Momentum shift in cotangent bundle reduction

Andrew D. Lewis*

2000/05/16
Last updated: 2007/03/31

We explore the basic idea of how momentum shift appears in cotangent bundle reduction, and illustrate these ideas with a simple example. For a detailed discussion of these matters, see [Marsden, Montgomery, and Ratiu 1990]. The idea here is to discuss the geometry of the situation rather than the dynamics. As such, no mention is made of a Hamiltonian, although one might very well be interested in the nature of such a Hamiltonian.

The cotangent bundle reduction scenario involves a manifold Q and, of course, its cotangent bundle $\pi_{T^*Q}\colon T^*Q \to Q$. We suppose that a Lie group G acts on Q from the left via a free and proper action Φ. Some relaxing of the assumptions on the action are possible, but this threatens to get into singular reduction; see [Ortega and Ratiu 2004]. But, supposing that the G-action is free and proper, we denote by $\pi\colon Q \to B = Q/G$ the corresponding principal G-bundle. If $\mathfrak{g}$ denotes the Lie algebra of G and if $\xi \in \mathfrak{g}$, then ξ_Q is the vector field on Q which is the infinitesimal generator for the action of the one-parameter subgroup associated with ξ.

For the action on Q we also have a lifted action on T^*Q, and this lifted action is by symplectic diffeomorphisms of the canonical symplectic structure ω_0. What's more, this action admits a natural Ad^*-equivariant momentum mapping $\boldsymbol{J}\colon T^*Q \to \mathfrak{g}^*$ defined by

$$\langle \boldsymbol{J}(\alpha_q); \xi\rangle = \langle \alpha_q; \xi_Q(q)\rangle.$$

For $\mu \in \mathfrak{g}^*$, G_μ denote the isotropy group of μ under the coadjoint action of G on $\mathfrak{g}^*$. By Ad^*-equivariance of the momentum map, G_μ also leaves invariant the momentum level set $\boldsymbol{J}^{-1}(\mu)$. We denote by $\boldsymbol{J}_\mu\colon T^*Q \to \mathfrak{g}_\mu^*$ the momentum map for the action of G_μ, and observe that $\boldsymbol{J}_\mu = p_\mu \circ \boldsymbol{J}$ where $p_\mu\colon \mathfrak{g}^* \to \mathfrak{g}_\mu^*$ is the canonical projection. If $Q_\mu = Q/G_\mu$, we may use the momentum map $\boldsymbol{J}_\mu$ to obtain a description of the cotangent bundle T^*Q_μ.

1 Proposition: $T^*Q_\mu \simeq \boldsymbol{J}_\mu^{-1}(0)/G_\mu$.

Proof: First note that by Ad^*-equivariance of $\boldsymbol{J}_\mu$, G_μ leaves invariant the submanifold $\boldsymbol{J}_\mu^{-1}(0)$. We have

$$\boldsymbol{J}_\mu^{-1}(0) = \{\alpha_q \in T^*Q \mid \alpha_q(\xi_Q(q)) = 0 \text{ for all } \xi \in \mathfrak{g}_\mu\}.$$

We define a mapping χ_μ from $\boldsymbol{J}_\mu^{-1}(0)/G_\mu$ to T^*Q_μ by

$$\chi_\mu([\alpha_q]) = \{v_{[q]} \mapsto \alpha_q(v_q)\}$$

where $v_q \in TQ$ is any vector which projects to $v_{[q]} \in T_{[q]}Q_\mu$.

We first show that χ_μ is well-defined. Thus we let $u_q \in T_qQ$ be another vector projecting to $v_{[q]}$. This means that $u_q = v_q + w_q$ where w_q projects to zero. If this is the case, then

*Professor, Department of Mathematics and Statistics, Queen's University, Kingston, ON K7L 3N6, Canada
Email: andrew.lewis@queensu.ca, URL: http://www.mast.queensu.ca/~andrew/

$\alpha_q(w_q) = 0$ for $\alpha_q \in \boldsymbol{J}_\mu^{-1}(0)$. We also need to show that χ_μ is independent of representative from $[\alpha_q]$. This follows, however, since $T^*_{[q]}Q_\mu \simeq (T_qQ/T_q(G_\mu \cdot q))^*$.

