Solutions 05

1. Let p be a prime number. Prove that a group of order $2p$ is either cyclic or dihedral.

Solution. From our classification of groups of small order, we know that a group of order 4 is either the cyclic group $\mathbb{Z}/\langle 4\rangle$ or the dihedral group $\mathbb{Z}/\langle 2\rangle \times \mathbb{Z}/\langle 2\rangle$, so we may assume that p is an odd prime.

Suppose that G is a group of order $2p$. We first show that G is generated by two elements. The number n_p of Sylow p-subgroups satisfies $2 \equiv 0 \bmod n_p$ and $n_p \equiv 1 \bmod p$, so we see that $n_p = 1$. Hence, G has a unique Sylow p-subgroup K and K is normal. Since $|K| = p$ and p is prime, the subgroup K is cyclic. Choose an element f in G such that $K = \langle f\rangle$. Let H be a Sylow 2-subgroup of G. Since $|H| = 2$, we may choose an element g in G such that $H = \langle g\rangle$. The elements in K have order 1 or p and the elements in H have order 1 or 2, so we have $H \cap K = \{e\}$. It follows that every element in the product KH has a unique expression as a product $f^i g^j$ where $0 \leqslant i < p$ and $0 \leqslant j < 2$. Thus, we obtain $G = KH = \langle f, g\rangle$.

We analyse the relations among these generators of the group G. Our choice of f and g yields the relations $f^p = e$ and $g^2 = e$. The normality of K implies that there exists $0 \leqslant r < p$ such that $gfg^{-1} = f^r$. Using these relations, we obtain

$$\begin{aligned} f &= g^2 f g^{-2} = g(gfg^{-1})g = gf^r g^{-1} \\ &= \underbrace{(gfg^{-1})(gfg^{-1})\cdots(gfg^{-1})}_{r\text{-times}} = \underbrace{(f^r)(f^r)\cdots(f^r)}_{r\text{-times}} = f^{r\cdot r} = f^{r^2}. \end{aligned}$$

It follows that $r^2 \equiv 1 \bmod p$ which means r is 1 or $p-1$. We have two cases:

($r = 1$) We see that $gfg^{-1} = f$ and $gf = fg$. Hence, G is an abelian group and $G \cong K \times H \cong \mathbb{Z}/\langle p\rangle \times \mathbb{Z}/\langle 2\rangle$. Since $\gcd(2, p) = 1$, we also have $G = \langle fg\rangle \cong \mathbb{Z}/\langle 2p\rangle$.

($r = p-1$) It follows that $gfg^{-1} = f^{-1}$ and, for all positive integers m, we obtain

$$gf^m g^{-1} = \underbrace{(gfg^{-1})(gfg^{-1})\cdots(gfg^{-1})}_{m\text{-times}} = \underbrace{(f^{-1})(f^{-1})\cdots(f^{-1})}_{m\text{-times}} = f^{-m}.$$

In particular, by choosing $0 < m < p$ such that $3m \equiv 1 \bmod p$, we have the relation $gf^m g^{-1} = f^{-m} = f^{2m} = (f^m)^2$. Let $h = f^m$. Since p is a prime number, we have $K = \langle h\rangle$ and

$$G = \{g^i h^j \mid 0 \leqslant i < 2, 0 \leqslant j < p, g^2 = e, h^p = e, hg = h^2 g\} = D_p.$$

Therefore, G isomorphic to the cyclic group $\mathbb{Z}/\langle 2p\rangle$ or the dihedral group D_p. □

2. Prove that there are no simple groups of order 80, 96, or 1000.

Solution. Suppose that G is a simple group of order $80 = 2^4 \cdot 5$. The number n_5 of Sylow 5-subgroups satisfies both $16 \equiv 0 \bmod n_5$ and $n_5 \equiv 1 \bmod 5$. Because G does not have a normal subgroup, we must have $n_5 \neq 1$ which means that $n_5 = 16$. Hence, the number of elements of order 5 is $(16)(4) = 64$. Similarly, the number n_2 of Sylow 2-subgroups also satisfies $5 \equiv 0 \bmod n_2$ and $n_2 \equiv 1 \bmod 2$. Since $n_2 \neq 1$, we have $n_2 = 5$. The number of elements of order 2^i with $i > 1$ is $(5)(15) = 75$, but $75 + 64 > 80$ is a contradiction. Therefore, there is no simple group of order 80.

