u'_{i-1} = 0$. If $f = \boldsymbol{x}^{\boldsymbol{u}'} - \boldsymbol{x}^{\boldsymbol{v}'}$, then the hypothesis on $>$ implies that $v'_1 = v'_2 = \cdots = v'_{i-1} = 0$. Our choice of the index i and the equation $\min(u'_i, v'_i) = 0$ imply that $u'_i > 0 = v'_i$ whence $\boldsymbol{x}^{\boldsymbol{u}} >_{\text{lex}} \boldsymbol{x}^{\boldsymbol{v}}$. Since $\boldsymbol{x}^{\boldsymbol{u}}$ and $\boldsymbol{x}^{\boldsymbol{v}}$ are arbitrary monomials, we conclude that $>$ equals $>_{\text{lex}}$.

ii. By definition, we have $\boldsymbol{x}^{\boldsymbol{u}} >_{\text{grevlex}} \boldsymbol{x}^{\boldsymbol{v}}$ if and only if either $\deg(\boldsymbol{x}^{\boldsymbol{u}}) > \deg(\boldsymbol{x}^{\boldsymbol{v}})$ or $\deg(\boldsymbol{x}^{\boldsymbol{u}}) = \deg(\boldsymbol{x}^{\boldsymbol{v}})$ and there exists an index $i \in \{1, 2, \dots, n\}$ such that $u_n = v_n$, $u_{n-1} = v_{n-1}$, …, $u_{i+1} = v_{i+1}$, and $u_i < v_i$. Set $\boldsymbol{x}^{\boldsymbol{u}} := \mathrm{LM}_{\text{grevlex}}(f)$ and let $\boldsymbol{x}^{\boldsymbol{v}}$ be any other monomial of the same total degree appearing in a polynomial f. The relation $\boldsymbol{x}^{\boldsymbol{u}} \in \langle x_i, x_{i+1}, \dots, x_n \rangle$ implies that $u_i + u_{i+1} + \cdots + u_n > 0$. Since $\boldsymbol{x}^{\boldsymbol{u}} >_{\text{grevlex}} \boldsymbol{x}^{\boldsymbol{v}}$, we have $v_i + v_{i+1} + \cdots + v_n \geqslant u_i + u_{i+1} + \cdots + u_n > 0$ and $\boldsymbol{x}^{\boldsymbol{v}} \in \langle x_i, x_{i+1}, \dots, x_n \rangle$.

Conversely, suppose that $>$ is a monomial order on S which refines total degree and, for any homogeneous polynomial f in S, the relation $\mathrm{LT}_>(f) \in \langle x_i, x_{i+1}, \dots, x_n \rangle$ implies that $f \in \langle x_i, x_{i+1}, \dots, x_n \rangle$. Consider monomials $\boldsymbol{x}^{\boldsymbol{u}}$ and $\boldsymbol{x}^{\boldsymbol{v}}$ in the ring S such that $\boldsymbol{x}^{\boldsymbol{u}} > \boldsymbol{x}^{\boldsymbol{v}}$ and $\deg(\boldsymbol{x}^{\boldsymbol{u}}) = \deg(\boldsymbol{x}^{\boldsymbol{v}})$. Setting $\boldsymbol{x}^{\boldsymbol{w}} := \gcd(\boldsymbol{x}^{\boldsymbol{u}}, \boldsymbol{x}^{\boldsymbol{v}})$, we have $\boldsymbol{x}^{\boldsymbol{u}} = \boldsymbol{x}^{\boldsymbol{w}}\, \boldsymbol{x}^{\boldsymbol{u}'}$ and $\boldsymbol{x}^{\boldsymbol{v}} = \boldsymbol{x}^{\boldsymbol{w}}\, \boldsymbol{x}^{\boldsymbol{v}'}$ where $\min(u'_j, v'_j) = 0$ for all $1 \leqslant j \leqslant n$. As $>$ is a monomial order, we see that $\boldsymbol{x}^{\boldsymbol{u}'} > \boldsymbol{x}^{\boldsymbol{v}'}$. Let i be the smallest integer such that $u'_n = u'_{n-1} = \cdots = u'_{i+1} = 0$. If $f = \boldsymbol{x}^{\boldsymbol{u}'} - \boldsymbol{x}^{\boldsymbol{v}'}$, then the hypothesis on $>$ implies that $v'_1 + v'_2 + \cdots + v'_i > 0$. Our choice of the index i and the equation $\min(u'_i, v'_i) = 0$ imply that $u'_i > 0 = v'_i$ whence $\boldsymbol{x}^{\boldsymbol{u}} >_{\text{grevlex}} \boldsymbol{x}^{\boldsymbol{v}}$. Since $\boldsymbol{x}^{\boldsymbol{u}}$ and $\boldsymbol{x}^{\boldsymbol{v}}$ are arbitrary monomials, we conclude that $>$ equals $>_{\text{grevlex}}$. □