To show that χ_μ is injective, let $[\alpha_{q_1}]$ and $[\beta_{q_2}]$ have the property that $\chi_\mu([\alpha_{q_1}]) = \chi_\mu([\beta_{q_2}])$. Thus for any $v_{[q]} \in T_{[q]}Q_\mu$ and for $u_{q_1} \in T_{q_1}Q$ and $v_{q_2} \in T_{q_2}Q$ which project to $v_{[q]}$, we have $\alpha_{q_1}(u_{q_1}) = \beta_{q_2}(v_{q_2})$. Since u_{v_1} and v_{q_2} both project to $v_{[q]}$ there exists $g \in G_\mu$ so that $v_{q_2} = T_q\Phi_g(u_{q_1})$. Therefore

$$\alpha_{q_1}(u_{q_1}) = \beta_{q_2}(T_q\Phi_g(u_{q_1}))$$

which implies that $\alpha_{q_1} = T_q^*\Phi_g(\beta_{q_2})$, and so $[\alpha_{q_1}] = [\beta_{q_2}]$.

Finally, we show that χ_μ is surjective. Let

$$\alpha_{[q]} \in T^*_{[q]}Q_\mu \simeq (T_qQ/T_q(G_\mu \cdot q))^* \simeq \operatorname{ann}(T_q(G_\mu \cdot q)) \subset T_q^*Q$$

and let $\alpha_q \in \operatorname{ann}(T_q(G_\mu \cdot q))$ be the image of $\alpha_{[q]}$ under these identifications. Evidently $\alpha_q \in \boldsymbol{J}_\mu^{-1}(0)$ and so take the projection of α_q to $\boldsymbol{J}_\mu^{-1}(0)/G_\mu$ to give an element of $\boldsymbol{J}_\mu^{-1}(0)/G_\mu$ mapping to $\alpha_{[q]}$ under χ_μ. ■

We also suppose that Q is equipped with a G-invariant Riemannian metric $\mathbb{G}$. Associated with such a metric is its ***mechanical connection***. This is the principal connection on the bundle π which is the $\mathbb{G}$-orthogonal complement to the vertical bundle $VQ = \ker(T\pi)$. The connection one-form for this connection we denote by $\alpha\colon TQ \to \mathfrak{g}$. For $\mu \in \mathfrak{g}^*$ we define a one-form α_μ on Q by $\alpha_\mu(v_q) = \langle \mu; \alpha(v_q)\rangle$.

2 Lemma: *The one-form α_μ is G_μ-invariant and $\boldsymbol{J}^{-1}(\mu)$-valued.*

Proof: Invariance of α_μ follows from equivariance of the connection one-form α. To show that α_μ is $\boldsymbol{J}^{-1}(\mu)$-valued, let $v_q \in TQ$. Since $\alpha(v_q)$ is defined to be that element $\xi \in \mathfrak{g}$ with the property that $\xi_Q(q)$ is the vertical part of v_q, we have

$$\alpha_\mu(v_q) = \mu(\xi).$$

By the definition of the momentum map, this is exactly the condition that $\alpha_\mu \in \boldsymbol{J}^{-1}(\mu)$. ■

Now let us recall without much backdrop the basic symplectic reduction theorem of Marsden and Weinstein [1974].

3 Theorem: *Let (P, ω) be a symplectic manifold with G a Lie group acting on P symplectically from the left, and admitting an Ad^*-equivariant momentum mapping $\boldsymbol{J}\colon P \to \mathfrak{g}^*$. Fix a regular value $\mu \in \mathfrak{g}^*$ for $\boldsymbol{J}$ and let G_μ denote its isotropy group. If G_μ acts freely and properly on $\boldsymbol{J}^{-1}(\mu)$ then the manifold $\boldsymbol{J}^{-1}(\mu)/G_\mu$ admits a natural symplectic form ω_μ which satisfies $p_\mu^*\omega_\mu = i_\mu^*\omega$ where $p_\mu\colon \boldsymbol{J}^{-1}(\mu) \to \boldsymbol{J}^{-1}(\mu)/G_\mu$ is the canonical projection and $i_\mu\colon \boldsymbol{J}^{-1}(\mu) \to P$ is the inclusion.*

The proof of this theorem is actually not difficult, but we refer to the reference.

