SPECIAL MANIFOLDS, ARITHMETIC AND HYPERBOLIC ASPECTS: A SHORT SURVEY

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ABSTRACT. We give a brief survey of the main results of [C04] and [C11], devoted to the bimeromorphic structure of compact Kähler manifolds X. Such manifolds are decomposed[1] by means of iterated fibrations into elementary components, which are *orbifold pairs* with a canonical bundle either positive, negative, or torsion. Towers of 'torsion and negative' components build however the new (unconditional) class of 'special manifolds', which are the ones which are in a precise sense 'opposite' to manifolds of general type. A single funtorial (unconditional) fibration (the 'core map') splits any X into its two components of 'opposite' geometry: 'special' (its fibres), and general type (its orbifold base). This geometric splitting is conjectured to split X at hyperbolic and hyperbolic levels as well, leading to natural generalisations (to arbitrary smooth orbifolds (X, D)) of Lang's conjectures, permitting to qualitatively describe in algebro-geometric terms the distribution of rational curves, rational points and entire curves on them.

1. INTRODUCTION

In the sequel, X (resp. Y) will denote a connected compact complex Kähler manifold[1] of complex dimension n (resp. p). We denote by K_X, Ω_X^p the usual sheaves of holomorphic differentials. A *fibration* $f : X \to Y$ will always denote a surjective meromorphic map with connected fibres (on some/any resolution of f).

We introduce in §2 the class of 'special' varieties by means of Bogomolov sheaves. This definition is short, but which does not allow a geometric insight, only obtained by orbifold base considerations, given in §3. Examples and conjectures are formulated. In §4 the conjecture $C_{n,m}^{orb}$, an orbifold variant of Iitaka's $C_{n,m}$, is formulated. This is proved when the base orbifold is of general type, following Viehweg's method. This is the technical core of the text. This result permits in §5 to construct the 'core map' c, which is our basic result. In order to decompose further any given X, we need to go further in the orbifold geometry in §6, and to define for them canonical reduction maps J (the Iitaka fibration) and r (a weak analog of the 'rational quotient'), which is conditional in $C_{n,m}^{orb}$. The decomposition $c = (Jr)^n$ of the core

[1]Or, more generally, in the class $\mathcal{C}$ of compact complex analytic spaces bimeromorphic to some compact Kähler manifold.

map as the canonical iteration of such fibrations is then obtained. The abundance conjecture optimally describes the (orbifold) fibres of the fibrations J and r. The LMMP appears here as aiming the construction of the elementary steps of our decomosition. In §7 we state the conjectures suggested by these decompositions. We refer to [C04] and [C11] for details not given here. The only new result not contained there is theorem 6.20.

2. Special Manifolds: Bogomolov sheaves

2.1. **Castelnuovo-de Franchis and Bogomolov theorems.**

Theorem 2.1. *([Bog]) Let $\mathcal{L} \subset \Omega^p_X$ be a rank-one coherent subsheaf of Ω^p_X. Then $\kappa(X, \mathcal{L}) \leq p^2$. Moreover, if equality holds, there exists a fibration[3] $f : X \to Y$ such that $\mathcal{L} = f^*(K_Y)$ over the generic point of Y (ie: $\mathcal{L}$ and $f^*(K_Y)$ have the same saturation in Ω^p_X).*

Remark 2.2.

1. A more precise description is given by Castelnuovo-De Franchis theorem when $p = 1$: if Ω^1_X has two linearly independent sections which wedge to zero, they are lifted from two sections of $\Omega^1_Y = K_Y$ for some curve Y of genus $g > 1$.

2. Bogomolov theorem (and its proof) extends to the case of sheaves of logarithmic differentials with poles on a normal crossing divisor, by Deligne theorem of closedness of such differentials.

3. We shall characterise geometrically below the situations in which $\kappa(X, \mathcal{L}) = p$: the condition $\kappa(Y) = p$ is sufficient, but not necesary by far, as shown by the following example.

4. Let E be an elliptic curve and C be a hyperelliptic curve with involution i such that $C' := C/ < i > \cong \mathbb{P}^1$. Let τ be a translation of order 2 on E, and let $f : X := (C \times E/ < (i, \tau >) \to C'$ be the Moishezon-Iitaka fibration of X. It is easy to see that the saturation $\mathcal{L}$ of $f^*(K_{C'})$ in Ω^1_X has Kodaira dimension 1. As we shall see below, this is due to the fact that f has sufficiently many multiple fibres (exactly $2(g(C) + 1)$ double fibres, actually).

2.2. **Bogomolov sheaves, Special Manifolds.**

Definition 2.3. *Let $\mathcal{L} \subset \Omega^p_X$ be saturated, coherent and of rank one. We say that it is a 'Bogomolov sheaf' of X if $\kappa(X, \mathcal{L}) = p > 0$.*

We say that X is 'special' if it has no Bogomolov sheaf. A compact complex analytic space is said to be 'special' if some (or any) of its resolutions is 'special'.

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[2] $\kappa(X, \mathcal{L})$ is defined in the usual way by considering, for $m > 0$, the meromorphic map defined by the space of sections of $Sym^m(\Omega^p_X)$ which take values in $\mathcal{L}^{\otimes m}$ at the generic point of X.

[3] Unique up to bimeromorphic equivalence.

Remark 2.4.

0. The sheaf $\mathcal{L}$ of example 2.2.4 above is a Bogomolov sheaf on X (in fact, it is the only one).

1. If $f : X \to Y$ is a fibration on some Y of general type of dimension $p > 0$, the saturation of $f^*(K_Y)$ in Ω^p_X is a Bogomolov sheaf of X, which is thus non-special. In particular, if X is of general type with $n > 0$, it is not 'special'.

2. We thus see easily that if X is a projective curve of genus $g \geq 0$, then X is special if and only if $g \leq 1$, that is: if and only if X is rational or elliptic.

3. 'Special' manifolds thus generalize rational and elliptic curves. We shall see in fact that they are, more precisely, the manifolds which are 'opposite' to manifolds of general type in a precise sense. We shall conjecture below that they are 'opposite' to manifolds of general type also for hyperbolicity and arithmetic properties.

4. It is certainly possible to show the finiteness of the set of Bogomolov sheaves on a given X[4] by adapting the proof of the theorem of Kobayashi-Ochiai for dominant meromorphic maps $f : X \dashrightarrow Y$, with Y of general type.

2.3. **Preservation of 'specialness'.**

1. Specialness is a birational property (by its very definition).

2. If X is 'special' and $f : X \to Z$ is a surjective meromorphic map to a complex analytic space, then Z is 'special', too. (Obvious).

3. If $X \to X'$ is a finite étale cover, and if X' is 'special', then X is 'special' too. We admit this (surprisingly) difficult result, which is proved (see 5.4) using the partial solution 4.2 of the conjecture $C^{orb}_{n,m}$ stated below.

4. If any two generic points of X can be joined by a chain of 'special' irreducible subvarieties, then X is 'special'. This can be proved by elementary means using the compactness of the components of the Barlet-Chow scheme of X. A direct proof (see 5.6) is obtained using the 'core map' defined below.

4'. In particular, if $f : X \to Y$ is a fibration with special fibres containing a 'special' subvariety $Z \subset X$ such that $f(Z) = Y$, then X is 'special'.

2.4. **Examples of 'special' manifolds.** We shall give here examples of 'special' manifolds. Most proofs cannot be given now, because they rest on theorem 4.2. A complete understanding of the class of 'special' manifolds can be gained only by the decomposition theorem **??** stated below, which requires working in the orbifold category.

1. Rationally connected manifolds X are 'special' (indeed, $Sym^m(\Omega^p_X) = 0$ for any $p, m > 0$. Hence $\kappa(X, \mathcal{L}) = 0$ for any $\mathcal{L} \subset \Omega^p_X$ and any $p > 0$).

