Solutions 5

P5.1. Let $\mathbb{K}$ be a field.

i. For any univariate polynomial $f := a_m\, x^m + a_{m-1}\, x^{m-1} + \cdots + a_1\, x + a_0$ of degree m in the ring $\mathbb{K}[x]$, define its ***homogenization*** in the ring $\mathbb{K}[x,y]$ to be

$$f^{\mathsf{h}} := a_m\, x^m + a_{m-1}\, x^{m-1}\, y + \cdots + a_1\, x\, y^{m-1} + a_0\, y^m\,.$$

Prove that the polynomial f has a root in the field $\mathbb{K}$ if and only if there exists a point (b,c) in $\mathbb{A}^2(\mathbb{K})$ such that $(b,c) \neq (0,0)$ and $f^{\mathsf{h}}(b,c) = 0$.

ii. Assume that the field $\mathbb{K}$ is not algebraically closed. Exhibit a bivariate polynomial h in the ring $\mathbb{K}[x,y]$ such that the affine subvariety $\mathrm{V}(h)$ in $\mathbb{A}^2(\mathbb{K})$ is just the origin $(0,0)$.

iii. Assume that the field $\mathbb{K}$ is not algebraically closed. For any positive integer n, demonstrate that there exists a polynomial f in the ring $\mathbb{K}[x_1, x_2, \dots, x_n]$ such that the affine subvariety $\mathrm{V}(f)$ in $\mathbb{A}^n(\mathbb{K})$ is the origin $(0,0,\dots,0)$.

iv. Assume that the field $\mathbb{K}$ is not algebraically closed. Prove that any affine subvariety $X = \mathrm{V}(g_1, g_2, \dots, g_r)$ in $\mathbb{A}^n(\mathbb{K})$ can be defined by a single equation.

Solution.

i. Suppose that an element b in the field $\mathbb{K}$ is a root of the polynomial f. It follows that $0 = f(b) = f^{\mathsf{h}}(b,1)$ and $(b,1) \in \mathbb{A}^2(\mathbb{K})$. Conversely, suppose that the point (b,c) in $\mathbb{A}^2(\mathbb{K})$ satisfies $(b,c) \neq (0,0)$ and $f^{\mathsf{h}}(b,c) = 0$. When $c = 0$, we would have $f^{\mathsf{h}}(b,0) = a_m\, b^m = 0$. Since $\deg(f) = m$, it follows that $a_m \neq 0$ and we deduce that $b = 0$. Hence, we must have $c \neq 0$. It follows that

$$\begin{aligned} 0 = f^h(b,c) &= a_m\, b^m + a_{m-1}\, b^{m-1}\, c + \cdots + a_1\, b\, c^{m-1} + a_0\, c^m \\ &= c^m \left(a_m \left(\frac{b}{c}\right)^m + a_{m-1} \left(\frac{b}{c}\right)^{m-1} + \cdots + a_1 \left(\frac{b}{c}\right) + a_0 \right) = c^m\, f\!\left(\frac{b}{c}\right). \end{aligned}$$

Therefore, the element b/c in the field $\mathbb{K}$ is a root of the polynomial f.

ii. As the field $\mathbb{K}$ is not algebraically closed, there exists a polynomial f in $\mathbb{K}[x]$ having positive degree and no root in $\mathbb{K}$. For the homogeneous bivariate polynomial $h := f^{\mathsf{h}}$, part **i** implies that the origin is the only solution of $h = 0$ in $\mathbb{A}^2(\mathbb{K})$.