P2.3. Let $\mathbf{M}$ be an $(m \times n)$-matrix with nonnegative real entries and let $\boldsymbol{r}_1, \boldsymbol{r}_2, \dots, \boldsymbol{r}_m$ denote the rows of $\mathbf{M}$. Assume that $\mathrm{Ker}(\mathbf{M}) \cap \mathbb{Z}^n = \{\mathbf{0}\}$. Define a binary relation $>_{\mathbf{M}}$ on the monomials in the polynomial ring $S := \mathbb{K}[x_1, x_2, \dots, x_n]$ as follows:

$x^u >_{\mathbf{M}} x^v$ if there is an positive integer i (at most m) such that $u \cdot r_i > v \cdot r_i$ and $u \cdot r_j = v \cdot r_j$ for all $1 \leqslant j \leqslant i - 1$.

i. Show that $>_{\mathbf{M}}$ is a monomial order on the polynomial ring S.

ii. When $\mathbf{M} := \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$, show that $>_{\mathbf{M}}$ equals $>_{\text{grevlex}}$ on $\mathbb{K}[x, y, z]$.

iii. For the $(n \times n)$-identity matrix $\mathbf{I}$, show that $>_{\text{lex}}$ equals $>_{\mathbf{I}}$.

Solution.

i. We check the three defining properties of a monomial order.

(total order) Suppose that $\boldsymbol{u}$ and $\boldsymbol{v}$ are distinct vectors in $\mathbb{N}^n$. Since we know that $\mathrm{Ker}(\mathbf{M}) \cap \mathbb{Z}^n = \{\mathbf{0}\}$, there exists a positive integer i such that $(\boldsymbol{u} - \boldsymbol{v}) \cdot \boldsymbol{r}_j = 0$ for all $1 \leqslant j \leqslant i - 1$, and $(\boldsymbol{u} - \boldsymbol{v}) \cdot \boldsymbol{r}_i \neq 0$. When $(\boldsymbol{u} - \boldsymbol{v}) \cdot \boldsymbol{r}_i > 0$, we have $\boldsymbol{x}^{\boldsymbol{u}} >_{\mathbf{M}} \boldsymbol{x}^{\boldsymbol{v}}$ and otherwise $\boldsymbol{x}^{\boldsymbol{v}} >_{\mathbf{M}} \boldsymbol{x}^{\boldsymbol{u}}$. Therefore, the binary relation $>_{\mathbf{M}}$ is a total order on $\mathbb{N}^n$.

CC BY NC ND

(multiplicative) Suppose that $\boldsymbol{x}^{\boldsymbol{u}} >_{\mathbf{M}} \boldsymbol{x}^{\boldsymbol{v}}$. By definition, there exists a positive integer i such that $\boldsymbol{u} \cdot \boldsymbol{r}_i = \boldsymbol{v} \cdot \boldsymbol{r}_j$ for all $1 \leqslant j \leqslant i-1$ and $\boldsymbol{u} \cdot \boldsymbol{r}_i > \boldsymbol{v} \cdot \boldsymbol{r}_i$. Since $\boldsymbol{x}^{\boldsymbol{w}}\, \boldsymbol{x}^{\boldsymbol{u}} = \boldsymbol{x}^{\boldsymbol{w}+\boldsymbol{u}}$ and $\boldsymbol{x}^{\boldsymbol{w}}\, \boldsymbol{x}^{\boldsymbol{v}} = \boldsymbol{x}^{\boldsymbol{w}+\boldsymbol{v}}$, it follows that $(\boldsymbol{w}+\boldsymbol{u}) \cdot \boldsymbol{r}_j = (\boldsymbol{w}+\boldsymbol{v}) \cdot \boldsymbol{r}_j$ for all $1 \leqslant j \leqslant i-1$ and $(\boldsymbol{w}+\boldsymbol{u}) \cdot \boldsymbol{r}_i > (\boldsymbol{w}+\boldsymbol{v}) \cdot \boldsymbol{r}_i$ which implies that $\boldsymbol{x}^{\boldsymbol{w}}\, \boldsymbol{x}^{\boldsymbol{u}} >_{\mathbf{M}} \boldsymbol{x}^{\boldsymbol{w}}\, \boldsymbol{x}^{\boldsymbol{u}}$.