[4] And, more generally, for 'smooth orbifolds' defined below.

2. Complex tori and Abelian varieties X are 'special', since $\kappa(X,\mathcal{L}) \leq 0$ for any $p, m > 0$, for any rank-one $\mathcal{L} \subset \Omega^p_X$.

3. More generally, if $c_1(X) = 0$, then X is 'special', since again $\kappa(X,\mathcal{L}) \leq 0$ for any $p, m > 0$, for any rank-one $\mathcal{L} \subset \Omega^p_X$, by either the existence of Ricci-flat Kähler metrics and the parallelism of holomorphic covariant tensors, or Miyaoka's generic semi-positivity theorem.

4. Still more generally, X is 'special' if $\kappa(X) = 0$.This is a consequence of 4.2. In this case, we cannot prove that $\kappa(X,\mathcal{L}) \leq 0$ for any $\mathcal{L}$ as above, although the abundance conjecture implies this.

5. For any pair $(n, k), k \in \{-\infty, 0, 1, ..., (n-1)\}$, there exists 'special' projective manifolds X of dimension n with $\kappa(X) = k$, as well as non-special manifolds with the same invariants (except for $k = 0$ of course). The notion of specialness is thus not determined by $\kappa(X)$, and certainly not restricted to the cases $\kappa(X) \leq 0$.

6. For surfaces, it is easy to characterise 'specialness' by the invariants κ and either $\tilde{q}$ or π_1. More precisely: a compact Kähler surface X is 'special' if and only if $\kappa(X) \leq 1$, and $\pi_1(X)$ is almost abelian (or equivalently, if $q(X') \leq 1$ for any finite étale cover X' of X). In particular: 'specialness' is preserved by deformation, and also by diffeomorphisms. One conjectures that deformation preserves specialness in higher dimensions too.

7. A compact Kähler manifold of algebraic dimension zero (or, equivalently containing only a finite set of irreducible compact divisors) is 'special'. More generally, the fibres of any algebraic reduction of a compact Kähler manifold are 'special'.

8. A quite different criterion for 'specialness' is derived from an orbifold version of a fundamental result of Kobayashi-Ochiai: if $h : \mathbb{C}^m \to X$ is a (transcendental) non-degenerate meromorphic map, then X is 'special'. Here non-degenerate means that it is holomorphic of rank n at some point $z \in \mathbb{C}^m$. More general versions exist.

2.5. **Conjectures about 'special manifolds.** 'Special manifolds' will below appear as exactly 'opposite' to manifolds of general type, and the structure results about them will naturally lead to formulate the following conjectures (which will be extended, strengthened and justified in conjectures 5.8, 7.2, 7.5,7.7):

Conjecture 2.5. *1. If X is special, $\pi_1(X)$ is almost abelian. (True when X is rationally connected and when $c_1(X) = 0$)*

2. X is special if and only if its Kobayashi pseudometric vanishes, or equivalently if and only if any two points can be joined by an entire holomorphic curve, or equivalently if and only if it contains a Zariski dense entire curve $h : \mathbb{C} \to X$.

3. Assume X is defined over a number field $k \subset \mathbb{C}$. Then X is special if and only if it is 'potentially dense' (ie: if $X(k')$ is Zariski dense for some finite extension k'/k).

4. 'Specialness' is closed under deformation and specialisation. In other words: let $X_s, s \in \mathbb{D}$ the unit disc, be a smooth family of compact Kähler manifolds. If one member X_s is 'special', all members are 'special' .

5. Let $F : X \to B$ be a fibration from $X \in \mathcal{C}$ onto a projective smooth curbe B. Assume F is not bimeromorphically isotrivial [5]. *Then: X is 'potentially dense' over $\mathbb{C}(B)$*[6] *if and only if its general fibre X_b is special.*

Of course, conjectures 2.5.2 and 2.5.3 are inspired by Lang's conjectures in hyperbolicity and arithmetics, of which they are the versions in the 'opposite' case.

3. Special Manifolds: Orbifold base

We will consider here a normal connected compact complex analytic space Z. An *orbifold* divisor is a finite linear combination $D := \sum_j c_j.D_j$, where the $D_j's$ are pairwise distinct irreducible closed divisors of Z, and $c_j \in [0,1] \cap \mathbb{Q}$ for any j.

To each *coefficient* c_j is associated a *multiplicity* $m_j := (1-c_j)^{-1} \in [1, +\infty[\cap \mathbb{Q} \cup \{+\infty\}$, or equivalently: $c_j = 1 - \frac{1}{m_j}$. Thus we can write also: $D = \sum_{\{F \subset X\}} (1 - \frac{1}{m_D(F)}).F$, where F ranges over all irreducible divisors or X, and $m_D(F) := m_j$ if $F = D_j$, while $m_D(F) := 1$ if F is none of the $D_j's$.

Such orbifold pairs (Z, D) interpolate between the compact case where $D = 0$ and $(Z, 0) = Z$ without orbifold structure, and the *open*, or *purely-logarithmic case* where $c_j = 1, \forall j$, where $(Z, D) = Z - Supp(D)$.

When Z is smooth and the support $Supp(D) := \cup D_j$ of D is of normal crossings, we say that (Z, D) is *smooth.* When all multiplicities m_j are integral or $+\infty$, we say that the *orbifold pair* (Z, D) is *integral*, and may be thought of a virtual ramified cover of Z ramifying at order m_j over each of the $D_j's$.

When D is integral, so is $D_{f,D}$. In particular, $D_f := D_{f,D=0}$ is integral.

There are (at least) 3 main reasons, apparently independent, to introduce this notion:

1. Moduli spaces (Deligne-Mumford 'stacks').
2. LMMP, in order to use inductive arguments on the dimension, by restriction to 'centers of log-canonical singularities'.
3. Orbifold base of fibrations, which is the main subject of the present survey.

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[5] That is: the generic fibres of F are not pairwise bimeromorphic.

[6] That is: $Z(C') \subset X' = X \times_B B'$ is Zariski-dense in X' for some finite (ramified) cover $u : B' \to B$, where $Z(B')$ is the union of the images of all sections $s' : B' \to X'$ of $F' := F \times u : X' \to B'$.

3.1. **Orbifold base of a fibration.**

Definition 3.1. *Let $f : X \to Z$ be a holomorphic fibration. Assume that an orbifold divisor D is given on X. We shall define an orbifold base $(Z, D_{f,D})$ of (f, D) as follows, by assigning to each irreducible Weil divisor $E \subset Z$ a multiplicity*[7] $m_{(f,D)}(E) := inf_k\{t_k.m_D(F_k)\}$, *where:* $f^*(E) = \sum_k t_k.F_k + R$, *where R is an f-exceptional divisor of X with $f(R) \subsetneq E$, while the F_k's range over all irreducible divisors of X surjectively mapped by f to E.*

Remark 3.2. *The geometric meaning of (Z, D_f) is that it is a virtual ramified cover of Z which eliminates by base-change the multiple fibres of f in codimension one*[8] *over Z. Moreover (on suitable bimeromorphic models), if $f : X \to Z$ and $g : Z \to W$ are two fibrations, then $D_{gf} = D_{g,D_f}$. This justifies the above formula for $m_{f,D}(E)$.*

3.2. **Orbifold canonical bundle.** Let (Z, D) be an orbifold pair. Assume that $K_Z + D$ is $\mathbb{Q}$-Cartier (this is the case if (Z, D) is smooth, for example). This will then be said to be the canonical bundle of (Z, D), and the *canonical dimension* $\kappa(Z, D)$ of (Z, D) will be defined, as usual, as being $\kappa(Z, K_Z + D)$. We say that (Z, D) is of *general type* if $\kappa(Z, D) = dim(Z)$.

