# Generalised subbundles and distributions: A comprehensive review

Andrew D. Lewis[*]

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**Abstract**

Distributions, i.e., subsets of tangent bundles formed by piecing together subspaces of tangent spaces, are commonly encountered in the theory and application of differential geometry. Indeed, the theory of distributions is a fundamental part of mechanics and control theory.

The theory of distributions is presented in a systematic way, and self-contained proofs are given of some of the major results. Parts of the theory are presented in the context of generalised subbundles of vector bundles. Special emphasis is placed on understanding the rôle of sheaves and understanding the distinctions between the smooth or finitely differentiable cases and the real analytic case. The Orbit Theorem and applications, including Frobenius's Theorem and theorems on the equivalence of families of vector fields, are considered in detail. Examples illustrate the phenomenon that can occur with generalised subbundles and distributions.

**Keywords.** Generalised subbundle, sheaf theory, distribution, Orbit Theorem, Frobenius's Theorem, equivalence of families of vector fields.

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[*]Professor, Department of Mathematics and Statistics, Queen's University, Kingston, ON K7L 3N6, Canada
Email: andrew.lewis@queensu.ca, URL: http://www.mast.queensu.ca/~andrew/

## 1. Introduction

Distributions arise naturally in differential geometry (for example, in the characterisation of a Poisson manifold as being a disjoint union of its symplectic leaves) and in many applications of differential geometry, such as mechanics (for example, as arising from nonholonomic constraints in mechanics), and control theory (for example, in characterisations of orbits of families of vector fields). As such, distributions have been widely studied and much is known about them. However, as is often the case with objects that are widely used, there has arisen certain conventions for handling distributions that are viable "a lot of the time," but which, in fact, do not have a basis in the general theory. For just one example, when distributions are used, there is often an unstated assumption of the distribution and all distributions arising from it having locally constant rank. However, this assumption is not always valid, and interesting phenomenon arise when it does not hold, e.g., the general class of abnormal sub-Riemannian minimisers described by Liu and Sussmann [1994] rely on the distribution generated by certain brackets being singular. For this reason, it seems that there may be some benefit to assembling the basic theory of distributions in one place, with complete proofs of important results, and this is what we do in this paper. The objective is to present, in one place, precise and general definitions and results relating to these definitions. Many of these definitions and results are most naturally presented in the setting of generalised subbundles of general vector bundles, and not just tangent bundles. Thus a substantial part of the paper deals with this.

A principal function of the paper is that of a review paper. However, there are also some other contributions which we now outline.

One of the contributions of the paper is to explicate clearly the rôle of sheaf theory in handling distributions. This contribution arises in four ways: (1) in understanding the difficulties of dealing with real analytic global sections of vector bundles, and the manner in which such global sections arise; (2) in understanding clearly the rôle of analyticity in results which require certain modules to be finitely generated; (3) in understanding some of the algebraic properties of generalised subbundles; (4) in properly characterising some local constructions involving distributions, e.g., invariance under vector fields and flows. To carry out these objectives in full detail requires a great deal to be drawn from sheaf theory, particularly as it relates to complex differential geometry. In this paper we make these connections explicit, we believe for the first time.

Apart from this explication of sheaves in the theory of generalised subbundles and distributions, another contribution of the paper is to clarify certain "folklorish" parts of the theory, i.e., things which are known to be true, but for which it is difficult (and for the author, impossible) to put together complete proofs in the existing literature. Here are a few such bits of folklore: (1) the fact that the rank of a real analytic generalised subbundle attains its maximal rank on the complement of an analytic set (proved here as Proposition 3.7); (2) the Serre–Swan Theorem for general vector bundles (proved here as Theorem 5.3); (3) the Noetherian properties of real analytic germs (proved here as Proposition 4.6); (4) the rôle of these Noetherian properties in the local finite generation of modules (proved here as Theorem 4.10); (5) all variants of the Orbit Theorem (i.e., all combinations of fixed-time and arbitrary-time, and finitely generated and non-finitely generated versions); (6) a counterexample that shows that smooth involutive distributions need not be integrable. The proofs of many of these facts are a matter of putting together known results; nonetheless, this has not been done to the best knowledge of the author.

There are also some new results in the paper. In Section 3.5 we introduce the class of "patchy" subsheaves of the sheaf of sections of a vector bundle. In Proposition **??** and Theorem 3.25 we show that the subsheaf of sections of a generalised subbundle is patchy. We subsequently show in Corollary 4.11 that patchy real analytic subsheaves are coherent, and this provides for these subsheaves access to the machinery of the cohomology of coherent real analytic sheaves presented in Section 2.5. Other new results are presented in Section 6.4, where we consider the notions of invariance of subsheaves of vector fields under diffeomorphisms and vector fields, and discuss the relationship between invariance under a vector field and invariance under the flow of the same vector field. Here the topology of stalks of sheaves discussed in Section 2.8 features prominently, and in the smooth case we reveal the rôle of the Whitney Spectral Theorem.

What follows is an outline of the paper. Many of the properties of distributions, particularly their smoothness, are prescribed locally. This raises the immediate question as to whether these local constructions give rise to meaningful global objects. The best way to systematically handle such an approach is via the use of sheaves. Thus in Section 2 we introduce the elements of sheaf theory that we will require. Many of the elementary constructions and results concerning distributions are just as easily done with vector bundles, rather than specifically with tangent bundles. For this reason, we devote a significant part of the paper to definitions and results for vector bundles. In Section 3 we give the basic definitions and properties of generalised subbundles, i.e., assignments of a subspace of each fibre of a vector bundle. The set of sections of a generalised subbundle is a submodule of the module of sections. This simple observation, appropriately parsed in terms of sheaf theory, gives rise to some interesting algebraic structure for generalised subbundles. In Section 4 we present some of this algebraic theory. In Section 5 we study the important question of global generators of generalised subbundles. This provides an instance of the importance of sheaf theory in the study of generalised subbundles. Next in the paper we turn particularly to distributions, i.e., generalised subbundles of tangent bundles. Here the additional structure of vector fields having a flow, and all the consequences of this, play a rôle. In particular, in Section 6 we look at invariant distributions and constructions related to the Lie bracket of vector fields. An important contribution of control theory to differential geometry is the Orbit Theorem. We study this theorem in some detail in Section 7, in particular giving equal emphasis to the so-called Fixed-time Orbit Theorem, something not normally done (but see [Jurdjevic 1997, §2.4]). Related to the Orbit Theorem, but not equivalent to it, is Frobenius's Theorem, which we present in Section 8.

**Notation.** Let us establish the notation we use in the paper.

We write $A \subseteq B$ if $A$ is a subset of $B$, allowing that $A = B$. If $A$ is a strict subset of $B$ we denote this by $A \subset B$. The image of a map $f\colon A \to B$ is denoted by $\mathrm{image}(f)$. The symbol "$\triangleq$" means "is defined to be equal to."

By $\mathbb{Z}$, $\mathbb{Z}_{\geq 0}$, and $\mathbb{Z}_{>0}$ we denote the sets of integers, nonnegative integers, and positive integers, respectively. By $\mathbb{R}$ and $\mathbb{C}$ we denote the sets of real and complex numbers, respectively. By $\mathbb{R}_{>0}$, $\mathbb{R}_{\geq 0}$, $\mathbb{R}_{<0}$, and $\mathbb{R}_{\leq 0}$ we denote the sets of positive real numbers, nonnegative real numbers, negative real numbers, and nonpositive real numbers, respectively. By $\mathbb{R}^n$ we denote $n$-dimensional real Euclidean space (with $\mathbb{C}^n$ being the complex analogue) and by $\mathbb{R}^{m\times n}$ we denote the set of real $m \times n$ matrices.

If $\mathsf{V}$ is a $\mathbb{R}$-vector space and if $S \subseteq \mathsf{V}$, by $\mathrm{span}_{\mathbb{R}}(S)$, $\mathrm{conv}(S)$, and $\mathrm{aff}(S)$ we denote the linear hull, i.e., the linear span, the convex hull, and the affine hull of $S$. The kernel of a linear map $L\colon \mathsf{U} \to \mathsf{V}$ is denoted by $\ker(L)$. Similarly, if $\mathsf{M}$ is a module over a commutative ring $\mathsf{R}$ and if $S \subseteq \mathsf{M}$, then $\mathrm{span}_{\mathsf{R}}(S)$ is the module generated by $S$.

By $\|\cdot\|$ we denote the standard Euclidean norm on $\mathbb{R}^n$. By $\mathsf{B}(r, \boldsymbol{x})$ we denote the open ball of radius $r$ centred at $\boldsymbol{x} \in \mathbb{R}^n$. If $\boldsymbol{f}\colon \mathcal{U} \to \mathbb{R}^m$ is a differentiable map from an open subset $\mathcal{U} \subseteq \mathbb{R}^n$, the $r$th derivative of $\boldsymbol{f}$ at $\boldsymbol{x} \in \mathcal{U}$ is denoted by $\boldsymbol{D}^r\boldsymbol{f}(\boldsymbol{x})$. If $\mathcal{U}_j \subseteq \mathbb{R}^{n_j}$, $j \in \{1, \dots, k\}$, and if

$$\boldsymbol{f}\colon \mathcal{U}_1 \times \cdots \times \mathcal{U}_k \to \mathbb{R}^m$$

is differentiable, we denote by $\boldsymbol{D}_j\boldsymbol{f}(\boldsymbol{x}_1,\dots,\boldsymbol{x}_k)$ the $j$th partial derivative of $\boldsymbol{x}$ at $(\boldsymbol{x}_1,\dots,\boldsymbol{x}_k)\in\mathcal{U}_1\times\cdots\times\mathcal{U}_k$, i.e., the derivative at $\boldsymbol{x}_j$ of the map

$$\boldsymbol{x}'_j\mapsto\boldsymbol{f}(\boldsymbol{x}_1,\dots,\boldsymbol{x}'_j,\dots,\boldsymbol{x}_k).$$

For the most part, we follow the differential geometric notations and conventions of [Abraham, Marsden, and Ratiu 1988]. The tangent bundle of a manifold $\mathsf{M}$ is denoted by $\mathsf{TM}$ and $\mathsf{T}_x\mathsf{M}$ denotes the tangent space at $x$. The cotangent bundle is denoted by $\mathsf{T}^*\mathsf{M}$ and $\mathsf{T}^*_x\mathsf{M}$ denotes the cotangent space at $x$. If $f\colon\mathsf{M}\to\mathsf{N}$ is a differentiable map between manifolds, we denote its derivative by $Tf\colon\mathsf{TM}\to\mathsf{TN}$, with $T_xf$ denoting the restriction of this derivative to $\mathsf{T}_x\mathsf{M}$. The differential of a differentiable function $f\colon\mathsf{M}\to\mathbb{R}$ is denoted by $\boldsymbol{d}f\colon\mathsf{M}\to\mathsf{T}^*\mathsf{M}$.

By $\mathsf{E}_x$ we denote the fibre at $x$ of a vector bundle $\pi\colon\mathsf{E}\to\mathsf{M}$. If $\pi\colon\mathsf{E}\to\mathsf{M}$ is a vector bundle and if $\mathcal{U}\subseteq\mathsf{M}$ is open, then $\mathsf{E}|\mathcal{U}$ denotes the restriction of this vector bundle to $\mathcal{U}$.

We shall speak of geometric objects of class $C^r$ for $r\in\mathbb{Z}_{\geq0}\cup\{\infty,\omega\}$, with objects of class $C^\infty$ being infinitely differentiable and objects of class $C^\omega$ being real analytic. We shall often use language like, "let $\mathsf{M}$ be a manifold of class $C^\infty$ or class $C^\omega$, as required." By this we mean that the reader should ascribe the attributes of smoothness or real analyticity as needed to make sense of the ensuing statements. For $r\in\mathbb{Z}_{\geq0}\cup\{\infty,\omega\}$, we denote the set of functions of class $C^r$ on a manifold $\mathsf{M}$ by $C^r(\mathsf{M})$ and the set of sections of class $C^r$ of a smooth or real analytic (as is required) vector bundle $\pi\colon\mathsf{E}\to\mathsf{M}$ by $\Gamma^r(\mathsf{E})$.

By $\mathscr{L}_Xf$ we denote the Lie derivative of the function $f$ by the vector field $X$. By $\Phi^X_t$ we denote the flow of a vector field $X$ on $\mathsf{M}$. Thus $t\mapsto\Phi^X_t(x)$ is the integral curve of $X$ through $x$.

## 2. A wee bit of sheaf theory

Our Definition 3.1 for what we mean by a generalised subbundle with prescribed smoothness will be local. One of the obvious questions arising from this sort of construction is whether there are in fact any globally defined vector fields taking values in the generalised subbundle. Sheaf theory, such as we study in this section, is designed to answer such questions. Sheaf theory is a large and complex subject, and here we only develop in a limited way those facets of the theory that we will use. There are various relevant references here, including [Kashiwara and Schapira 1990, Tennison 1976]. The discussion in [Warner 1983, Chapter 5] is also useful and concise. The presentation of sheaf cohomology in [Ramanan 2005] is at a level appropriate for someone with a good background in differential geometry. Coherent analytic sheaves are the subject of the book of Grauert and Remmert [1984].

### 2.1. Presheaves and sheaves of sets.

Although we will be interested almost exclusively in this paper with sheaves of rings and modules, it is convenient to first define sheaves of sets. The starting point for the definition is that of presheaves.

**2.1 Definition: (Presheaf of sets)** Let $\mathsf{M}$ be a smooth manifold. A ***presheaf of sets*** over $\mathsf{M}$ is an assignment to each open set $\mathcal{U}\subseteq\mathsf{M}$ a set $F(\mathcal{U})$ and, to each pair of open sets $\mathcal{V},\mathcal{U}\subseteq\mathsf{M}$ with $\mathcal{V}\subseteq\mathcal{U}$, a map $r_{\mathcal{U},\mathcal{V}}\colon F(\mathcal{U})\to F(\mathcal{V})$ called the ***restriction map***, with these assignments having the following properties:

(i) $r_{\mathcal{U},\mathcal{U}}$ is the identity map;

(ii) if $\mathcal{W}, \mathcal{V}, \mathcal{U} \subseteq \mathsf{M}$ are open with $\mathcal{W} \subseteq \mathcal{V} \subseteq \mathcal{U}$, then $r_{\mathcal{U},\mathcal{W}} = r_{\mathcal{V},\mathcal{W}} \circ r_{\mathcal{U},\mathcal{V}}$.

We shall frequently use a single symbol, like $\mathscr{F}$, to refer to a presheaf, with the understanding that $\mathscr{F} = (F(\mathcal{U}))_{\mathcal{U}\,\mathrm{open}}$, and that the restriction maps are understood. An element $s \in F(\mathcal{U})$ is called a ***section over $\mathcal{U}$*** and an element of $F(\mathsf{M})$ is called a ***global section***. •

Presheaves can be restricted to open sets.

**2.2 Definition: (Restriction of a presheaf)** Let $\mathscr{F} = (F(\mathcal{U}))_{\mathcal{U}\,\mathrm{open}}$ be a presheaf of sets over a smooth or real analytic manifold $\mathsf{M}$. If $\mathcal{U} \subseteq \mathsf{M}$ is open, then we denote by $\mathscr{F}|\mathcal{U}$ the ***restriction*** of $\mathscr{F}$ to $\mathcal{U}$, which is the presheaf over $\mathcal{U}$ whose sections over $\mathcal{V} \subseteq \mathcal{U}$ are simply $F(\mathcal{V})$. •

Let us give the examples of presheaves that will be of interest to us here.

**2.3 Examples: (Presheaves)** Let $\mathsf{M}$ be a smooth or real analytic manifold, as is required, let $\pi\colon \mathsf{E} \to \mathsf{M}$ be a smooth or real analytic vector bundle, as is required, and let $r \in \mathbb{Z}_{\geq 0} \cup \{\infty, \omega\}$.

1. Let us denote by $\mathscr{C}^r_{\mathsf{M}}$ the presheaf over $\mathsf{M}$ for which the sections over an open subset $\mathcal{U} \subseteq \mathsf{M}$ is the set $C^r(\mathcal{U})$ of functions of class $C^r$ on $\mathcal{U}$. The restriction maps are the natural restrictions of functions. This presheaf we call the ***presheaf of functions of class $C^r$*** on $\mathsf{M}$.
2. Let us denote by $\mathscr{G}^r_{\mathsf{E}}$ the presheaf over $\mathsf{M}$ whose sections over an open subset $\mathcal{U} \subseteq \mathsf{M}$ is the set $\Gamma^r(\mathcal{U})$ of sections of $\mathsf{E}|\mathcal{U}$ of class $C^r$. The restriction maps, again, are the natural restrictions. This presheaf we call the ***presheaf of sections of $\mathsf{E}$ of class $C^r$***. •

The notion of a presheaf has built into it a global character; for example, the specification of global sections is part of the definition. The power of sheaf theory, however, is that it gives a framework for extending local constructions to global ones. (A good model to have in mind is using analytic continuation to patch together locally defined holomorphic functions to arrive at a globally defined holomorphic function.) In order to do this in a self-consistent way, one must place some conditions on the presheaves one uses. In this way we arrive at the notion of a sheaf, as in the following definition.

**2.4 Definition: (Sheaf of sets)** Let $\mathsf{M}$ be a smooth manifold and suppose that we have a presheaf $\mathscr{F} = (F(\mathcal{U}))_{\mathcal{U}\,\mathrm{open}}$ of sets with restriction maps $r_{\mathcal{U},\mathcal{V}}$ for $\mathcal{U}, \mathcal{V} \subseteq \mathsf{M}$ open and satisfying $\mathcal{V} \subseteq \mathcal{U}$.

(i) The presheaf $\mathscr{F}$ is ***separated*** when, if $\mathcal{U} \subseteq \mathsf{M}$ is open, if $(\mathcal{U}_a)_{a\in A}$ is an open covering of $\mathcal{U}$, and if $s, t \in F(\mathcal{U})$ satisfy $r_{\mathcal{U},\mathcal{U}_a}(s) = r_{\mathcal{U},\mathcal{U}_a}(t)$ for every $a \in A$, then $s = t$;

(ii) The presheaf $\mathscr{F}$ has the ***gluing property*** when, if $\mathcal{U} \subseteq \mathsf{M}$ is open, if $(\mathcal{U}_a)_{a\in A}$ is an open covering of $\mathcal{U}$, and if, for each $a \in A$, there exists $s_a \in F(\mathcal{U}_a)$ with the family $(s_a)_{a\in A}$ satisfying

$$r_{\mathcal{U}_{a_1},\mathcal{U}_{a_1}\cap\mathcal{U}_{a_2}}(s_{a_1}) = r_{\mathcal{U}_{a_2},\mathcal{U}_{a_1}\cap\mathcal{U}_{a_2}}(s_{a_2})$$

for each $a_1, a_2 \in A$, then there exists $s \in F(\mathcal{U})$ such that $s_a = r_{\mathcal{U},\mathcal{U}_a}(s)$ for each $a \in A$.

(iii) The presheaf $\mathscr{F}$ is a ***sheaf of sets*** if it is separated and has the gluing property. •

One fairly easily verifies that the presheaves $\mathscr{C}^r_{\mathsf{M}}$ and $\mathscr{G}^r_{\mathsf{E}}$ from Example 2.3 are, in fact, sheaves. Examples of presheaves failing to be sheaves arise, of course, by failing either of the conditions (i) or (ii) of the definition. Presheaves failing condition (i) do not often arise in settings such as that in this paper, and we refer the reader to the references for a discussion of this phenomenon. However, presheaves failing to satisfy the gluing property (ii) can arise, and the failure of a presheaf to satisfy this property is one that one is often forced to deal with. As an elementary example of a presheaf failing to satisfy the gluing property, let $\mathsf{M}$ be a noncompact smooth manifold and denote by $\mathscr{C}^\infty_{\mathrm{bdd}}(\mathsf{M})$ the presheaf whose sections over an open subset $\mathcal{U}$ is the set $C^\infty_{\mathrm{bdd}}(\mathcal{U})$ of bounded smooth functions on $\mathcal{U}$. Because it is possible to patch together locally defined bounded smooth functions to arrive at a globally defined unbounded function (we leave the straightforward construction of a counterexample to the reader), $\mathscr{C}^\infty_{\mathrm{bdd}}(\mathsf{M})$ is not a sheaf.

**2.2. Germs and étale spaces.** Associated to every presheaf is a topological space that captures the local behaviour of the presheaf. To construct this space, the notion of a germ is essential. We work with a presheaf $\mathscr{F}$ of sets on a manifold $\mathsf{M}$. We let $x \in \mathsf{M}$ and let $\mathscr{N}_x$ be the collection of open subsets of $\mathsf{M}$ containing $x$. We define an equivalence relation in $(F(\mathcal{U}))_{\mathcal{U}\in\mathscr{N}_x}$ by saying that $s_1 \in F(\mathcal{U}_1)$ and $s_2 \in F(\mathcal{U}_2)$ are ***equivalent*** if there exists $\mathcal{V} \in \mathscr{N}_x$ such that $\mathcal{V} \subseteq \mathcal{U}_1$, $\mathcal{V} \subseteq \mathcal{U}_2$, and $r_{\mathcal{U}_1,\mathcal{V}}(s_1) = r_{\mathcal{U}_2,\mathcal{V}}(s_2)$. The equivalence class of a section $s \in F(\mathcal{U})$ we denote by $r_{\mathcal{U},x}(s)$, by $[(s,\mathcal{U})]_x$, or simply by $[s]_x$ if we are able to forget about the neighbourhood on which $s$ is defined. With this construction, we make the following definition.

**2.5 Definition: (Stalk, germ of a section)** Let $\mathsf{M}$ be a smooth manifold and let $\mathscr{F} = (F(\mathcal{U}))_{\mathcal{U}\,\mathrm{open}}$ be a presheaf of sets over $\mathsf{M}$. For $x \in \mathsf{M}$, the ***stalk*** of $\mathscr{F}$ at $x$ is the set of equivalence classes under the equivalence relation defined above, and is denoted by $\mathscr{F}_x$. The equivalence class $r_{\mathcal{U},x}(s)$ of a section $s \in F(\mathcal{U})$ is called the ***germ*** of $s$ at $x$. •

The germs at $x$ of the presheaves $\mathscr{C}^r_{\mathsf{M}}$ and $\mathscr{G}^r_{\mathsf{E}}$ from Example 2.3 are denoted by $\mathscr{C}^r_{x,\mathsf{M}}$ and $\mathscr{G}^r_{x,\mathsf{M}}$, respectively.

With stalks at hand, we can make another useful construction associated with a presheaf.

