The bundle of infinite jets

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Abstract

The set of infinite jets is defined and given the structure of a Fréchet manifold.

1. Introduction

It is convenient in the formal theory of partial differential equations to be able to talk about infinite jets, since these can be used to represent Taylor series of various objects. Not surprisingly, the set of infinite jets is an infinite-dimensional manifold. The most natural model space, as it turns out, is not a Banach space, but a Fréchet space. Therefore, we begin our discussion with a quick review of Fréchet spaces. This is followed by a definition of the manifold of infinite jets of a fibred manifold.

A good account of the differentiable structure of infinite jets is given in [Saunders 1989, Chapter 7]. We therefore refrain from proving all of the facts we state here since the reader can refer to Saunders. There are also many interesting things about the bundle of infinite jets that we do not mention, and again [Saunders 1989] is a reference for some of these.

2. Fréchet spaces

While Banach spaces form the starting point for infinite-dimensional analysis, often the norm structure of a Banach space is insufficient to describe a desired topology on a vector space. The notion of a Fréchet space is a relatively simple extension of that of a Banach space. We refer to [Rudin 1991] for more details on functional analysis, and for proofs of the theorems we state here.

2.1. Seminorms, multinorms, and Fréchet spaces. The starting point is the notion of a seminorm.

1 Definition: (Seminorm) A ***seminorm*** of a $\mathbb{R}$-vector space V is a map $\lambda\colon \mathsf{V} \to \mathbb{R}_{\geq 0}$ with the following properties:

(i) $\lambda(av) = |a|\,\lambda(v)$ for $a \in \mathbb{R}$ and $v \in \mathsf{V}$ (***homogeneity***);

(ii) $\lambda(v_1 + v_2) \leq \lambda(v_1) + \lambda(v_2)$ (***triangle inequality***).

A ***seminormed vector space*** is a pair (V, λ) where λ is a seminorm on V. •

The only property of a norm missing is the requirement that only the zero vector have zero norm; for a seminorm nonzero vectors may have zero norm.

Rather than a normed vector space where one norm does the job of defining a topology, we consider a vector space with a collection of seminorms.

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2 DEFINITION: (Multinormed space) A ***multinorm*** on a $\mathbb{R}$-vector space V is a family $\{\lambda_a\}_{a\in A}$ of seminorms on V with the property that, if $\lambda_a(v) = 0$ for each $a \in A$, then $v = 0_{\mathsf{V}}$. A ***multinormed vector space*** is a pair $(\mathsf{V}, \{\lambda_a\}_{a\in A})$ where $\{\lambda_a\}_{a\in A}$ is a multinorm on V. •

A multinormed vector space comes with a natural topology which generalises the norm topology of a normed vector space. Recall that a ***basis*** for a topological space $(X, \mathscr{O})$ is a collection $\mathscr{B} \subset \mathscr{O}$ of open sets such that if $O \in \mathscr{O}$ then $O = \cup_{i\in I} B_i$ for $B_i \in \mathscr{B}$, $i \in I$. Also, a ***subbasis*** for $(X, \mathscr{O})$ is a collection $\mathscr{S} \subset \mathscr{O}$ of open sets such that the collection

$$\{ S_1 \cap \cdots \cap S_k \mid \ S_1, \dots, S_k \in \mathscr{S}\}$$

of finite intersections of sets from $\mathscr{S}$ forms a basis for $(X, \mathscr{O})$. For a multinormed vector space $(\mathsf{V}, \{\lambda_a\}_{a\in A})$ define

$$B_a(v, r) = \{ v' \in \mathsf{V} \mid \ \lambda_a(v' - v) < r\}.$$

for $a \in A$, $v \in \mathsf{V}$, and $r \in \mathbb{R}_{>0}$.

3 DEFINITION: (Multinorm topology) Let $(\mathsf{V}, \{\lambda_a\}_{a\in A})$ be a multinormed vector space. The ***multinorm topology*** is the finest topology which has $\{ B_a(v, r) \mid \ v \in \mathsf{V},\ r > 0,\ a \in A\}$ as a subbasis. •

4 DEFINITION: (Complete multinormed vector space) Let $(\mathsf{V}, \{\lambda_a\}_{a\in A})$ be a multinormed vector space. A sequence $\{v_j\}_{j\in\mathbb{N}}$ is a ***Cauchy sequence*** if, for every neighbourhood $\mathcal{U}$ of 0_{V} (neighbourhoods being understood to be with respect to the multinorm topology), there exists $N \in \mathbb{N}$ such that $v_j - v_k \in U$. A multinormed vector space $(\mathsf{V}, \{\lambda_a\}_{a\in A})$ is ***complete*** if every Cauchy sequence converges. •

Finally we may say what we mean by a Fréchet space.