If we have a holomorphic fibration $f : (X, D) \to Z$ such that $K_Z + D_{f,D}$ is $\mathbb{Q}$-Cartier, we can define thus define $\kappa(Z, D_{f,D})$. This will however not be a bimeromorphic invariant of $(X, D), f$ in general.

More precisely: if $g : (X', D') \to (X, D)$ is a bimeromorphic map from X' to X such that $g_*(D') = D$, it is easy to see that $D_{fg,D'} = D_f$. But if we have a bimeromorphic map $h : Z' \to Z$ and a factorisation $f = hf'$ for some holomorphic fibration $f' : (X, D) \to Z'$, we have also: $D_{f,D} = h_*(D_{f',D})$. We thus only get: $\kappa(Z', D_{f',D}) \leq \kappa(Z, D_{f,D})$, and simple examples show that strict inequality may occur.

3.3. **Bimeromorphic equivalence of fibrations.** We shall say that $f : X \to Z$ and $f' : X' \to Z'$ are bimeromorphically equivalent if there exists bimeromorphic meromorphic maps $u : X' \to X$ and $v : Z' \to Z$ such that $fu = vf' : X' \to Z$. By suitable modifications of X' and Z', we can and shall assume that u, v, f, f' are holomorphic, and that $K_Z + D_f$ and $K_{Z'} + D_{f'}$ are $\mathbb{Q}$-Cartier. We write: $f' \sim f$

We shall then define $\kappa(f) := inf_{\{f' \sim f\}}\kappa(Z', D_{f'}) \in \{-\infty, 0, ..., dim(Z)\}$.

This is independent on the bimeromorphic model of f which is chosen, and is thus defined for any meromorphic fibration $f : X \to Z$, with X, Z arbitrarily singular (provided X has a Kähler smooth model).

[7] Note that the integers t_k are well-defined, even if X is only assumed to be normal.

[8] This is actually true only for the 'classical' orbifold base $D^*_f := \sum_E(1 - \frac{1}{m^*_f(E)}).E$, with $m^*_{(f)}(E) := gcd_k\{t_k\}$, which we shall however not consider here, for reasons given below (see Remark 3.8).

Definition 3.3. *We say that f is of general type if $\kappa(f) = dim(Z)$.*

Fibrations of general type enjoy a certain regularity:

Proposition 3.4. *Let $f : X \to Z$ be a meromorphic fibration of general type. It is 'almost holomorphic'* [9] *if X is smooth.*

We now describe fibrations $f : X \to Z$ such that $\kappa(f) = \kappa(Z, D_f)$.

3.4. **Neat models.**

Definition 3.5. *The holomorphic fibration $f : X \to Z$ will be said to be 'neat' if X and Z are smooth, and if there exists a bimeromorphic holomorphic map $u : X \to X_0$, with X_0 smooth, such that any irreducible divisor $E \subset X$ which is f-exceptional* [10] *is also u-exceptional.*

By Raynaud's flattening and Hironaka desingularisation theorems, any f has a bimeromorphic model which is 'neat' (first flatten f by modifying Z, then desingularise).

Theorem 3.6. *Let $f : X \to Z$ be 'neat'. Then $\kappa(f) = \kappa(Z, D_f)$.*

Idea of proof: Let $p := dim(Z)$, and $\mathcal{L} \subset \Omega^p_X$ be the saturation of $f^*(K_Z)$. Notice that $\kappa(X, \mathcal{L})$ is a bimeromorphic invariant of f. Then, for any $m > 0$ sufficiently divisible (by the lowest common multiple of the multiplicities of the components of D_f, precisely), $f^*(m.(K_Z+D_f)) \subset Sym^m(\Omega^p_X)$ has the same saturation as $\mathcal{L}^{\otimes m}$. Moreover, because the f-exceptional divisors of X are also u-exceptional, Hartog's theorem implies that any section of $f^*(m.(K_Z + D_f))$ defined ouside the union of these divisors extends holomorphically to X. Since the support of $[\mathcal{L}^{\otimes m}/f^*(m.(K_Z + D_f))]$ does not contain $f^{-1}(G)$ for any irreducible divisor $G \subset Z$, we see that the sections of $\mathcal{L}^{\otimes m}$ and of $f^*(m.(K_Z + D_f))$ coincide for any sufficiently divisible m $\square$

We obtain the following crucial:

Corollary 3.7. *In the notations of theorem 3.6, let $\mathcal{L}_f \subset \Omega^p_X$ be the saturation of $f^*(K_Z)$. Then:*

1. The correspondance between f and $\mathcal{L}_f$ induces a bijection between Bogomolov sheaves on X and fibrations of general type up to bimeromorphic equivalence on X.

2. X is special if and only if it admits no fibration of general type.

We can thus say that X is 'special' precisely if it does not admit a neat fibration 'onto an orbifold of general type'.

Remark 3.8. *This correspondance between Bogomolov sheaves and fibrations of general type needs the use of the infimum (as opposed to the gcd) in the multiplicities defining the orbifold base of a fibration (see*

[9] Which means that its indeterminacy locus does not meet its generic fibre.

[10] This means that $f(E)$ has codimension at least 2 in Z.

definition 3.1). This is why 'classical' multiplicities are not considered here.

The following property is elementary:

Proposition 3.9. *Let $f : X \to Y$ and $g : X \to Z$ be fibrations. Assume that the 'general'* [11] *fibres of f are 'special', and that g is of general type. There then exists a factorisation $h : Y \to Z$ such that $g = h \circ f$.*

4. The Orbifold version of $C_{n,m}$ Conjecture

Conjecture 4.1. *Let (X, D) be smooth, and $f : X \to Z$ be a 'neat' fibration. Then $\kappa(X, D) \geq \kappa(X_z, D_z) + \kappa(Z, D_{f,D})$, where (X_z, D_z) is the generic (smooth) orbifold fibre of $f : (X, D) \to Z$.*

This conjecture coincides with Iitaka's $C_{n,m}$ conjecture when $D = 0$, if one ignores the strengthening term $D_{f,D}$. We call Conjecture 4.1 the '$C^{orb}_{n,m}$-conjecture'.

The main technical result of the present text is the solution of $C^{orb}_{n,m}$ for fibrations of general type:

Theorem 4.2. *If (X, D) is smooth and $f : X \to Z$ is a 'neat' fibration of general type, then: $\kappa(X, D) \geq \kappa(X_z, D_z) + dim(Z)$.*

The proof of this important result is an orbifold adaptation of Viehweg's proof of $C_{n,m}$ when Z is of general type. An immediate but important consequence[12] is:

Corollary 4.3. *If $\kappa(X) = 0$, then X is 'special'.*

A second basic application is the core map, which we now expose.

5. The Core Fibration

Theorem 5.1. *Let X be a compact Kähler manifold. There exists a unique*[13] *almost holomorphic fibration $c_X : X \to C(X)$, called the 'core map' such that:*

1. Its 'general' fibre is special.
2. Its orbifold base $(C(X), D_{c_X}) := (C, D_c)$ is of general type.

Remark 5.2. *1. There are two extreme cases: X is 'special' (resp. of general type) if and only if $C(X)$ is a point (resp. $X = C(X)$). 2. The proof shows that if $dim(C(X)) := p \geq 0$, there is a unique saturated rank-one subsheaf $\mathcal{L} \subset \Omega^p_X$ with $\kappa(X, \mathcal{L}) = p$. One has thus $\mathcal{L} = \mathcal{L}_{c_X}$.*

[11] That is: those not mapped to a countable union of proper Zariski-closed subsets of Y.

[12] We give only the stement for $D = 0$, although it holds, with its proof, in general.

[13] Up to bimeromorphic equivalence.