**2.6 Definition: (Étale space of a presheaf)** Let $\mathsf{M}$ be a smooth manifold and let $\mathscr{F} = (F(\mathcal{U}))_{\mathcal{U}\,\mathrm{open}}$ be a presheaf of sets over $\mathsf{M}$. The ***étale space*** of $\mathscr{F}$ is the disjoint union of the stalks of $\mathscr{F}$:

$$\mathrm{Et}(\mathscr{F}) = \mathring{\bigcup_{x\in\mathsf{M}}} \mathscr{F}_x.$$

The ***étale topology*** on $\mathrm{Et}(\mathscr{F})$ is that topology whose basis consists of subsets of the form

$$\mathcal{B}(\mathcal{U}, s) = \{r_{\mathcal{U},x}(s) \mid x \in \mathcal{U}\}, \qquad \mathcal{U} \subseteq \mathsf{M}\,\mathrm{open},\ s \in F(\mathcal{U}).$$

By $\pi_{\mathscr{F}} \colon \mathrm{Et}(\mathscr{F}) \to \mathsf{M}$ we denote the canonical projection $\pi_{\mathscr{F}}(r_{\mathcal{U},x}(s)) = x$ which we call the ***étale projection***. •

One verifies the following properties of étale spaces, the proofs for which we refer to the referenced texts.

**2.7 Proposition: (Properties of the étale topology)** *Let $\mathsf{M}$ be a smooth manifold with $\mathscr{F} = (F(\mathcal{U}))_{\mathcal{U}\,open}$ a presheaf of sets over $\mathsf{M}$. The étale topology on $\mathrm{Et}(\mathscr{F})$ has the following properties:*

*(i) the sets $\mathcal{B}(\mathcal{U}, s)$, $\mathcal{U} \subseteq \mathsf{M}$ open, $s \in F(\mathcal{U})$, form a basis for a topology;*

*(ii) the projection $\pi_{\mathscr{F}}$ is a local homeomorphism, i.e., about every $[s]_x \in \mathrm{Et}(\mathscr{F})$ there exists a neighbourhood $\mathcal{O} \subseteq \mathrm{Et}(\mathscr{F})$ such that $\pi_{\mathscr{F}}|\mathcal{O}$ is a homeomorphism onto its image.*

The way in which one should think of the étale topology is depicted in Figure 1. The

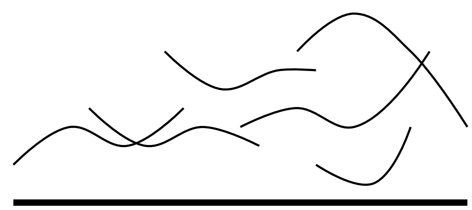

**Figure 1.** How to think of open sets in the étale topology

point is that open sets in the étale topology can be thought of as the "graphs" of local sections. It is a fun exercise to show that the étale topologies for the étale spaces of the sheaves $\mathscr{C}^r_{\mathsf{M}}$ and $\mathscr{G}^r_{\mathsf{E}}$ from Example 2.3 are Hausdorff if and only if $r = \omega$.

There is a natural notion of a local section of the étale space of a presheaf.

**2.8 Definition: (Section of the étale space of a presheaf)** Let $\mathsf{M}$ be a smooth manifold and let $\mathscr{F} = (F(\mathcal{U}))_{\mathcal{U}\,\mathrm{open}}$ be a presheaf of sets over $\mathsf{M}$. For $\mathcal{U} \subseteq \mathsf{M}$ open, a ***section*** of $\mathrm{Et}(\mathscr{F})$ over $\mathcal{U}$ is a continuous mapping $\sigma\colon \mathcal{U} \to \mathrm{Et}(\mathscr{F})$ with the property that $\pi_{\mathscr{F}} \circ \sigma = \mathrm{id}_{\mathcal{U}}$. The set of sections of $\mathrm{Et}(\mathscr{F})$ over $\mathcal{U}$ we denote by $\Gamma(\mathcal{U}; \mathrm{Et}(\mathscr{F}))$. •

Note that $(\Gamma(\mathcal{U}; \mathrm{Et}(\mathscr{F})))_{\mathcal{U}\,\mathrm{open}}$ is a presheaf if we use the natural restriction maps, i.e., the set theoretic restrictions. This presheaf can be verified to always be a sheaf. Moreover, if $\mathscr{F}$ is itself a sheaf, then there exists a natural isomorphism from $\mathscr{F}$ to the presheaf of local sections of $\mathrm{Et}(\mathscr{F})$. Explicitly, $s \in F(\mathcal{U})$ is mapped by this natural isomorphism to the local section $x \mapsto [s]_x$.

The upshot of this section is the following. A sheaf $\mathscr{F}$ is in natural correspondence with the local sections of its étale space $\mathrm{Et}(\mathscr{F})$. In particular, the attributes of a sheaf $\mathscr{F}$ are determined by the germs used in constructing its étale space. Said otherwise, a presheaf that is a sheaf is determined by its germs. For this reason, we shall adopt the usual convention and abandon the distinction between a sheaf and its étale space, and write $\mathscr{F}$ for both the presheaf and its étale space.

**2.3. Sheaves of rings and modules.** The sheaves in which we are most interested are the sheaf $\mathscr{C}^r_{\mathsf{M}}$ of functions and the sheaf $\mathscr{G}^r_{\mathsf{E}}$ of sections of a vector bundle. Just like the set $\Gamma^r(\mathsf{E})$ of sections is a module over the ring $C^r(\mathsf{M})$, the corresponding sheaves inherit some algebraic structure. Let us first give a general definition.

**2.9 Definition: (Presheaves of rings and modules)**

(i) A ***presheaf of rings*** over a smooth manifold $\mathsf{M}$ is a presheaf $\mathscr{R} = (R(\mathcal{U}))_{\mathcal{U}\,\mathrm{open}}$ whose local sections are rings and for which the restriction maps $r_{\mathcal{U},\mathcal{V}}\colon R(\mathcal{U}) \to R(\mathcal{V})$, $\mathcal{U}, \mathcal{V} \subseteq \mathsf{M}$ open, $\mathcal{V} \subseteq \mathcal{U}$, are homomorphisms of rings.

(ii) If $\mathscr{R} = (R(\mathcal{U}))_{\mathcal{U}\,\mathrm{open}}$ is a presheaf of rings over a smooth manifold $\mathsf{M}$, a ***presheaf of $\mathscr{R}$-modules*** is a presheaf $\mathscr{F} = (F(\mathcal{U}))_{\mathcal{U}\,\mathrm{open}}$ of sets such that $F(\mathcal{U})$ is a module over

$R(\mathcal{U})$ and such that the restriction maps $r^{\mathscr{R}}_{\mathcal{U},\mathcal{V}}$ and $r^{\mathscr{F}}_{\mathcal{U},\mathcal{V}}$ satisfy

$$\begin{aligned} r^{\mathscr{F}}_{\mathcal{U},\mathcal{V}}(s+t) &= r^{\mathscr{F}}_{\mathcal{U},\mathcal{V}}(s) + r^{\mathscr{F}}_{\mathcal{U},\mathcal{V}}(t), && s,t \in F(\mathcal{U}), \\ r^{\mathscr{F}}_{\mathcal{U},\mathcal{V}}(f\,s) &= r^{\mathscr{R}}_{\mathcal{U},\mathcal{V}}(f) r^{\mathscr{F}}_{\mathcal{U},\mathcal{V}}(s), && f \in R(\mathcal{U}),\ s \in F(\mathcal{U}). \end{aligned}$$ •

Of course, if $\pi\colon \mathsf{E} \to \mathsf{M}$ is a smooth or real analytic vector bundle, as required, and if $r \in \mathbb{Z}_{\geq 0} \cup \{\infty, \omega\}$, then $\mathscr{G}^r_{\mathsf{E}}$ is a sheaf of modules over the sheaf of rings $\mathscr{C}^r_{\mathsf{M}}$.

### 2.4. Morphisms and subsheaves.

Next we introduce maps between presheaves.

**2.10 Definition: (Morphism of presheaves of sets)** Let $\mathsf{M}$ be a smooth manifold and let $\mathscr{F} = (F(\mathcal{U}))_{\mathcal{U}\,\mathrm{open}}$ and $\mathscr{G} = (G(\mathcal{U}))_{\mathcal{U}\,\mathrm{open}}$ be presheaves of sets over $\mathsf{M}$. A ***morphism*** of the presheaves $\mathscr{F}$ and $\mathscr{G}$ is an assignment to each open set $\mathcal{U} \subseteq \mathsf{M}$ a map $\Phi_{\mathcal{U}}\colon F(\mathcal{U}) \to G(\mathcal{U})$ such that the diagram

$$\begin{tikzcd} F(\mathcal{U}) \arrow[r, "\Phi_{\mathcal{U}}"] \arrow[d, "r_{\mathcal{U},\mathcal{V}}"'] & G(\mathcal{U}) \arrow[d, "r_{\mathcal{U},\mathcal{V}}"] \\ F(\mathcal{V}) \arrow[r, "\Phi_{\mathcal{V}}"'] & G(\mathcal{V}) \end{tikzcd} \tag{2.1}$$

commutes for every open $\mathcal{U}, \mathcal{V} \subseteq \mathsf{M}$ with $\mathcal{V} \subseteq \mathcal{U}$. We shall often use the abbreviation $\Phi = (\Phi_{\mathcal{U}})_{\mathcal{U}\,\mathrm{open}}$. If $\mathscr{F}$ and $\mathscr{G}$ are sheaves, $\Phi$ is called a ***morphism*** of sheaves. •

Our interest is mainly in, not morphisms between presheaves of sets, but in morphisms between presheaves of $\mathscr{C}^r_{\mathsf{M}}$-modules.

**2.11 Definition: (Morphism of presheaves of modules)** Let $\mathsf{M}$ be a smooth or real analytic manifold as required, let $r \in \mathbb{Z}_{\geq 0} \cup \{\infty, \omega\}$, and let $\mathscr{F} = (F(\mathcal{U}))_{\mathcal{U}\,\mathrm{open}}$ and $\mathscr{G} = (G(\mathcal{U}))_{\mathcal{U}\,\mathrm{open}}$ be sheaves of $\mathscr{C}^r_{\mathsf{M}}$-modules. A morphism $(\Phi_{\mathcal{U}})_{\mathcal{U}\,\mathrm{open}}$ of the presheaves $\mathscr{F}$ and $\mathscr{G}$ is a ***morphism of $\mathscr{C}^r_{\mathsf{M}}$-modules*** if $\Phi_{\mathcal{U}}\colon F(\mathcal{U}) \to G(\mathcal{U})$ is a homomorphism of $C^r(\mathcal{U})$ modules for each open set $\mathcal{U} \subseteq \mathsf{M}$. •

Another form of morphism, one that maps from one manifold to another, will also be useful for us. In order to state the definition, we need some notation. We let $\mathsf{M}$ be a manifold and let $\mathscr{F} = (F(\mathcal{U}))_{\mathcal{U}\,\mathrm{open}}$ be a presheaf over $\mathsf{M}$. Let $A \subseteq \mathsf{M}$. Let $\mathcal{U}, \mathcal{V} \subseteq \mathsf{M}$ be neighbourhoods of $A$. Sections $s \in F(\mathcal{U})$ and $t \in F(\mathcal{V})$ are ***equivalent*** if there exists a neighbourhood $\mathcal{W} \subseteq \mathcal{U} \cap \mathcal{V}$ of $A$ such that $r_{\mathcal{U},\mathcal{W}}(s) = r_{\mathcal{V},\mathcal{W}}(t)$. Let $\mathscr{F}_A$ denote the set of equivalence classes under this equivalence relation. Let us denote an equivalence class by $[(s,\mathcal{U})]_A$ or by $[s]_A$ if the subset $\mathcal{U}$ is of no consequence. Restriction maps can be defined between such sets of equivalence classes as well. Thus we let $A, B \subseteq \mathsf{M}$ be subsets for which $A \subseteq B$. If $[(s,\mathcal{U})]_B \in \mathscr{F}_B$ then, since $\mathcal{U}$ is also a neighbourhood of $A$, $[(s,\mathcal{U})]_B \in \mathscr{F}_A$, and we denote by $r_{B,A}([(s,\mathcal{U})]_B)$ the equivalence class in $\mathscr{F}_A$. One can readily verify that these restriction maps are well-defined.

**2.12 Definition: (Direct image and inverse image presheaves)** Let $r \in \mathbb{Z}_{\geq 0} \cup \{\infty, \omega\}$, let $\mathsf{M}$ and $\mathsf{N}$ be smooth or real analytic manifolds, as required, let $\Phi \in C^r(\mathsf{M};\mathsf{N})$ be a $C^r$-map, and let $\mathscr{F} = (F(\mathcal{U}))_{\mathcal{U}\,\mathrm{open}}$ be a presheaf of $\mathscr{C}^r_{\mathsf{M}}$-modules and $\mathscr{G} = (G(\mathcal{V}))_{\mathcal{V}\,\mathrm{open}}$ be a presheaf of $\mathscr{C}^r_{\mathsf{N}}$-modules.

(i) The ***direct image presheaf*** of $\mathscr{F}$ by $\Phi$ is the presheaf $\Phi_*\mathscr{F}$ on $\mathsf{N}$ given by $\Phi_*\mathscr{F}(\mathcal{V}) = F(\Phi^{-1}(\mathcal{V}))$ for $\mathcal{V} \subseteq \mathsf{N}$ open. If $r_{\mathcal{U},\mathcal{V}}$ denote the restriction maps for $\mathscr{F}$, the restriction maps $\Phi_* r_{\mathcal{U},\mathcal{V}}$ for $\Phi_*\mathscr{F}$ satisfy, for $\mathcal{U},\mathcal{V} \subseteq \mathsf{N}$ open with $\mathcal{V} \subseteq \mathcal{U}$,

$$\Phi_* r_{\mathcal{U},\mathcal{V}}(s) = r_{\Phi^{-1}(\mathcal{U}),\Phi^{-1}(\mathcal{V})}(s)$$

for $s \in \Phi_*\mathscr{F}(\mathcal{U}) = F(\Phi^{-1}(\mathcal{U}))$.

(ii) The ***inverse image presheaf*** of $\mathscr{F}$ by $\Phi$ is the presheaf $\Phi^{-1}\mathscr{F}$ over $\mathsf{M}$ defined by $\Phi^{-1}\mathscr{F}(\mathcal{U}) = \mathscr{F}_{\Phi(\mathcal{U})}$. The restriction maps for $\Phi^{-1}\mathscr{F}$ are defined by $\Phi^{-1} r_{\mathcal{U},\mathcal{V}}([s]) = r_{\Phi(\mathcal{U}),\Phi(\mathcal{V})}([s])$. •

If $\mathscr{F}$ is a sheaf, one readily verifies that $\Phi_*\mathscr{F}$ is also a sheaf. From this one readily deduces that, if $\Phi$ is a diffeomorphism, then $\Phi^{-1}\mathscr{G}$ is a sheaf if $\mathscr{G}$ is a sheaf. The asymmetry in the notation for the direct and inverse image (i.e., the fact that $\Phi^{-1}$ seems like it should be $\Phi^*$) is explained by the fact that $\Phi^*$ is the notation used in a related, but not exactly identical, situation. We refer to [Taylor 2002, §7.3] for details.

We can also talk about subsheaves in a more or less obvious way.

**2.13 Definition: (Subpresheaf, subsheaf)** Let $\mathsf{M}$ be a smooth manifold, and let $\mathscr{F} = (F(\mathcal{U}))_{\mathcal{U}\text{ open}}$ and $\mathscr{G} = (G(\mathcal{U}))_{\mathcal{U}\text{ open}}$ be presheaves of sets over $\mathsf{M}$. The presheaf $\mathscr{F}$ is a ***subpresheaf*** of $\mathscr{G}$ if, for each open set $\mathcal{U} \subseteq \mathsf{M}$, $F(\mathcal{U})$ is a subset of $G(\mathcal{U})$ and if the inclusion maps $i_{\mathscr{F},\mathcal{U}} \colon F(\mathcal{U}) \to G(\mathcal{U})$, $\mathcal{U} \subseteq \mathsf{M}$ open, define a morphism $i_{\mathscr{F}} = (i_{\mathscr{F},\mathcal{U}})_{\mathcal{U}\text{ open}}$ of presheaves of sets. If $\mathscr{F}$ and $\mathscr{G}$ are sheaves, we say that $\mathscr{F}$ is a ***presheaf*** of $\mathscr{G}$. •

Of course, if one replaces "presheaf of sets" with "presheaf of $\mathscr{C}^r_{\mathsf{M}}$-modules" in the above definition one arrives at the notions of a subpresheaf and subsheaf of $\mathscr{C}^r_{\mathsf{M}}$-modules.

**2.5. Coherent real analytic sheaves.** As we have mentioned several times, one of the uses of sheaf theory is that it allows one to systematically address global existence questions. One such question is the following. Suppose that we are given a vector bundle $\pi \colon \mathsf{E} \to \mathsf{M}$. Clearly, about any $x \in \mathsf{M}$ there are many local sections. One can legitimately ask whether global sections are plentiful. If the vector bundle is smooth, then one can use constructions with cutoff functions and partitions of unity to construct a section having any "reasonable" property. (In sheaf language, this is because the sheaf of smooth sections has a property called "softness.") However, if the vector bundle is real analytic, the question is not so easy to answer. To motivate this a little further, let us recall that a holomorphic vector bundle over a compact base has few global sections; precisely, the space of global sections is finite-dimensional over $\mathbb{C}$. For example, the dimension of the $\mathbb{C}$-vector space of sections of the so-called tautological bundle over complex projective space is zero [Smith, Kahanpää, Kekäaläainen, and Traves 2000, page 133]. This immediately makes one think an analogous situation likely holds for real analytic vector bundles. However, this is not so, but the reasons for this are not trivial. In this section we outline some of the historical developments leading to a few main results that we shall make use of.

First of all, let us deal with the fact that real analytic manifolds are not exactly analogous to holomorphic manifolds. In fact, [Grauert 1958] shows that real analytic manifolds are analogous to the class of holomorphic manifolds known as Stein manifolds. Stein manifolds, unlike general holomorphic manifolds, possess many holomorphic functions. For example, about any point in a Stein manifold, one can find globally defined functions that, in a neighbourhood of the point, form the components of a holomorphic coordinate chart. Thus we have some hope that the question about the plenitude of global sections of a real analytic vector bundle has an answer unlike that for holomorphic vector bundles over a compact base.

However, there is still much work to be done. In the holomorphic case, the big result here, proved by Cartan [1951-52] and known as "Cartan's Theorem A" (there is also a "Theorem B" which we will get to in time), has as a consequence that the module of germs at $x$ of sections of a holomorphic vector bundle over a Stein base is generated by germs of global sections. In [Cartan 1957] these holomorphic results are extended to the real analytic case. Thus, as a consequence of Cartan's Theorem A in the real analytic case, the module $\mathscr{G}^{\omega}_{x,\mathsf{E}}$ is generated by germs of global sections. However, Cartan's results extend far beyond sheaves of sections of vector bundles to the setting of coherent analytic sheaves. We shall actually access these more general results, so in this section we give the definitions and sketch the results to which we shall subsequently make reference.

We begin with the notion of a locally finitely generated sheaf of modules.

**2.14 Definition: (Locally finitely generated sheaf of modules)** Let $\mathsf{M}$ be a smooth or real analytic manifold, as required, let $r \in \mathbb{Z}_{\geq 0} \cup \{\infty, \omega\}$, and let $\mathscr{F} = (F(\mathcal{U}))_{\mathcal{U}\,\text{open}}$ be a sheaf of $\mathscr{C}^r_{\mathsf{M}}$-modules. The sheaf $\mathscr{F}$ is ***locally finitely generated*** if, for each $x_0 \in \mathsf{M}$, there exists a neighbourhood $\mathcal{U}$ of $x_0$ and sections $s_1, \dots, s_k \in F(\mathcal{U})$ such that $[s_1]_x, \dots, [s_k]_x$ generate the $\mathscr{C}^r_{x,\mathsf{M}}$-module $\mathscr{F}_x$ for every $x \in \mathcal{U}$. •

Next we turn to the other property required of a coherent sheaf. We let $\mathsf{M}$ be a smooth or real analytic manifold, as required, let $r \in \mathbb{Z}_{\geq 0} \cup \{\infty, \omega\}$, and let $\mathscr{F} = (F(\mathcal{U}))_{\mathcal{U}\,\text{open}}$ be a sheaf of $\mathscr{C}^r_{\mathsf{M}}$-modules. Let $\mathcal{U} \subseteq \mathsf{M}$ be open and let $s_1, \dots, s_k \in F(\mathcal{U})$. We define a morphism $\varrho(s_1, \dots, s_k)$ of sheaves from $(\mathscr{C}^r_{\mathcal{U}})^k$ to $\mathscr{F}|\mathcal{U}$ by defining it stalkwise:

$$\varrho(s_1, \dots, s_k)_x([f^1]_x, \dots, [f^k]_x) = \sum_{j=1}^{k} [f^j]_x [s_j]_x, \qquad x \in \mathcal{U}.$$

The kernel $\ker(\varrho(s_1, \dots, s_k))$ of this morphism we call the ***sheaf of relations*** of the sections $s_1, \dots, s_k$ over $\mathcal{U}$.

With the preceding construction, we now make the following definition.

**2.15 Definition: (Coherent sheaf)** Let $\mathsf{M}$ be a smooth or real analytic manifold, as required, let $r \in \mathbb{Z}_{\geq 0} \cup \{\infty, \omega\}$, and let $\mathscr{F} = (F(\mathcal{U}))_{\mathcal{U}\,\text{open}}$ be a sheaf of $\mathscr{C}^r_{\mathsf{M}}$-modules. The sheaf $\mathscr{F}$ is ***coherent***

(i) if it is locally finitely generated and

(ii) if, for every open $\mathcal{U} \subseteq \mathsf{M}$ and $s_1, \dots, s_k \in F(\mathcal{U})$, $\ker(\varrho(s_1, \dots, s_k))$ is locally finitely generated. •

Now we can define the objects of interest to us.

**2.16 Definition: (Coherent real analytic sheaf)** Let $\mathsf{M}$ be a real analytic manifold. A ***coherent real analytic sheaf*** is a coherent sheaf $\mathscr{F}$ of $\mathscr{C}^{\omega}_{\mathsf{M}}$-modules. •

We can give an important example of a coherent real analytic sheaf.

**2.17 Theorem: (Oka's Theorem)** *If $\pi\colon \mathsf{E} \to \mathsf{M}$ is a real analytic vector bundle then $\mathscr{G}^\omega_{\mathsf{E}}$ is coherent.*

Outline of proof: Proofs of this result in the holomorphic case can be found in many texts on several complex variables, see [e.g., Hörmander 1966, Theorem 6.4.1]. The proofs are all lengthy inductive arguments based on the Weierstrass Preparation Theorem. The Weierstrass Preparation Theorem in the real analytic case is given in [Krantz and Parks 2002, Theorem 6.1.3]. With this version of the theorem, the standard holomorphic proofs of Oka's Theorem apply to the real analytic case. ■

Now we are in a position to state an important result concerning global sections of coherent real analytic sheaves.