Suppose that G is a simple group of order $96 = 2^4 \cdot 3$. Let P denote a Sylow 2-subgroup, so $[G : P] = 3$. Left multiplication of G on coset space G/P gives a group homomorphism $\varphi \colon G \to \mathfrak{S}_{G/P} \cong \mathfrak{S}_3$ and the kernel $\operatorname{Ker}(\varphi)$ is a subgroup of P. Since G is simple, we must have $\operatorname{Ker}(\varphi) = \{e\}$, so the map φ is injective. Hence, the First Isomorphism Theorem establishes that $\varphi(G)$ is a subgroup of $\mathfrak{S}_3$. However, the inequality $|G| = 96 > 6 = |\mathfrak{S}_3|$ provides a contradiction. Thus, we conclude that there is no simple group of order 96.

Let G be a group of order $1000 = 2^3 \cdot 5^3$. The number n_5 of Sylow 5-subgroups satisfies $8 \equiv 0 \bmod n_5$ and $n_5 \equiv 1 \bmod 5$. It follows that $n_5 = 1$ and the unique Sylow 5-subgroup is normal. Therefore, there is no simple group of order 1000. □

3. Let $\widehat{\mathbb{C}} := \mathbb{C} \sqcup \{\infty\}$ be the extended complex plane. Consider the functions $f : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ and $g : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ defined by $f(z) := z + 2$ and $g(z) := z/(2z+1)$ respectively.

i. Prove that the functions f and g are bijections and, thereby, elements of the symmetric group on the set $\widehat{\mathbb{C}}$.

ii. Show that any nonzero power of f maps the interior of the unit circle $|z| = 1$ to the exterior. Similarly, show that any nonzero power of g maps the exterior of the unit circle to the punctured interior (a point is removed from the interior).

iii. Demonstrate that the subgroup of the symmetric group on $\widehat{\mathbb{C}}$ generated by functions f and g is free.

Solution.

i. Since $f(z) - 2 = z = f(z+2)$ and

$$\frac{g(z)}{1 - 2g(z)} = \frac{\frac{z}{2z+1}}{1 - \frac{2z}{2z+1}} = z = \frac{\frac{z}{1-2z}}{\frac{2z}{1-2z} + 1} = g\Big(\frac{z}{1-2z}\Big),$$

we see that $f^{-1}(z) = z - 2$ and $g^{-1}(z) = z/(1-2z)$. Hence, the functions f and g are bijections and, thereby, elements of the symmetric group on $\widehat{\mathbb{C}}$.

ii. Since $f^n(z) = z + 2n$ for any integer n, the inequality $|z| < 1$ implies that, for any nonzero integer n, we have

$$|f^n(z)| = |z + 2n| = |2n - (-z)| \geqslant 2\,|n| - |z| \geqslant 2\,|n| - 1 \geqslant 1\,.$$

Hence, any nonzero power of f maps the interior of the unit circle $|z| = 1$ to the exterior. Observe that the function f fixes the point ∞.

For any integer n, induction shows that $g^n(z) = z/(2nz+1)$. Moreover, observe that $g^n(-1/2n) = \infty$ and $g^n(\infty) = 1/2n$. For any nonzero integer n, the inequality $|z| > 1$ yields $1/\,|z| < 1$ and

$$|g^n(z)| = \frac{|z|}{|2nz + 1|} \leqslant \frac{1}{\left|\frac{1}{|z|} - 2\,|n|\right|} < 1\,,$$

so any nonzero power of g maps the exterior of the unit circle to the punctured interior.

iii. Let $G := \langle f, g \rangle$ denote the subgroup of the symmetric group on the set $\widehat{\mathbb{C}}$ generated by the functions f and g and let F be the free group generated by two elements. The universal mapping property for free groups gives a surjective group homomorphism $\varphi\colon F \to G$. The kernel of φ contains all reduced words in $\{f, g\}$ which equal the identity map $\mathrm{id}_{\widehat{\mathbb{C}}}$. However, part *ii* implies that no non-trivial reduced word in $\{f, g\}$ can equal the identity map. Therefore, the map φ is injective and $F \cong G$. □