5 DEFINITION: (Fréchet space) A ***Fréchet space*** is a complete multinormed vector space $(\mathsf{V}, \{\lambda_a\}_{a\in A})$ for which the set A is countable. •

The following result explains the importance of countability of the collection of seminorms for a Fréchet space.

6 THEOREM: (Fréchet spaces are metrisable) *If* $(\mathsf{V}, \{\lambda_j\}_{j\in\mathbb{N}})$ *is a multinormed vector space with countably many seminorms, then the map* $d\colon \mathsf{V} \times \mathsf{V} \to \mathbb{R}_{\geq 0}$ *defined by*

$$d(v_1, v_2) = \sum_{j=1} \frac{1}{2^j} \frac{\lambda_j(v_1 - v_2)}{1 + \lambda_j(v_1 - v_2)}$$

is a metric on V*, and the metric topology of* d *agrees with the multinorm topology.*

2.2. Fréchet spaces arising as inverse limits of Banach spaces. In this section we detail a very particular sort of Fréchet space; it is this construction which we shall encounter below in our definition of the differentiable structure on the set of infinite jets.

It is convenient for the first part of the discussion to have at hand the general notion of a topological vector space.

7 Definition: (Topological vector space) A $\mathbb{R}$-topological vector space is a pair $(\mathsf{V}, \mathscr{O})$ with V a $\mathbb{R}$-vector space and $\mathscr{O}$ a topology on V with the property that the maps

$$\mathbb{R} \times \mathsf{V} \ni (a, v) \mapsto av \in \mathsf{V}$$
$$\mathsf{V} \times \mathsf{V} \ni (v_1, v_2) \mapsto v_1 + v_2 \in \mathsf{V}$$

are continuous. •

The key idea is that of an inverse limit of topological vector spaces.

8 Definition: Let $\{\mathsf{V}_j\}_{j\in\mathbb{N}}$ be a sequence of topological vector spaces with a collection $f_{j+1,j}\colon \mathsf{V}_{j+1} \to \mathsf{V}_j$, $j \in \mathbb{N}$, of continuous linear maps. An ***inverse limit*** for this sequence is

(i) a topological vector space V_∞ and

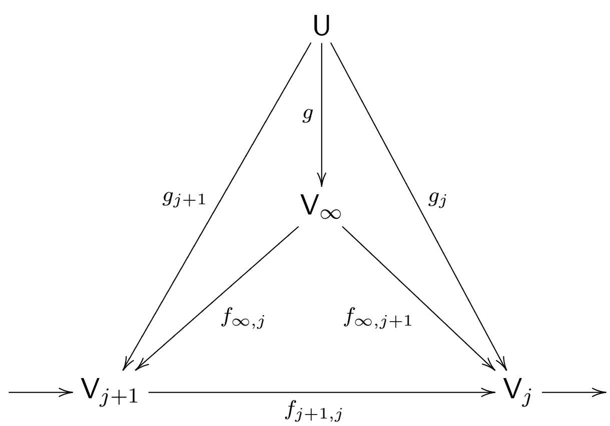
(ii) a sequence of continuous linear maps $\{f_{\infty,j}\colon \mathsf{V}_\infty \to \mathsf{V}_j\}_{j\in\mathbb{N}}$

with the properties that

(iii) $f_{\infty,j} = f_{j+1,j} \circ f_{\infty,j+1}$ for each $j \in \mathbb{N}$ and

(iv) if U is a topological vector space with $g_j\colon \mathsf{U} \to \mathsf{V}_j$, $j \in \mathbb{N}$, continuous maps satisfying $g_j = f_{j+1,j} \circ g_{j+1}$, then there exists a unique continuous linear map $g\colon \mathsf{U} \to \mathsf{V}_\infty$ satisfying $g_j = f_{\infty,j} \circ g$ for each $j \in \mathbb{N}$. •

The following diagram illustrates the relationships between the maps in the definition:



It turns out that the inverse limit V_∞, if it exists, is unique.

Now let us consider a specific instance of an inverse limit of Banach spaces.