Idea of proof: Let $p \geq 0$ be maximum such that there exist $\mathcal{L} \subset \Omega^p_X$ saturated, of rank one, with $\kappa(X, \mathcal{L}) = p$, and let $f : X \to Z$ be the associated fibration. It is thus almost holomorphic (by 3.4), and its orbifold base (on a 'neat' model) is of general type. We need to prove that its general fibres are 'special'. We may assume that X is not 'special' and proceed by induction on $n := dim(X)$. Assume that the general fibre X_z of f is not 'special'. We can thus, using the compactness of the components of the Barlet-Chow scheme of X construct a relative 'core map' $c_f : X \to Y$, together with a factorisation $F : Y \to Z$ such that $F \circ c_f = f : X \to Z$, and over the 'general' $z \in Z$, we get by restriction to X_z, the 'core map' $c_z : X_z \to Y_z$ of X_z. Let now (Y, D_{c_f}) be the orbifold base of c_f. For $z \in Z$ general, $(Y_z, D_{c_f|X_z}) = (Y_z, D_{c_f})_{|Y_z}$ is of general type. From theorem 4.2 we see that (Y, D_{c_f}) is of general type, contradicting the definition of $p = dim(Z)$, since $dim(Y) > dim(Z)$. We chose $c = f$. The uniqueness of c follows from proposition 3.9 $\square$

Remark 5.3. *If X is defined over the field $K \subset \mathbb{C}$, so is c_X, by an easy Galoisian argument and the uniqueness of c_X.*

We give some additional properties of the 'core map' using its uniqueness:

Corollary 5.4. *Let $u : X' \to X$ be a finite étale cover, let $c_X : X \to C$ be the core map of X, and $c' : X' \to C'$ the Stein factorisation of $c_X \circ u : X' \to C$, with $c_u : C' \to C$ finite (ramified in general). Then c' is the core map of X'.*

In particular: if X is 'special', so is X'.

Idea of proof: We can assume that u is Galoisian, of group G. The family of fibres of $c_{X'} : X' \to C_{X'}$ is G-invariant and there exists a factorisation $f : X \to (C_{X'}/G)$. which is of general type since so is $c_{X'}$ (just consider the saturation of $f^*(K_{(C_{X'}/G)})$ in Ω^*_X, and its inverse image in $\Omega^*_{X'}$. Use the fact that u is étale.). Since the fibres of $c_{X'}$ are 'special', so are their images by u, the fibres of f. Thus $f = c_X$ as claimed $\square$

From proposition 3.9 we get immediately:

Corollary 5.5. *Let X be smooth, and $c_X : X \to C$ its core map. Let $f : X \to Z$ be a fibration. If f is of general type (resp. if its general fibres are special), there exists a factorisation: $h : C \to Z$ (resp. $h : Z \to X$) such that $f = h \circ c_X$ (resp. $c_X = h \circ f$).*

Another elementary consequence is the following.

Corollary 5.6. *Let X be smooth. Assume that there exists a nompty open (analytic) subset U of X any two points of which can be joined by a chain of 'special' subvarieties* [14] *of X. Then X is 'special'.*

[14] That is: irreducible compact analytic subsets.

Remark 5.7. *The smoothness assumption is essential, here (consider the cone over a general type manifold). On the other hand, a 'special' manifold may contain no proper 'special' subvariety, except for points (consider a 'simple' abelian variety).*

5.1. **Conjectures about the core.** Using the 'core', we may now formulate conjectures about all manifolds $X \in \mathcal{C}$, not only for 'special' ones as in 2.5. Indeed, the 'core' splits X geometrically into its two 'opposite' parts: 'special' (the fibres), and 'general type' (the orbifold base). We conjecture that it also splits X arithmetically and 'hyperbolically', following Lang's conjectures relating geometry, arithmetics and hyperbolicity.

Let $c : X \to C$ be the 'core' of some $X \in \mathcal{C}$ (on some 'neat' model). Let (C, D) be its orbifold base: it is of general type.

We shall associate in §7 to (C, D) a Kobayashi pseudometric $d_{(C,D)}$ and a set $(C, D)(k) \subset C$ of k-rational points $(C, D)(k)$ if X (and so (C, D)) is defined over a number field $k \subset \mathbb{C}$.

These definitions are functorial, so that $d_X \leq c^*(d_{(C,D)})$ and $c(X(k)) \subset (C, D)(k)$.

Even without these precise definitions (given in §7), the conjectures below give a qualitative description of d_X and $X(kk)$.

Conjecture 5.8. *1. Let d_X be the Kobayashi pseudo-metric of X. Then $d_X = c^*(d_{(C,D)})$, and there exists a proper closed algebraic subset $W \subset C$ such that $d_{(C,D)}$ is a metric on $C - W$.*

2. Assume that X is defined over the number field $k \subset \mathbb{C}$. Then $(C, D)(k) \cap (C - W)$ is finite, $c(X(k)) \subset (C, D)(k)$. Moreover, there exists a finite extension k'/k such that $X(k') \cap (c^{-1}(C - W)$ is Zariski-dense in $c^{-1}[(C, D)(k) \cap (C - W)]$.

Remark 5.9. *These conjectures strengthen the combination of conjectures 2.5.(2,3) together with versions for orbifolds of general type of Lang's conjectures in hyperbolicity ($d_{(C,D)}$ is a metric on $C - W$) and arithmetics $(C, D)(k)$ is finite outside W). One may conjecture that W is the union of all 'suborbifolds' of (C, D) which are not of general type.*

We give a generalisation of conjecture 2.5.(4) as well:

Conjecture 5.10. *The dimension of the core is invariant under deformation and specialisation. In other words: let $X_s, s \in \mathbb{D}$ the unit disc, be a smooth family of compact Kähler manifolds. The dimension $dim(C(X_s))$ is then independent of $s \in \mathbb{D}$.*

6. The decomposition $c = (Jr)^n$ of the core

We shall show, conditionally in $C^{orb}_{n,m}$, that the core fibration can be written as the $n-th$ iterate of the composition $J \circ r$ of two canonically defined fibrations: J and r, respectively the Moishezon-Iitaka fibration and a weak version of the 'rational quotient' of [C92] (and called the MRC-fibration in [KMM]).

This decomposition cannot take place in the bimeromorphic category of varieties without orbifold structure. We thus consider orbifold pairs, and define some of their geometric invariants as well as bimeromorphic maps between them. These notions are delicate and still not defined in complete generality. For these reasons, we shall restrict to *smooth* orbifold pairs (X, D) and to the situations needed for the present exposition.

6.1. **Kodaira dimension of an orbifold fibration.** We need to extend the definitions and results of §3.3 to the case where X is equipped with an orbifold divisor D. The proofs of the relevant results are entirely the same.

We shall denote by $f : (X, D) \to Z$ the data consisting of a fibration $f : X \to Z$, together with an orbifold divisor D on X. We have already defined the orbifold base $(Z, D_{f,D})$ and its canonical dimension $\kappa(Z, D_{f,D})$ in §3.3.

Definition 6.1. *Let $u : X' \to X$ be a bimeromorphic map, and let D' and D be orbifold divisors on X' and X respectively such that $u_*(D') = D$. We shall say that $u : (X', D') \to (X, D)$ is 'weakly bimeromorphic' (or a 'weak modification' if $K_{X'} + D' \geq u^*(K_X + D)$. We then have: $\kappa(X', D') = \kappa(X, D)$*[15] .

In this situation, we shall say that the fibrations $f : X \to Z$ and $f' : X' \to Z'$ are 'bimeromorphically equivalent' if there exists a bimeromorphic meromorphic map $v : Z' \to Z$ such that $fu = vf' : X' \to Z$. By suitable modifications of (X', D') and Z', we can and shall assume that u, v, f, f' are holomorphic, and that $K_Z + D_{f,D}$ and $K_{Z'} + D_{f',D'}$ are $\mathbb{Q}$-Cartier. We write: $(f', D') \sim (f, D)$.