iii. We proceed by induction on n. When $n = 1$, the hypothesis that the field $\mathbb{K}$ is not algebraically closed establishes the claim. When $n = 2$, the assertion follows from part **ii**. Suppose that the claim holds for some positive integer n: there exists a polynomial g in the ring $\mathbb{K}[x_1, x_2, \dots, x_n]$ such that the only solution of $g = 0$ in $\mathbb{A}^n(\mathbb{K})$ is the origin. By part **ii**, there also exists a polynomial h in the ring $\mathbb{K}[x_{n+1}, y]$ such that the only solution of $h = 0$ is the origin in $\mathbb{A}^2(\mathbb{K})$. Thus, the composite polynomial $f(x_1, x_2, \dots, x_{n+1}) = h\big(x_{n+1}, g(x_1, x_2, \dots, x_n)\big)$ in $\mathbb{K}[x_1, x_2, \dots, x_{n+1}]$ equals zero if and only if $x_{n+1} = 0$ and $g(x_1, x_2 \dots, x_n) = 0$, which is is equivalent to

$$x_1 = x_2 = \cdots = x_n = x_{n+1} = 0$$

completing the induction.

iv. Part **iii** shows that there is a polynomial f in $\mathbb{K}[y_1, y_2, \dots, y_r]$ such that the only solution to $f = 0$ in $\mathbb{A}^r(\mathbb{K})$ is the origin. The composite polynomial $h := f(g_1, g_2, \dots, g_r)$ in

$\mathbb{K}[x_1, x_2, \ldots, x_n]$ vanishes if and only if we have $g_1 = g_2 = \cdots = g_r = 0$. We conclude that $X = \mathrm{V}(g_1, g_2, \ldots, g_r) = \mathrm{V}(h)$. □

P5.2. For any ideal I in the ring $S := \mathbb{K}[x_1, x_2, \ldots, x_n]$ and any polynomial f in S, the *saturation* of I with respect to f is the set

$$(I : f^\infty) := \left\{ g \in S \;\middle|\; \text{there exists a positive integer } m \text{ such that } f^m g \in I \right\}.$$

i. Prove that $(I : f^\infty)$ is an ideal in the ring S.
ii. Prove that there is an ascending chain of ideals $(I : f) \subseteq (I : f^2) \subseteq (I : f^3) \subseteq \cdots$.
iii. For any positive integer ℓ, prove that we have the equality $(I : f^\infty) = (I : f^\ell)$ if and only if we have the equality $(I : f^\ell) = (I : f^{\ell+1})$.

Solution.

i. Suppose that the polynomials g_1 and g_2 belong to $(I : f^\infty)$. By definition, there exists positive integers m_1 and m_2 such that $f^{m_1} g_1 \in I$ and $f^{m_2} g_2 \in I$. Consider polynomials h_1 and h_2 in S. Setting $m := \max(m_1, m_2)$, we have

$$f^m(h_1 g_1 + h_2 g_2) = h_1 f^{m-m_1} (f^{m_1} g_1) + h_2 f^{m-m_2} (f^{m_2} g_2) \in I$$

so $h_1 g_1 + h_2 g_2 \in (I : f^\infty)$. We deduce that $(I : f^\infty)$ is an ideal in S.

ii. Let ℓ be a positive integer and suppose that $g \in (I : f^\ell)$. By definition, we have $f^\ell g \in I$. As I is an ideal, it follows that $f(f^\ell g) = f^{\ell+1} g \in I$. Since $g \in (I : f^{\ell+1})$, we conclude that $(I : f^\ell) \subseteq (I : f^{\ell+1})$.

iii. For any positive integer ℓ, the definiton of saturation and part **ii** establish the inclusions $(I : f^\ell) \subseteq (I : f^\infty)$ and $(I : f^\ell) \subseteq (I : f^{\ell+1})$.

Suppose that $(I : f^\infty) \subseteq (I : f^\ell)$ and consider an element g in S. It follows that the existence a positive integer m such that $f^m g \in I$ implies that $f^\ell g \in I$. Hence, the relation $f^{\ell+1} g \in I$ implies that $f^\ell g \in I$ which demonstrates that $(I : f^{\ell+1}) \subseteq (I : f^\ell)$.