**2.18 Theorem: (Cartan's Theorem A)** *Let $\mathsf{M}$ be a paracompact Hausdorff real analytic manifold and let $\mathscr{F}$ be a coherent real analytic sheaf. Then, for $x \in \mathsf{M}$, the $\mathscr{C}^\omega_{x,\mathsf{M}}$-module $\mathscr{F}_x$ is generated by germs of global sections of $\mathscr{F}$.*

Outline of proof: The holomorphic case, i.e., for coherent complex analytic sheaves over Stein manifolds, was first proved by Cartan [1951-52]. Proofs of this result are often found in texts on several complex variables [e.g., Hörmander 1966, Theorem 7.2.8]. The real analytic case we state here was proved by Cartan [1957] using the fact that real analytic manifolds can be, in an appropriate sense, be approximated by Stein manifolds. In [Cartan 1957] the theorems are stated for real analytic submanifolds of $\mathbb{R}^n$. However, by the real analytic embedding theorem of Grauert [1958], this assumption holds for any paracompact Hausdorff real analytic manifold. ■

We note that the sheaf $\mathscr{G}^r_{\mathsf{E}}$ of $C^r$-sections of a smooth vector bundle $\pi\colon \mathsf{E} \to \mathsf{M}$ is *not* coherent. Thus coherence is really an analytic tool, and indeed is the device one uses to compensate for the fact that one does not have analytic partitions of unity.

**2.6. Real analytic spaces.** The study of analytic spaces is fundamental in the theory of complex analytic geometry. As such, references for this theory abound, with [Taylor 2002] as an example. In the real analytic case, there are fewer references. The study of these spaces was really initiated with the work of Cartan [1957] and of Whitney and Bruhat [1959]. Monographs on results in this area are those of Narasimhan [1966] and Guaraldo, Macrí, and Tancredi [1986]. Since this theory has some important repercussions for us, we shall review it here.

We begin with the definition.

**2.19 Definition: (Real analytic space, ideal sheaf of a real analytic space)** If $\mathsf{M}$ is a real analytic manifold, a ***real analytic space*** in $\mathsf{M}$ is a subset $\mathsf{S} \subseteq \mathsf{M}$ such that, for each $x_0 \in \mathsf{S}$, there is a neighbourhood $\mathcal{U}$ of $x_0$ and $f_1, \dots, f_k \in C^\omega(\mathcal{U})$ such that

$$\mathsf{S} \cap \mathcal{U} = \{x \in \mathcal{U} \mid \ f_1(x) = \cdots = f_k(x)\}.$$

The ***ideal sheaf*** of a real analytic space $\mathsf{S}$ is the subsheaf $\mathscr{I}_{\mathsf{S}} = (I_{\mathsf{S}}(\mathcal{U}))_{\mathcal{U}\,\mathrm{open}}$ of $\mathscr{C}^\omega_{\mathsf{M}}$ defined by

$$I_{\mathsf{S}}(\mathcal{U}) = \{f \in C^\omega(\mathcal{U}) \mid \ f(x) = 0 \text{ for all } x \in \mathsf{S} \cap \mathcal{U}\}. \qquad \bullet$$

Characterisations of the ideal sheaf are important in the theory of analytic spaces. As an example of a question of interest, one wonders whether, given an analytic space, its ideal sheaf is generated by globally defined functions. One is not surprised to learn that coherence of the ideal sheaf is important for answering such questions, and indeed Cartan [1950] proves that the ideal sheaf of a complex analytic space is coherent. However, in the real analytic case, it is no longer true that ideal sheaves are coherent, as the following example of Cartan shows.

**2.20 Example: (Cartan's umbrella)** We take $\mathsf{M} = \mathbb{R}^3$ with $(x_1, x_2, x_3)$ the standard coordinates. Consider the analytic function

$$\begin{aligned} f\colon \mathbb{R}^3 &\to \mathbb{R} \\ (x_1, x_2, x_3) &\mapsto x_3(x_1^2 + x_2^2) - x_2^3. \end{aligned}$$

In Figure 2 we show the 0-level set of $f$, which is thus an analytic space that we denote

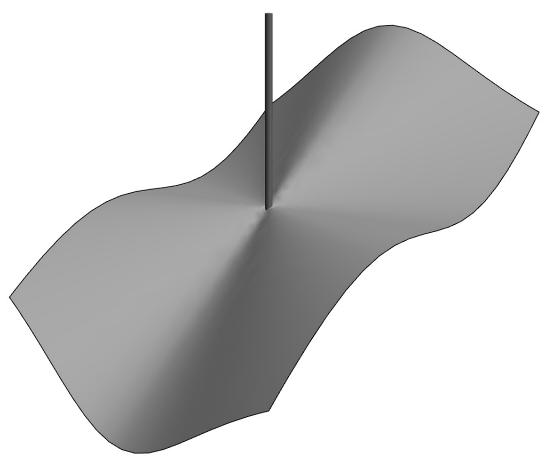

**Figure 2.** Cartan's umbrella

by $\mathsf{C}$. We claim that $\mathscr{I}_{\mathsf{C}}$ is not coherent. To show this, we first look at the stalk of $\mathscr{I}_{\mathsf{C}}$ at $\mathbf{0} = (0,0,0)$. We claim that $[f]_{\mathbf{0}}$ generates this stalk.

To see now that $\mathscr{I}_{\mathsf{C}}$ is not locally finitely generated, let $\mathcal{U}$ be a neighbourhood of $\mathbf{0}$ and let $\boldsymbol{x}_0 = (0,0,a) \in \mathcal{U}$ with $a \neq 0$. For a sufficiently small neighbourhood $\mathcal{V}$ of $\boldsymbol{x}_0$ (specifically, require that $\mathcal{V}$ be a ball not containing $\mathbf{0}$) define $g \in C^\omega(\mathcal{V})$ by $g(x_1, x_2, x_3) = x_1$, and note that $[(g,\mathcal{V})]_{\boldsymbol{x}_0} \in \mathscr{I}_{\mathsf{C},\boldsymbol{x}_0}$. We claim that $[(g,\mathcal{V})]_{\boldsymbol{x}_0}$ is not in the ideal generated by $[f]_{\boldsymbol{x}_0}$. Indeed, note that if $g = hf$ in some neighbourhood of $\boldsymbol{x}_0$, then we must have

$$h(x_1, x_2, x_3) = \frac{x_3(x_1^2 + x_2^2) - x_2^3}{x_1},$$

which can easily seen to not be real analytic in any neighbourhood of $\boldsymbol{x}_0$. Finally, by Lemma 4.17 we can now conclude that $\mathscr{I}_{\mathsf{C}}$ is not locally finitely generated.

A consequence of this is that real analytic functions on $\mathsf{C}$ cannot generally be extended to real analytic functions away from $\mathsf{C}$. Consider, for example, the function

$$g(x_1, x_2, x_3) = \frac{x_1}{x_1^2 + x_2^2 + (x_3 - 1)^2}.$$

One readily checks that $g$ be extended to a real analytic function in a neighbourhood of any point, but cannot be extended to a real analytic function on a neighbourhood of $\mathsf{C}$. •

**2.7. The beginnings of sheaf cohomology.** Sheaf cohomology is a powerful tool for dealing systematically with the problems concerning the "local to global" passage. The theory has a reputation for being difficult to learn. This is in some sense true, but is also exacerbated by many treatments of the subject which provide a purely category theory based treatment which is difficult for a beginner to penetrate. Here we sketch the beginnings of sheaf cohomology in a fairly concrete manner, and state a weak form of Cartan's Theorem B that we shall subsequently use. A readable, but not comprehensive, introduction to sheaf cohomology may be found in the book of Ramanan [2005]. A less readable (by non-experts) but comprehensive account may be found in the book of Kashiwara and Schapira [1990].

We let $r \in \mathbb{Z}_{\geq 0} \cup \{\infty, \omega\}$, let $\mathsf{M}$ be a smooth or real analytic manifold, as required, and let $\mathscr{F} = (F(\mathcal{U}))_{\mathcal{U}\,\mathrm{open}}$ be a sheaf of $\mathscr{C}^r_{\mathsf{M}}$-modules. We suppose that we are given an open cover $\mathscr{U} = (\mathcal{U}_a)_{a \in A}$ for $\mathsf{M}$, and we let $\mathrm{C}^0(\mathscr{U}; \mathscr{F})$ be the set of all sections over all open sets in $\mathscr{U}$. Thus an element of $\mathrm{C}^0(\mathscr{U}; \mathscr{F})$ is a family $(s_a)_{a \in A}$ where $s_a \in F(\mathcal{U}_a)$. Now let $\mathrm{Z}^0(\mathscr{U}; \mathscr{F})$ be the elements of $\mathrm{C}^0(\mathscr{U}; \mathscr{F})$ that agree on their intersection. Thus $(s_a)_{a \in A} \in \mathrm{Z}^0(\mathscr{U}; \mathscr{F})$ if the restrictions of $s_a$ and $s_b$ to $\mathcal{U}_a \cap \mathcal{U}_b$ agree whenever $\mathcal{U}_a \cap \mathcal{U}_b \neq \varnothing$. Since $\mathscr{F}$ is a sheaf, if $(s_a)_{a \in A} \in \mathrm{Z}^0(\mathscr{U}; \mathscr{F})$ then there exists a unique $s \in \Gamma(\mathsf{M}; \mathscr{F})$ such that the restriction of $s$ to $\mathcal{U}_a$ agrees with $s_a$ for each $a \in A$. Thus we naturally identify $\mathrm{Z}^0(\mathscr{U}; \mathscr{F})$ with $\Gamma(\mathsf{M}; \mathscr{F})$. Let us also define $\mathrm{B}^0(\mathscr{U}; \mathscr{F}) = 0$ by convention. We take $\mathrm{H}^0(\mathscr{U}; \mathscr{F}) = \mathrm{Z}^0(\mathscr{U}; \mathscr{F})/\mathrm{B}^0(\mathscr{U}; \mathscr{F})$ so that $\mathrm{H}^0(\mathscr{U}; \mathscr{F})$ is naturally identified with $\Gamma(\mathsf{M}; \mathscr{F})$. This is the ***zeroth cohomology group*** of $\mathscr{F}$ for the cover $\mathscr{U}$.

The preceding constructions are related to restricting global sections to sets from the open cover. Now we restrict further. Let $\mathrm{C}^1(\mathscr{U}; \mathscr{F})$ be the set of sections over $\mathcal{U}_a \cap \mathcal{U}_b$, $a, b \in A$. Thus an element of $\mathrm{C}^1(\mathscr{U}; \mathscr{F})$ is a family $(s_{ab})_{a,b \in A}$ such that $s_{ab} \in F(\mathcal{U}_a \cap \mathcal{U}_b)$. Given $(s_a)_{a \in A} \in \mathrm{C}^0(\mathscr{U}; \mathscr{F})$ we have an induced element $(s_{ab})_{a,b \in A} \in \mathrm{C}^1(\mathscr{U}; \mathscr{F})$ defined by $s_{ab} = s_b - s_a$ (to keep things simple, we omit the restrictions which are really required here). Let us denote by $\mathrm{B}^1(\mathscr{U}; \mathscr{F})$ the elements of $\mathrm{C}^1(\mathscr{U}; \mathscr{F})$ obtained in this way. We define $\mathrm{Z}^1(\mathscr{U}; \mathscr{F}) \subseteq \mathrm{C}^1(\mathscr{U}; \mathscr{F})$ by an algebraic condition that is vacuously satisfied by elements of $\mathrm{B}^1(\mathscr{U}; \mathscr{F})$. If we take $\mathrm{Z}^1(\mathscr{U}; \mathscr{F})$ to be those elements $(s_{ab})_{a,b \in A} \in \mathrm{C}^1(\mathscr{U}; \mathscr{F})$ for which

$$s_{bc} - s_{ac} + s_{ab} = 0, \qquad a, b, c \in A,$$

(again, we omit restrictions for brevity), we can see that this condition is satisfied by elements of $\mathrm{B}^1(\mathscr{U}; \mathscr{F})$. We thus define $\mathrm{H}^1(\mathscr{U}; \mathscr{F}) = \mathrm{Z}^1(\mathscr{U}; \mathscr{F})/\mathrm{B}^1(\mathscr{U}; \mathscr{F})$, which is the ***first cohomology group*** of $\mathscr{F}$ for the cover $\mathscr{U}$. The vanishing of the first cohomology group is intimately connected with the capacity of the sheaf to support the patching together of local constructions to form a global construction.

The path from the preceding rather elementary constructions to higher cohomology groups now typically proceeds in one of two equivalent directions. One can continue with the constructions with open covers and prove, that for suitable open covers, one arrives at a cover-independent theory. This gives what is known as Čech cohomology. This approach can often be used to compute the cohomology of a concrete sheaf. Another approach, more abstract and so more difficult to understand, realises cohomology groups as “the right derived functors for the global section functor.” About this we shall say nothing more, but refer to the references.

For us, the following results are useful. The first result is useful in the smooth case.

**2.21 Theorem: (Cohomology of sheaves of modules over the ring of continuous or differentiable functions)** *If $r \in \mathbb{Z}_{\geq 0} \cup \{\infty\}$, if $\mathsf{M}$ is a smooth paracompact Hausdorff manifold, if $\mathscr{F}$ is a sheaf of $\mathscr{C}^r_{\mathsf{M}}$-modules, and if $\mathscr{U} = (\mathcal{U}_a)_{a\in A}$ is an open cover of $\mathsf{M}$, then $\mathrm{H}^1(\mathscr{U};\mathscr{F}) = 0$.*

Outline of proof: The sheaf of rings $\mathscr{C}^r_{\mathsf{M}}$ is easily shown to have the property of “fineness” [Wells Jr. 2008, Definition 3.3]; this amounts to the fact that a smooth manifold possesses a $C^r$-partition of unity [Abraham, Marsden, and Ratiu 1988, Theorem 5.5.7]. One then can show [Wells Jr. 2008, Proposition 3.5] that fine sheaves have the property of “softness” [Wells Jr. 2008, Definition 3.1]. A sheaf of modules over a soft sheaf of rings can be shown to be soft [Wells Jr. 2008, Lemma 3.16]. Finally, the cohomology of soft sheaves may be shown to vanish at orders larger than zero [Wells Jr. 2008, Theorem 3.11]. ■

In the real analytic case, the following result is the one we shall find useful.

**2.22 Theorem: (A consequence of Cartan's Theorem B)** *If $\mathsf{M}$ is a paracompact Hausdorff real analytic manifold, if $\mathscr{F}$ is a coherent sheaf of $\mathscr{C}^\omega_{\mathsf{M}}$-modules, and if $\mathscr{U} = (\mathcal{U}_a)_{a\in A}$ is an open cover of $\mathsf{M}$, then $\mathrm{H}^1(\mathscr{U};\mathscr{F}) = 0$.*

Outline of proof: The history of the proof is rather like that for Cartan's Theorem A given above. ■

**2.8. Topologies on stalks of sheaves of sections.** We will require topologies on the stalks $\mathscr{G}^r_{x,\mathsf{E}}$ of the sheaf of sections of a vector bundle $\pi\colon \mathsf{E} \to \mathsf{M}$ of class $C^r$. This is done differently in the cases $r = \infty$ and $r = \omega$.

**The smooth case.** Let $\pi\colon \mathsf{E} \to \mathsf{M}$ be a smooth vector bundle. Without loss of generality (since we are only topologising stalks, so all constructions need only be local) we suppose that $\mathsf{M}$ is Hausdorff and paracompact. If $\mathcal{U} \subseteq \mathsf{M}$ is open, we recall the weak topology on $\Gamma^\infty(\mathsf{E}|\mathcal{U})$ [Michor 1980]. This is most easily described by assigning a smooth vector bundle metric $\mathbb{G}$ to $\mathsf{E}$. Thus $\mathbb{G}_x$ is an inner product on $\mathsf{E}_x$.[1] We let $\|\cdot\|_x$ denote the induced norm on $\mathsf{E}_x$. Note that the $r$th jet bundle $\pi^r\colon \mathsf{J}^r\mathsf{E} \to \mathsf{M}$ of $\pi\colon \mathsf{E} \to \mathsf{M}$ is a vector bundle [Kolář, Michor, and Slovák 1993, §12.17]. Thus we may define a vector bundle metric on this bundle

[1]The construction of $\mathbb{G}$ in the smooth case follows from standard arguments using partitions of unity, cf. the proof of the existence of a Riemannian metric on a smooth, paracompact, Hausdorff manifold [Abraham, Marsden, and Ratiu 1988, Corollary 5.5.13].

that we denote by $\mathbb{G}^r$. The corresponding norm on the fibre over $x$ we denote by $\|\cdot\|^r_x$. For $K \subseteq \mathcal{U}$ compact and for $r \in \mathbb{Z}_{\geq 0}$ we can then define a seminorm $\|\cdot\|_{r,K}$ on $\Gamma^\infty(\mathsf{E}|\mathcal{U})$ by

$$\|\xi\|_{r,K} = \sup\{\|j_r\xi(x)\|^r_x \mid x \in K\}.$$

If $(K_j)_{j\in\mathbb{Z}_{>0}}$ is a sequence of compact sets such that $\mathcal{U} = \cup_{j\in\mathbb{Z}_{>0}} K_j$ (by [Aliprantis and Border 2006, Lemma 2.76]) then the locally convex topology defined by the family of seminorms $\|\cdot\|_{r,K}$, $r \in \mathbb{Z}_{\geq 0}$, $K \subseteq \mathcal{U}$ compact, is the same as the locally convex topology defined by the countable family of seminorms $\|\cdot\|_{r,K_j}$, $r \in \mathbb{Z}_{\geq 0}$, $j \in \mathbb{Z}_{>0}$. Moreover, this topology can be easily verified to be Hausdorff and complete. Thus $\Gamma^\infty(\mathsf{E}|\mathcal{U})$ is a Fréchet space with this topology. The topology can also be shown to be independent of the choices of the vector bundle metrics $\mathbb{G}^r$, $r \in \mathbb{Z}_{\geq 0}$.

Now let $x \in \mathsf{M}$ and let $\mathscr{N}_x$ be the set of neighbourhoods of $x$, noting that $\mathscr{N}_x$ is a directed set under inclusion. Note that $\mathscr{G}^\infty_{x,\mathsf{E}}$ is the direct limit (in the category of $\mathbb{R}$-vector spaces) of $(\Gamma^\infty(\mathsf{E}|\mathcal{U}))_{\mathcal{U}\in\mathscr{N}_x}$ with respect to the mappings $r_{\mathcal{U},x}$. If $(\mathcal{U}_j)_{j\in\mathbb{Z}_{>0}}$ is a sequence of neighbourhoods of $x$ such that $\mathcal{U}_{j+1} \subseteq \mathcal{U}_j$ and such that $\cap_{j\in\mathbb{Z}_{>0}}\mathcal{U}_j = \{x\}$, then this family is cofinal in $\mathscr{N}_x$ and so the resulting direct limit topology on $\mathscr{G}^\infty_{x,\mathsf{E}}$ induced by the mappings $r_{\mathcal{U}_j,x}$, $j \in \mathbb{Z}_{>0}$, gives $\mathscr{G}^\infty_{x,\mathsf{E}}$ the structure of an (LF)-space; see [Köthe 1969, §19.5].

We will be interested in closed submodules of $\mathscr{G}^\infty_{x,\mathsf{E}}$. These can be described with the aid of Whitney's Spectral Theorem [Whitney 1948]. By $j_\infty\xi(x)$ we denote the infinite jet of a section $\xi$ at $x$.

**2.23 Theorem: (Whitney's Spectral Theorem)** *Let $\pi\colon \mathsf{E} \to \mathsf{M}$ be a smooth vector bundle with bounded fibre dimension and with $\mathsf{M}$ smooth, second countable, and Hausdorff. If $\mathscr{M} \subseteq \Gamma^\infty(\mathsf{E})$ is a submodule, then the closure of $\mathscr{M}$ in the weak topology on $\Gamma^\infty(\mathsf{E})$ is*

$$\mathrm{cl}(\mathscr{M}) = \{\xi \in \Gamma^\infty(\mathsf{E}) \mid j_\infty\xi(x) \in \{j_\infty\eta(x) \mid \eta \in \mathscr{M}\} \text{ for each } x \in \mathsf{M}\}.$$

With this, we can characterise closed submodules of the stalks $\mathscr{G}^\infty_{x,\mathsf{E}}$.

**2.24 Proposition: (Closed submodules of stalks of smooth sections)** *Let $\pi\colon \mathsf{E} \to \mathsf{M}$ be a smooth vector bundle and let $x \in \mathsf{M}$. If $\mathscr{F}_x \subseteq \mathscr{G}^\infty_{x,\mathsf{E}}$ is a submodule, then the closure of $\mathscr{F}_x$ in the (LF)-topology on $\mathscr{G}^\infty_{x,\mathsf{E}}$ is*

$$\begin{aligned}\mathrm{cl}(\mathscr{F}_x) = \{[(\xi,\mathcal{U})]_x \in \mathscr{G}^\infty_{x,\mathsf{E}} \mid\ & \text{there exists a neighbourhood } \mathcal{V} \subseteq \mathcal{U} \text{ of } x \text{ such that} \\ & j_\infty\xi(y) \in \{j_\infty\eta(y) \mid \eta \in r^{-1}_{\mathcal{V},x}(\mathscr{F}_x)\} \text{ for every } y \in \mathcal{V}\}.\end{aligned}$$

Proof: First let $[(\xi,\mathcal{U})]_x \in \mathscr{G}^\infty_{x,\mathsf{E}}$ be such that there exists a neighbourhood $\mathcal{V} \subseteq \mathcal{U}$ of $x$ for which

$$j_\infty\xi(y) \in \{j_\infty\eta(y) \mid \eta \in r^{-1}_{\mathcal{V},x}(\mathscr{F}_x)\}$$

for every $y \in \mathcal{V}$. Let $(\mathcal{U}_j)_{j\in\mathbb{Z}_{>0}}$ be a sequence of neighbourhoods of $x$ such that $\mathcal{U}_{j+1} \subseteq \mathcal{U}_j$ and such that $\cap_{j\in\mathbb{Z}_{>0}}\mathcal{U}_j = \{x\}$, and note that $r^{-1}_{\mathcal{U}_j,x}(\mathscr{F}_x)$ is a submodule of $\Gamma^\infty(\mathsf{E}|\mathcal{U}_j)$ for each $j \in \mathbb{Z}_{>0}$. Let $N \in \mathbb{Z}_{>0}$ be sufficiently large that $\mathcal{U}_j \subseteq \mathcal{V}$ for all $j \geq N$. Then

$$j_\infty\xi(y) \in \{j_\infty\eta(y) \mid \eta \in r^{-1}_{\mathcal{U}_j,x}(\mathscr{F}_x)\}$$

for all $y \in \mathcal{U}_j$ and $j \geq N$. Thus $r_{\mathcal{U},\mathcal{U}_j}(\xi) \in \mathrm{cl}(r^{-1}_{\mathcal{U}_j,x}(\mathscr{F}_j))$ for $j \geq N$ by the Whitney Spectral Theorem. It then follows that, for every $j \geq N$,

$$r_{\mathcal{U}_j,x}(r_{\mathcal{U},\mathcal{U}_j}(\xi)) = r_{\mathcal{U},x}(\xi) = [(\xi,\mathcal{U})]_x \in \mathrm{cl}(\mathscr{F}_x),$$

cf. [Köthe 1969, §19.5].

Next let $[(\xi, \mathcal{U})]_x$ be in the closure of $\mathscr{F}_x$. It follows that $r_{\mathcal{U},x}^{-1}([(\xi,\mathcal{U})]_x)$ is in the closure of $r_{\mathcal{U},x}^{-1}(\mathscr{F}_x)$. Therefore, by the Whitney Spectral Theorem,

$$j_\infty \xi(y) \in \{ j_\infty \eta(y) \mid \eta \in r_{\mathcal{U},x}^{-1}(\mathscr{F}_x) \}$$

for all $y \in \mathcal{U}$, giving the desired characterisation of $\operatorname{cl}(\mathscr{F}_x)$. ■

The following example shows that there can be finitely generated submodules of germs of sections that are not closed. We refer to [Roth 1970] for further discussion along these lines.