We shall then define:

$$\kappa(f, D) := inf_{\{(f',D')\sim(f,D)\}}\{\kappa(Z', D_{f',D'})\} \in \{-\infty, 0, ..., dim(Z)\}.$$

This is independent on the bimeromorphic model of f which is chosen, and is thus defined for any meromorphic fibration $f : (X, D) \to Z$, with X, Z arbitrarily singular (provided X has a Kähler smooth model).

Theorem 3.6 holds in this more general situation as well:

Theorem 6.2. *Let (X, D) be smooth, and let $f : X \to Z$ be a 'neat' fibration. Then $\kappa(f, D) = \kappa(Z, D_{f,D})$.*

6.2. **The orbifold canonical fibration** J**.** Let (X, D) be smooth, and assume that $\kappa(X, D) \geq 0$. There then exists as usual a 'canonical fibration' $J = J_{X,D} : (X, D) \to J(X, D)$, which we may assume to be

[15]More precisely: the m-th plurigenera of (X', D') and (X, D) coincide for m sufficiently divisible.

'neat', given (on some weakly bimeromorphic model) by the sections of some multiple of $K_X + D$. We have, if J is neat:

$$dim(Z) = \kappa(X, D) \geq \kappa(f, D) = \kappa(Z, D_{f,D}) \geq -\infty.$$

6.3. **The 'k-rational quotient' r.**

Definition 6.3. *Let (X, D) be smooth. We define:*
$dim(X) \geq \kappa_+(X, D) := max_{\{f:X\to Z\}}\{\kappa(Z, D_{f,D})\} \geq -\infty$.

The basic example is:

Theorem 6.4. *Let (X, D) be birationally Fano, smooth. Then:*

$$\kappa_+(X, D) = -\infty.$$

Proof: We refer to lemma 6.20 for the proof and the relevant definition, just mentioning that (X, D) is 'birationally Fano' if $-(K_X + D)$ is ample on X) □

Remark 6.5. *If X is rationally connected and $D = 0$, then $\kappa_+(X) = -\infty$, and conjecturally, the converse is true as well. More generally, if $r_X : X \to R(X)$ is the rational quotient (with rationally connected fibres and non-uniruled base by* [GHS]*), one conjectures that $\kappa(R(X) \geq 0$, and r_X is (up to bimeromorphic equivalence) characterised by these two properties.*

Using $C^{orb}_{n,m}$, we shall extend this construction to the orbifold situation.

Proposition 6.6. *Assume $C^{orb}_{n,m}$. Let (X, D) be smooth. There exists a unique*[16] *fibration $r := r_{X,D} : (X, D) \to R := R(X, D)$ such that:*

1. Its general orbifold fibres have $\kappa_+ = -\infty$.

2. $\kappa(r, D) \geq 0$.

Moreover, r is almost holomorphic. It is called the 'k-rational quotient' of (X, D)[17].

Idea of proof: If $\kappa_+(X, D) = -\infty$, we take R to be a point. If $\kappa(X, D) \geq 0$, we take $R = X$. It is then easy to see that (X, D) is not covered by a family of suborbifolds (becoming smooth on a suitable weak modification of (X, D)) with $\kappa = -\infty$. Assume now that $\kappa(X, D) = -\infty$, but that there exists some fibration $f : (X, D) \to Z$ with $dim(Z) > 0$ and $\kappa(f, D) \geq 0$. Choose $dim(Z)$ to be maximum with this property. We may assume that f is 'neat'. We then claim that $\kappa_+(X_z, D_z) = -\infty$. Otherwise, using induction on dimension and the compactness of the Barlet-Chow space, we can construct a relative 'k-rational quotient' $r_f : X \to Y$ and $\rho : Y \to Z$ such that $\rho \circ r_f = f$, and the restriction $r_{f,z} : (X_z, D_z) \to Y_z$ of r_f to X_z is the 'k-rational quotient' of (X_z, D_z). We have: $dim(Y) > dim(Z)$ by assumption, and

[16]Up to bimeromorphic equivalence.

[17]The term will find its justification in corollary 6.19 below.

$\kappa(Y_z, D_{r_{f,z},D_z} = D_{r_f,D}|Y_z)) \geq 0$. Because we may assume that both r_f and ρ are neat, we deduce from $C^{orb}_{n,m}$ that:

$$\kappa(Y, D_{r_f,D}) \geq \kappa(Y_z, D_{r_f,D}|Y_z) + \kappa(Z, D_{f,D}) \geq 0,$$

contradicting the maximality of $dim(Z)$

The uniqueness of r follows from the following elementary lemma 6.7. □

Lemma 6.7. *Let $f : (X, D) toZ$ be a 'neat' fibration with orbifold base (Z, D_Z) such that $\kappa(Z, D_Z) \geq 0$. Let $g : (XD) \to Y$ be a holomorphic fibration with $\kappa(X_y, D_y) = -\infty$ for $y \in Y$ general. There exists a factorisation $h : Y \to Z$ such that $hg = f$.*

Remark 6.8. *From their constructions, we see that J and r are preserved by 'weak modifications' $u : (X', D') \to (X, D)$ as defined in 6.1.*

6.4. **Special orbifolds.**

Definition 6.9. *Let (X, D) be a smooth orbifold. We say that (X, D) is 'special' if $\kappa(f, D) < dim(Z)$ for any fibration $f : (X, D) \to Z$ with $dim(Z) > 0$. Orbifold 'specialness' is preserved by 'weak modifications'.*

Remark 6.10. *There is an alternative definition of orbifold specialness in terms of D-Bogomolov sheaves on X, similar to definition 2.3 and theorem 3.6. We shall not give them here, and refer to* [C11] *for details. The same arguments as when $D = 0$ show the existence of a 'core fibration' $c_{X,D} : (X, D) \to C(X, D)$ for any smooth orbifold (X, D).*

One gets immediately from theorem 4.2, as in the proof of corollary 4.3:

Theorem 6.11. *Let (X, D) be smooth. Then (X, D) is 'special' if either $\kappa(X, D) = 0$, or if $\kappa_+(X, D) = -\infty$.*

The following result is elementary.

Proposition 6.12. *Let $f : (X, D) \to Z$ be a 'neat' holomorphic fibration such that its 'general' orbifold fibre (X_z, D_z) and its orbifold base $(Z, D_{f,D})$ are special. Then (X, D) is 'special'.*

Remark 6.13. *This is (another) justification of the consideration of orbifold pairs: it is not true that X is special if it has a fibration $f : X \to Z$ with base Z and fibres X_z special (see example 2.2.4).*

Corollary 6.14. *Assume $C^{orb}_{n,m}$. Let (X, D) be smooth.*

1. Define $r : (X, D) \to R$ its 'k-rational quotient' on some 'neat' model. Then $\kappa(R, D_{r,D} := D_R) \geq 0$, and so $J : (R, D_R) \to J(R, D_R)$ is well-defined (on some neat model again). Making a weak modification of (X, D), we may thus assume that $J \circ r : (X, D) \to J(R, D_R)$ is well-defined and 'neat'. Moreover (from proposition 6.12), its orbifold fibres are special.

2. Thus $(J\circ r)^k$ is a uniquely and well-defined fibration, for any $k\geq 0$. Its orbifold fibres are special (from 6.12 and induction)
3. $J(R,D_R)=X$ if and only if (X,D) is of general type.

