Solutions 11

1. Let K be a field and let U, V, and W be K-vector spaces. Consider K-linear maps $\varphi\colon U \to V$ and $\psi\colon V \to W$. The map φ has ***finite index*** if both of the K-modules $\operatorname{Ker}(\varphi)$ and $\operatorname{Coker}(\varphi)$ are finite-dimensional. The ***index*** of φ is

$$\operatorname{ind}(\varphi) := \dim \operatorname{Ker}(\varphi) - \dim \operatorname{Coker}(\varphi)\,.$$

i. Prove that U decomposes into a direct sum of $\operatorname{Ker}(\varphi)$ and two K-modules U' and U'' such that $\operatorname{Ker}(\psi \circ \varphi) = \operatorname{Ker}(\varphi) \oplus U'$ and $\operatorname{Im}(\psi \circ \varphi) = \psi\big(\varphi(U'')\big)$.

ii. Prove that if two of the three K-linear maps φ, ψ, and $\psi \circ \varphi$ are of finite index, then so is the third and $\operatorname{ind}(\psi \circ \varphi) = \operatorname{ind}(\varphi) + \operatorname{ind}(\psi)$.

Solution.

i. Let $\{u_j\}_{j \in J}$ be a basis for the K-module $\operatorname{Ker}(\varphi)$. Since $\operatorname{Ker}(\varphi) \subseteq \operatorname{Ker}(\psi \circ \varphi)$, the linearly independent family $\{u_j\}_{j\in J}$ extends to a basis $\{u_j\}_{j \in J \cup J'}$ of $\operatorname{Ker}(\psi \circ \varphi)$. Let U' denote the submodule of U with basis $\{u_j\}_{j \in J'}$. By construction, we have $\operatorname{Ker}(\psi \circ \varphi) = \operatorname{Ker}(\varphi) \oplus U'$. We can also extend the linearly independent family $\{u_j\}_{j \in J \cup J'}$ to a basis $\{u_j\}_{j \in J \cup J' \cup J''}$ of U. Let U'' be the submodule of U with basis $\{u_j\}_{j \in J''}$. Again by construction, U'' is a complementary submodule of $\operatorname{Ker}(\psi \circ \varphi)$ in U, $\operatorname{Im}(\psi \circ \varphi) = \psi\big(\varphi(U'')\big)$, and $U = \operatorname{Ker}(\varphi) \oplus U' \oplus U''$.

ii. By part *i*, there is a basis $\{u_j\}_{j \in J_0 \cup J_1 \cup J_2}$ of U such that the family $\{u_j\}_{j \in J_0}$ is basis for $\operatorname{Ker}(\varphi)$, the family $\{u_j\}_{j \in J_0 \cup J_1}$ is a basis for $\operatorname{Ker}(\psi \circ \varphi)$, the family $\{\varphi(u_j)\}_{j \in J_1}$ is a basis for the submodule $\operatorname{Im}(\varphi) \cap \operatorname{Ker}(\psi)$ of V, the family $\{\varphi(u_j)\}_{j \in J_2}$ is a basis for a complementary submodule of $\operatorname{Im}(\varphi) \cap \operatorname{Ker}(\psi)$ in $\operatorname{Im}(\varphi) \subseteq V$, and the family $\big((\psi \circ \varphi)(u_j)\big)_{j \in J_2}$ is a basis for the submodule $\operatorname{Im}(\psi \circ \varphi)$ of W. There is a basis $\{v_j\}_{j \in J_1 \cup J_3}$ for $\operatorname{Ker}(\psi)$ such that $v_j = \varphi(u_j)$ for all $j \in J_1$ and there is a basis $\{v_j\}_{j \in J_2 \cup J_4}$ for a complementary submodule of $\operatorname{Ker}(\psi)$ in V such that $v_j = \varphi(u_j)$ for all $j \in J_2$. Thus, the family $\{v_j\}_{j \in J_1 \cup J_2 \cup J_3 \cup J_4}$ is a basis for V. Similarly, there is a basis $\{w_j\}_{j \in J_2 \cup J_4 \cup J_5}$ for W such that $w_j = (\psi \circ \varphi)(u_j) = \psi(v_j)$ for all $j \in J_2$ and $w_j = \psi(v_j)$ for all $j \in J_4$. Hence, we obtain