**2.25 Example: (A finitely generated submodule of smooth germs that is not closed)** We take $\mathsf{M} = \mathbb{R}$ with coordinate $x$. We take $f \in C^\infty(\mathbb{R})$ to be defined by

$$f(x) = \begin{cases} \mathrm{e}^{-1/x^4}, & x \neq 0, \\ 0, & x = 0. \end{cases}$$

We claim that the submodule $\langle f \rangle_{\mathcal{U}}$ of $C^\infty(\mathcal{U})$ generated by $f|\mathcal{U}$ is not closed for any neighbourhood $\mathcal{U}$ of 0. Indeed, let $\mathcal{U}$ be a neighbourhood of 0. Note that the function $g \in C^\infty(\mathcal{U})$ defined by

$$g(x) = \begin{cases} \mathrm{e}^{-1/x^2}, & x \neq 0, \\ 0, & x = 0 \end{cases}$$

has the property that its Taylor series at 0 agrees with that of $f$: both Taylor series are identically zero. However, since

$$\lim_{x \to 0} \frac{g(x)}{f(x)} = \infty,$$

there is no function $h \in C^\infty(\mathcal{U})$ such that $g = hf$, and so $g \notin \langle f \rangle_{\mathcal{U}}$. Moreover, by Proposition 2.24, the argument shows that the module $\langle [f]_0 \rangle$ generated by $[f]_0$ is not closed. •

**The real analytic case.** The topology on the stalks of the sheaf of sections of a real analytic vector bundle is more difficult to describe than the smooth case. In the real analytic case, we must first extend real analytic objects to holomorphic objects on a complexification of the vector bundle.

We let $\pi \colon \mathsf{E} \to \mathsf{M}$ be a real analytic vector bundle, supposing that $\mathsf{M}$ is paracompact and Hausdorff. As in the smooth case, our definition of the appropriate topologies is facilitated by the introduction of a vector bundle metric on $\mathsf{E}$. Let us be sure we understand how to construct such a metric in the real analytic case. First of all, by the real analytic embedding theorem of Grauert [1958], we analytically embed $\mathsf{E}$ into $\mathbb{R}^N$ for sufficiently large $N$. Then $\mathsf{M}$ and the fibres $\mathsf{E}_x$ of $\mathsf{E}$ are real analytic submanifolds of $\mathbb{R}^N$. Moreover, $\mathsf{E}$ is naturally isomorphic to the normal bundle of $\mathsf{M}$ in $\mathsf{E}$. Using the Euclidean inner product $\mathbb{G}_x$ on the fibres of the normal bundle at $x \in \mathsf{M}$, we define a real analytic vector bundle metric $\mathbb{G}$ on $\mathsf{E}$. Following Whitney and Bruhat [1959], we can regard $\mathsf{M}$ as the real part of a corresponding holomorphic manifold $\overline{\mathsf{M}}$. Moreover, using our observation above that $\mathsf{E}$ is isomorphic to the normal bundle of $\mathsf{M} \subseteq \mathsf{E} \subseteq \mathbb{R}^N$, we can extend $\mathsf{E}$ to be a holomorphic vector bundle

$\overline{\pi}\colon \overline{\mathsf{E}} \to \overline{\mathsf{M}}$. The restriction of $\overline{\mathsf{E}}$ to $\mathsf{M} \subseteq \overline{\mathsf{M}}$ agrees with the complexification $\mathsf{E} \otimes_{\mathbb{R}} \mathbb{C}$ of $\mathsf{E}$, and we denote this by $\mathsf{E}^{\mathbb{C}}$. That is,

$$\mathsf{E}^{\mathbb{C}} \triangleq \overline{\mathsf{E}}|\mathsf{M} \simeq \mathsf{E} \otimes_{\mathbb{R}} \mathbb{C}.$$

By shrinking the holomorphic extension $\overline{\mathsf{M}}$ if necessary, we can suppose that $\mathbb{G}$ extends to a Hermitian metric $\overline{\mathbb{G}}$ on the fibres of $\overline{\mathsf{E}}$. Denote by $\|\cdot\|_{\overline{\mathbb{G}}}$ the norm induced on the fibres of $\overline{\mathsf{E}}$.

With these holomorphic extensions at hand, we can now begin to describe the topology on the stalks of $\mathscr{G}^{\omega}_{\mathsf{E}}$. As in the smooth case, we first describe the topology of $\Gamma^{\omega}(\mathsf{E}|\mathcal{U})$ for open sets $\mathcal{U} \subseteq \mathsf{M}$. The construction of this topology is done in a few steps.

1. *Topologise the holomorphic sections:* We first consider the complexification. Thus let $\overline{\mathcal{U}} \subseteq \overline{\mathsf{M}}$ be open and let $\Gamma^{\mathrm{hol}}(\overline{\mathsf{E}}|\overline{\mathcal{U}})$ denote the holomorphic sections of $\overline{\mathsf{E}}|\overline{\mathcal{U}}$. Define a topology on $\Gamma^{0}(\overline{\mathsf{E}}|\overline{\mathcal{U}})$ as that defined by the family of seminorms $\|\cdot\|_K$, $K \subseteq \overline{\mathcal{U}}$ compact, given by
$$\|\overline{\xi}\|_K = \sup\{\|\overline{\xi}(z)\|_{\overline{\mathbb{G}}} \mid\ z \in K\}.$$
This defines the compact-open, or weak $C^0$-, topology on $\Gamma^{0}(\overline{\mathsf{E}}|\overline{\mathcal{U}})$. It is well-known that $\Gamma^{\mathrm{hol}}(\overline{\mathsf{E}}|\overline{\mathcal{U}})$ is a *closed* subspace of $\Gamma^{0}(\overline{\mathsf{E}}|\overline{\mathcal{U}})$ with this topology [Gunning and Rossi 1965, Theorem V.B.5]. Thus $\Gamma^{\mathrm{hol}}(\overline{\mathsf{E}}|\overline{\mathcal{U}})$ has a natural Fréchet topology.
2. *Restrict from holomorphic to real analytic sections of* $\mathsf{E}^{\mathbb{C}}$*:* Now let $\mathcal{U} \subseteq \mathsf{M}$ be open and let $\overline{\mathcal{U}} \subseteq \overline{\mathsf{M}}$ be a neighbourhood of $\mathcal{U}$. If $\overline{\xi} \in \Gamma^{\mathrm{hol}}(\overline{\mathsf{E}}|\overline{\mathcal{U}})$ then $\overline{\xi}|\mathcal{U} \in \Gamma^{\omega}(\mathsf{E}^{\mathbb{C}}|\mathcal{U})$, cf. [Krantz 1992, Corollary 2.3.7]. Conversely, if $\mathcal{U} \subseteq \mathsf{M}$ is open and if $\xi \in \Gamma^{\omega}(\mathsf{E}^{\mathbb{C}}|\mathcal{U})$, then there exists a neighbourhood $\overline{\mathcal{U}}$ of $\mathcal{U}$ and $\overline{\xi} \in \Gamma^{\mathrm{hol}}(\overline{\mathsf{E}}|\overline{\mathcal{U}})$ such that $\overline{\xi}|\mathcal{U} = \xi$. (To see this, it is easiest to think in terms of Taylor series. The Taylor series for $\xi$ will be a series with coordinates $(x^1,\dots,x^n)$ for $\mathsf{M}$ as indeterminates and with complex coefficients. The Taylor series for the corresponding section $\overline{\xi}$ will be given by replacing the real indeterminates $(x^1,\dots,x^n)$ with complex indeterminates $(z^1,\dots,z^n)$ representing coordinates for $\overline{\mathsf{M}}$.) Moreover, $\overline{\xi}$ is unique in that any two such extensions will agree on any connected neighbourhood of $\mathcal{U}$. For $\mathcal{U} \subseteq \mathsf{M}$ open and $\overline{\mathcal{U}}$ a neighbourhood of $\mathcal{U}$ in $\overline{\mathsf{M}}$, let us denote by
$$\overline{\rho}_{\overline{\mathcal{U}},\mathcal{U}}\colon \Gamma^{\mathrm{hol}}(\overline{\mathsf{E}}|\overline{\mathcal{U}}) \to \Gamma^{\omega}(\mathsf{E}^{\mathbb{C}}|\mathcal{U}) \tag{2.2}$$
the restriction map, noting that this map is injective if $\overline{\mathcal{U}}$ is connected. Moreover, if $\mathcal{U} \subseteq \mathsf{M}$ is open and if $\mathscr{N}_{\mathcal{U}}$ denotes the set of neighbourhoods of $\mathcal{U}$ in $\overline{\mathsf{M}}$, then
$$\Gamma^{\omega}(\mathsf{E}^{\mathbb{C}}|\mathcal{U}) = \cup_{\overline{\mathcal{U}}\in\mathscr{N}_{\mathcal{U}}} \operatorname{image}(\overline{\rho}_{\overline{\mathcal{U}},\mathcal{U}}).$$
3. *Projection to real analytic sections of* $\mathsf{E}$*:* We let $\mathrm{Re}_{\mathcal{U}}\colon \mathsf{E}^{\mathbb{C}}|\mathcal{U} \to \mathsf{E}|\mathcal{U}$ be the projection onto the real part of the fibres. Let us also abuse notation slightly and let
$$\mathrm{Re}_{\mathcal{U}}\colon \Gamma^{\omega}(\mathsf{E}^{\mathbb{C}}|\mathcal{U}) \to \Gamma^{\omega}(\mathsf{E}|\mathcal{U})$$
be the induced map on sections. We then define
$$\rho_{\overline{\mathcal{U}},\mathcal{U}} \triangleq \mathrm{Re}_{\mathcal{U}} \circ \overline{\rho}_{\overline{\mathcal{U}},\mathcal{U}}\colon \Gamma^{\mathrm{hol}}(\overline{\mathsf{E}}|\overline{\mathcal{U}}) \to \Gamma^{\omega}(\mathsf{E}|\mathcal{U}).$$

Let us see how this homomorphism reacts with the module structures on the domain and codomain. We let $\mathcal{U} \subseteq \mathsf{M}$ be open and let $\overline{\mathcal{U}} \in \mathscr{N}_{\mathcal{U}}$. Making a slight abuse of notation, we denote by

$$\rho_{\overline{\mathcal{U}},\mathcal{U}} \colon C^{\mathrm{hol}}(\overline{\mathcal{U}}) \to C^{\omega}(\mathcal{U})$$

the restriction map, where $C^{\mathrm{hol}}(\overline{\mathsf{E}}|\overline{\mathcal{U}})$ denotes the holomorphic functions on $\overline{\mathcal{U}}$. We claim that if $\mathscr{M} \subseteq \Gamma^{\omega}(\mathsf{E}|\mathcal{U})$ is a submodule over $C^{\omega}(\mathcal{U})$ then $\rho^{-1}_{\overline{\mathcal{U}},\mathcal{U}}(\mathscr{M})$ is a submodule over $\rho^{-1}_{\overline{\mathcal{U}},\mathcal{U}}(C^{\omega}(\mathcal{U}))$. First of all, let us show that $\rho^{-1}_{\overline{\mathcal{U}},\mathcal{U}}(C^{\omega}(\mathcal{U}))$ is a ring. If $\overline{f}$ and $\overline{g}$ are such that $\rho_{\overline{\mathcal{U}},\mathcal{U}}(\overline{f}) = f$ and $\rho_{\overline{\mathcal{U}},\mathcal{U}}(\overline{g}) = g$ for $f, g \in C^{\omega}(\mathcal{U})$, then clearly

$$\rho_{\overline{\mathcal{U}},\mathcal{U}}(\overline{f} + \overline{g}) = f + g, \quad \rho_{\overline{\mathcal{U}},\mathcal{U}}(\overline{f}\overline{g}) = fg,$$

and so $\overline{f}+\overline{g}, \overline{f}\overline{g} \in \rho^{-1}_{\overline{\mathcal{U}},\mathcal{U}}(C^{\omega}(\mathcal{U}))$. Now let $\overline{\xi}$ and $\overline{\eta}$ be such that $\rho_{\overline{\mathcal{U}},\mathcal{U}}(\overline{\xi}) = \xi$ and $\rho_{\overline{\mathcal{U}},\mathcal{U}}(\overline{\eta}) = \eta$ for $\xi, \eta \in \mathscr{M}$. It is clear that

$$\rho_{\overline{\mathcal{U}},\mathcal{U}}(\overline{\xi} + \overline{\eta}) = \xi + \eta$$

and so $\overline{\xi} + \overline{\eta} \in \rho^{-1}_{\overline{\mathcal{U}},\mathcal{U}}(\mathscr{M})$. Also, since the restriction of the product $\overline{f}\,\overline{\xi}$ agrees with the products of the restrictions of $\overline{f}$ and $\overline{\xi}$, we have $\rho_{\overline{\mathcal{U}},\mathcal{U}}(\overline{f\xi}) = f\xi$ and so $\overline{f}\,\overline{\xi} \in \rho^{-1}_{\overline{\mathcal{U}},\mathcal{U}}(\mathscr{M})$, giving our claim.

4. *Use the inductive limit topology:* Now let $\mathcal{U} \subseteq \mathsf{M}$ be open and let $\mathscr{N}_{\mathcal{U}}$ be the set of neighbourhoods of $\mathcal{U}$ in $\overline{\mathsf{M}}$. Note that $\mathscr{N}_{\mathcal{U}}$ is a directed set under inclusion. The topology on $\Gamma^{\omega}(\mathsf{E}^{\mathbb{C}}|\mathcal{U})$ is the inductive limit topology with respect to the family of restriction mappings (2.2), i.e., the finest topology for which all of these maps are continuous. Thus $\Gamma^{\omega}(\mathsf{E}^{\mathbb{C}}|\mathcal{U})$ is an inductive limit of Fréchet spaces, but it need not be an (LF)-space since this limit need not be countable. One verifies that $\Gamma^{\omega}(\mathsf{E}|\mathcal{U})$ is a closed subspace of $\Gamma^{\omega}(\mathsf{E}^{\mathbb{C}}|\mathcal{U})$. Thus the induced topology gives us the desired topology on $\Gamma^{\omega}(\mathsf{E}|\mathcal{U})$. This topology has some not so friendly properties; we refer to the work of Martineau [1966] and the discussions of this in [Krantz and Parks 2002, §2.6] and [Domański and Vogt 2000] for details.

Let us consider some related constructions with stalks, as this notation will be of use to us in the proof of Theorem 2.26 below. With our notation above, we have the sheaf $\mathscr{G}^{\mathrm{hol}}_{\overline{\mathsf{E}}}$ of $\mathscr{C}^{\mathrm{hol}}_{\overline{\mathsf{M}}}$-modules. We also have the sheaves $\mathscr{G}^{\omega}_{\mathsf{E}}$ and $\mathscr{G}^{\omega}_{\mathsf{E}^{\mathbb{C}}}$ of $\mathscr{C}^{\omega}_{\mathsf{M}}$-modules. Our constructions above ensure that restriction of germs to $\mathsf{M} \subseteq \overline{\mathsf{M}}$ gives a bijection from $\mathscr{G}^{\mathrm{hol}}_{\overline{x},\overline{\mathsf{E}}}$ to $\mathscr{G}^{\omega}_{x,\mathsf{E}^{\mathbb{C}}}$ for each $x \in \mathsf{M}$, where $\overline{x}$ denotes the image of $x \in \mathsf{M}$ in $\overline{\mathsf{M}}$. We thus have a homomorphism

$$\overline{\rho}_{\overline{x},x} \colon \mathscr{G}^{\mathrm{hol}}_{\overline{\mathsf{E}}} \to \mathscr{G}^{\omega}_{\mathsf{E}^{\mathbb{C}}}$$

of $\mathbb{R}$-vector spaces (topologies will be considered shortly). By taking real parts, we further get a homomorphism

$$\rho_{\overline{x},x} \colon \mathscr{G}^{\mathrm{hol}}_{\overline{\mathsf{E}}} \to \mathscr{G}^{\omega}_{\mathsf{E}}.$$

We may argue as in step 3 above that, if $\mathscr{F}_x \subseteq \mathscr{G}^{\omega}_{x,\mathsf{E}}$ is a $\mathscr{C}^{\omega}_{x,\mathsf{M}}$-submodule, then $\rho^{-1}_{\overline{x},x}(\mathscr{F}_x)$ is a $\mathscr{C}^{\mathrm{hol}}_{\overline{x},\overline{\mathsf{M}}}$-submodule of $\mathscr{G}^{\mathrm{hol}}_{\overline{x},\overline{\mathsf{E}}}$.

Finally, we topologise the stalks of $\mathscr{G}^{\mathrm{hol}}_{\overline{\mathsf{E}}}$ and $\mathscr{G}^{\omega}_{\mathsf{E}}$. Let $x \in \mathsf{M}$ and let $\overline{x} \in \overline{\mathsf{M}}$. Let $\mathscr{N}_x$ and $\mathscr{N}_{\overline{x}}$ denote the families of neighbourhoods of $x$ and $\overline{x}$ in $\mathsf{M}$ and $\overline{\mathsf{M}}$, respectively. Note that these are both directed sets under inclusion. The topologies on $\mathscr{G}^{\omega}_{x,\mathsf{E}}$ and $\mathscr{G}^{\mathrm{hol}}_{\overline{x},\overline{\mathsf{E}}}$ are then the inductive limit topologies with respect to the families of mappings $r_{\mathcal{U},x}$, $\mathcal{U} \in \mathscr{N}_x$, and $r_{\overline{\mathcal{U}},\overline{x}}$, $\overline{\mathcal{U}} \in \mathscr{N}_{\overline{x}}$, respectively.

Next we need to describe the closed submodules of $\mathscr{G}^{\omega}_{x,\mathsf{E}}$.

**2.26 Theorem: (Closed submodules of stalks of real analytic sections)** *Let $\pi\colon \mathsf{E} \to \mathsf{M}$ be a real analytic vector bundle and let $x \in \mathsf{M}$. If $\mathscr{F}_x \subseteq \mathscr{G}^{\omega}_{x,\mathsf{E}}$ is a submodule, then it is closed in the inductive limit topology on $\mathscr{G}^{\omega}_{x,\mathsf{E}}$.*

Proof: We first claim that the map $\rho_{\overline{x},x}$ is continuous. By Proposition 2 from Section 4.1 of [Grothendieck 1973] it suffices to show that, for any neighbourhood $\overline{\mathcal{U}}$ of $\overline{x}$ in $\overline{\mathsf{M}}$, the composition

$$\rho_{\overline{x},x} \circ r_{\overline{\mathcal{U}},\overline{x}}\colon \Gamma^{\mathrm{hol}}(\overline{\mathsf{E}}|\overline{\mathcal{U}}) \to \mathscr{G}^{\omega}_{x,\mathsf{E}}$$

is continuous. Note that the diagram

$$\begin{tikzcd}
\Gamma^{\mathrm{hol}}(\overline{\mathsf{E}}|\overline{\mathcal{U}}) \arrow[r, "\rho_{\overline{\mathcal{U}},\mathcal{U}}"] \arrow[d, "r_{\overline{\mathcal{U}},\overline{x}}"'] & \Gamma^{\omega}(\mathsf{E}|\mathcal{U}) \arrow[d, "r_{\mathcal{U},x}"] \\
\mathscr{G}^{\mathrm{hol}}_{\overline{x},\overline{\mathsf{E}}} \arrow[r, "\rho_{\overline{x},x}"'] & \mathscr{G}^{\omega}_{x,\mathsf{E}}
\end{tikzcd} \tag{2.3}$$

commutes for every $\overline{\mathcal{U}} \in \mathscr{N}_{\overline{x}}$, where $\mathcal{U} = \overline{\mathcal{U}} \cap \mathsf{M}$. Therefore, it suffices to show that

$$r_{\mathcal{U},x} \circ \rho_{\overline{\mathcal{U}},\mathcal{U}}\colon \Gamma^{\mathrm{hol}}(\overline{\mathsf{E}}|\overline{\mathcal{U}}) \to \mathscr{G}^{\omega}_{x,\mathsf{E}}$$

is continuous for every $\mathcal{U} \in \mathscr{N}_x$ and $\overline{\mathcal{U}} \in \mathscr{N}_{\mathcal{U}}$. However, again by Proposition 2 from Section 4.1 of [Grothendieck 1973], the homomorphisms $\rho_{\overline{\mathcal{U}},\mathcal{U}}$ and $r_{\mathcal{U},x}$ are continuous, and so the claim follows.

Now let $[(\xi,\mathcal{U})]_x$ be in the closure of $\mathscr{F}_x$ in $\mathscr{G}^{\omega}_{x,\mathsf{E}}$. Let $\overline{\mathcal{U}} \in \mathscr{N}_{\mathcal{U}}$ be such that $\xi$ extends to a section $\overline{\xi}$ of $\overline{\mathsf{E}}|\overline{\mathcal{U}}$. Then $r^{-1}_{\mathcal{U},x}([(\xi,\mathcal{U})]_x)$ is in the closure of $r^{-1}_{\mathcal{U},x}(\mathscr{F}_x)$ in $\Gamma^{\omega}(\mathsf{E}|\mathcal{U})$ and $\rho^{-1}_{\overline{\mathcal{U}},\mathcal{U}}(r^{-1}_{\mathcal{U},x}([(\xi,\mathcal{U})]_x))$ is in the closure of $\rho^{-1}_{\overline{\mathcal{U}},\mathcal{U}}(r^{-1}_{\mathcal{U},x}(\mathscr{F}_x))$ in $\Gamma^{\mathrm{hol}}(\overline{\mathsf{E}}|\overline{\mathcal{U}})$. By the commutativity of the diagram (2.3), this implies that $r^{-1}_{\overline{\mathcal{U}},\overline{x}}([(\overline{\xi},\overline{\mathcal{U}})]_{\overline{x}})$ is in the closure of $r^{-1}_{\overline{\mathcal{U}},\mathcal{U}}(\overline{\mathscr{F}}_{\overline{x}})$ in $\overline{\Gamma}^{\mathrm{hol}}(\overline{\mathsf{E}}|\overline{\mathcal{U}})$. Consequently, $r^{-1}_{\overline{\mathcal{U}},\overline{x}}([(\overline{\xi},\overline{\mathcal{U}})]_{\overline{x}})$ is in the closure of $r^{-1}_{\overline{\mathcal{U}},\mathcal{U}}(\rho^{-1}_{\overline{x},x}(\mathscr{F}_x))$ in $\Gamma^{\mathrm{hol}}(\overline{\mathsf{E}}|\overline{\mathcal{U}})$. As we argued before the statement of the theorem, $\rho^{-1}_{\overline{x},x}(\mathscr{F}_x)$ is a $\mathscr{C}^{\mathrm{hol}}_{\overline{x},\overline{\mathsf{M}}}$-submodule of $\mathscr{G}^{\mathrm{hol}}_{\overline{x},\overline{\mathsf{E}}}$. Therefore, by [Taylor 2002, Proposition 11.2.2], $r^{-1}_{\overline{\mathcal{U}},\mathcal{U}}(\rho^{-1}_{\overline{x},x}(\mathscr{F}_x))$ is closed in $\Gamma^{\mathrm{hol}}(\overline{\mathsf{E}}|\overline{\mathcal{U}})$ and so contains $r^{-1}_{\overline{\mathcal{U}},\overline{x}}([(\overline{\xi},\overline{\mathcal{U}})]_{\overline{x}})$. Consequently, $\rho^{-1}_{\overline{\mathcal{U}},\mathcal{U}}(r^{-1}_{\mathcal{U},x}(\mathscr{F}_x))$ contains $\rho^{-1}_{\overline{\mathcal{U}},\mathcal{U}}(r^{-1}_{\mathcal{U},x}([(\xi,\mathcal{U})]_x))$, and so $[(\xi,\mathcal{U})]_x \in \mathscr{F}_x$, as desired. ■

Note that, unsurprisingly given the topology on $\mathscr{G}^{\omega}_{x,\mathsf{E}}$, the proof relies in an essential way on the holomorphic analogue of the theorem. This holomorphic analogue is an essential ingredient in the proofs of Cartan's Theorems A and B.