Conversely, suppose that $(I : f^{\ell+1}) \subseteq (I : f^\ell)$ and consider an element g in S. It follows that the relation $f^{\ell+1} g \in I$ implies that $f^\ell g \in I$. Assume that there exists a positive integer m such that $f^m g \in I$. When $m \leqslant \ell$, we have $f^\ell g = f^{\ell-m}(f^m g) \in I$. When $m > \ell$, we have $f^{\ell+1}(f^{m-\ell-1} g) = f^m g \in I$ and the assumption implies that $f^\ell(f^{m-\ell-1} g) = f^{m-1} g \in I$. Repeating this process, we obtain $f^\ell g \in I$. We conclude that $(I : f^\infty) \subseteq (I : f^\ell)$. □

P5.3. The ideals I and J in the ring $S := \mathbb{K}[x_1, x_2, \ldots, x_n]$ are ***comaximal*** if $I + J = S$.

i. Over an algebraically closed field, show that the ideals I and J are comaximal if and only if we have $\mathrm{V}(I) \cap \mathrm{V}(J) = \varnothing$. Without the algebraically closed hypothesis, show that this can be false.
ii. When the ideals I and J are comaximal, show that $I J = I \cap J$.
iii. When the ideals I and J are comaximal, show that, for all positive integers i and j, the ideals I^i and J^j are comaximal.

Solution.

i. Suppose that $I + J = \langle 1 \rangle$. We have $\mathrm{V}(I) \cap \mathrm{V}(J) = \mathrm{V}(I + J) = \mathrm{V}(1) = \varnothing$. For the converse, suppose that $\varnothing = \mathrm{V}(I) \cap \mathrm{V}(J) = \mathrm{V}(I + J)$. When $\mathbb{K}$ is an algebraically closed field, the Weak Nullstellensatz implies that $I + J = \langle 1 \rangle$. When the field $\mathbb{K}$ is not algebraically closed, there exists a polynomial f in the ring $\mathbb{K}[x_1]$ having positive degree and no root in $\mathbb{K}$. We see that $\langle f \rangle + \langle f \rangle = \langle f \rangle \neq \langle 1 \rangle$, but $\mathrm{V}(f) \cap \mathrm{V}(f) = \varnothing$.

ii. We always have $I\,J \subseteq I \cap J$. Suppose that I and J are comaximal. It follows that there exists elements $f \in I$ and $g \in J$ such that $f + g = 1$. For any $h \in I \cap J$, we have $h \in I$ and $h \in J$. It follows that $h = h(f+g) = h\,f + h\,g \in I\,J$ and $I\,J \supseteq I \cap J$. We conclude that $I\,J = I \cap J$ whenever I and J are comaximal.

iii. Suppose that I and J are comaximal. There exists elements $f \in I$ and $g \in J$ such that $f + g = 1$. For any positive integers i and j, the binomial theorem gives

$$1 = (f+g)^{i+j-1} = \sum_{k=0}^{i+j-1} \binom{i+j-1}{k} f^k\, g^{i+j-1-k} .$$

Since the first i summands (those index by $0 \leqslant k < i$) are divisible by $g^j \in J^j$ and the last j summands (those index by $i \leqslant k \leqslant i+j-1$) are divisible by $f^i \in I^i$, it follows that $1 = (f+g)^{i+j-1} \in I^i + J^j$. Therefore, the ideals I^i and J^j are comaximal. □

P5.4. **i.** Consider the affine subvariety $X := \mathrm{V}(xy - yz - y, x^2 - y^2 - z^2)$ in $\mathbb{A}^3$. Show that X is a union of three irreducible components. Describe them and find their prime ideals.

ii. Show that the set of real points on the irreducible complex surface

$$\mathrm{V}(x^2 y - xz^2 + yz^2) \subset \mathbb{A}^3$$

is connected but is not equidimensional; it is the union of a closed curve and a closed surface in the induced Euclidean topology.