Idea of proof: For claim 3. only, since 1,2 follow directly from 6.12: If $J(R,D_R)=X$, we have $R=X$, $(X,D)=(R,D_R)$ and so $\kappa(X,D)\geq 0$, and next: $J(R,D_R)=R$, which means that $\kappa(X,D)=\kappa(R,D_R)=dim(J)=dim(R)=dim(X)$ $\square$

6.5. **The decomposition $c=(J\circ r)^n$ of the core.**

Theorem 6.15. *Assume $C^{orb}_{n,m}$. Let (X,D) be a smooth orbifold, and $c=(X,D)\to C(X,D)$ be its 'core fibration'. Then $c=(J\circ r)^n$, with $n:=dim(X)$.*

Proof: The orbifold fibres of $(J\circ r)^n$ are special, by 6.14. Let $(Jr)^n:(X,D)\to Z_n$. We just need to show that $(Z,D_{(Jr)^n,D})$ is of general type, since c is characterised by these two properties. This follows from 6.12.3, and the equality: $(Jr)^{n+1}=(Jr)^n$, since the dimension d_k of the image Z_k of $(Jr)^k:(X,D)\to Z_k$ decreases with k, and stabilizes precisely when the orbifold base is of general type $\square$

Corollary 6.16. *Assume $C^{orb}_{n,m}$. Let X be a connected compact Kähler manifold. Then X is 'special' if and only if $(Jr)^n$ is the constant map (in other words: X is 'special' if and only if X is a tower of fibrations with orbifold fibres having either $\kappa=0$, or $\kappa_+=-\infty$).*

6.6. **Interpreting $\kappa=0$ and $\kappa_+=-\infty$ using Abundance.**
Recall (see [KM]) that an orbifold pair (Z,D) is *log-canonical* (l.c for short) if K_Z+D is $\mathbb{Q}$-Cartier, and if there exists a weakly bimeromorphic map $u:(X,D_X)\to(Z,D)$ (in the sense of definition 6.1 above) with (X,D) smooth. Note that K_X+D_X is pseudo-effective (*pseff* for short) if and only if so is K_Z+D.
Recall the central:

Conjecture 6.17. *("Abundance conjecture") Let (Z,D) be a l.c orbifold pair in $\mathcal{C}$. If K_Z+D is pseff, there exists a composition of divisorial contractions and log-flips $g:(Z,D)\to(Z',D')$ with $D':=g_*(D)$ and (Z',D') l.c such that $K_{Z'}+D'$ is semi-ample.*

We abbreviate this by saying that K_X+D is 'birationally semi-ample' (and 'birationally torsion' if $\kappa(X,D)=0$).

Let $\psi:Z'\to W$ be the fibration given by $K_{Z'}+D'$.

In this situation, we can make a 'weak modification $u:(X',D')\to(X,D)$ and a modification $v:W'\to W$ in such a way that $\psi':=v^{-1}\circ\psi\circ u^{-1}:X'\to W'$ is holomorphic, and 'neat', with $(W',D_{\psi',D'})$ smooth. We shall call it a 'neat birationally K-semi-ample fibration'.

One has also (in particular):

Theorem 6.18. *([BCHM]) Let (Z,D) be l.c projective. If K_Z+D is not pseff, there exists a composition of divisorial contractions and log-flips $g:(Z,D)\to(Z',D')$ with $D':=g_*(D)$, and a ('Fano'-) fibration $\varphi:(Z',D')\to W$ such that: (Z',D') is l.c, $-(K_{Z'}+D')$ is φ-ample and $dim(W)<dim(Z)$.*

We abbreviate this by saying that $\varphi:(X,D)\to W$ is a 'birationally Fano fibration'. It is birationally a projective morphism, hence a Moishezon morphism, and X is Moishezon if so is W.

In this situation, we can make a 'weak modification $u:(X',D')\to(X,D)$ and a modification $v:W'\to W$ in such a way that $\varphi':=v^{-1}\circ\varphi\circ u^{-1}:X'\to W'$ is holomorphic, and 'neat', with $(W',D_{\varphi',D'})$ smooth. We shall call it a 'neat birationally Fano fibration'.

Corollary 6.19. *Assume the Abundance conjecture 6.17.*

Let (X,D) be a smooth orbifold pair in $\mathcal{C}$.

1. If $\kappa(X,D)\geq 0$, then K_X+D is 'birationally semi-ample' (and 'birationally torsion' if $\kappa(X,D)=0$).

2. $\kappa_+(X,D)=-\infty$ if and only if there exists a finite sequence of 'birationally Fano fibrations' $\varphi_i:(X_i,D_i)\to(X_{i+1},D_{i+1})$ $(i=0,1,...,k\leq(n-1))$ such that $(X_0,D_0)=(X,D)$, and X_k is one point.

In this case X is Moishezon (and projective if Kähler).

2. The core map of (X,D) is (after a 'weak modification' of (X,D)) a composition of 'neat birationally K-semi-ample' and of 'neat birationally Fano' fibrations.

Proof: Assertions 1 and 2 are clear, from assertions 1 and 2, together with theorem 6.15. We show assertion 2. Assume first that $\kappa_+(X,D)=-\infty$, so that K_X+D is not pseff. There thus exists a non-trivial 'neat birationally Fano fibration' $\varphi':(X',D')\to W'$ (after a 'weak modification' of (X,D)). And we have $\kappa(W',D_{\varphi',D'})=-\infty$. By induction on $dim(X)$, we may iterate to get the conclusion.

Assume conversely that we have such a sequence of 'neat birationally Fano fibrations'. By induction on the number of terms, and using proposition 6.7, the claim follows from the next theorem 6.20 □

Theorem 6.20. *Let (X,D) be birationally Fano, and smooth. Then:*

$$\kappa_+(X,D)=-\infty.$$

Proof: Let $g:(X,D)\to(Z',D')$ be a sequence of divisorial contractions and flips, with (Z',D') Fano and l.c. Assume there exists a 'neat' fibration $f:(X,D)\to Y$ with $\kappa(Y,D_{f,D})\geq 0$ and $dim(Y)=p>0$. We thus get[18] a non-zero section of $H^0(X,S^m(\Omega^p_X(D)))$ for some large and divisible integer $m>0$. If $C\subset Z'$ is however a generic Mehta-Ramanathan curve on Z' for some (any) polarization H, then: $H^0(C,S^m(\Omega^p_X(D))_{|C})=0$, by [CP], theorem 3.1.(2), which implies the

[18]We refer to [CP] and [C11] for the definition of the sheaves $S^m(\Omega^p_X(D))$.

vanishing of $H^0(X, S^m(\Omega^p_X(D)))$, and contradicts the existence of f with the asserted properties. $\square$

6.7. **Lifting properties using the $c = (Jr)^n$ decomposition.**

We wish to reduce the verification of certain properties P related to hyperbolicity, arithmetics, topology ,... which are conjectured for special manifolds to the classes of orbifold pairs with either $\kappa = 0$, or $\kappa_+ = -\infty$. The following is an immediate deduction from theorem 6.15.

Corollary 6.21. *Assume $C^{orb}_{n,m}$. Let P be a property of smooth orbifolds (X, D).*

Assume the following properties:

1. P is satisfied if either $\kappa(X, D) = 0$, or $\kappa_+(X, D) = -\infty$.

2. P is preserved under weak modifications of smooth orbifolds.

3. P is satisfied by (X, D) if there is a 'neat' fibration $f : (X, D) \to Z$, and if P is satisfied both by the general orbifold fibre and the base orbifold of f.

Then: P is satisfied by any 'special' smooth (X, D).

Assume moreover that:

4. P is not satisfied by any smooth orbifold of general type and positive dimension.

5. P is satisfied by $(Z, D_{f,D})$ if it is by (X, D) and there exists a neat fibration $f : (X, D) \to Z$.

Then P is satisfied by (X, D) if and only if (X, D) is special.

We shall give in the next sections some examples of properties P conjecturally stable under the above operations.