$$\begin{aligned}
&\dim \operatorname{Ker}(\varphi) + \dim \operatorname{Ker}(\psi) + \dim \operatorname{Coker}(\psi \circ \varphi) \\
&= |J_0| + \big(|J_1| + |J_3|\big) + \big(|J_4| + |J_5|\big) \\
&= \big(|J_0| + |J_1|\big) + \big(|J_3| + |J_4|\big) + |J_5| \\
&= \dim \operatorname{Ker}(\psi \circ \varphi) + \dim \operatorname{Coker}(\varphi) + \dim \operatorname{Coker}(\psi)\,.
\end{aligned}$$

Therefore, if two of the three linear maps φ, ψ, and $\psi \circ \varphi$ are of finite index, then so is the third and $\operatorname{ind}(\psi \circ \varphi) = \operatorname{ind}(\varphi) + \operatorname{ind}(\psi)$. □

Remark. If U, V and W are finite-dimensional, then we have

$$\begin{aligned}
\operatorname{ind}(\varphi) &= \dim \operatorname{Ker}(\varphi) - \dim \operatorname{Coker}(\varphi) = \dim U - \dim V \\
\operatorname{ind}(\psi) &= \dim \operatorname{Ker}(\psi) - \dim \operatorname{Coker}(\psi) = \dim V - \dim W \\
-\operatorname{ind}(\psi \circ \varphi) &= -\dim \operatorname{Ker}(\psi \circ \varphi) + \dim \operatorname{Coker}(\psi \circ \varphi) = -\dim U + \dim W\,.
\end{aligned}$$

Adding these three equations establishes that $\operatorname{ind}(\psi \circ \varphi) = \operatorname{ind}(\varphi) + \operatorname{ind}(\psi)$.

2. Let $\mathbb{F}_q$ be a finite field with q elements.

i. For any nonnegative integer n, calculate the number of elements in the $\mathbb{F}_q$-vector space $\mathbb{F}_q^n$.

ii. Let $\mathrm{GL}(n, \mathbb{F}_q)$ denote the group of all invertible $(n \times n)$-matrices over the field $\mathbb{F}_q$. Determine the order of the group $\mathrm{GL}(n, \mathbb{F}_q)$.

iii. Let $\mathrm{SL}(n, \mathbb{F}_q)$ be the subgroup of $\mathrm{GL}(n, \mathbb{F}_q)$ consisting of matrices having determinant 1. Find the order of the group $\mathrm{SL}(n, \mathbb{F}_q)$.

Solution.

i. Let $e_1, e_2, \dots, e_n$ be a basis of the vector space $\mathbb{F}_q^n$. Every element of $\mathbb{F}_q^n$ can be expressed uniquely as $a_1\, e_1 + a_2\, e_2 + \cdots + a_n\, e_n$ where a_j in $\mathbb{F}_q$ for all $1 \leqslant j \leqslant n$. Since finite field $\mathbb{F}_q$ has q elements, it follows that the vector space $\mathbb{F}_q^n$ has q^n elements.