9 Example: (Inverse limit of Banach spaces) Let $\{\mathsf{V}_j, \|\cdot\|_j\}_{j\in\mathbb{N}}$ be a sequence of Banach spaces such that we have a sequence $\pi_{j+1,j}\colon \mathsf{V}_{j+1} \to \mathsf{V}_j$ of continuous epimorphisms. In this case the inverse limit of the sequence $\{\mathsf{V}_j\}_{j\in\mathbb{N}}$ exists and can moreover be explicitly described. We take V_∞ to be the set of maps $f\colon \mathbb{N} \to \cup_{j\in\mathbb{N}}\mathsf{V}_j$ such that

1. $f(j) \in \mathsf{V}_j$ and
2. $\pi_{j+1,j}(f(j+1)) = f(j)$.

There is then a natural projection $\pi_{\infty,j}\colon \mathsf{V}_\infty \to \mathsf{V}_j$ defined by $\pi_{\infty,j}(f) = f(j)$. The topology on V_∞ can be described as follows. On V_∞ define a family $\{\lambda_j\}_{j\in\mathbb{N}}$ of seminorms by $\lambda_j(f) = \|\pi_{\infty,j}(f)\|_j$. This is easily seen to be a multinorm, and can moreover be check to be a complete multinorm since the norms $\|\cdot\|_j$, $j \in \mathbb{N}$, are complete. This gives V_∞ the structure of a Fréchet space. It is now just tedium to check that V_∞ equipped with the Fréchet space topology is an inverse limit of $\{\mathsf{V}_j\}_{j\in\mathbb{N}}$. •

2.3. Differentiability of maps between Fréchet spaces. Since we wish to use Fréchet spaces as model spaces for manifolds, we need to be sure we understand how smoothness of maps between Fréchet spaces is defined and characterised. In differential geometry, as one makes the transition from finite to infinite dimensions, the simplest extension is to Banach manifolds, where most of the important concepts from finite dimensions carry over. However, for manifolds modelled on Fréchet spaces, one needs to exercise some caution, since, for example, one loses the Inverse Function Theorem in its most general form. However, for our discussion here, many of these difficulties will not materialise.

We begin with the definition of what it means for a map between Fréchet spaces to be differentiable of arbitrary order.

10 Definition: (Differentiability of maps between Fréchet spaces) Let $(\mathsf{U}, \{\mu_j\}_{j\in\mathbb{N}})$ and $(\mathsf{V}, \{\lambda_j\}_{j\in\mathbb{N}})$ be Fréchet spaces, let $\mathcal{U} \subset \mathsf{U}$ be an open set, and let $f\colon \mathcal{U} \to \mathsf{V}$.

(i) The map f is of ***class C^1*** if, for each $x \in \mathcal{U}$ and $u \in \mathsf{U}$, the limit

$$\boldsymbol{D}f(x;u) \triangleq \lim_{t\to 0} \tfrac{1}{t}(f(x+tu) - f(x))$$

exists, and if the resulting map $\mathcal{U} \times \mathsf{U} \ni (x,u) \mapsto \boldsymbol{D}f(x;u) \in \mathsf{V}$ is continuous.

We define higher-order differentiability inductively. We thus suppose that the map f being of class C^r implies the existence of a map

$$\mathcal{U} \oplus \Big(\prod_{j=1}^{r} \mathsf{U}\Big) \ni (x, u_1, \dots, u_r) \mapsto \boldsymbol{D}^r f(x; u_1, \dots, u_r) \in \mathsf{V}$$

which is continuous.

(ii) The map f is of ***class C^{r+1}*** if it is class C^r and if the map $(x, u_1, \dots, u_r) \mapsto \boldsymbol{D}^r f(x; u_1, \dots, u_r)$ is of class C^1. We then define

$$\boldsymbol{D}^{r+1} f(x; u_0, u_1, \dots, u_r) = \lim_{t\to 0} \tfrac{1}{t}(\boldsymbol{D}^r f(x + tu_0; u_1, \dots, u_r) - \boldsymbol{D}^r f(x; u_1, \dots, u_r)).$$

(iii) The map f is of ***class C^∞***, or is ***infinitely differentiable***, if it is of class C^r for each $r \in \mathbb{N}$. •

Here is perhaps not the place to give a complete exposition of differential calculus on Fréchet spaces. However, it is certainly worth pointing out the most significant difference between the Fréchet differential calculus and the Banach differential calculus. In Banach differential calculus, the set $L(\mathsf{U};\mathsf{V})$ of continuous linear maps between Banach spaces is itself a Banach space. However, the set of continuous linear maps between Fréchet spaces may not be a Fréchet space itself. Therefore, there are some problems to be encountered in requiring that the map $x \mapsto \boldsymbol{D}f(x)$ be continuous. Thus in the Fréchet definition of derivative we do not ask that this map be continuous. It is here where problems with the Inverse Function Theorem arise. Indeed, it was the management of these technical issues that led to the Nash–Moser Inverse Function Theorem, which Nash used to prove his famous embedding theorem for Riemannian manifolds; [Nash 1956]. Moser [1966] generalised Nash's ideas; an excellent account of the resulting general theory can be found in the paper of Hamilton [1982].