Remark 6.22. *Using Abundance, we can even replace in condition 1 of corollary 6.21 $\kappa = 0$ and $\kappa_+ = -\infty$ for smooth orbifolds by: $K_Z + D$ torsion and $-(K_Z + D)$ ample, but for l.c orbifold pairs, instead of smooth ones. The definition of the properties P we are interested in below is however far from obvious in this larger class of singular orbifold pairs.*

7. Conjectures for smooth orbifolds

We formulated conjectures 2.5 and 5.8 concerning the qualitative geometry (π_1, deformations) as well as hyperbolicity and arithmetics of compact Kähler manifolds. The (conditional) decomposition $c = (J \circ r)^n$ of the core shows that their solution should be reduced to formulate and establishing them for smooth orbifolds with either $\kappa = 0$, or with $\kappa_+ = -\infty$. This will be done next.

The definitions are however much more delicate in this orbifold context. In particular, the 'classical' version also appears naturally, and leads to a different version. The two versions behave functorially, but (probably) differently. They lead to two versions of 'specialness' and 'core'. We chose the 'non-classical' version because it is the one which is compatible with the definition by means of Bogomolov sheaves. But the 'classical' version leads to stronger conjectures in the case of orbifolds with $\kappa = 0$ or $\kappa_+ = -\infty$, which are the ones we state in these two cases.

Here are some other conjectures.

Let (X, D) be an *integral* smooth orbifold pair in $\mathcal{C}$. Most invariants of varieties and manifolds can be defined in a natural way for such pairs. Let us mention (see below, we refer to [C11] for more details):

1. The fundamental group $\pi_1(X, D)$.
2. The Kobayashi pseudo-metric $d_{(X,D)}$, and the notion of (orbifold) entire curve $h : \mathbb{C} \to (X, D)$.
3. The notion of D-rational curve.
4. If (X, D) is defined over (say) a number field k, the notion of D-integral point (over (k, S), once a model over $Spec(\mathcal{O}_{k,S})$ has been chosen, together with a finite set of places S of $\mathcal{O}_k$).
5. The function field version of the preceding arithmetic notion.

In the following subsections, we shall give the relevant definitions. For this, we write: $D := \sum_j (1 - \frac{1}{m_j}).D_j$ with $m_j > 0$ either integers or $+\infty$, and $Supp(D) := \cup_j D_j$. Recall that D is 'finite' if so are all the $m_j's$.

7.1. **Fundamental group.**

Definition 7.1. *For (X, D) as above, it is defined as the quotient of $\pi_1(X - Supp(D) - S)$[19] divided by the normal subgroup generated by the classes γ_j^m, if γ_j is a small loop around D_j .*

Conjecture 7.2. *$\pi_1(X, D)$ is almost abelian if (X, D) is special, and finite if (X, D) is Fano and if D is 'finite'.*[20]

This conjecture is known in a certain number of significant cases when D is 'finite'. For example, if $D = 0$, it is known if X is either rationally connected, or with $c_1(X) = 0$ ([Y]). It is also known that the image of a linear representation of $\pi_1(X))$ is almost abelian if X is special ([C04]), and that $\pi_1(X)$ is almost abelian if $X \in \mathcal{C}$ is special with $dim(X) \leq 3$ ([CC]).

Finally, if $f : (X, D) \to Z$ is 'neat', and $\pi_1(X, z, D_z)$ and $\pi_1(Z, D^*_{f,D})$ (its 'classical' orbifold base[21]) are almost abelian, then so is $\pi_1(X, D)$

[19] S is the union of the singular sets of X and $Supp(D)$ if we do not assume (X, D) to be smooth, but just l.c

[20] This should also hold respectively for l.c and klt 'integral' pairs, in particular when $K_X + D$ is torsion and anti-ample respectively.

[21] The statement remains true, but is weaker with the 'non-classical' orbifold base.

([**?**]). In other words, the property 6.21.3 is true for the property $P :=$ '$\pi_1(X, D)$ is almost abelian'.

A related question is the structure of the Albanese map $a_X : X \to Alb_X$ for special manifolds. It is known ([C04]) that it is surjective with connected fibres, which extends theorems of Ueno (when $a(X) = 0$) and Kawamata (when $\kappa(X) = 0$). Moreover, the fibres of a_X are special ([CC14]).

7.2. **Orbicurves, Kobayashi pseudometrics.** We shall give a definition which will be used for orbifold curves (either rational, entire or discs) and corresponding respectively to $C = \mathbb{P}^1, \mathbb{C}, \mathbb{D}$ the unit disc.

Definition 7.3. *A 'D-orbicurve' is a holomorphic map $h : C \to X$ from a connected smooth complex analytic curve C such that, for any j, $h^*(D_j) \geq m_j.h^{-1}(D_j) \neq C$ (ie: if the order of contact of $h(C)$ with D_j is at least m_j at each intersection point, and if $h(C) \subsetneq Supp(D)$). We denote by $Hol(C, (X, D))$ the set of such orbicurves with given C.*

The orbicurve $h : C \to X$ is 'classical' if, moreover, the order of contact of $h(C)$ with each D_j is divisible by m_j at each intersection point. We denote with $Hol^*(C, (X, D) \subset hol(C, (X, D)$ the set of 'classical' orbicurves with given C (because when $D = Supp(D)$ (ie: when all $m_j = +\infty$ for evey j), the D-orbicurves h are the ones whose images avoid $Supp(D)$.

A D-orbicurve $h : C \to X$ is a D-rational curve (resp. a D-entire curve, a D-disc) when $C = \mathbb{P}^1$ (resp. $C = \mathbb{C}$, resp. $c = \mathbb{D}$)). Observe that the classical and non-classical notions coincide in the two extreme cases where $D = 0$ and when $D = Supp(D)$.

We say that (X, D) is C-connected (resp. that (X, D) is classically C-connected) if any two generic points of X are joined by some C-orbicurve (resp. by some classical C-orbicurve). When $C = \mathbb{P}^1$ (resp. $C = \mathbb{C}$), we say that (X, D) is rationally connected (resp. $\mathbb{C}$-connected), and that (X, D) is 'classically' rationally connected when we deal with the 'classical' D-rational curves, and is 'classically' $\mathbb{C}$-connected when $C = \mathbb{C}$.

Let $P_{\mathbb{D}}$ be the Poincaré metric on $\mathbb{D}$. The Kobayashi pseudometric $d_{(X,D)}$ on X is the smallest pseudometric δ on X such that $\delta \geq h^*(P_{\mathbb{D}})$ for any $h \in Hol(\mathbb{D}, (X, D))$. The 'classical' version, denoted $d^*_{(X,D)}$, is defined similarly, replacing $Hol(\mathbb{D}, (X, D))$ by $Hol^*(\mathbb{D}, (X, D))$. Obviously: $d_{(X,D)} \leq d^*_{(X,D)}$.

Functoriality: let $f : (X, D) \to Z$ be a fibration with smooth orbifold base (Z, D_Z): it induces a natural map $f_* : Hol(C, (X, D)) \to Hol(C, (Z, D_Z))$ for every C[22]. Thus: $d_{(X,D)} \geq f^*(d_{(Z,D_Z)})$ (the Kobayashi

[22]Strictly speaking, this true only if f is an orbifold morphism, which is realised if the multiplicities on the f-exceptional divors is sufficiently great (or divisible in

pseudometric decreases). We have the same property for the classical notion, but applied to the classical orbifold base (Z, D^*_Z).