ii. An $(n \times n)$-matrix $\mathbf{A}$ over $\mathbb{F}_q$ is invertible if and only if its columns are linearly independent vectors in $\mathbb{F}_q^n$. The first column $\mathbf{a}_1$ of $\mathbf{A}$ can be any nonzero vector in $\mathbb{F}_q^n$, so there are $q^n - 1$ possibilities. Once the first column is chosen, the second column $\mathbf{a}_2$ of $\mathbf{A}$ can be any vector which is not a multiple of the first. Hence, $\mathbf{a}_1 \neq c\, \mathbf{a}_2$ where c in $\mathbb{F}_q$, so there are $q^n - q$ choices for $\mathbf{a}_2$. In general, the ith column $\mathbf{a}_i$ of $\mathbf{A}$ can be any vector which cannot be written in the form $c_1\, \mathbf{a}_1 + c_2\, \mathbf{a}_2 + \cdots + c_{i-1}\, \mathbf{a}_{i-1}$ where c_i in $\mathbb{F}_q$. Thus, there are $q^n - q^{i-1}$ possibilities for $\mathbf{a}_i$. By multiplying these together, we see that the order of $\mathrm{GL}(n, \mathbb{F}_q)$ is

$$(q^n - 1)(q^n - q) \cdots (q^n - q^{n-1}) = q^{\binom{n}{2}} \prod_{j=1}^{n} (q^j - 1)\,.$$

iii. The determinant function defines a group homomorphism from the general linear group $\mathrm{GL}(n, \mathbb{F}_q)$ onto the multiplicative group $\mathbb{F}_q^\times$ which as $q-1$ elements. Since $\mathrm{SL}(n, \mathbb{F}_q)$ is the kernel of this group homomorphism, it follows that

$$|\mathbb{F}_q^\times| = \frac{|\mathrm{GL}_n(\mathbb{F}_q)|}{|\mathrm{SL}_n(\mathbb{F}_q)|}$$

so we obtain

$$|\mathrm{SL}_n(\mathbb{F}_q)| = \frac{(q^n - 1)(q^n - q) \cdots (q^n - q^{n-1})}{q - 1} = q^{\binom{n}{2}} \prod_{j=2}^{n} (q^j - 1)\,. \qquad \square$$

3. Consider the ring $\mathbb{Q}[x]$. Find a basis for the submodule of $\mathbb{Q}[x]^3$ generated by

$$f_1 := \begin{bmatrix} 2x - 1 \\ x \\ x^2 + 3 \end{bmatrix}, \qquad f_2 := \begin{bmatrix} x \\ x \\ x^2 \end{bmatrix}, \qquad \text{and} \qquad f_3 := \begin{bmatrix} x + 1 \\ 2x \\ 2x^2 - 3 \end{bmatrix}.$$

Solution. Since

$$f_1 - 3f_2 + f_3 = \begin{bmatrix} 2x - 1 \\ x \\ x^2 + 3 \end{bmatrix} - 3 \begin{bmatrix} x \\ x \\ x^2 \end{bmatrix} + \begin{bmatrix} x + 1 \\ 2x \\ 2x^2 - 3 \end{bmatrix} = 0\,,$$

the set $\{f_1, f_2, f_3\}$ is not $\mathbb{Q}$-linear independent. Setting $g_1 := f_1 - f_2$, $g_2 := f_3 - f_2$, we obtain

$$g_1 + g_2 = f_1 - 2\,f_2 + f_3 = f_2\,, \qquad 2\,g_1 + g_2 = f_1\,, \qquad \text{and} \qquad g_1 + 2\,g_2 = f_3\,,$$

so $\langle f_1, f_2, f_3\rangle = \langle g_1, g_2\rangle$. If $p\,g_1 + q\,g_2 = 0$ for some p and q in $\mathbb{Q}[x]$, then each coordinate in $p\,g_1 + q\,g_2$ is zero:

$$(x-1)\,p + q = 0, \qquad x\,q = 0, \qquad \text{and} \qquad 3\,p + (x^2 - 3)\,q = 0$$

which implies that $p = q = 0$. Therefore, we see that

$$g_1 = \begin{bmatrix} x-1 \\ 0 \\ 3 \end{bmatrix}, \qquad \text{and} \qquad g_2 = \begin{bmatrix} 1 \\ x \\ x^2 - 3 \end{bmatrix}$$

form a basis for the submodule $\langle f_1, f_2, f_3\rangle$. □