## 3. Generalised subbundles of vector bundles

In this section we introduce the major player in this paper in the general setting of vector bundles; later in the paper we shall specialise to tangent bundles. We also introduce the connections between these constructions and subsheaves of the sheaf of sections of a vector bundle.

**3.1. Generalised subbundles.** We begin by giving the definitions we shall use throughout the paper.

**3.1 Definition:** **(Generalised subbundle)** Let $\pi\colon \mathsf{E} \to \mathsf{M}$ be a vector bundle of class $C^\infty$ or $C^\omega$, as is required. A ***generalised subbundle*** of $\mathsf{E}$ is a subset $\mathsf{F} \subseteq \mathsf{E}$ such that, for each $x \in \mathsf{M}$, the subset $\mathsf{F}_x = \mathsf{F} \cap \mathsf{E}_x$ is a subspace (and so, in particular, is nonempty). The subspace $\mathsf{F}_x$ is the ***fibre*** of $\mathsf{F}$ at $x$. Associated with the notion of a generalised subbundle we have the following.

(i) A generalised subbundle $\mathsf{F}$ is of ***class $C^r$***, $r \in \mathbb{Z}_{\geq 0} \cup \{\infty, \omega\}$, if, for each $x_0 \in \mathsf{M}$, there exists a neighbourhood $\mathcal{N}$ of $x_0$ and a family $(\xi_j)_{j \in J}$ of $C^r$-sections, called ***local generators***, of $\mathsf{E}|\mathcal{N}$ such that

$$\mathsf{F}_x = \mathrm{span}_{\mathbb{R}}(\xi_j(x) \mid \; j \in J)$$

for each $x \in \mathcal{N}$.

(ii) A generalised subbundle $\mathsf{F}$ of class $C^r$, $r \in \mathbb{Z}_{\geq 0} \cup \{\infty, \omega\}$, is ***locally finitely generated*** if, for each $x_0 \in \mathsf{M}$, there exists a neighbourhood $\mathcal{N}$ of $x_0$ and a family $(\xi_1, \dots, \xi_k)$ of $C^r$-sections, called ***local generators***, of $\mathsf{E}|\mathcal{N}$ such that

$$\mathsf{F}_x = \mathrm{span}_{\mathbb{R}}(\xi_1(x), \dots, \xi_k(x))$$

for each $x \in \mathcal{N}$.

(iii) A generalised subbundle $\mathsf{F}$ of class $C^r$, $r \in \mathbb{Z}_{\geq 0} \cup \{\infty, \omega\}$, is ***finitely generated*** if there exists a family $(\xi_1, \dots, \xi_k)$ of $C^r$-sections, called ***generators***, of $\mathsf{E}$ such that

$$\mathsf{F}_x = \mathrm{span}_{\mathbb{R}}(\xi_1(x), \dots, \xi_k(x))$$

for each $x \in \mathsf{M}$.

The nonnegative integer $\dim(\mathsf{F}_x)$ is called the ***rank*** of $\mathsf{F}$ at $x$ and is sometimes denoted by $\mathrm{rank}(\mathsf{F}_x)$. •

Let us also give a few related standard definitions that we shall use.

**3.2 Definition:** **(Restriction of a generalised subbundle)** Let $\pi\colon \mathsf{E} \to \mathsf{M}$ be a vector bundle of class $C^\infty$ or $C^\omega$, as is required, let $r \in \mathbb{Z}_{\geq 0} \cup \{\infty, \omega\}$, and let $\mathsf{F} \subseteq \mathsf{E}$ be a $C^r$-generalised subbundle. If $\mathcal{U} \subseteq \mathsf{M}$ is open, the ***restriction*** of $\mathsf{F}$ to $\mathcal{U}$ is

$$\mathsf{F}|\mathcal{U} = \cup_{x \in \mathcal{U}} \mathsf{F}_x. \qquad \bullet$$

**3.3 Definition:** **(Section of a generalised subbundle)** Let $\pi\colon \mathsf{E} \to \mathsf{M}$ be a vector bundle of class $C^\infty$ or $C^\omega$, as is required, let $r \in \mathbb{Z}_{\geq 0} \cup \{\infty, \omega\}$, and let $\mathsf{F} \subseteq \mathsf{E}$ be a $C^r$-generalised subbundle. If $\mathcal{U} \subseteq \mathsf{M}$ is open, a ***local section*** of $\mathsf{F}$ over $\mathcal{U}$ is a section $\xi\colon \mathcal{U} \to \mathsf{E}$ such that $\xi(x) \in \mathsf{F}_x$ for every $x \in \mathcal{U}$. A local section $\xi$ of $\mathsf{F}$ is of ***class $C^k$***, $k \leq r$, if it is of class $C^k$ as a local section of $\mathsf{E}$. The set of local sections of $\mathsf{F}$ over $\mathcal{U}$ of class $C^k$ is denoted by $\Gamma^k(\mathsf{F}|\mathcal{U})$, or simply by $\Gamma^k(\mathsf{F})$ when $\mathcal{U} = \mathsf{M}$. •

Of particular interest to us in the paper are generalised subbundles of tangent bundles. These we give a special name.

**3.4 Definition: (Distribution)** Let $\mathsf{M}$ be a smooth or real analytic manifold, as required. A ***distribution*** on $\mathsf{M}$ is a generalised subbundle $\mathsf{D}$ of $\mathsf{TM}$, and a distribution $\mathsf{D}$ is of class $C^r$, $r \in \mathbb{Z}_{\geq 0} \cup \{\infty, \omega\}$, if it is of class $C^r$ as a generalised subbundle. •

For the first few sections of the paper we shall focus on generalised subbundles of vector bundles, turning especially to distributions in Section 6.

**3.2. Regular and singular points of generalised subbundles.** One of the complications ensuing from the notion of a generalised subbundle arises if the dimensions of the subspaces $\mathsf{F}_x$, $x \in \mathsf{M}$, are not locally constant. The following definition associates some language with this.

**3.5 Definition: (Regular point, singular point)** Let $\mathsf{F}$ be a generalised subbundle of a vector bundle $\pi\colon \mathsf{E} \to \mathsf{M}$. A point $x_0 \in \mathsf{M}$

(i) is a ***regular point*** for $\mathsf{F}$ if there exists a neighbourhood $\mathcal{N}$ of $x_0$ such that $\operatorname{rank}(\mathsf{F}_x) = \operatorname{rank}(\mathsf{F}_{x_0})$ for every $x \in \mathcal{N}$ and

(ii) is a ***singular point*** for $\mathsf{F}$ if it is not a regular point for $\mathsf{F}$.

A generalised subbundle $\mathsf{F}$ is ***regular*** if every point in $\mathsf{M}$ is a regular point for $\mathsf{F}$, and is ***singular*** otherwise. •

Although our definition of regular and singular points is made for arbitrary generalised subbundles, these definitions only have real value in the case when the generalised subbundle has some smoothness. A regular generalised subbundle of class $C^r$, $r \in \mathbb{Z}_{\geq 0} \cup \{\infty, \omega\}$, is often called ***subbundle*** of $\mathsf{E}$ of class $C^r$.

For continuous generalised subbundles one can make some statements about the character of rank and the character of the set of regular and singular points. In the following result, if $\mathsf{F}$ is a generalised subbundle of a vector bundle $\pi\colon \mathsf{E} \to \mathsf{M}$, then we denote by $\operatorname{rank}_{\mathsf{F}}\colon \mathsf{M} \to \mathbb{Z}_{\geq 0}$ the function defined by $\operatorname{rank}_{\mathsf{F}}(x) = \operatorname{rank}(\mathsf{F}_x)$.

**3.6 Proposition: (Rank and regular points for continuous generalised subbundles)** *If $\mathsf{F}$ is a generalised subbundle of class $C^0$ of a vector bundle $\pi\colon \mathsf{E} \to \mathsf{M}$, then the function $\operatorname{rank}_{\mathsf{F}}$ is lower semicontinuous and the set of regular points of $\mathsf{F}$ is open and dense.*

Proof: Let $a \in \mathbb{R}$ and let $x_0 \in \operatorname{rank}_{\mathsf{F}}^{-1}((a, \infty))$. Thus $k \triangleq \operatorname{rank}_{\mathsf{F}}(x_0) > a$. This means that there are $k$ sections $\xi_1, \dots, \xi_k$ of class $C^0$ defined in a neighbourhood $\mathcal{N}$ of $x_0$ such that $\mathsf{F}_{x_0} = \operatorname{span}_{\mathbb{R}}(\xi_1(x_0), \dots, \xi_k(x_0))$. Now choose a vector bundle chart $(\mathcal{V}, \psi)$ for $\mathsf{E}$ about $x_0$ so that the local sections $\xi_1, \dots, \xi_k$ have local representatives

$$\boldsymbol{x} \mapsto (\boldsymbol{x}, \boldsymbol{\xi}_j(\boldsymbol{x})), \qquad j \in \{1, \dots, k\}.$$

Let $(\mathcal{U}, \phi)$ be the induced chart for $\mathsf{M}$ and let $\boldsymbol{x}_0 = \phi(x_0)$. The vectors $(\boldsymbol{\xi}_1(\boldsymbol{x}_0), \dots, \boldsymbol{\xi}_k(\boldsymbol{x}_0))$ are then linearly independent. Therefore, there exist $j_1, \dots, j_k \in \{1, \dots, n\}$ (supposing that $n$ is the dimension of $\mathsf{M}$) such that the matrix

$$\begin{bmatrix} \boldsymbol{\xi}_1^{j_1}(\boldsymbol{x}_0) & \cdots & \boldsymbol{\xi}_k^{j_1}(\boldsymbol{x}_0) \\ \vdots & \ddots & \vdots \\ \boldsymbol{\xi}_1^{j_k}(\boldsymbol{x}_0) & \cdots & \boldsymbol{\xi}_k^{j_k}(\boldsymbol{x}_0) \end{bmatrix}$$

has nonzero determinant, where $\boldsymbol{\xi}_i^{j_l}(\boldsymbol{x}_0)$ is the $j_l$th component of $\boldsymbol{\xi}_i$, $i, l \in \{1, \ldots, k\}$. By continuity of the determinant there exists a neighbourhood $\mathcal{U}'$ of $\boldsymbol{x}_0$ such that the matrix

$$\begin{bmatrix} \boldsymbol{\xi}_1^{j_1}(\boldsymbol{x}) & \cdots & \boldsymbol{\xi}_k^{j_1}(\boldsymbol{x}) \\ \vdots & \ddots & \vdots \\ \boldsymbol{\xi}_1^{j_k}(\boldsymbol{x}) & \cdots & \boldsymbol{\xi}_k^{j_k}(\boldsymbol{x}) \end{bmatrix}$$

has nonzero determinant for every $\boldsymbol{x} \in \mathcal{U}'$. Thus the vectors $(\boldsymbol{\xi}_1(\boldsymbol{x}), \ldots, \boldsymbol{\xi}_k(\boldsymbol{x}))$ are linearly independent for every $\boldsymbol{x} \in \mathcal{U}'$. Therefore, the local sections $\xi_1, \ldots, \xi_k$ are linearly independent on $\phi^{-1}(\mathcal{U}')$, and so $\phi^{-1}(\mathcal{U}') \subseteq \mathrm{rank}_{\mathsf{F}}^{-1}((a, \infty))$ which gives lower semicontinuity of $\mathrm{rank}_{\mathsf{F}}$.

Let us denote by $R_{\mathsf{F}}$ the set of regular points of $\mathsf{F}$ and let $x_0 \in R_{\mathsf{F}}$. Then, by definition of $R_{\mathsf{F}}$, there exists a neighbourhood $\mathcal{U}$ of $x_0$ such that $\mathcal{U} \subseteq R_{\mathsf{F}}$. Thus $R_{\mathsf{F}}$ is open. Now let $x_0 \in \mathsf{M}$ and let $\mathcal{U}$ be a connected neighbourhood of $x_0$. Since the function $\mathrm{rank}_{\mathsf{F}}$ is locally bounded, there exists a least integer $N$ such that $\mathrm{rank}_{\mathsf{F}}(x) \leq N$ for each $x \in \mathcal{U}$. Moreover, since $\mathrm{rank}_{\mathsf{F}}$ is integer-valued, there exists $x' \in \mathcal{U}$ such that $\mathrm{rank}_{\mathsf{F}}(x') = N$. Now, by lower semicontinuity of $\mathrm{rank}_{\mathsf{F}}$, there exists a neighbourhood $\mathcal{U}'$ of $x'$ such that $\mathrm{rank}_{\mathsf{F}}(x) \geq N$ for all $x \in \mathcal{U}'$. By definition of $N$ we also have $\mathrm{rank}_{\mathsf{F}}(x) \leq N$ for each $x \in \mathcal{U}'$. Thus $x' \in R_{\mathsf{F}}$, and so $x_0 \in \mathrm{cl}(R_{\mathsf{F}})$. Therefore, $R_{\mathsf{F}}$ is dense. ■

For real analytic generalised subbundles, one can say much more about the set of regular points.

**3.7 Proposition: (Rank and regular points for real analytic generalised subbundles)** *If $\mathsf{F}$ is a generalised subbundle of class $C^\omega$ of a real analytic vector bundle $\pi\colon \mathsf{E} \to \mathsf{M}$ over a paracompact analytic base $\mathsf{M}$, then the following statements hold:*

- *(i) the set of singular points for $\mathsf{F}$ is a locally analytic set, i.e., for each $x \in \mathsf{M}$ there exists a neighbourhood $\mathcal{U}$ of $x$ and real analytic functions $f_1, \ldots, f_k \in C^\omega(\mathcal{U})$ such that the set of singular points of $\mathsf{F}$ in $\mathcal{U}$ is given by $\cap_{j=1}^k f_j^{-1}(0)$;*
- *(ii) if $x_1, x_2 \in \mathsf{M}$ are regular points for $\mathsf{F}$ in the same connected component of $\mathsf{M}$, then $\mathrm{rank}_{\mathsf{F}}(x_1) = \mathrm{rank}_{\mathsf{F}}(x_2)$.*

Proof: (i) Let $x_0 \in \mathsf{M}$. By Corollary 4.13 we assume that $(\mathcal{U}, \phi)$ is a coordinate chart about $x_0$ with $\mathcal{U}$ connected and that we have real analytic sections $\xi_1, \ldots, \xi_k$ of $\mathsf{E}|\mathcal{U}$ such that

$$\mathsf{F}_x = \mathrm{span}_{\mathbb{R}}(\xi_1(x), \ldots, \xi_k(x)), \qquad x \in \mathcal{U}.$$

Denote by $\boldsymbol{x} \mapsto (\boldsymbol{x}, \boldsymbol{\xi}_j(\boldsymbol{x}))$, $j \in \{1, \ldots, k\}$, the local representatives of the sections $\xi_1, \ldots, \xi_k$. Define $\boldsymbol{\xi}\colon \mathcal{U} \to \mathbb{R}^{n \times k}$ by

$$\boldsymbol{\xi}(x) = \left[\ \boldsymbol{\xi}_1(x) \ \middle|\ \cdots \ \middle|\ \boldsymbol{\xi}_k(x)\ \right].$$

Let us denote

$$\mathrm{grank}(\boldsymbol{\xi}) = \max\{\mathrm{rank}(\boldsymbol{\xi}(x)) \mid\ x \in \mathcal{U}\}.$$

For $m \in \{1, \ldots, \mathrm{grank}(\boldsymbol{\xi})\}$ define

$$\mathcal{U}_m = \{x \in \mathcal{U} \mid\ \mathrm{rank}(\boldsymbol{\xi}(x)) < m\}.$$

Fix $m \in \{1, \dots, \operatorname{grank}(\boldsymbol{\xi})\}$. We claim that if $\mathcal{U}_m \neq \varnothing$ then it is closed with empty interior. Indeed, $x \in \mathcal{U}_m$ if and only if the determinants of all $m \times m$ submatrices of $\boldsymbol{\xi}(x)$ vanish. Thus $\mathcal{U}_m$ is analytic. Note that the set of points where the determinant of a fixed $m \times m$ submatrix vanishes is closed, being the preimage of $0 \in \mathbb{R}$ under the real analytic (and so continuous) determinant function. Thus $\mathcal{U}_m$ is the intersection of a finite collection of closed sets, and so is closed. Suppose that $\mathcal{U}_m$ has a nonempty interior. By the Identity Theorem (this is proved in the holomorphic case in [Gunning 1990a, Theorem A.3], and the same proof applies to the real analytic case) and connectedness of $\mathcal{U}$ it follows that all the determinants of all $m \times m$ submatrices of $\boldsymbol{\xi}$ vanish on $\mathcal{U}$. This contradicts the fact that $m < \operatorname{grank}(\boldsymbol{\xi})$ and the definition of $\operatorname{grank}(\boldsymbol{\xi})$.

To complete this part of the theorem, we claim that $x \in \mathcal{U}$ is a singular point for $\mathsf{F}|\mathcal{U}$ if and only if $x \in \mathcal{U}_{\operatorname{grank}(\boldsymbol{\xi})}$. The assertion is trivial if $\mathcal{U}_{\operatorname{grank}(\boldsymbol{\xi})} = \varnothing$, so we suppose otherwise. If $x \in \mathcal{U}_{\operatorname{grank}(\boldsymbol{\xi})}$ then the fact that $\mathcal{U}_{\operatorname{grank}(\boldsymbol{\xi})}$ has empty interior ensures that $x$ is not a regular point for $\mathsf{F}|\mathcal{U}$. Conversely, if $x \notin \mathcal{U}_{\operatorname{grank}(\boldsymbol{\xi})}$ then $\operatorname{rank}(\mathsf{F}_x) = \operatorname{grank}(\boldsymbol{\xi})$ and by Proposition 3.6 we conclude that $x$ is a regular point for $\mathsf{F}|\mathcal{U}$.

(ii) We suppose that $\mathsf{M}$ is connected so, consequently, $\mathsf{M}$ has a well-defined dimension. By our above constructions, for each $x \in \mathsf{M}$ let $(\mathcal{U}_x, \phi_x)$ be a chart for $\mathsf{M}$ about $x$ and define

$$\operatorname{grank}_{\mathsf{F}}(x) = \max\{\operatorname{rank}(\mathsf{F}_y) \mid \ y \in \mathcal{U}_x\}.$$

Let $x_1, x_2 \in \mathsf{M}$ and since $\mathsf{M}$ is path connected by [Abraham, Marsden, and Ratiu 1988, Proposition 1.6.7], let $\gamma\colon [0,1] \to \mathsf{M}$ be a continuous curve for which $\gamma(0) = x_1$ and $\gamma(1) = x_2$. Define $\phi\colon [0,1] \to \mathbb{Z}_{\geq 0}$ by

$$\phi(s) = \operatorname{grank}_{\mathsf{F}}(\gamma(s)).$$

We claim that $\phi$ is locally constant. Indeed, let $s_0 \in [0,1]$ and let $\delta \in \mathbb{R}_{>0}$ be such that $\gamma(s) \in \mathcal{U}_{\gamma(s_0)}$ for $s \in [0,1] \cap (s_0 - \delta, s_0 + \delta)$. From the first part of the proof, the sets

$$\{x \in \mathcal{U}_{\gamma(s_0)} \mid \ \operatorname{rank}(\mathsf{F}_x) < \phi(s_0)\}$$

and

$$\{x \in \mathcal{U}_{\gamma(s)} \mid \ \operatorname{rank}(\mathsf{F}_x) < \phi(s)\}$$

are closed subsets with empty interior in $\mathcal{U}_{\gamma(s_0)}$ and $\mathcal{U}_{\gamma(s)}$, respectively. It follows that the sets

$$\{x \in \mathcal{U}_{\gamma(s)} \cap \mathcal{U}_{\gamma(s_0)} \mid \ \operatorname{rank}(\mathsf{F}_x) < \phi(s_0)\}$$

and

$$\{x \in \mathcal{U}_{\gamma(s)} \cap \mathcal{U}_{\gamma(s_0)} \mid \ \operatorname{rank}(\mathsf{F}_x) < \phi(s)\}$$

are closed with empty interior in $\mathcal{U}_{\gamma(s)} \cap \mathcal{U}_{\gamma(s_0)}$. Therefore, for

$$x \in \mathcal{U}_{\gamma(s)} \cap \mathcal{U}_{\gamma(s_0)} \setminus \big(\{x \in \mathcal{U}_{\gamma(s)} \cap \mathcal{U}_{\gamma(s_0)} \mid \ \operatorname{rank}(\mathsf{F}_x) < \phi(s_0)\} \\ \cup \{x \in \mathcal{U}_{\gamma(s)} \cap \mathcal{U}_{\gamma(s_0)} \mid \ \operatorname{rank}(\mathsf{F}_x) < \phi(s)\}\big),$$

$\phi(s) = \operatorname{rank}(\mathsf{F}_x) = \phi(s_0)$. Thus $\phi$ is indeed locally constant, and so constant since $[0,1]$ is connected. ■

The notions of regularity and singularity bear on the character of local generators for a generalised subbundle.