7.3. **Rational Curves.**

Conjecture 7.4. *$\kappa_+(X,D) = -\infty$ if and only if (X,D) is classically rationally connected.*[23]

It is easy to see that $\kappa_+(X,D) = -\infty$ if (X,D) is rationally connected (see [C11]). The other direction is considerably deeper. The only known particular case is that X is rationally connected if it is Fano. It is also known ([K-McK]) that (X,D) is uniruled (ie: X is covered by D-rational curves if (X,D) is Fano). Even in dimension 2, it is not known whether smooth integral Fano orbifolds (X,D) are covered by classical D-rational cuves (see [C10], §7 for a detailed discussion). An example where the question is open is $(\mathbb{P}^2, D)$ where D consists of 4 lines in general position with multiplicities $2, 3, 7, 41$.

7.4. **Hyperbolicity.**

Conjecture 7.5. *0. If $\kappa(X,D) = 0$, then $d^*_{(X,D)} \equiv 0$ on X.*

Moreover, (X,D) is special if and only if the following (conjecturally) equivalent properties are satisfied:

1. $d_{(X,D)} \equiv 0$ on X.

2. Any two points of X are connected by a chain of D-entire curves.

3. Any two points of X are connected by a D-entire curve.[24]

4. There exists a Zariski-dense D-entire curve on X.

In general, let $c : (X,D) \to C$ be a 'neat' model of the 'core fibration', and (C, D_C) its orbifold base. Then:

5. $d_{(X,D)} = c^*(d_{(C,D_C)})$.

6. $d_{(C,D_C)}$ is a non-degenerate metric on some Zariski-dense open subset of C.

The statement 6 in the preceding conjecture is an orbifold version of Lang's conjecture. It also extends the qualitative version of Vojta's conjecture. Notice also that conjecture 7.4 implies that $d^*_{(X,D)} \equiv 0$ if $\kappa_+(X,D) = 0$.

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the 'classical' case). We shall ignore this point here, which in general should not affect the qualitative aspect of the conjectures stated below

[23] If one does not assume that D is integral the condition $\kappa_+(X,D) = -\infty$ may be conjectured to be equivalent to (X,D) being weakly rationally connected, meaning that generic pairs of points of X are connected by rational curves G such that $(K_X + D).G < 0$.

[24] Such a curve may be seen as the analogue of a D-rational curve with ample normal bundle (relative to $T_X(D)$) in conjecture 7.4.

7.5. **Function fields.** Let $F : (X, D) \to C$ be a fibration onto a connected smooth projective curve B. Let $\mathbb{C}(B)$ be the field of meromorphic functions on B. We still write $D := \sum(1 - \frac{1}{m_j}).D_j$, with $m_j > 0$ either integers or $+\infty$ and $F(D_j) = B$ for every j.

Let $u : B' \to B$ be a finite (possibly ramified) cover. A $\mathbb{C}'$-rational point of (X, D) is an orbicurve $s : B' \to X$ such that $F \circ s' = u$. The 'classical' $\mathbb{C}'$-rational points are defined by considering classical orbicurves instead. The corresponding sets are denoted by: $(X, D)(\mathbb{C}(B')))$ (resp. $(X, D)^*(\mathbb{C}(B'))$.

We say that (X, D) is 'potentially dense' (resp. 'classically potentially dense') over $\mathbb{C}(B)$ if $(X, D)(\mathbb{C}(B'))$ (resp. if $(X, D)^*(\mathbb{C}(B'))$ has a Zariski-dense image in X for some B'.

We have the same functoriality properties for sets of rational points under fibrations as in the case of D-curves (or D^*-curves).

We say that (X, D) is 'special' (resp. of general type) over $\mathbb{C}(B)$ if so is its generic fibre (X_b, D_b).

Conjecture 7.6. *0. If $\kappa(X_b, D_b) = 0$, or if $\kappa_+(X_b, D_b) = -\infty$, then (X, D) is classically potentially dense over $\mathbb{C}(B)$.*

1. If (X, D) is 'special' over $K := \mathbb{C}(B)$, then (X, D) is potentially dense over K.

2. If (X, D) is potentially dense over K and not bimeromorphically isotrivial[25], then X is special over K.

3. If (X, D) is of general type over K, then (X, D) is not potentially dense over K.

7.6. **Arithmetic.** We consider here a smooth projective orbifold (X, D) defined over a number field k, and consider a model of (X, D) over $\mathcal{O}_{k,S}$ if S is a finite set of places of k. We define for each $x \in X(\mathcal{O}_{k,S})$, not in $Supp(D)(\mathcal{O}_{k,S})$, and each j and $v \in Spec(\mathcal{O}_{k,S})$ the arithmetic intersection number of x with D_j at v as being the largest integer $t = t_{x,D_j,v}$ such that any local equation defining D_j at x vanishes at order t modulo v.

Such an x is a D-integral point of $X(\mathcal{O}_{k,S})$ if $t_{x,D_j,v} \geq m_j$ for each j and each $x \in D_j$. It is a 'classical' integral point if $t_{x,D_j,v}$ is divisible by m_j for each j. The set of such points are denoted $(X, D)(\mathcal{O}_{k,S})$ and $(X, D)^*(\mathcal{O}_{k,S})$ respectively. We thus have:

$$(X, D)^*(\mathcal{O}_{k,S}) \subset (X, D)(\mathcal{O}_{k,S}) \subset X(\mathcal{O}_{k,S}).$$

Again, the classical and non-classical notions coincide when $D = 0$ and when $D = Supp(D)$. We have the same functoriality properties as for orbicurves above. We have the analogs of the conjectures above.

[25] We say that (X, D) is bimeromorphically isotrivial if, possibly after some finite base change, there exists a modification $\mu : (X', D') \to (X, D)$ with $\mu_*(D') = D$ and a trivialisation $(X', D') = (F, D_F) \times B$ over B.

Conjecture 7.7. *0. If $\kappa(X, D) = 0$, or if $\kappa_+(X, D) = -\infty$, $(X, D)^*(\mathcal{O}_{k',S'})$ is Zariski dense for some finite extension k'/k, with S' the inverse image of S.*

1. (X, D) is special if and only if $(X, D)(\mathcal{O}_{k',S'})$ is Zariski dense for some finite extension k'/k, with S' the inverse image of S.

2. If (X, D) is of general type, $(X, D)(\mathcal{O}_{k,S})$ is not Zariski dense.

Remark 7.8. *The following simplest example shows the difference beween the classical and non-classical versions. Let $X = \mathbb{P}^1$, and let $D := (1 - \frac{1}{p}).\{0\} + (1 - \frac{1}{q}).\{\infty\} + (1 - \frac{1}{r}).\{1\}$, for positive integers p, q, r with $\frac{1}{p} + \frac{1}{q} + \frac{1}{r} < 1$. The corresponding orbifold $(\mathbb{P}^1, D)$ is thus of general type, defined over $k = \mathbb{Q}$.*

It is easy to check that $(\mathbb{P}^1, D)^*(\mathbb{Q})$ consists of the $x = \frac{u}{v} \in \mathbb{Q}$ such that $\pm u = a^p, \pm v = b^q, a^p = b^q + c^r$ for some triple of integers a, b, c.

On the other hand, $(\mathbb{P}^1, D)(\mathbb{Q})$ consists of the $x = \frac{u}{v} \in \mathbb{Q}$ such that $\pm u$ is 'p-full', $\pm v$ is 'q-full', and $\pm u - \pm v = w$ is 'r-full', where a positive integer w is said to be 'r-full' if each prime appears in its decomposition as a product of primes with exponent either 0, or at least r.

It is now known by [DG] *that $(\mathbb{P}^1, D)^*(\mathbb{Q})$ is finite. This is shown using Falting's solution of Mordell's conjecture, and an orbifold-étale cover which preserves rational points after Chevalley-Weil's theorem.*

It is however unknown whether $(\mathbb{P}^1, D)(\mathbb{Q})$ is finite (as conjectured in 7.7). This however follows immediately from the abc-conjecture (as noticed by P. Colmez).

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