When P is a cotangent bundle and G acts by cotangent lifts, then the symplectic reduction theorem takes on more structure which can be described in terms of the technology introduced above. The basic result is this.

4 Theorem: *Suppose that G acts freely and properly on Q by isometries of $\mathbb{G}$. Let μ be a regular value of the momentum map $\boldsymbol{J}$. The two-form $\Omega_\mu = \omega_0 + \pi^*_{T^*Q}\boldsymbol{d}\alpha_\mu$ is a G_μ-invariant symplectic form on T^*Q which descends to a symplectic form $\tilde{\Omega}_\mu$ on T^*Q_μ. Furthermore, there exists a canonical symplectic embedding of the reduced symplectic manifold P_μ as a subbundle of T^*Q_μ with the symplectic form $\tilde{\Omega}_\mu$.*

Proof: That Ω_μ is symplectic is easily checked in coordinates. That Ω_μ descends to a symplectic form on T^*Q_μ follows since

1. G_μ-invariance of α_μ implies that there exists a one-form $\tilde{\alpha}_\mu$ on Q_μ with the property that $\pi^*_\mu \tilde{\alpha_\mu} = \alpha_\mu$ and
2. the symplectic form ω_0 has the property that its restriction to $\boldsymbol{J}_\mu^{-1}(0)$ is equal to the pull-back of the canonical symplectic form on T^*Q_μ via the projection $\boldsymbol{J}_\mu^{-1}(0) \to \boldsymbol{J}_\mu^{-1}(0)/G_\mu$,

where we have used our description of T^*Q_μ given by Proposition 1.

Define a map $\psi_\mu \colon T^*Q \to T^*Q$ by $\psi_\mu(\alpha_q) = \alpha_q - \alpha_\mu(q)$. We claim that

1. $\psi_\mu | \boldsymbol{J}^{-1}(\mu) \subset \boldsymbol{J}_\mu^{-1}(0)$ and
2. ψ_μ is a symplectic mapping from (T^*Q, ω_0) to (T^*Q, Ω_μ).

Part 1 follows since if $\alpha_q \in \boldsymbol{J}^{-1}(\mu)$ then $\alpha_q(\xi_Q(q)) = \mu(\xi)$ for every $\xi \in \mathfrak{g}$. Therefore, if $\xi \in \mathfrak{g}_\mu$,

$$\begin{aligned} \langle \psi_\mu(\alpha_q); \xi_Q(q) \rangle &= \alpha_q(\xi_Q(q)) - \alpha_\mu(\xi_Q(q)) \\ &= \mu(\xi) - \mu(\xi) = 0 \end{aligned}$$

since α_μ is $\boldsymbol{J}^{-1}(\mu)$-valued by Lemma 2. Part 2 is determined readily from a coordinate computation. Since α_μ is G_μ-invariant, the map ψ_μ is G_μ-equivariant, and so descends to a map ϕ_μ on the quotient $\boldsymbol{J}^{-1}(\mu)/G_\mu = P_\mu$ to the quotient $\boldsymbol{J}_\mu^{-1}(0)/G_\mu \simeq T^*Q_\mu$. That ϕ_μ is symplectic follows from our claim 2 above. ■

Let us look now at an elementary example.

5 Example: The first example takes $Q = \mathbb{R}^2 \setminus \{0\}$ and $G = SO(2)$ acting via rotations. This action is clearly free and proper. The natural momentum map is $\boldsymbol{J}(r, \theta, p_r, p_\theta) = p_\theta$, using the canonical isomorphism $\mathfrak{so}(2)^* \simeq \mathbb{R}$. The level set $\boldsymbol{J}^{-1}(\mu)$ is given by

$$\boldsymbol{J}^{-1}(\mu) = \{(r, \theta, p_r, p_\theta) \mid \; p_\theta = \mu\},$$