But onto more mundane matters. We wish to characterise differentiable maps between Fréchet spaces in the particular case when the Fréchet space structure is the inverse limit of a sequence of Banach spaces as in Example 9.

11 PROPOSITION: (Differentiability for inverse limits) *Let* $\{\mathsf{V}_j, \|\cdot\|_j\}_{j\in\mathbb{N}}$ *be a sequence of Banach spaces with continuous epimorphisms* $\pi_{j+1,j}\colon \mathsf{V}_{j+1} \to \mathsf{V}_j$, $j \in \mathbb{N}$. *Let* V_∞ *be the inverse limit of* $\{\mathsf{V}_j\}_{j\in\mathbb{N}}$ *as defined in Example 9. If* $\mathcal{U} \subset \mathsf{V}_\infty$ *is an open set then* $f\colon \mathcal{U} \to \mathsf{V}_\infty$ *is of class* C^1 *if and only if the maps* $\pi_{\infty,j} \circ f$, $j \in \mathbb{N}$, *are of class* C^1.

3. Spaces of infinite jets

The set of infinite sets is often defined as the projective or inverse limit of the system formed by the collection of finite jets. This characterisation of infinite jets is not of much value unless it is accompanied by a description of what the infinite jets are as a set. Thus we bypass the limit characterisation and simply define the infinite jets directly.

3.1. The differentiable structure of the bundle of infinite jets. First let us define what we mean by an infinite jet. The reader will wish to understand this definition in the context of Example 9 perhaps.

12 DEFINITION: (Bundle of infinite jets) Let $\pi\colon \mathsf{Y} \to \mathsf{X}$ be a fibred manifold and let $x \in \mathsf{X}$. An ***infinite jet*** at x is a map $p_\infty\colon \mathbb{N}_0 \to \cup_{k\in\mathbb{N}_0} \mathsf{J}_k\pi_x$ with the following properties:

(i) $p_\infty(k) \in \mathsf{J}_k\pi_x$ for each $k \in \mathbb{N}_0$;

(ii) $\pi^k_l(p_\infty(k)) = p_\infty(l)$ for all $k, l \in \mathbb{N}_0$ such that $l \leq k$.

The set of infinite sets at x we denote by $\mathsf{J}_\infty\pi_x$ and we denote $\mathsf{J}_\infty\pi = \cup_{x\in\mathsf{X}}\mathsf{J}_\infty\pi_x$. The latter set we call the ***bundle of infinite jets*** of $\pi\colon \mathsf{Y} \to \mathsf{X}$. •

We define projections $\pi_\infty\colon \mathsf{J}_\infty\pi \to \mathsf{X}$ and $\pi^\infty_k\colon \mathsf{J}_\infty\pi \to \mathsf{J}_k\pi$ by asking that $\pi_\infty(p_\infty) = x$ if $p_\infty \in \mathsf{J}_\infty\pi_x$ and that $\pi^\infty_k(p_\infty) = p_\infty(k)$ for $k \in \mathbb{N}_0$.

Let us now give $\mathsf{J}_\infty\pi$ a differentiable structure. As we shall see, the natural model vector space for charts for $\mathsf{J}_\infty\pi$ is $\mathbb{R}^n \oplus (\prod_{k\in\mathbb{N}_0} L^k_{\mathrm{sym}}(\mathbb{R}^n;\mathbb{R}^m;))$ where $n = \dim(\mathsf{X})$, $n+m = \dim(\mathsf{Y})$, and where $L^k_{\mathrm{sym}}(\mathbb{R}^n;\mathbb{R}^m)$ denotes the symmetric k-multilinear maps from $\prod_{j=1}^k \mathbb{R}^n$ to $\mathbb{R}^m$. We need to equip this infinite-dimensional $\mathbb{R}$-vector with a topology, and we shall do this by providing $\prod_{k\in\mathbb{N}_0} L^k_{\mathrm{sym}}(\mathbb{R}^n;\mathbb{R}^m;)$ with the structure of a Fréchet space. We first note that for fixed $k \in \mathbb{N}_0$ we have a norm on $L^k_{\mathrm{sym}}(\mathbb{R}^n;\mathbb{R}^m)$ defined by