Solution.

i. The equation $0 = xy - yz - y = y(x - z - 1)$ implies that $y = 0$ or $x - z = 1$. When $y = 0$, the equation $0 = x^2 - y^2 - z^2$ implies that $0 = x^2 - z^2 = (x+z)(x-z)$ so $x + z = 0$ or $x - z = 0$. When $y \neq 0$, we have $x - z = 1$ and

$$0 = x^2 - y^2 - z^2 = (x-z)(x+z) - y^2 = x + z - y^2 = 2z + 1 - y^2 .$$

It follows that

$$\mathrm{V}(xy - yz - y, x^2 - y^2 - z^2) = \mathrm{V}(x - z, y) \cup \mathrm{V}(x + z, y) \cup \mathrm{V}(x - z - 1, y^2 - 2z - 1) .$$

Since each of these components is clearly rational, we see that they are irreducible. Therefore, the affine subvariety $X = \mathrm{V}(xy - yz - y, x^2 - y^2 - z^2)$ is the union of three irreducible curves: the $x = z$ diagonal line in the xz-plane, the $x = -z$ antidiagonal line in the xz-plane, and a parabola lying in the $x - z = 1$ plane.

ii. We observe that $x^2 y - xz^2 + yz^2 = (x^2 + z^2)y - xz^2$. Over the real numbers, the equation $x^2 + z^2 = 0$ implies that $x = z = 0$. In this situation, any $y \in \mathbb{R}$ satisfies $(x^2 + z^2)y - xz^2$, so the y-axis contained the set of real points of $\mathrm{V}(x^2 y - xz^2 + yz^2)$. When $x^2 + z^2 \neq 0$, the surface has the rational parametrization $\rho\colon \mathbb{A}^2 \setminus \{(0,0)\} \to \mathbb{A}^3$ defined by

$$\rho(s,t) = \left(s, \frac{st^2}{s^2 + t^2}, t \right) .$$

Hence, the real points of the variety $\mathrm{V}(x^2y - xz^2 + yz^2)$ are the union of the y-axis and surface $\rho(\mathbb{A}^2 \setminus \{(0,0)\})$. Since we have

$$\lim_{(s,t)\to(0,0)} \frac{st^2}{s^2+t^2} = 0\,,$$

the origin lies in Zariski closure of the surface. Therefore, the set of real points on affine subvariety $\mathrm{V}(x^2y - xz^2 + yz^2)$ in $\mathbb{A}^2(\mathbb{R})$ is connected, but is the union of two proper closed subsets in the induced Euclidean topology. □

P5.5. **i.** Let I be a monomial ideal in the ring $S := \mathbb{K}[x_1, x_2, \dots, x_n]$. Suppose that the monomial $x^{\boldsymbol{u}}$ is a minimal generator of the ideal I and satisfies $x^{\boldsymbol{u}} = x^{\boldsymbol{v}_1} x^{\boldsymbol{v}_2}$ for some relative prime monomials $x^{\boldsymbol{v}_1}$ and $x^{\boldsymbol{v}_2}$. Show $I = \big(I + \langle x^{\boldsymbol{v}_1}\rangle\big) \cap \big(I + \langle x^{\boldsymbol{v}_2}\rangle\big)$.

ii. Using part **i**, find an irredundant primary decomposition of the monomial ideal

$$\langle x^2y^2, x^2yz, xy^2z, x^2z^2, xyz^2, y^2z^2\rangle\ .$$

Solution.