(artinian) Let $\boldsymbol{e}_1, \boldsymbol{e}_2, \dots, \boldsymbol{e}_n$ be the standard basis of $\mathbb{Z}^n$, so $x_j = \boldsymbol{x}^{\boldsymbol{e}_j}$ for all $1 \leqslant j \leqslant n$. Since we have $\operatorname{Ker}(\mathbf{M}) \cap \mathbb{Z}^n = \{\mathbf{0}\}$, there exists a positive integer i (for each $\boldsymbol{e}_k$) such that $\boldsymbol{e}_k \cdot \boldsymbol{r}_j = 0$ for all $1 \leqslant j \leqslant i-1$ and $\boldsymbol{e}_k \cdot \boldsymbol{r}_j \neq 0$. Because $\mathbf{M}$ has nonnegative entries, we have $\boldsymbol{e}_k \cdot \boldsymbol{r}_i > 0$. Therefore, we see that $x_k >_{\mathbf{M}} 1$ for all $1 \leqslant k \leqslant n$.

ii. We have

$$x^{u_1}y^{u_2}z^{u_3} >_{\mathbf{M}} x^{v_1}y^{v_2}z^{v_3} \iff \begin{cases} u_1+u_2+u_3 > v_1+v_2+v_3 \\ \text{or} \begin{cases} u_1+u_2+u_3 &= v_1+v_2+v_3 \\ u_1+u_2 &> v_1+v_2 \end{cases} \\ \text{or} \begin{cases} u_1+u_2+u_3 &= v_1+v_2+v_3 \\ u_1+u_2 &= v_1+v_2 \\ u_1 &> v_1 \end{cases} \end{cases}$$

$$\Leftrightarrow \begin{cases} u_1+u_2+u_3 > v_1+v_2+v_3 \\ \text{or} \begin{cases} u_1+u_2+u_3 &= v_1+v_2+v_3 \\ u_3 &< v_3 \end{cases} \\ \text{or} \begin{cases} u_1+u_2+u_3 &= v_1+v_2+v_3 \\ u_3 &= v_3 \\ u_2 &< v_2 \end{cases} \end{cases}$$

$$\Leftrightarrow x^{u_1}y^{u_2}z^{u_3} >_{\text{grevlex}} x^{v_1}y^{v_2}z^{v_3}\,.$$

iii. We have

$$\begin{aligned} \boldsymbol{x}^{\boldsymbol{u}} >_{\mathbf{I}} \boldsymbol{x}^{\boldsymbol{v}} &\Leftrightarrow \text{there exist } i \text{ such that } u_j = v_j \text{ for all } 1 \leqslant j \leqslant i-1 \text{ and } u_i > v_i \\ &\Leftrightarrow \boldsymbol{x}^{\boldsymbol{u}} >_{\text{lex}} \boldsymbol{x}^{\boldsymbol{v}}\,. \end{aligned}$$

□

P2.4. Let $\mathbb{F}_2$ be a finite field with 2 elements and consider the ideal I in $\mathbb{F}_2[x,y,z]$ consisting of all polynomials that vanish at every point in $\mathbb{A}^3(\mathbb{F}_2)$.

i. Show that $\langle x^2-x, y^2-y, z^2-z \rangle \subseteq I$.

ii. For any coefficients $a_0, a_1, \dots, a_7$ in $\mathbb{F}_2$, show that the polynomial

$$f := a_0\, xyz + a_1\, xy + a_2 x\, z + a_3\, yz + a_4\, x + a_5\, y + a_6\, z + a_7$$

belongs to the ideal I if and only if we have $a_0 = a_1 = \cdots = a_7 = 0$.

iii. Show that $I = \langle x^2-x, y^2-y, z^2-z \rangle$.