$$\|\boldsymbol{A}\|'_k = \sup\{\, \|\boldsymbol{A}(\boldsymbol{v},\dots,\boldsymbol{v})\| \mid \ \boldsymbol{v} \in \mathbb{R}^n,\ \|\boldsymbol{v}\| = 1\},$$

where $\|\cdot\|$ denotes the Euclidean norm. We next put a norm on $\prod_{j=0}^k L^j_{\mathrm{sym}}(\mathbb{R}^n;\mathbb{R}^m)$ by

$$\|\boldsymbol{A}_0 + \dots + \boldsymbol{A}_k\|_k = \max\{\|\boldsymbol{A}_0\|'_0, \dots, \|\boldsymbol{A}_k\|'_k\}.$$

Now define a projection $\Pi_k\colon \prod_{k\in\mathbb{N}_0} L^k_{\mathrm{sym}}(\mathbb{R}^n;\mathbb{R}^m) \to \prod_{j=0}^k L^j_{\mathrm{sym}}(\mathbb{R}^n;\mathbb{R}^m)$ by

$$\Pi_k(\boldsymbol{A}) = (\boldsymbol{A}(0),\dots,\boldsymbol{A}(k))$$

(here we recall that an element of $\prod_{k\in\mathbb{N}_0} L^k_{\mathrm{sym}}(\mathbb{R}^n;\mathbb{R}^m)$ is, by definition, a map $\boldsymbol{A}$ from $\mathbb{N}_0$ to $\cup_{k\in\mathbb{N}_0} L^k_{\mathrm{sym}}(\mathbb{R}^n;\mathbb{R}^m)$ with the property that $\boldsymbol{A}(k) \in L^k_{\mathrm{sym}}(\mathbb{R}^n;\mathbb{R}^m)$). We now define a countable family of seminorms, i.e., a countable multinorm, $\{\lambda_k\}_{k\in\mathbb{N}_0}$ on $\prod_{k\in\mathbb{N}_0} L^k_{\mathrm{sym}}(\mathbb{R}^n;\mathbb{R}^m)$ by

$$\lambda_k(\boldsymbol{A}) = \|\Pi_k(\boldsymbol{A})\|_k\,.$$

This indeed defines a multinorm since if $\boldsymbol{A} \neq 0$ then $\Pi_k(\boldsymbol{A}) \neq 0$ for some $k \in \mathbb{N}_0$, and so it follows that $\lambda_k(\boldsymbol{A}) \neq 0$. This defines a Fréchet space structure on $\prod_{k\in\mathbb{N}_0} L^k_{\mathrm{sym}}(\mathbb{R}^n;\mathbb{R}^m)$.

We observe that this Fréchet space structure is of the form given in Example 9. Indeed, if we define $\mathsf{V}_k = \prod_{j=0}^k L^j_{\mathrm{sym}}(\mathbb{R}^n;\mathbb{R}^m)$ then we have a sequence $\{\mathsf{V}_k, \|\cdot\|_k\}_{k\in\mathbb{N}}$ of Banach spaces and we also have natural epimorphisms from V_{k+1} to V_k for $k \in \mathbb{N}_0$. It is then easy to see that $\prod_{k\in\mathbb{N}_0} L^k_{\mathrm{sym}}(\mathbb{R}^n;\mathbb{R}^m)$, equipped with the Fréchet space structure above, is the inverse limit of this sequence.

We then have an induced Fréchet space structure on $\mathbb{R}^n \oplus \big(\prod_{k\in\mathbb{N}_0} L^k_{\mathrm{sym}}(\mathbb{R}^n;\mathbb{R}^m)\big)$ defined by adding to the family $\{\lambda_k\}_{k\in\mathbb{N}_0}$ of seminorms the seminorm λ' defined by $\lambda'(\boldsymbol{x}\oplus\boldsymbol{A}) = \|\boldsymbol{x}\|$.