i. Since I is a monomial ideal, it is enough to show that $\big(I + \langle x^{\boldsymbol{v}_1}\rangle\big) \cap \big(I + \langle x^{\boldsymbol{v}_2}\rangle\big)$ and I contain the same monomials. A monomial $x^{\boldsymbol{w}}$ belongs to $\big(I + \langle x^{\boldsymbol{v}_j}\rangle\big)$ if and only if $x^{\boldsymbol{w}} \in I$ or $x^{\boldsymbol{v}_j}$ dividies $x^{\boldsymbol{w}}$. Because $x^{\boldsymbol{v}_2}$ and $x^{\boldsymbol{v}_2}$ are relatively prime, we have

$$x^{\boldsymbol{w}} \in \big(I + \langle x^{\boldsymbol{v}_1}\rangle\big) \cap \big(I + \langle x^{\boldsymbol{v}_2}\rangle\big) \quad\Leftrightarrow\quad x^{\boldsymbol{w}} \in I \text{ or } x^{\boldsymbol{u}} = x^{\boldsymbol{v}_1+\boldsymbol{v}_2} \text{ divides } x^{\boldsymbol{w}} \quad\Leftrightarrow\quad x^{\boldsymbol{w}} \in I\,.$$

ii. Repeated applications of part **i** give

$$\begin{aligned}
&\langle x^2y^2, x^2yz, xy^2z, x^2z^2, xyz^2, y^2z^2\rangle \\
&= \langle x^2, x^2yz, xy^2z, x^2z^2, xyz^2, y^2z^2\rangle \cap \langle y^2, x^2yz, xy^2z, x^2z^2, xyz^2, y^2z^2\rangle \\
&= \langle x^2, xy^2z, xyz^2, y^2z^2\rangle \cap \langle y^2, x^2yz, x^2z^2, xyz^2\rangle \\
&= \langle x^2, xy^2z, xyz^2, y^2\rangle \cap \langle x^2, xy^2z, xyz^2, z^2\rangle \cap \langle y^2, x^2yz, x^2, xyz^2\rangle \cap \langle y^2, x^2yz, z^2, xyz^2\rangle \\
&= \langle x^2, y^2, xyz^2\rangle \cap \langle x^2, xy^2z, z^2\rangle \cap \langle x^2, y^2, xyz^2\rangle \cap \langle x^2yz, y^2, z^2\rangle \\
&= \langle x^2, y^2, x\rangle \cap \langle x^2, y^2, y\rangle \cap \langle x^2, y^2, z^2\rangle \cap \langle x^2, x, z^2\rangle \cap \langle x^2, y^2, z^2\rangle \cap \langle x^2, z, z^2\rangle \\
&\qquad \cap \langle x^2, y^2, z^2\rangle \cap \langle y, y^2, z^2\rangle \cap \langle z, y^2, z^2\rangle \\
&= \langle x, y^2\rangle \cap \langle x^2, y\rangle \cap \langle x^2, y^2, z^2\rangle \cap \langle x, z^2\rangle \cap \langle x^2, z\rangle \cap \langle y, z^2\rangle \cap \langle y^2, z\rangle
\end{aligned}$$

For any monomial ideal J generated by pure powers of a subset of the variables, every zerodivisor in the quotient ring S/J is nilpotent, so the ideal J is primary. Hence, $\langle x, y^2\rangle$ and $\langle x^2, y\rangle$ are both $\langle x, y\rangle$-primary ideals, $\langle x, z^2\rangle$ and $\langle x^2, z\rangle$ are both $\langle x, z\rangle$-primary ideals, and $\langle y, z^2\rangle$ and $\langle y^2, z\rangle$ are both $\langle y, z\rangle$-primary ideals. Thus, the irredundant irreducible decomposition is

$$\begin{aligned}
\langle x^2y^2, x^2yz, xy^2z, x^2z^2, xyz^2, y^2z^2\rangle &= \langle x^2, xy, y^2\rangle \cap \langle x^2, y^2, z^2\rangle \cap \langle x^2, xz, z^2\rangle \cap \langle y^2, yz, z^2\rangle \\
&= \langle x, y\rangle^2 \cap \langle x^2, y^2, z^2\rangle \cap \langle x, z\rangle^2 \cap \langle y, z\rangle^2\ .
\end{aligned}$$

□