**3.8 Proposition: (A useful class of local generators for generalised subbundles)** *Let* $\mathsf{M}$ *be a manifold of class* $C^\infty$ *or* $C^\omega$*, as is required, let* $r \in \mathbb{Z}_{\geq 0} \cup \{\infty, \omega\}$*, and let* $\mathsf{F}$ *be a* $C^r$*-generalised subbundle on* $\mathsf{M}$*. Then, for each* $x_0 \in \mathsf{M}$*, there exists a neighbourhood* $\mathcal{N}$ *of* $x_0$ *and local generators* $(\xi_1, \dots, \xi_k)$ *for* $\mathsf{F}$ *on* $\mathcal{N}$ *with the following properties:*

*(i)* $(\xi_1(x_0), \dots, \xi_m(x_0))$ *is a basis for* $\mathsf{F}_{x_0}$*;*

*(ii)* $\xi_{m+1}(x_0) = \cdots = \xi_k(x_0) = 0_{x_0}$*.*

*In particular, if* $x_0$ *is a regular point for* $\mathsf{F}$*, the sections* $(\xi_1, \dots, \xi_m)$ *are local generators for* $\mathsf{F}$ *in some neighbourhood (possibly smaller than* $\mathcal{N}$*) of* $x_0$*.*

Proof: Let $(\eta_1, \dots, \eta_k)$ be local generators for $\mathsf{F}$ defined on a neighbourhood $\mathcal{N}$ of $x_0$. We can assume there are finitely many of these by Theorem 5.1 in the case when $r \in \mathbb{Z}_{\geq 0} \cup \{\infty\}$ and by Corollary 4.13 in case $r = \omega$. We may rearrange the sections $(\eta_1, \dots, \eta_k)$ so that $(\eta_1(x_0), \dots, \eta_m(x_0))$ forms a basis for $\mathsf{F}_{x_0}$. We then let $(\boldsymbol{v}_{m+1}, \dots, \boldsymbol{v}_k) \subseteq \mathbb{R}^k$ be a basis for the kernel of the linear map $L_{x_0} \colon \mathbb{R}^k \to \mathsf{F}_{x_0}$ defined by

$$L_{x_0}(\boldsymbol{v}) = \sum_{j=1}^{k} v_j \eta_j(x_0).$$

Define an invertible $k \times k$ matrix $\boldsymbol{R}$ by

$$\boldsymbol{R} = \left[\ \boldsymbol{e}_1 \ \middle|\ \cdots \ \middle|\ \boldsymbol{e}_m \ \middle|\ \boldsymbol{v}_{m+1} \ \middle|\ \cdots \ \middle|\ \boldsymbol{v}_k\ \right],$$

where $\boldsymbol{e}_j \in \mathbb{R}^k$, $j \in \{1, \dots, k\}$, is the $j$th standard basis vector, and define $\xi_1, \dots, \xi_k$ by

$$\xi_j = \sum_{l=1}^{k} R_{lj} \eta_l, \qquad j \in \{1, \dots, k\}.$$

Then $\xi_j = \eta_j$ for $j \in \{1, \dots, k\}$, and $\xi_j(x_0) = L_{x_0}(\boldsymbol{v}_j) = 0_{x_0}$ for $j \in \{m+1, \dots, k\}$. This gives the first conclusion of the proposition.

For the second, if $x_0$ is a regular point for $\mathsf{F}$, then let $\mathsf{F}'$ be the generalised subbundle on $\mathcal{N}$ generated by $(\xi_1, \dots, \xi_m)$. Then, for $x$ in a neighbourhood $\mathcal{N}'$ of $x_0$, $\mathrm{rank}_{\mathsf{F}'}(x) = m$ by lower semicontinuity of $\mathrm{rank}_{\mathsf{F}'}$. Since $\mathsf{F}' \subseteq \mathsf{F}$ it follows that $\mathsf{F}'|\mathcal{N} = \mathsf{F}|\mathcal{N}$. ■

**3.3. Generalised subbundles and subsheaves of sections.** One of the features of this paper is that the rôle of sheaves in distribution theory is made explicit and we illustrate how the theory leads to nontrivial fundamental results, particularly for real analytic distributions. In this section we make connections with sheaf theory to our definitions for generalised subbundles in the preceding section. We shall see that there is a relationship between generalised subbundles of $\pi \colon \mathsf{E} \to \mathsf{M}$ and subsheaves of $\mathscr{G}^r_{\mathsf{E}}$, but the relationship is not always a perfect correspondence.

Let us first consider the natural subsheaf arising from a generalised subbundle.

**3.9 Definition: (Sheaf of sections of a generalised subbundle)** Let $\pi \colon \mathsf{E} \to \mathsf{M}$ be a smooth or real analytic vector bundle, as is required, let $r \in \mathbb{Z}_{\geq 0} \cup \{\infty, \omega\}$, and let $\mathsf{F} \subseteq \mathsf{E}$ be a generalised subbundle of class $C^r$. The ***sheaf of sections*** of $\mathsf{F}$ is the sheaf $\mathscr{G}^r_{\mathsf{F}}$ whose local sections over the open set $\mathcal{U} \subseteq \mathsf{M}$ is the set of sections of $\mathsf{F}|\mathcal{U}$. •

Clearly $\mathscr{G}^r_{\mathsf{F}}$ is a subsheaf of $\mathscr{C}^r_{\mathsf{M}}$-modules of $\mathscr{G}^r_{\mathsf{E}}$. Note that a $C^r$-generalised subbundle $\mathsf{F}$, by definition, is constructed using local generators. It is clear that each local generator is a local section of the sheaf $\mathscr{G}^r_{\mathsf{F}}$. What is not generally true is that the local generators for the distribution generate the stalks of the sheaf $\mathscr{G}^r_{\mathsf{F}}$, as the following example shows.

**3.10 Example: (Generators for a generalised subbundle may not generate the stalks of the corresponding sheaf)** We take $\mathsf{M} = \mathbb{R}$ and $\mathsf{E} = \mathbb{R} \times \mathbb{R}$ with the vector bundle projection $\pi \colon \mathsf{E} \to \mathsf{M}$ being projection onto the first factor: $\pi(x, v) = x$. We let $r \in \mathbb{Z}_{>0} \cup \{\infty, \omega\}$. For $x \in \mathbb{R}$ we consider the neighbourhood $\mathcal{N}_x = \mathbb{R}$ of $x$ and on $\mathcal{N}_x$ we take the local (in fact, global in this case) section $\xi_x(y) = (y, y^2)$. Thus, in this case, the neighbourhood of $x$ and the local generators defined on this neighbourhood are the same for each $x$. This cannot be expected to be the usual situation, but in this simple example it turns out to be possible. In any case, this data then defines a $C^r$-generalised subbundle $\mathsf{F}$ by

$$\mathsf{F}_x = \begin{cases} \{x\} \times \mathbb{R}, & x \neq 0, \\ \{x\} \times \{0\}, & x = 0. \end{cases}$$

Associated to this generalised subbundle we have the subsheaf $\mathscr{G}^r_{\mathsf{F}}$ of sections of $\mathsf{F}$. We can describe this subsheaf explicitly. Indeed, we claim that if $\mathcal{U} \subseteq \mathbb{R}$ is open, then any section of $\mathsf{F}|\mathcal{U}$ is a $C^r(\mathcal{U})$-multiple of the section $\xi \colon x \mapsto (x, x)$. This is clear if $0 \notin \mathcal{U}$ since $\xi$ is then a nonvanishing section of a one-dimensional vector bundle. So we need only consider the case when $0 \in \mathcal{U}$. In this case, $\mathcal{U}$, being an open subset of $\mathbb{R}$, is a union of open intervals, one of which contains 0. On the other open intervals any section is clearly a multiple of $\xi$, again since $\xi$ is a nonvanishing section of a one-dimensional vector bundle. Thus we need only show that, if $a < 0 < b$, then any $C^r$-section $\eta$ of $\mathsf{E}|(a, b)$ is a multiple of $\xi$. In the following calculation, we identify $\eta$ with a $\mathbb{R}$-valued function on $(a, b)$. For $x \in (a, b)$ compute

$$\eta(x) = \int_0^x \eta'(t)\,\mathrm{d}t = x \int_0^1 \eta'(xs)\,\mathrm{d}s.$$

The function $\bar{\eta} \colon x \mapsto \int_0^1 \eta'(xs)\,\mathrm{d}s$ is of class $C^r$ and so $\eta = \mathrm{id}_{(a,b)} \cdot \bar{\eta}$. Thus every $C^r$-section $\eta$ on $(a, b)$ vanishing at 0 is a product of $\mathrm{id}_{(a,b)}$ and a function of class $C^r$, i.e., $\eta(x) = x\bar{\eta}(x)$. This characterises local sections of $\mathscr{G}^r_{\mathsf{F}}$ over $\mathcal{U}$ as multiples of $\xi$, as claimed.

Note, however, that the generator $\xi_0$ for $\mathsf{F}$ about $x = 0$ does not generate the stalk $\mathscr{G}^r_{0,\mathsf{F}}$ since, for example, the germ of the local section $y \mapsto (y, y)$ is not a $\mathscr{C}^r_{0,\mathsf{M}}$-multiple of $[\xi_0]_0$. •

By our above considerations, we can proceed from generalised subbundles to subsheaves, keeping in mind the caution that the local generators for the generalised subbundles are not necessarily generators for stalks of the sheaf. We can also proceed from subsheaves to generalised subbundles of $\mathsf{E}$.

**3.11 Definition: (The generalised subbundle associated to a subsheaf)** Let $\pi \colon \mathsf{E} \to \mathsf{M}$ be a smooth or real analytic vector bundle, as is required, let $r \in \mathbb{Z}_{\geq 0} \cup \{\infty, \omega\}$, and let $\mathscr{F}$ be a subsheaf of $\mathscr{C}^r_{\mathsf{M}}$-modules of $\mathscr{G}^r_{\mathsf{E}}$. The ***generalised subbundle generated by*** $\mathscr{F}$ is defined by

$$\mathsf{F}(\mathscr{F})_x = \mathrm{span}_{\mathbb{R}}(\xi(x) \mid\ [\xi]_x \in \mathscr{F}_x).$$

If $\mathscr{F} \subseteq \Gamma^r(\mathsf{TM})$ is a subsheaf of vector fields, we will use the notation $\mathsf{D}(\mathscr{F})$ to denote the ***distribution generated by*** $\mathscr{F}$, i.e.,

$$\mathsf{D}(\mathscr{F})_x = \mathrm{span}_{\mathbb{R}}(X(x) \mid\ [X]_x \in \mathscr{F}_x). \qquad \bullet$$

There are two obvious questions related to this inclusion that come to mind at this point.

1. Is every generalised subbundle generated by some subsheaf of $\mathscr{G}^r_{\mathsf{E}}$ of class $C^r$?
2. It is clear that $\mathscr{F} \subseteq \mathscr{G}^r_{\mathsf{F}(\mathscr{F})}$. If some generalised subbundle $\mathsf{F}$ is generated by a subsheaf $\mathscr{F}$, is it true that $\mathscr{F} = \mathscr{G}^r_{\mathsf{F}}$?

Let us address these questions in order.

The answer to the first question is, "Yes," in the smooth case. In fact, in the smooth case one can say much more.

**3.12 Proposition: (Smooth subsheaves possess global generators for stalks)** *Let $r \in \mathbb{Z}_{\geq 0} \cup \{\infty\}$ and let $\pi\colon \mathsf{E} \to \mathsf{M}$ be a smooth vector bundle with $\mathsf{M}$ paracompact and Hausdorff. If $\mathscr{F} = (F(\mathcal{U}))_{\mathcal{U}\ open}$ is a subsheaf of $\mathscr{G}^r_{\mathsf{E}}$ then the map*

$$F(\mathsf{M}) \ni \xi \mapsto [\xi]_x \in \mathscr{F}_x$$

*is surjective for each $x \in \mathsf{M}$. In particular, the generalised subbundle $\mathsf{F}(\mathscr{F})$ generated by $\mathscr{F}$ is of class $C^r$.*

Proof: Let $x_0 \in \mathsf{M}$ and let $[(\xi, \mathcal{U})]_{x_0} \in \mathscr{F}_{x_0}$. Let $\mathcal{V} \subseteq \mathcal{W}$ be relatively compact neighbourhoods of $x_0$ such that

$$\mathrm{cl}(\mathcal{V}) \subseteq \mathcal{W} \subseteq \mathrm{cl}(\mathcal{W}) \subseteq \mathcal{U},$$

and, by the Tietze Extension Theorem [Abraham, Marsden, and Ratiu 1988, Proposition 5.5.8], let $f \in C^\infty(\mathsf{M})$ be such that $f$ takes the value 1 on $\mathcal{V}$ and vanishes on the complement of $\mathcal{W}$. We then define

$$\bar{\xi}(x) = \begin{cases} f(x)\xi(x), & x \in \mathcal{U}, \\ 0, & \text{otherwise}, \end{cases}$$

We claim that $\bar{\xi} \in F(\mathsf{M})$. Let $x \in \mathsf{M}$. If $x \in \mathrm{cl}(\mathcal{W})$ then let $\mathcal{U}_x \subseteq \mathcal{U}$ be a neighbourhood of $x$ and let $\xi_x = \bar{\xi}|\mathcal{U}_x$, noting that $\xi_x \in F(\mathcal{U}_x)$. If $x \in \mathsf{M} \setminus \mathrm{cl}(\mathcal{W})$ then let $\mathcal{U}_x$ be a neighbourhood of $x$ such that $\mathcal{U}_x \subseteq \mathsf{M} \setminus \mathrm{cl}(\mathcal{W})$ and let $\xi_x$ be the zero section of $\mathcal{U}_x$. Note that $\xi_x \in F(\mathcal{U}_x)$. Since $\mathscr{F}$ is a sheaf, it follows that there exists $\eta \in F(\mathsf{M})$ such that $\eta|\mathcal{U}_x = \xi_x$ for every $x \in \mathsf{M}$. Again since $\mathscr{F}$ is a sheaf, $\eta = \bar{\xi}$, showing that $\bar{\xi} \in F(\mathsf{M})$, as desired. Finally, since $[(\xi, \mathcal{U})]_{x_0} = [(\bar{\xi}, \mathsf{M})]_x$, the first assertion of the proposition follows.

The second assertion follows since the first assertion implies that we can choose global smooth generators for $\mathsf{F}(\mathscr{F})$. ■

Now let us show that the analogue of the preceding result does not hold in the real analytic case.

**3.13 Example: (Analytic subsheaves may not generate analytic generalised subbundles)** We take $\mathsf{M} = \mathbb{R}$ and consider the trivial vector bundle $\mathsf{E} = \mathbb{R} \times \mathbb{R}$ with projection $\pi(x, v) = x$. We take $S = \{0\} \cup \{\frac{1}{j} \mid j \in \mathbb{Z}_{>0}\}$ and define a subsheaf $\mathscr{F}_S$ of $\mathscr{G}^\omega_{\mathsf{E}}$ by

$$\mathscr{F}_S(\mathcal{U}) = \{\xi \in \Gamma^\omega(\mathsf{E}|\mathcal{U}) \mid \xi(x) = 0 \text{ for all } x \in \mathcal{U} \cap S\}.$$

It is clear that

$$\mathsf{F}(\mathscr{F}_S)_x = \begin{cases} \{x\} \times \mathbb{R}, & x \notin S, \\ \{(x, 0)\}, & x \in S. \end{cases}$$

By the Identity Theorem for real analytic functions [Krantz and Parks 2002, Corollary 1.2.6], the stalk $\mathscr{F}_{S,0}$ of $\mathscr{F}_S$ at 0 consists of germs of functions that vanish in some neighbourhood of 0, i.e., $\mathscr{F}_{S,0} = \{0\}$. More precisely, if $\mathcal{N}$ is a connected neighbourhood of 0, then $\mathscr{F}_S(\mathcal{N}) = \{0\}$. This precludes the existence of real analytic local generators in any neighbourhood of 0, and so $\mathsf{F}(\mathscr{F}_S)$ is not real analytic. •

This addresses the first of our questions. As to the second question concerning the relationship between $\mathscr{F}$ and $\mathscr{G}^r_{\mathsf{F}(\mathscr{F})}$, we consider the following example.

**3.14 Example: (A subsheaf strictly contained in the subsheaf of sections of the generalised subbundle it generates)** We take $\mathsf{M} = \mathbb{R}$ and $\mathsf{E} = \mathbb{R} \times \mathbb{R}$ with the vector bundle projection $\pi\colon \mathsf{E} \to \mathsf{M}$ being projection onto the first factor. We consider the real analytic subsheaf $\mathscr{F}$ generated by the global section $x \mapsto (x, x^2)$. That is, the local sections of $\mathscr{F}$ over $\mathcal{U} \subseteq \mathbb{R}$ are the $\mathscr{C}^\omega_{\mathcal{U}}$-multiples of this section restricted to $\mathcal{U}$. As in Example 3.10, the generalised subbundle generated by this subsheaf is

$$\mathsf{F}(\mathscr{F})_x = \begin{cases} \{x\} \times \mathbb{R}, & x \neq 0, \\ \{x\} \times \{0\}, & x = 0. \end{cases}$$

The section $x \mapsto (x, x)$ is a section of $\Gamma^\omega(\mathsf{F}(\mathscr{F}))$ that is not a global section of $\mathscr{F}$, showing that $\mathscr{F} \subset \mathscr{G}^\omega_{\mathsf{F}(\mathscr{F})}$. •

**3.4. Generators for submodules and subsheaves of sections.** One often wishes to define a submodule of sections, or a subsheaf of sections, of a vector bundle by using generators. In this section we give the notation for doing this, and establish a few elementary consequences of these constructions.

Let us consider first the nonsheaf situation where we have families of globally defined sections.

**3.15 Definition: (Generalised subbundles and modules of sections generated by a family of sections)** Let $r \in \mathbb{Z}_{\geq 0} \cup \{\infty, \omega\}$ and let $\pi\colon \mathsf{E} \to \mathsf{M}$ be a smooth or real analytic manifold, as required. If $\mathscr{X} = (\xi_j)_{j \in J}$ is a family of sections of $\mathsf{E}$,

(i) the ***generalised subbundle generated by*** $\mathscr{X}$ is

$$\mathsf{F}(\mathscr{X})_x = \operatorname{span}_{\mathbb{R}}(\xi_j(x)|\ j \in J)$$

and

(ii) the ***module of sections generated by*** $\mathscr{X}$ is

$$\langle \mathscr{X} \rangle = \operatorname{span}_{C^r(\mathsf{M})}(\xi_j|\ j \in J).$$

In the special case $\mathsf{E} = \mathsf{TM}$ and where $\mathscr{X} = (X_j)_{j \in J}$ is a family of vector fields, the ***distribution generated by*** $\mathscr{X}$ is denoted by $\mathsf{D}(\mathscr{X})$. •

The following elementary result is often useful to make certain assumptions without losing generality.

**3.16 Proposition: (Generalised subbundles generated by families of sections)** *Let $r \in \mathbb{Z}_{\geq 0} \cup \{\infty, \omega\}$, let $\pi\colon \mathsf{E} \to \mathsf{M}$ be a smooth or real analytic vector bundle, as required, and let $\mathscr{X} \subseteq \Gamma^r(\mathsf{E})$. Then the generalised subbundles*

*(i)* $\mathsf{F}(\mathscr{X})$,

*(ii)* $\mathsf{F}(\mathrm{span}_{\mathbb{R}}(\mathscr{X}))$, *and*

*(iii)* $\mathsf{F}(\langle\mathscr{X}\rangle)$

*agree.*

Proof: Since we clearly have

$$\mathsf{F}(\mathscr{X}) \subseteq \mathsf{F}(\mathrm{span}_{\mathbb{R}}(\mathscr{X})) \subseteq \mathsf{F}(\langle\mathscr{X}\rangle),$$

it suffices to show that $\mathsf{F}(\langle\mathscr{X}\rangle) = \mathsf{F}(\mathscr{X})$. We have

$$\begin{aligned}
&\mathsf{F}(\langle\mathscr{X}\rangle)_x \\
&\quad = \mathrm{span}_{\mathbb{R}}((f^1\xi_1 + \cdots + f^k\xi_k)(x) \mid\ k \in \mathbb{Z}_{>0},\ f^1, \dots, f^k \in C^r(\mathsf{M}),\ \xi_1, \dots, \xi_k \in \mathscr{X}) \\
&\quad = \mathrm{span}_{\mathbb{R}}(a^1\xi_1(x) + \cdots + a^k\xi_k(x) \mid\ k \in \mathbb{Z}_{>0},\ a^1, \dots, a^k \in \mathbb{R},\ \xi_1, \dots, \xi_k \in \mathscr{X}) \\
&\quad = \mathrm{span}_{\mathbb{R}}(\xi(x) \mid\ \xi \in \mathscr{X}) = \mathsf{F}(\mathscr{X})_x,
\end{aligned}$$

giving the result. ■

Let us now turn to sheaves generated by families of locally defined sections.

**3.17 Definition: (Sheaf of modules and generalised subbundles generated by a sheaf of sets of sections)** Let $r \in \mathbb{Z}_{\geq 0} \cup \{\infty, \omega\}$ and let $\pi\colon \mathsf{E} \to \mathsf{M}$ be a smooth or real analytic manifold, as required. Let $\mathscr{X} = (X(\mathcal{U}))_{\mathcal{U}\,\mathrm{open}}$ be a subsheaf of sets of the sheaf $\mathscr{G}^r_{\mathsf{E}}$—i.e., an assignment to each open set $\mathcal{U} \subseteq \mathsf{M}$ a subset $X(\mathcal{U}) \subseteq \Gamma^r(\mathsf{E}|\mathcal{U})$ with the assignment satisfying $X(\mathcal{V}) = r_{\mathcal{U},\mathcal{V}}(X(\mathcal{U}))$ for every pair of open sets $\mathcal{U}, \mathcal{V}$ for which $\mathcal{V} \subseteq \mathcal{V}$.

(i) The ***sheaf of $\mathscr{C}^r_{\mathsf{M}}$-modules generated by $\mathscr{X}$*** is the subsheaf $\langle\mathscr{X}\rangle = (\langle X\rangle(\mathcal{U}))_{\mathcal{U}\,\mathrm{open}}$ of $\mathscr{C}^r_{\mathsf{M}}$-modules defined by $\langle X\rangle(\mathcal{U}) = \langle X(\mathcal{U})\rangle$.

(ii) The ***generalised subbundle generated by $\mathscr{X}$*** is defined by

$$\mathsf{F}(\mathscr{X})_x = \mathrm{span}_{\mathbb{R}}(\xi(x) \mid\ [\xi]_x \in \mathscr{X}_x).$$

(iii) If $\mathsf{E} = \mathsf{TM}$ and so $\mathscr{X}$ is a subsheaf of sets of $\mathscr{G}^r_{\mathsf{TM}}$, then the ***distribution generated by $\mathscr{X}$*** is defined by $\mathsf{D}(\mathscr{X}) = \mathsf{F}(\mathscr{X})$. •

We can also adapt Proposition 3.16 to the sheaf setting.

**3.18 Proposition: (Generalised subbundles generated by families of sections)** *Let $r \in \mathbb{Z}_{\geq 0} \cup \{\infty, \omega\}$, let $\pi\colon \mathsf{E} \to \mathsf{M}$ be a smooth or real analytic vector bundle, as required, and let $\mathscr{X} = (X(\mathcal{U}))_{\mathcal{U}\,open}$ be a subsheaf of sets of $\mathscr{G}^r_{\mathsf{E}}$. Then, the generalised subbundles*

*(i)* $\mathsf{F}(\mathscr{X})$ *and*

*(ii)* $\mathsf{F}(\langle\mathscr{X}\rangle)$

*agree.*

Proof: This follows immediately from Proposition 3.16. ■

Of course, it is also true that both generalised subbundles in the statement of the preceding result agree with the generalised subbundle whose fibre at $x$ is given by

$$\operatorname{span}_{\mathbb{R}}(\xi(x)|\ [\xi]_x \in \mathscr{X}_x),$$

the setting for this being to think of $\mathscr{G}^r_{\mathsf{E}}$ as a sheaf of $\mathbb{R}$-vector spaces. We leave it to the reader to develop the attendant definitions as we shall not make use of any of these.