With the topology of the model space defined, let us see how to construct charts for $\mathsf{J}_\infty\pi$ which take values in this model space. Let $(\mathcal{V},\psi)$ be an adapted chart for Y with $(\mathcal{U},\phi)$ the induced chart for X. Let $x \in \mathcal{U}$ and let $p_\infty \in \mathsf{J}_\infty\pi_x$. Define $\boldsymbol{x} \oplus \boldsymbol{p}_k \in \mathbb{R}^n \oplus \big(\prod_{k\in\mathbb{N}_0} L^k_{\mathrm{sym}}(\mathbb{R}^n;\mathbb{R}^m)\big)$ by asking that $(\boldsymbol{x}, \boldsymbol{p}_k(0),\dots,\boldsymbol{p}_k(k)) \in \mathbb{R}^n \oplus \big(\prod_{j=0}^k\big)$ be the local representative of $\pi^\infty_k(p_\infty)$ in the natural chart for $\mathsf{J}_k\pi$. Now we define a chart $(j_\infty\mathcal{U}, j_\infty\psi)$ by $j_\infty\mathcal{U} = \cup_{x\in\mathcal{U}}\mathsf{J}_\infty\pi_x$ and

$$j_\infty\psi(p_\infty) = \boldsymbol{x} \oplus \boldsymbol{p}_\infty \in \mathbb{R}^n \oplus \big(\textstyle\prod_{k\in\mathbb{N}_0} L^k_{\mathrm{sym}}(\mathbb{R}^n;\mathbb{R}^m)\big),$$

with $\boldsymbol{p}_\infty(k) = \boldsymbol{p}_k(k)$, using the definition above for $\boldsymbol{p}_k(k)$. Condition (ii) in Definition 12 ensures that $\Pi_k(j_\infty\psi(p_\infty)) = j_k\psi(\pi^\infty_k(p_\infty))$, and so $j_\infty\psi$ is an injection.

To show that this defines a differentiable structure on $\mathsf{J}_\infty\pi$ we need to verify the that overlap maps between charts are diffeomorphisms. This, however, can be checked with the aid of Proposition 11 using the fact that the Fréchet space structure on $\mathbb{R}^n \oplus \big(\prod_{k\in\mathbb{N}_0} L^k_{\mathrm{sym}}(\mathbb{R}^n;\mathbb{R}^m)\big)$ is the inverse limit of a sequence of Banach spaces.

One can then verify that the projections π_∞ and π^∞_k, $k \in \mathbb{N}_0$, are surjective submersions.

3.2. The relationship between infinite jets and sections. For finite-order jet bundles, one has the intuitive idea that, if ξ is a section of $\pi\colon \mathsf{Y} \to \mathsf{X}$, then $j_k\xi$ provides the kth-order approximation to ξ. Naïvely, one would then anticipate that $j_\infty\xi$ should, in some sense, agree with ξ. Of course, life is not so pleasant in general.

Of course, it is the case that given a smooth local section $(\xi,\mathcal{U})$ of $\pi\colon \mathsf{Y} \to \mathsf{X}$ one can define a local section $(j_\infty\xi,\mathcal{U})$ of $\pi_\infty\colon \mathsf{J}_\infty\pi \to \mathsf{X}$ by $j_\infty\xi(x)(k) = j_k\xi(x)$. The converse question can also be answered with the aid of the following result of Borel.

13 THEOREM: (Borel's Theorem) *If $\boldsymbol{A} \in \prod_{k\in\mathbb{N}_0} L^k_{\mathrm{sym}}(\mathbb{R}^n;\mathbb{R}^m)$ then there exists a smooth function $f\colon \mathcal{U} \to \mathbb{R}^m$ from a neighbourhood $\mathcal{U}$ of $\boldsymbol{0} \in \mathbb{R}^n$ such that $\frac{1}{k!}\boldsymbol{D}^k f(\boldsymbol{0}) = \boldsymbol{A}(k)$, i.e., $\boldsymbol{A}$ defines the Taylor series of f.*

Note that the theorem does not include any hypotheses about the convergence of the series

$$\sum_{k=0}^{\infty} \boldsymbol{A}(k)(\boldsymbol{x}, \dots, \boldsymbol{x}).$$

Indeed, this series can be expected to diverge, in general. Thus the theorem tells us that for smooth functions, the possible behaviour of their Taylor series can be as bad as is possible. However, it also gives the following result.

14 COROLLARY: (Infinite jets are jets of sections) *If* $\pi\colon \mathsf{Y} \to \mathsf{X}$ *is a fibred manifold and if* $p_\infty \in \mathsf{J}_\infty\pi_\infty$*, then there exists a local section* $(\xi, \mathcal{U})$ *such that* $p_\infty = j_\infty\xi(x)$*.*

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