Problems 5

Due: Monday, 21 November 2022 before 17:00 EST

P5.1 For any R-complex C, prove that the following are equivalent.

(a) There exists a homotopy equivalence $\varphi\colon C \to \mathrm{H}(C)$.

(b) There exists a homomorphism $\sigma\colon C \to C$ having degree 1 such that $\partial^C = \partial^C \sigma \partial^C$.

(c) The cycle complex $\mathrm{Z}(C)$ and the boundary complex $\mathrm{B}(C)$ are both direct summands of C.

(d) The R-complex C is a direct sum of subcomplexes concentrated in one homological degree and subcomplexes concentrated in two homological degrees having zero homology.

P5.2 Let P be a semi-projective R-complex and consider parallel morphisms $\psi\colon P \to B$ and $\psi'\colon P \to B$ of R-complexes. Whenever there is a quasi-isomorphism $\varphi\colon B \to C$ such that $\varphi\,\psi \sim \varphi\,\psi'$, demonstrate that $\psi \sim \psi'$.

P5.3 For any morphism $\varphi\colon B \to C$ of R-complexes and any quasi-isomorphism $\gamma\colon C' \to C$, prove that there exists a morphism $\varphi'\colon B' \to C'$ of R-complexes and a quasi-isomorphism $\beta\colon B' \to B$ such that $\gamma\varphi' \sim \varphi\,\beta$. In other words, the diagram

$$\begin{array}{ccc} C' & \xleftarrow{\;\varphi'\;} & B' \\ {\scriptstyle\gamma}\downarrow{\scriptstyle\simeq} & \searrow\!\!\searrow & {\scriptstyle\simeq}\downarrow{\scriptstyle\beta} \\ C & \xleftarrow{\;\varphi\;} & B \end{array}$$

of R-complexes commutes up to homotopy.

P5.4 Let R be a commutative ring and let I and J be two R-ideals. For any positive integer i, verify that $\mathrm{Tor}_{i+1}(R/I, R/J) \cong \mathrm{Tor}_i(R/I, J)$ and $\mathrm{Tor}_1(R/I, R/J) \cong (I \cap J)/IJ$.