Note that the specification of a family $\mathscr{X} = (\xi_j)_{j\in J}$ of global sections of $\pi\colon \mathsf{E} \to \mathsf{M}$ also prescribes a subsheaf of $\mathscr{G}^r_{\mathsf{E}}$.

**3.19 Definition: (Subsheaves defined by a family of global sections)** Let $r \in \mathbb{Z}_{\geq 0} \cup \{\infty, \omega\}$, let $\pi\colon \mathsf{E} \to \mathsf{M}$ be a smooth or real analytic vector bundle, as required, and let $\mathscr{X} \subseteq \Gamma^r(\mathsf{E})$.

(i) For $\mathcal{U} \subseteq \mathsf{M}$ open, the ***restriction*** of $\mathscr{X}$ to $\mathcal{U}$ is

$$\mathscr{X}|\mathcal{U} = \{\xi|\mathcal{U} \mid \xi \in \mathscr{X}\}.$$

(ii) The ***subsheaf generated by*** $\mathscr{X}$ is the subsheaf $\mathscr{F}_{\mathscr{X}} = (F_{\mathscr{X}}(\mathcal{U}))_{\mathcal{U}\ \text{open}}$ of $\mathscr{G}^r_{\mathsf{E}}$ defined by

$$F_{\mathscr{X}}(\mathcal{U}) = \operatorname{span}_{C^r(\mathcal{U})}(\mathscr{X}|\mathcal{U}).$$ •

Thus prescribing the data of a subsheaf is more general than prescribing a globally defined submodule.

**3.20 Definition: (Finitely generated submodule, locally finitely generated submodule)** Let $r \in \mathbb{Z}_{\geq 0} \cup \{\infty, \omega\}$, let $\pi\colon \mathsf{E} \to \mathsf{M}$ be a smooth or real analytic vector bundle, as required, and let $\mathscr{M} \subseteq \Gamma^r(\mathsf{E})$ be a submodule of sections.

(i) The submodule $\mathscr{M}$ is ***finitely generated*** if it is finitely generated in the usual sense.

(ii) The submodule $\mathscr{M}$ is ***locally finitely generated*** if, for each $x \in \mathsf{M}$, there exists a neighbourhood $\mathcal{U}$ of $x$ such that the submodule $F_{\mathscr{M}}(\mathcal{U})$ of the $C^r(\mathcal{U})$-module $\Gamma^r(\mathsf{E}|\mathcal{U})$ is finitely generated. •

We have on hand now a few different situations where we can apply the notion of being locally finitely generated. We refer to Section 4.4 for a discussion of the property of local finite generation for various objects.

### 3.5. Patchy subsheaves.

Example 3.10 indicates that generators for a generalised subbundle do not generally serve as generators for stalks of the associated subsheaf of sections. In this section we describe a construction where local generators *can* be used to define a subsheaf. This requires that we place some compatibility conditions on local generators.

**3.21 Proposition: (A class of subsheaves of the module of sections of a vector bundle)** *Let $r \in \mathbb{Z}_{\geq 0} \cup \{\infty, \omega\}$ and let $\pi\colon \mathsf{E} \to \mathsf{M}$ be a smooth or real analytic vector bundle, as is required. Consider the following data:*

*(i) an open cover $\mathscr{U} = (\mathcal{U}_a)_{a\in A}$ for $\mathsf{M}$;*

*(ii) for each $a \in A$, a family $\mathscr{X}_a = (\xi_b)_{b\in B_a}$ of $C^r$-sections of $\mathsf{E}|\mathcal{U}_a$ such that, if $\mathcal{U}_{a_1} \cap \mathcal{U}_{a_2} \neq \varnothing$, then*

$$\langle \mathscr{X}_{a_1} | \mathcal{U}_{a_1} \cap \mathcal{U}_{a_2} \rangle = \langle \mathscr{X}_{a_2} | \mathcal{U}_{a_1} \cap \mathcal{U}_{a_2} \rangle;$$

*(iii) for each $a \in A$, the sheaf $\mathscr{F}_a = (F_a(\mathcal{U}))_{\mathcal{U} \subseteq \mathcal{U}_a \text{ open}}$ of $\mathscr{C}^r_{\mathcal{U}_a}$-modules given by*

$$F_a(\mathcal{U}) = \langle \mathscr{X}_a | \mathcal{U} \rangle.$$

*Then there exists a unique subsheaf $\mathscr{F}_{\mathscr{U}} = (F_{\mathscr{U}}(\mathcal{U}))_{\mathcal{U} \text{ open}}$ of $\mathscr{G}^r_{\mathsf{E}}$ with the property that $\mathscr{F}_{\mathscr{U}} | \mathcal{U}_a = \mathscr{F}_a$ for each $a \in A$.*

Proof: For $\mathcal{U} \subseteq \mathsf{M}$ open we define

$$F(\mathcal{U}) = \big\{ (\xi_a)_{a \in A} \big| \ \xi_a \in F_a(\mathcal{U} \cap \mathcal{U}_a), \ a \in A, \\ \xi_{a_1} | \mathcal{U} \cap \mathcal{U}_{a_1} \cap \mathcal{U}_{a_2} = \xi_{a_2} | \mathcal{U} \cap \mathcal{U}_{a_1} \cap \mathcal{U}_{a_2}, \ a_1, a_2 \in A \big\}.$$

For $\mathcal{U}, \mathcal{V} \subseteq \mathsf{M}$ open and satisfying $\mathcal{V} \subseteq \mathcal{U}$, define $r_{\mathcal{U},\mathcal{V}} \colon F(\mathcal{U}) \to F(\mathcal{V})$ by

$$r_{\mathcal{U},\mathcal{V}}((\xi_a)_{a \in A}) = (\xi_a | \mathcal{V} \cap \mathcal{U}_a)_{a \in A}.$$

We will verify that $\mathscr{F} = (F(\mathcal{U}))_{\mathcal{U} \text{ open}}$ is a sheaf.

Let $\mathcal{W} \subseteq \mathsf{M}$ be open and let $(\mathcal{W}_i)_{i \in I}$ be an open cover for $\mathcal{W}$. Let $\xi, \eta \in F(\mathcal{W})$ satisfy $r_{\mathcal{W},\mathcal{W}_i}(\xi) = r_{\mathcal{W},\mathcal{W}_i}(\eta)$ for each $i \in I$. We write $\xi = (\xi_a)_{a \in A}$ and $\eta = (\eta_a)_{a \in A}$ and note that we have

$$\xi_a | \mathcal{W}_i \cap \mathcal{U}_a = \eta_a | \mathcal{W}_i \cap \mathcal{U}_a, \qquad a \in A, \ i \in I.$$

Since $\mathscr{F}_a$ is separated, $\xi_a = \eta_a$ for each $a \in A$ and so $\xi = \eta$.

Let $\mathcal{W} \in \mathscr{O}$ and let $(\mathcal{W}_i)_{i \in I}$ be an open cover for $\mathcal{W}$. For each $i \in I$ let $\xi_i \in F(\mathcal{W}_i)$ and suppose that $r_{\mathcal{W}_i,\mathcal{W}_i \cap \mathcal{W}_j}(\xi_i) = r_{\mathcal{W}_j,\mathcal{W}_i \cap \mathcal{W}_j}(\xi_j)$ for each $i, j \in I$. We write $\xi_i = (\xi_{i,a})_{a \in A}$, $i \in I$, and note that

$$\xi_{i,a} | \mathcal{W}_i \cap \mathcal{W}_j \cap \mathcal{U}_a = \xi_{j,a} | \mathcal{W}_i \cap \mathcal{W}_j \cap \mathcal{U}_a, \qquad i, j \in I, \ a \in A.$$

Since $\mathscr{F}_a$ satisfies the gluing property, there exists $\xi_a \in F_a(\mathcal{W} \cap \mathcal{U}_a)$ such that

$$\xi_a | \mathcal{W}_i \cap \mathcal{U}_a = \xi_{i,a}, \qquad i \in I, \ a \in A.$$

Let us define $\xi = (\xi_a)_{a \in A}$. By (ii) we have

$$\xi_{i,a_1} | \mathcal{W}_i \cap \mathcal{U}_{a_1} \cap \mathcal{U}_{a_2} = \xi_{i,a_2} | \mathcal{W}_i \cap \mathcal{U}_{a_1} \cap \mathcal{U}_{a_2}, \qquad i \in A, \ a_1, a_2 \in A.$$

Thus

$$s_{a_1} | \mathcal{W} \cap \mathcal{U}_{a_1} \cap \mathcal{U}_{a_2} = s_{a_2} | \mathcal{W} \cap \mathcal{U}_{a_1} \cap \mathcal{U}_{a_2}, \qquad a_1, a_2 \in A,$$

and so $s$ as constructed is an element of $F(\mathcal{W})$.

The preceding shows that $\mathscr{F}$ is a sheaf of sets. To verify that it is a sheaf $\mathscr{C}^r_{\mathsf{M}}$-modules we define the algebraic operations in the obvious way by defining them in $F(\mathcal{U})$ by

$$(\xi_a)_{a \in A} + (\eta_a)_{a \in A} = (\xi_a + \eta_a)_{a \in A}, \qquad f\big((\xi_a)_{a \in A}\big) = ((f | \mathcal{U} \cap \mathcal{U}_a)\xi_a \cdot t_a)_{a \in A},$$

respectively. One easily verifies that these operations are well-defined, and that the restriction morphisms for $\mathscr{F}$ are $\mathscr{C}^r_{\mathsf{M}}$-module homomorphisms.

The uniqueness assertion of the proposition follows from the fact that our constructions obviously prescribe the stalks of $\mathscr{F}$, and so uniquely prescribe $\mathscr{F}$ since it is a sheaf. ■

Let us give a name to the sheaf defined as in the preceding result.

**3.22 Definition: (Patchy subsheaf of modules)** Let $r \in \mathbb{Z}_{\geq 0} \cup \{\infty, \omega\}$ and let $\pi\colon \mathsf{E} \to \mathsf{M}$ be a smooth or real analytic vector bundle, as is required. Consider the data $\mathscr{U} = (\mathcal{U}_a)_{a \in A}$ and, for each $a \in A$, $\mathscr{X}_a$ as in Proposition 3.21. We call the subsheaf $\mathscr{F}_{\mathscr{U}}$ of $\mathscr{C}^r_{\mathsf{M}}$-modules defined as in Proposition 3.21 the ***patchy subsheaf*** defined by the above data. •

Note that the specification of a submodule $\mathscr{M} \subseteq \Gamma^r(\mathsf{M})$ is a particular case of a patchy subsheaf corresponding to the cover of $\mathsf{M}$ by the single open set $\mathsf{M}$. Thus the specification of a submodule of $\Gamma^r(\mathsf{E})$ is a special case of specifying a subsheaf of $\mathscr{G}^r_{\mathsf{E}}$.

Given our definition of a generalised subbundle, the motivation for studying patchy sheaves is clear. Specifically, the notions of smoothness we provide for generalised subbundles are defined in terms of an open cover of the manifold, on each subset of which there are sections of the prescribed smoothness that generate the generalised subbundle. The patchy condition is simply one of ensuring that there is some compatibility on overlapping open sets. What is not clear is the extent to which a given subsheaf is patchy, and, if a subsheaf is patchy, what are the implications of this.

As to the patchiness of subsheaf, let us first consider the smooth case. $\mathscr{F}_{\mathscr{U}} \subseteq \mathscr{G}^r_{\mathsf{F}_{\mathscr{U}}}$, the opposite inclusion will not generally hold.

**3.23 Proposition: (Subsheaves of modules of smooth sections are patchy)** *If $r \in \mathbb{Z}_{\geq 0} \cup \{\infty\}$, if $\pi\colon \mathsf{E} \to \mathsf{M}$ is a smooth vector bundle, and if $\mathscr{F}$ is a subsheaf of $\mathscr{G}^r_{\mathsf{E}}$, then $\mathscr{F}$ is a patchy subsheaf.*

Proof: We will prove that we can take as our patchy subsheaf data the following:

1. the trivial open cover $\mathscr{U} = (\mathsf{M})$;
2. for the single element $\mathsf{M}$ of the open cover, the family $F(\mathsf{M})$ of global sections.

Thus we are in the situation of Definition 3.19 where we are considering a sheaf defined by a submodule of global sections. To show that $\mathscr{F} = \mathscr{F}_{\mathscr{U}}$, it is sufficient by [Hartshorne 1977, Proposition II.1.1] to prove that the stalk $\mathscr{F}_{x_0}$ is generated by global sections for every $x_0 \in \mathsf{M}$. This, however, has been shown in Proposition 3.12. ■

In the real analytic case, however, there are subsheaves that are not patchy.

**3.24 Example: (A subsheaf of real analytic sections that is not patchy)** We consider $\mathsf{M} = \mathbb{R}$, $\mathsf{E} = \mathbb{R} \times \mathbb{R}$, and $\pi(x, v) = x$. Let

$$S = \{\tfrac{1}{j} \mid j \in \mathbb{Z}_{>0}\} \cup \{0\}.$$

Consider the presheaf $\mathscr{I}_S = (I_S(\mathcal{U}))_{\mathcal{U}\ \mathrm{open}}$ given by

$$I_S(\mathcal{U}) = \{f \in \Gamma^\omega(\mathcal{U}) \mid f(x) = 0 \text{ for } x \in \mathcal{U} \cap S\}.$$

One can easily verify that $\mathscr{I}_S$ is a subsheaf of $\mathscr{G}^\omega_{\mathsf{E}}$. We claim $\mathscr{I}_S$ is not patchy. The sneakiest way to see this is as follows. Note that $\mathscr{I}_{S,0} = \{0\}$ since every section defined on a connected neighbourhood of 0 and vanishing on $S$ must be identically zero, as we saw in Example 3.13. However, note that if $x \neq 0$ then $\mathscr{I}_{S,x} \neq \{0\}$ and so, by Lemma 4.17, it follows that $\mathscr{I}_S$ cannot be locally finitely generated. By Theorem 4.10 it then follows that $\mathscr{I}_S$ is not patchy. •

However, in the real analytic case, one does have the following result, establishing a correspondence of sorts between patchy real analytic subsheaves and real analytic distributions.

**3.25 Theorem: (Patchy real analytic subsheaves)** *If $\pi\colon \mathsf{E} \to \mathsf{M}$ is a real analytic vector bundle, then the following statements hold:*

*(i) if $\mathsf{F}$ is a real analytic generalised subbundle of $\mathsf{E}$, then $\mathscr{G}^{\omega}_{\mathsf{F}}$ is patchy;*

*(ii) if $\mathscr{F}$ is a patchy subsheaf of $\mathscr{G}^{\omega}_{\mathsf{E}}$, then $\mathsf{F}(\mathscr{F})$ is a real analytic generalised subbundle.*

Proof: (i)

1. Fix some point $p \in \mathsf{M}$.
2. Let $r_1 < r_2 < \cdots < r_s$ be the ranks of the $\mathsf{F}_x$ (the images of the sections of $\mathsf{F}$ in the fibres of $\mathsf{E}$)
3. Let $\mathsf{M}_k$ be the (closed) set of points in the manifold $\mathsf{M}$ where the rank of $\mathsf{F}_x$ is $\leq r_k$ (so $\mathsf{M}_{k-1}$ is contained in $\mathsf{M}_k$). We only really care about those $\mathsf{M}_k$ which contain $p$, so let's ignore any that don't, and just assume that $p$ is in all of them, i.e., that $\mathsf{M}_1, \mathsf{M}_2, \dots, \mathsf{M}_s$ all contain $p$.
4. We now work inductively (descending on $k$), constructing subsheaves of $\mathscr{G}^{\omega}_{\mathsf{E}}$ that are equal to $\mathscr{F}$ away from $\mathsf{M}_{k-1}$. At each step we'll check that the resulting sheaf is finitely generated near $p$. At the last stage we'll be out of subsets $\mathsf{M}$, and the subsheaf constructed will be $\mathscr{G}$.
5. Let $\mathscr{G}_s$ be the subsheaf of sections of $\mathsf{E}$ such that their restriction to $\mathsf{M}_s \setminus \mathsf{M}_{s-1}$ lies in the rank $r_s$ bundle corresponding to the fibres of $\mathsf{F}$. Thinking in coordinates shows that this condition can be expressed as linear conditions on the sections of $\mathsf{E}$, and that the linear conditions extend across $\mathsf{M}_{s-1}$. (The linear conditions are the vanishing of certain $(r_s + 1) \times (r_s + 1)$ minors, where the matrix in question has rows local generators of $\mathsf{F}$ and one more row for a section of $\mathsf{E}$, thought of as a variable. The columns are local coordinates of $\mathsf{E}$. The condition that the minors are zero is linear in the coordinates of the section of $\mathsf{E}$).

   That means that we can express $\mathscr{G}_s$ as the kernel of a map between locally free sheaves. Since the structure sheaf is coherent, the kernel is finitely generated. Let $\mathsf{H}_s$ be the trivial bundle of the rank equal to the number of local generators of $\mathscr{G}_s$, and $\mathsf{H}_s \to \mathsf{E}$ the map whose image is $\mathscr{G}_s$.
6. Now let $\mathscr{G}_{s-1}$ be the subsheaf of sections of $\mathscr{G}_s$ whose restrictions to $\mathsf{M}_{s-1} \setminus \mathsf{M}_{s-2}$ lie in the rank $r_{s-1}$ vector bundle corresponding to the fibres of $\mathsf{F}$. It suffices to show that the subsheaf of $\mathscr{G}^{\omega}_{\mathsf{H}_s}$ which maps to $\mathscr{G}_{s-1}$ is locally finitely generated, since then its image $\mathscr{G}_{s-1}$ will be too.

   Let $\mathscr{Q}_s$ be the restriction of $\mathscr{G}^{\omega}_{\mathsf{H}_s}$ to $\mathsf{M}_s$, and note that $\mathscr{Q}_s$ is a quotient of $\mathscr{G}^{\omega}_{\mathsf{H}_s}$. By the same local coordinates idea used in part 4, we'll be able to express the condition that the sections of $\mathscr{Q}_s$ lie in the rank $r_{s-1}$ bundle corresponding to the sections of $\mathsf{F}$ as linear conditions on the sections of $\mathscr{Q}_s$ (restricted to $\mathsf{M}_s$), and so conclude that we have local finite generation (of a particular subsheaf of $\mathscr{Q}_s$). The subsheaf of $\mathscr{G}^{\omega}_{\mathsf{H}_s}$ we want is those sections which, upon restriction to $\mathsf{M}_s$, lie in the (locally finitely generated) subsheaf of $\mathscr{Q}_s$ constructed above. This subsheaf of $\mathscr{G}^{\omega}_{\mathsf{H}_s}$ should also be finitely generated (since $\mathscr{Q}_s$ is a quotient of $\mathscr{G}^{\omega}_{\mathsf{H}_s}$, this must follow from a general principle, but I should think about the argument for a bit).

   Let $\mathsf{H}_{s-1}$ be the trivial bundle whose rank is equal to the number of generators of the subsheaf of $\mathscr{G}^{\omega}_{\mathsf{H}_s}$ we just constructed, and $\mathsf{H}_{s-1} \to \mathsf{H}_s \to \mathsf{E}$ the induced map, with image $\mathsf{G}_{s-1}$.

7. Now repeat step 5 until we get down to the bottom of the list, having constructed $\mathscr{G}_1$ which is $\mathscr{G}$, and a trivial bundle $\mathsf{H}_1$ with a map $\mathsf{H}_1 \to \mathsf{E}$ whose image is $\mathsf{G}_1 = \mathsf{G}$.

(ii) Suppose that $\mathscr{F}$ is the patchy subsheaf corresponding to the data of an open cover $\mathscr{U} = (\mathcal{U}_a)_{a\in A}$ with corresponding subsets $(\mathscr{X}_a)_{a\in A}$ of real analytic sections over the subsets of the open cover. Let $x_0 \in \mathsf{M}$ and let $a \in A$ be such that $x_0 \in \mathcal{U}_a$. For $x \in \mathcal{U}_a$ we have

$$\mathsf{F}(\mathscr{F})_x = \{\xi(x) \mid [\xi]_x \in \mathscr{F}_x\} = \{\xi(x) \mid \xi \in \langle \mathscr{X}_a \rangle\},$$

showing that $\mathsf{F}(\mathscr{F})$ is generated, as a generalised subbundle, by locally defined real analytic sections, and so is real analytic. ■

The preceding results and example should be interpreted as follows. The result tells us that the attribute of being patchy is not interesting in the smooth or finitely differentiable case. The example tells us that the attribute of being patchy is at least not vacuous in the real analytic case. It may not be apparent at this point that patchiness is something useful to study. However, as we shall see in Corollary 4.11, patchy real analytic sheaves are coherent and so Cartan's Theorems A and B are available to be used for these sheaves.

## 4. Algebraic constructions associated to generalised subbundles

In this section we consider some algebraic constructions associated with generalised subbundles. It is often the case that such considerations are phrased in terms of the ring structure of functions on a manifold and the corresponding module structure of the sections of a vector bundle. This is more naturally carried out using sheaves, and so much of what we say in this section is based on sheaves.

**4.1. From stalks of a sheaf to fibres.** Let $r \in \{\infty, \omega\}$ and let $\pi\colon \mathsf{E} \to \mathsf{M}$ be a vector bundle of class $C^r$. The stalk of the sheaf $\mathscr{G}^r_{\mathsf{E}}$ at $x \in \mathsf{M}$ is the set $\mathscr{G}^r_{x,\mathsf{E}}$ of germs of sections which is a module over the ring $\mathscr{C}^r_{x,\mathsf{M}}$ of germs of functions. The stalk is *not* the same as the fibre $\mathsf{E}_x$, however, the fibre can be obtained from the stalk, and in this section we see how this is done. We shall couch this in a brief general algebraic construction, just to add colour.

Recall that if $\mathsf{R}$ is a commutative unit ring, if $\mathsf{I} \subseteq \mathsf{R}$ is an ideal, and if $\mathsf{A}$ is a unital $\mathsf{R}$-module, $\mathsf{IA}$ is the submodule of $\mathsf{A}$ generated by elements of the form $rv$ where $r \in \mathsf{I}$ and $v \in \mathsf{A}$.