Solution.

i. Since the univariate polynomial $t^2 - t = t(t-1)$ has both 0 and 1 as roots for any $t \in \{x,y,z\}$, it follows that $\langle x^2-x, y^2-y, z^2-z \rangle \subseteq I$.

cc BY NC ND

ii. When $a_0 = a_1 = \cdots = a_7 = 0$, the polynomial f is the zero polynomial which vanishes at every point. Now, suppose that f vanishes at every point in $\mathbb{A}^3(\mathbb{F}_2)$. It follows that $f(0,0,0) = a_7 = 0$, $f(1,0,0) = a_4 = 0$, $f(0,1,0) = a_5 = 0$, and $f(0,0,1) = a_6 = 0$. We deduce that $f(1,1,0) = a_1 = 0$, $f(1,0,1) = a_2 = 0$, and $f(0,1,1) = a_3 = 0$. Finally, we have $f(1,1,1) = a_0 = 0$.

iii. Fix a monomial order $>$ on $\mathbb{F}_2[x,y,z]$ and consider a polynomial g in I. The division algorithm implies that there exists polynomials $h_1, h_2, h_3 \in \mathbb{F}_2[x,y,z]$ and scalars $a_0, a_1, \ldots, a_7 \in \mathbb{F}_2$ such that

$$g = h_1(x^2 - x) + h_2(y^2 - y) + h_3(z^2 - z) + a_0xyz + a_1xy + a_2xz + a_3yz + a_4x + a_5y + a_6z + a_7$$

Since part **i** yields $g - h_1(x^2 - x) - h_2(y^2 - y) - h_3(z^2 - z) \in I$, part **ii** establishes that $a_0 = a_1 = \cdots = a_7 = 0$. We conclude that $g \in \langle x^2 - x, y^2 - y, z^2 - z \rangle$ and $I = \langle x^2 - x, y^2 - y, z^2 - z \rangle$. $\square$

P2.5. A ring R satisfies the ***artinian*** if any descending sequence of ideals in R stabilizes. In other words, for any descending sequence $I_0 \supseteq I_1 \supseteq I_2 \supseteq \cdots$ of ideals in R, there exists a nonnegative integer m such that $I_m = I_{m+1} = I_{m+2} = \cdots$.

i. For any positive integer n, show that the quotient rings $\mathbb{Z}/\langle n \rangle$ and $\mathbb{K}[x]/\langle x^n \rangle$ are artinian.

ii. Show that rings $\mathbb{Z}$ and $\mathbb{K}[x]$ are not artinian.

iii. Show that every prime ideal in an artinian ring is maximal.

Solution.

i. Since $\mathbb{Z}/\langle n \rangle$ has only n distinct elements, every descending chain of ideals can have at most n distinct ideals, so must stabilize.

Regarding the quotient $\mathbb{K}[x]/\langle x^n \rangle$ as $\mathbb{K}$-vector space, the monomials $1, x, \ldots, x^{n-1}$ form a basis, so $\dim_{\mathbb{K}} \mathbb{K}[x]/\langle x^n \rangle = n$. Moreover, every ideal in $\mathbb{K}[x]/\langle x^n \rangle$ is also a $\mathbb{K}$-vector subspace. It follows that every descending chain of ideals can have at most $n+1$ distinct ideals.

ii. Since $\langle 2 \rangle \supset \langle 2^2 \rangle \supset \langle 2^3 \rangle \supset \cdots$ and $\langle x \rangle \supset \langle x^2 \rangle \supset \langle x^3 \rangle \supset \cdots$ are infinite descending chains of distinct ideals in $\mathbb{Z}$ and $\mathbb{K}[x]$ respectively, neither ring is artinian.

iii. Let I be a prime ideal in an artinian ring R. Since I is prime, the quotient ring R/I is a domain. A descending chain of ideals in the quotient ring R/I pulls back to a descending chain of ideals in R. Since R is artinian, this the chain in R stabilizes which implies that the chain in R/I also stabilizes. In other words, the quotient ring R/I is also artinian.

Let f be a nonzero element in the quotient ring R/I. Since R/I is artinian, it follows that $\langle f^m \rangle = \langle f^{m+1} \rangle$ for some positive integer m, so $f^m = g\, f^{m+1}$ for some $g \in R/I$. Since R/I is a domain and $f \neq 0$, we may cancel f^m from both sides of this equation to obtain $g\, f = 1_R$. It follows that f is a unit. Therefore, R/I is a field and I is a maximal ideal. $\square$

cc BY NC ND