and so is an affine subbundle of T^*Q, and in particular a subbundle when $\mu = 0$. Since G is Abelian, the coadjoint action is trivial and so for any $\mu \in \mathfrak{so}(2)^*$, $G_\mu = G$. One can then choose coordinates (r, p_r) for $P_\mu = \boldsymbol{J}^{-1}(\mu)$, and the projection p_μ is then defined by

$$p_\mu(r, \theta, p_r, \mu) = (r, p_r).$$

The symplectic form ω_μ on P_μ satisfies $p^*_\mu \omega_\mu = i^*_\mu \omega_0$. If we use coordinates (r, θ, p_r) for $\boldsymbol{J}^{-1}(\mu)$ we compute

$$i^*_\mu \omega_0 = \mathrm{d}r \wedge \mathrm{d}p_r,$$

and so we conclude that $\omega_\mu = \mathrm{d}r \wedge \mathrm{d}p_r$.

Let's see how this sits with Theorem 4. As a G-invariant Riemannian metric we take the Euclidean metric which in polar coordinates is

$$\mathbb{G} = \mathrm{d}r \otimes \mathrm{d}r + r^2 \mathrm{d}\theta \otimes \mathrm{d}\theta.$$

Clearly the action of G is by isometries. The vertical subbundle here is $VQ = \mathrm{span}(\frac{\partial}{\partial\theta})$, and the $\mathbb{G}$-orthogonal complement is just $HQ = \mathrm{span}(\frac{\partial}{\partial r})$. The connection one-form is then $\alpha(r, \theta, v_r, v_\theta) = v_\theta$, using the canonical isomorphism of $\mathfrak{so}(2) \simeq \mathbb{R}$. For $\mu \in \mathfrak{so}(2)^*$ we then have $\alpha_\mu = \mu \mathrm{d}\theta$. As α_μ is closed we have $\Omega_\mu = \omega_0$. Since $\boldsymbol{J}_\mu = \boldsymbol{J}$ we have

$$\boldsymbol{J}_\mu^{-1}(0) = \{(r, \theta, p_r, p_\theta) \mid\ p_\theta = 0\},$$

so that we may use coordinates (r, θ, p_r) for $\boldsymbol{J}_\mu^{-1}(0)$ and coordinates (r, p_r) for $T^*Q_\mu = \boldsymbol{J}_\mu^{-1}(0)/G_\mu$. Guided by the proof of Theorem 4 we define a symplectic diffeomorphism ψ_μ from (T^*Q, ω_0) to (T^*Q, Ω_μ) by

$$\psi_\mu(r, \theta, p_r, p_\theta) = (r, \theta, p_r, p_\theta - \mu),$$

and note that ψ_μ is an embedding—in fact a diffeomorphism—from $\boldsymbol{J}^{-1}(\mu)$ to $\boldsymbol{J}_\mu^{-1}(0)$. By equivariance this map drops to a map ϕ_μ from P_μ to T^*Q_μ, and in our given coordinates this map is defined as

$$\phi_\mu(r, p_r) = (r, p_r).$$

Thus in this case, nothing really happens since α_μ is closed. In fact, with this example, *any* choice of G-invariant Riemannian metric will lead to one-forms α_μ which are closed. The reason for this is that any principal connection HQ will be 1-dimensional and so have zero curvature. The exterior derivative of α_μ has a relationship with the curvature form of such a nature that the former is zero when the latter is zero. •

References

Marsden, J. E., Montgomery, R., and Ratiu, T. S. [1990] *Reduction, symmetry and phases in mechanics*, Memoirs of the American Mathematical Society, **88**(436), ISSN: 0065-9266.

Marsden, J. E. and Weinstein, A. [1974] *Reduction of symplectic manifolds with symmetry*, Reports on Mathematical Physics, **5**(1), pages 121–130, ISSN: 0034-4877, DOI: 10.1016/0034-4877(74)90021-4.

Ortega, J.-P. and Ratiu, T. S. [2004] *Momentum Maps and Hamiltonian Reduction*, number 222 in Progress in Mathematics, Birkhäuser: Boston/Basel/Stuttgart, ISBN: 978-0-8176-4307-2.