**4.1 Proposition: (Vector spaces from modules over local rings)** *Let* $\mathsf{R}$ *be a commutative unit ring that is local, i.e., possess a unique maximal ideal* $\mathfrak{m}$*, and let* $\mathsf{A}$ *be a unital* $\mathsf{R}$*-module. Then* $\mathsf{A}/\mathfrak{m}\mathsf{A}$ *is a vector space over* $\mathsf{R}/\mathfrak{m}$*. Moreover, this vector space is naturally isomorphic to* $(\mathsf{R}/\mathfrak{m}) \otimes_{\mathsf{R}} \mathsf{A}$*.*

Proof: We first prove that $\mathsf{R}/\mathfrak{m}$ is a field. Denote by $\pi_{\mathfrak{m}}\colon \mathsf{R} \to \mathsf{R}/\mathfrak{m}$ the canonical projection. Let $\mathsf{I} \subseteq \mathsf{R}/\mathfrak{m}$ be an ideal. We claim that

$$\tilde{\mathsf{I}} = \{r \in \mathsf{R} \mid \pi_{\mathfrak{m}}(r) \in \mathsf{I}\}$$

is an ideal in $\mathsf{R}$. Indeed, let $r_1, r_2 \in \tilde{\mathsf{I}}$ and note that $\pi_{\mathfrak{m}}(r_1 - r_2) = \pi_{\mathfrak{m}}(r_1) - \pi_{\mathfrak{m}}(r_2) \in \mathsf{I}$ since $\pi_{\mathfrak{m}}$ is a ring homomorphism and since $\mathsf{I}$ is an ideal. Thus $r_1 - r_2 \in \tilde{\mathsf{I}}$. Now let $r \in \tilde{\mathsf{I}}$ and

$s \in \mathsf{R}$ and note that $\pi_{\mathfrak{m}}(sr) = \pi_{\mathfrak{m}}(s)\pi_{\mathfrak{m}}(r) \in \mathsf{I}$, again since $\pi_{\mathfrak{m}}$ is a ring homomorphism and since $\mathsf{I}$ is an ideal. Thus $\tilde{\mathsf{I}}$ is an ideal. Clearly $\mathfrak{m} \subseteq \tilde{\mathsf{I}}$ so that either $\tilde{\mathsf{I}} = \mathfrak{m}$ or $\tilde{\mathsf{I}} = \mathsf{R}$. In the first case $\mathsf{I} = \{0_{\mathsf{R}} + \mathfrak{m}\}$ and in the second case $\mathsf{I} = \mathsf{R}/\mathfrak{m}$. Thus the only ideals of $\mathsf{R}/\mathfrak{m}$ are $\{0_{\mathsf{R}} + \mathfrak{m}\}$ and $\mathsf{R}/\mathfrak{m}$. To see that this implies that $\mathsf{R}/\mathfrak{m}$ is a field, let $r + \mathfrak{m} \in \mathsf{R}/\mathfrak{m}$ be nonzero and consider the ideal $(r + \mathfrak{m})$. Since $(r + \mathfrak{m})$ is nontrivial we must have $(r + \mathfrak{m}) = \mathsf{R}/\mathfrak{m}$. In particular, $1 = (r + \mathfrak{m})(s + \mathfrak{m})$ for some $s + \mathfrak{m} \in \mathsf{R}/\mathfrak{m}$, and so $r + \mathfrak{m}$ is a unit.

Now we show that $\mathsf{A}/\mathfrak{m}\mathsf{A}$ is a vector space over $\mathsf{R}/\mathfrak{m}$. This amounts to showing that the natural vector space operations

$$(u + \mathfrak{m}\mathsf{A}) + (v + \mathfrak{m}\mathsf{A}) = u + v + \mathfrak{m}\mathsf{A}, \quad (r + \mathfrak{m})(u + \mathfrak{m}\mathsf{A}) = ru + \mathfrak{m}\mathsf{A}$$

make sense. The only possible issue is with scalar multiplication, so suppose that

$$r + \mathfrak{m} = s + \mathfrak{m}, \quad u + \mathfrak{m}\mathsf{A} = v + \mathfrak{m}\mathsf{A}$$

so that $s = r + a$ for $a \in \mathfrak{m}$ and $v = u + w$ for $w \in \mathfrak{m}\mathsf{A}$. Then

$$sv = (r + a)(u + w) = ru + au + rw + aw,$$

and we observe that $au, rw, aw \in \mathfrak{m}\mathsf{A}$, and so the sensibility of scalar multiplication is proved.

For the last assertion, note that we have the exact sequence

$$0 \longrightarrow \mathfrak{m} \longrightarrow \mathsf{R} \longrightarrow \mathsf{R}/\mathfrak{m} \longrightarrow 0$$

By right exactness of the tensor product [Hungerford 1980, Proposition IV.5.4] this gives the exact sequence

$$\mathfrak{m} \otimes_{\mathsf{R}} \mathsf{A} \longrightarrow \mathsf{A} \longrightarrow (\mathsf{R}/\mathfrak{m}) \otimes_{\mathsf{R}} \mathsf{A} \longrightarrow 0$$

noting that $\mathsf{R} \otimes_{\mathsf{R}} \mathsf{A} \simeq \mathsf{A}$. By this isomorphism, the image of $\mathfrak{m} \otimes_{\mathsf{R}} \mathsf{A}$ in $\mathsf{A}$ is simply generated by elements of the form $rv$ for $r \in \mathfrak{m}$ and $v \in \mathsf{A}$. That is to say, the image of $\mathfrak{m} \otimes_{\mathsf{R}} \mathsf{A}$ in $\mathsf{A}$ is simply $\mathfrak{m}\mathsf{A}$. Thus we have the induced commutative diagram

$$\begin{tikzcd}
 & \mathfrak{m} \otimes_{\mathsf{R}} \mathsf{A} \arrow[r] \arrow[d] & \mathsf{A} \arrow[r] \arrow[d] & (\mathsf{R}/\mathfrak{m}) \otimes_{\mathsf{R}} \mathsf{A} \arrow[r] \arrow[d, dashed] & 0 \\
0 \arrow[r] & \mathfrak{m}\mathsf{A} \arrow[r] & \mathsf{A} \arrow[r] & \mathsf{A}/\mathfrak{m}\mathsf{A} \arrow[r] & 0
\end{tikzcd}$$

with exact rows. We claim that there is an induced mapping as indicated by the dashed arrow, and that this mapping is an isomorphism. To define the mapping, let $\alpha \in (\mathsf{R}/\mathfrak{m}) \otimes_{\mathsf{R}} \mathsf{A}$ and let $v \in \mathsf{A}$ project to $\alpha$. The image of $\beta$ is then taken to be $v + \mathfrak{m}\mathsf{A}$. It is a straightforward exercise to show that this mapping is well-defined and is an isomorphism, using exactness of the diagram. ■

With this simple algebraic construction as background, we can then indicate how to recover the fibres of a vector bundle from the stalks of its sheaf of sections. To do this, the notation

$$\mathfrak{m}_x = \{[f]_x \in \mathscr{C}^r_{x,\mathsf{M}} \mid f(x) = 0\}$$

will be useful for $r \in \mathbb{Z}_{\geq 0} \cup \{\infty, \omega\}$. Algebraically, $\mathfrak{m}_x$ is the unique maximal ideal of the local ring $\mathscr{C}^r_{x,\mathsf{M}}$ [Navarro González and Sancho de Salas 2003, Nestruev 2003].

**4.2 Proposition: (From stalks to fibres)** *Let $r \in \{\infty, \omega\}$ and let $\pi \colon \mathsf{E} \to \mathsf{M}$ be a vector bundle of class $C^r$. For $x \in \mathsf{M}$ let $\mathfrak{m}_x$ denote the unique maximal ideal in $\mathscr{C}^r_{x,\mathsf{M}}$. Then the following statements hold:*

(i) *the field $\mathscr{C}^r_{x,\mathsf{M}}/\mathfrak{m}_x$ is isomorphic to $\mathbb{R}$ via the isomorphism*

$$[f]_x + \mathfrak{m}_x \mapsto f(x);$$

(ii) *the $\mathscr{C}^r_{x,\mathsf{M}}/\mathfrak{m}_x$-vector space $\mathscr{G}^r_{x,\mathsf{E}}/\mathfrak{m}_x\mathscr{G}^r_{x,\mathsf{E}}$ is isomorphic to $\mathsf{E}_x$ via the isomorphism*

$$[\xi]_x + \mathfrak{m}_k\mathscr{G}^r_{x,\mathsf{E}} \mapsto \xi(x);$$

(iii) *the map from $(\mathscr{C}^r_{x,\mathsf{M}}/\mathfrak{m}_x) \otimes_{\mathscr{C}^r_{x,\mathsf{M}}} \mathscr{G}^r_{x,\mathsf{E}}$ to $\mathsf{E}_x$ defined by*

$$([f]_x + \mathfrak{m}_x) \otimes [\xi]_x \mapsto f(x)\xi(x)$$

*is an isomorphism of $\mathbb{R}$-vector spaces.*

Proof: (i) The map is clearly a homomorphism of fields. To show that it is surjective, if $a \in \mathbb{R}$ then $a$ is the image of $[f]_x + \mathfrak{m}_x$ for any germ $[f]_x$ for which $f(x) = a$. To show injectivity, if $[f]_x + \mathfrak{m}_x$ maps to 0 then clearly $f(x) = 0$ and so $f \in \mathfrak{m}_x$.

(ii) The map is clearly linear, so we verify that it is an isomorphism. Let $v_x \in \mathsf{E}_x$. Then $v_x$ is the image of $[\xi]_x + \mathfrak{m}_x\mathscr{G}^r_{x,\mathsf{E}}$ for any germ $[\xi]_x$ for which $\xi(x) = v_x$. Also suppose that $[\xi]_x + \mathfrak{m}_x\mathscr{G}^r_{x,\mathsf{E}}$ maps to zero. Then $\xi(x) = 0$. Since $\mathscr{G}^r_{\mathsf{E}}$ is locally free (see the next section in case the meaning here is not patently obvious), it follows that we can write

$$\xi(y) = f_1(y)\eta_1(y) + \cdots + f_m(y)\eta_m(y)$$

for sections $\eta_1, \dots, \eta_m$ of class $C^r$ in a neighbourhood of $x$ and for functions $f_1, \dots, f_m$ of class $C^r$ in a neighbourhood of $x$. Moreover, the sections may be chosen such that $(\eta_1(y), \dots, \eta_m(y))$ is a basis for $\mathsf{E}_y$ for every $y$ in some suitably small neighbourhood of $x$. Thus

$$\xi(x) = 0 \quad \implies \quad f_1(x) = \cdots = f_m(x) = 0,$$

giving $\xi \in \mathfrak{m}_x\mathscr{G}^r_{x,\mathsf{E}}$, as desired.

(iii) The $\mathbb{R}$-linearity of the stated map is clear, and the fact that the map is an isomorphism follows from the final assertion of Proposition 4.1. ■

The preceding result relates stalks to fibres. Subsequently, specifically in Theorem 4.9, we shall take a more global view towards relating vector bundles and sheaves.

In the preceding result we were able to rebuild the fibre of a vector bundle from the germs of sections. There is nothing keeping one from making this construction for a general sheaf.

**4.3 Definition: (Fibres for sheaves of $\mathscr{C}^r_{\mathsf{M}}$-modules)** Let $r \in \mathbb{Z}_{\geq 0} \cup \{\infty, \omega\}$, let $\mathsf{M}$ be a smooth or real analytic manifold, as required, and let $\mathscr{F}$ be a sheaf of $\mathscr{C}^r_{\mathsf{M}}$-modules. The ***fibre*** of $\mathscr{F}$ is the $\mathbb{R}$-vector space $\mathsf{E}(\mathscr{F})_x = \mathscr{F}_x/\mathfrak{m}_x\mathscr{F}_x$. •

By Proposition 4.2, the fibres of $\mathscr{G}^r_{\mathsf{E}}$ are isomorphic to the usual fibres. However, if $\mathscr{F}$ is merely a subsheaf of the sheaf of sections of a vector bundle, the relationship to the usual notion of fibre is not generally what one expects, as the following example illustrates.

**4.4 Example: (Fibres for a non-vector bundle sheaf)** Let $r \in \mathbb{Z}_{\geq 0} \cup \{\infty, \omega\}$. Let us take $\mathsf{M} = \mathbb{R}$ and define a presheaf $\mathscr{I}_0^r = (I_0(\mathcal{U}))_{\mathcal{U}\,\mathrm{open}}$ by

$$I_0^r(\mathcal{U}) = \begin{cases} C^r(\mathcal{U}), & 0 \notin \mathcal{U}, \\ \{f \in C^r(\mathcal{U}) \mid f(0) = 0\}, & x \in \mathcal{U}. \end{cases}$$

One directly verifies that $\mathscr{I}_0^r$ is a sheaf. Moreover, $\mathscr{I}_0^r$ is a sheaf of $\mathscr{C}_{\mathbb{R}}^r$-modules; this too is easily verified. Let us compute the fibres associated with this sheaf. The germs of this sheaf at $x \in \mathbb{R}$ are readily seen to be given by

$$\mathscr{I}_{0,x}^r = \begin{cases} \mathscr{C}_{x,\mathbb{R}}^r, & x \neq 0, \\ \mathfrak{m}_0 = \{[f]_0 \in \mathscr{C}_{0,\mathbb{R}}^r \mid f(x) = 0\}, & x = 0. \end{cases}$$

Thus we have

$$\mathsf{E}(\mathscr{I}_0^r)_x = \begin{cases} \mathscr{C}_{x,\mathbb{R}}^r / \mathfrak{m}_x \mathscr{C}_{x,\mathbb{R}}^r \simeq \mathbb{R}, & x \neq 0, \\ \mathfrak{m}_0 / \mathfrak{m}_0^2 \simeq \mathbb{R}, & x = 0. \end{cases}$$

Note that the fibre at 0 is "bigger" than we expect it to be. •

The next result shows that the sheaf fibre is always larger than the usual fibre, a fact that will be useful for us in our proof of Theorem 5.2 below.

**4.5 Proposition: (Relationships between the two notions of fibre)** *Let $r \in \mathbb{Z}_{\geq 0} \cup \{\infty, \omega\}$, let $\pi \colon \mathsf{E} \to \mathsf{M}$ be a smooth or real analytic vector bundle, as required, and let $\mathscr{F}$ be a subsheaf of $\mathscr{G}_{\mathsf{E}}^r$. Then, for each $x \in \mathsf{M}$, there exists a natural epimorphism from the fibre $\mathscr{F}_x / \mathfrak{m}_x \mathscr{F}_x$ onto the fibre $\mathsf{F}(\mathscr{F})_x$. If the function $x \mapsto \dim(\mathsf{F}(\mathscr{F})_x)$ is locally constant at $x$, then the natural epimorphism is an isomorphism.*

Proof: Let $x \in \mathsf{M}$, let $\mathcal{N}$ be a neighbourhood of $x$ and let $(\xi_j)_{j \in J}$ be local generators for $\mathsf{F}$ about $x$, defined on $\mathcal{N}$. Let us define a morphism $\Psi$ from $\oplus_{j \in J} \mathscr{C}_{\mathcal{N}}^r$ to $\mathscr{G}_{\mathsf{E}|\mathcal{N}}^r$, a morphism $\Phi$ from $\oplus_{j \in J} \mathscr{C}_{\mathcal{N}}^r$ to $\mathscr{G}_{\mathsf{F}}^r|\mathcal{N}$, and a morphism $\iota$ from $\mathscr{G}_{\mathsf{F}}^r|\mathcal{N}$ to $\mathscr{G}_{\mathsf{E}|\mathcal{N}}^r$ by

$$\begin{aligned} \Psi_{\mathcal{U}}\Big(\oplus_{j \in J} f^j\Big) &= \sum_{j \in J} f^j \hat{\xi}_j, \\ \Phi_{\mathcal{U}}\Big(\oplus_{j \in J} f^j\Big) &= \sum_{j \in J} f^j \xi_j, \\ \iota_{\mathcal{U}}\Big(\sum_{j \in J} f^j \xi_j\Big) &= \sum_{j \in J} f^j \hat{\xi}_j, \end{aligned}$$

for $\mathcal{U} \subseteq \mathcal{N}$ open. Here $\xi_j$ denotes an element of $\Gamma^r(\mathsf{F}|\mathcal{U})$ and $\hat{\xi}_j$ denotes the element of $\Gamma^r(\mathsf{E}|\mathcal{U})$ which is image of $\xi_j$ under the obvious inclusion. At the stalk level we thus have the commutative diagram

$$\begin{tikzcd} \oplus_{j \in J} \mathscr{C}_{x,\mathsf{M}}^r \arrow[rr, "\Psi_x"] \arrow[dr, "\Phi_x"'] & & \mathscr{G}_{x,\mathsf{E}}^r \\ & \mathscr{F}_x \arrow[ur, "\iota_x"'] & \end{tikzcd}$$

We now take the tensor product of the diagram of $\mathscr{C}^r_{x,\mathsf{M}}$-modules with $\mathscr{C}^r_{x,\mathsf{M}}/\mathfrak{m}_x$, and use the commuting of tensor product with direct sums [Bourbaki 1989, Proposition II.3.7.7] and Proposition 4.1 to give the commutative diagram

$$\begin{tikzcd} \oplus_{j\in J}\mathbb{R} \arrow[rr, "\tilde{\Psi}_x"] \arrow[dr, "\tilde{\Phi}_x"'] & & \mathsf{E}_x \\ & \mathscr{F}_x/\mathfrak{m}_x\mathscr{F}_x \arrow[ur, "\tilde{\iota}_x"'] & \end{tikzcd}$$

Here the homomorphisms $\tilde{\Psi}_x$, $\tilde{\Phi}_x$, and $\tilde{\iota}_{a,x}$ are defined by

$$\begin{aligned} \tilde{\Psi}_x\Big(\oplus_{b\in B_a}\alpha_b\Big) &= \sum_{b\in B_a}\alpha_b\xi_b(x),\\ \tilde{\Phi}_x\Big(\oplus_{b\in B_a}\alpha_b\Big) &= \sum_{b\in B_a}\alpha_b([\xi_b]_x+\mathfrak{m}_x\mathscr{F}_{a,x}),\\ \tilde{\iota}_{a,x}\Big(\sum_{b\in B_a}\alpha_b([\xi_b]_x+\mathfrak{m}_x\mathscr{F}_{a,x})\Big) &= \sum_{b\in B_a}\alpha_b\xi_b(x). \end{aligned}$$

Note that $\mathrm{image}(\tilde{\Psi}_x)=\mathsf{F}(\mathscr{F})_x$ by definition of $\mathsf{F}(\mathscr{F})_x$. By the commuting of the preceding diagram, $\mathrm{image}(\tilde{\iota}_x)=\mathsf{F}(\mathscr{F})_x$, and so $\tilde{\iota}_x$ is an epimorphism.

For the final assertion, note that if $x\mapsto\dim(\mathsf{F}(\mathscr{F})_x)$ is locally constant, then $\mathsf{F}(\mathscr{F})_x$ are the fibres of a subbundle of $\mathsf{E}$. In this case, the final assertion then follows from the correspondence between the notions of fibre for sheaves of sections of vector bundles. ■

**4.2. Subsheaves of sections of a real analytic vector bundle.** Now we turn to the consideration of sheaves associated to real analytic generalised subbundles. It is here that we will rely on some of the important known results about coherent real analytic sheaves.

We begin with some a few more or less standard observations about germs of functions and sections.

**4.6 Proposition: (Noetherian property of germs of real analytic functions and sections)** *Let $\pi\colon\mathsf{E}\to\mathsf{M}$ be an analytic vector bundle. For $x\in\mathsf{M}$ the following statements hold:*

*(i) the ring $\mathscr{C}^\omega_{x,\mathsf{M}}$ is Noetherian, i.e., if we have a sequence $(\mathsf{I}_j)_{j\in\mathbb{Z}_{>0}}$ of ideals of $\mathscr{C}^\omega_{x,\mathsf{M}}$ for which $\mathsf{I}_j\subseteq\mathsf{I}_{j+1}$, $j\in\mathbb{Z}_{>0}$, then there exists $N\in\mathbb{Z}_{>0}$ such that $\mathsf{I}_j=\mathsf{I}_N$ for $j\geq N$;*

*(ii) the module $\mathscr{G}^\omega_{x,\mathsf{E}}$ is Noetherian, i.e., if we have a sequence $(\mathsf{A}_j)_{j\in\mathbb{Z}_{>0}}$ of submodules of $\mathscr{G}^\omega_{x,\mathsf{E}}$ for which $\mathsf{A}_j\subseteq\mathsf{A}_{j+1}$, $j\in\mathbb{Z}_{>0}$, then there exists $N\in\mathbb{Z}_{>0}$ such that $\mathsf{A}_j=\mathsf{A}_N$ for $j\geq N$;*

Outline of proof: In the holomorphic case, the first part of the result is well-known and can be found in most any text on several complex variables [e.g., Hörmander 1966, Theorem 6.3.3]. The proof is an inductive one based on an induction on $\dim(\mathsf{M})$ using the Weierstrass Preparation Theorem. The real analytic Weierstrass Preparation Theorem is proved by Krantz and Parks [2002, Theorem 6.1.3], and with this the holomorphic proof of the Noetherian property applies in the real analytic case. The second part of the result follows since every finitely generated module over a Noetherian ring is Noetherian [Hungerford 1980, Theorem VIII.1.8]. ■

General properties of Noetherian modules then give the following result.

**4.7 Corollary: (Stalks of real analytic subsheaves are finitely generated)** *Let $\pi\colon \mathsf{E} \to \mathsf{M}$ be an analytic vector bundle and let $\mathscr{F}$ be a subsheaf of $\mathscr{G}^{\omega}_{\mathsf{E}}$. Then, for $x \in \mathsf{M}$, $\mathscr{F}_x$ is finitely generated.*

Proof: This follows from the fact that submodules of finitely generated Noetherian modules are finitely generated [Hungerford 1980, Theorem VIII.1.9]. ■

This property of stalks of real analytic subsheaves is often not used properly. Specifically, this Noetherian property of stalks of real analytic subsheaves is used as a standin for the much stronger property of being locally finitely generated (see Definition 2.14). It is actually this latter property we will use in this paper, and that is most useful in general. So we now turn to this question of local finite generation.

**4.3. Locally finitely generated subsheaves of modules.** First let us recall the correspondence between vector bundles, or more generally regular generalised subbundles, with locally free, locally finitely generated sheaves. We need, therefore, to complement Definition 2.14 of locally finitely generated with the following.

**4.8 Definition: (Locally free sheaf)** Let $\mathsf{M}$ be a smooth or real analytic manifold, as required, let $r \in \mathbb{Z}_{\geq 0} \cup \{\infty, \omega\}$, and let $\mathscr{F}$ be a sheaf of $\mathscr{C}^r_{\mathsf{M}}$-modules over $\mathsf{M}$. The sheaf $\mathscr{F}$ is ***locally free*** if, for each $x_0 \in \mathsf{M}$, there exists a neighbourhood $\mathcal{U}$ of $x_0$ such that $\mathscr{F}|\mathcal{U}$ is isomorphic to a direct sum $\oplus_{a\in A}(\mathscr{R}|\mathcal{U})$. •

The following theorem then gives the correspondence we are after. The result is often found in the holomorphic case in texts on algebraic geometry [e.g., Taylor 2002, Proposition 7.6.5], but is seldom given in the setting here (but see [Ramanan 2005, §2.2] for some discussion).

**4.9 Theorem: (Correspondence between regular generalised subbundles and locally free, locally finitely generated sheaves)** *Let $r \in \mathbb{Z}_{\geq 0} \cup \{\infty, \omega\}$, let $\pi\colon \mathsf{E} \to \mathsf{M}$ be a smooth or real analytic vector bundle, as required, and let $\mathsf{F} \subseteq \mathsf{E}$ be a regular generalised subbundle of class $C^r$. Then $\mathscr{G}^r_{\mathsf{E}}$ is a locally free, locally finitely generated sheaf of $\mathscr{C}^r_{\mathsf{M}}$-modules.*

*Conversely, if $\mathscr{F}$ is a locally free, locally finitely generated sheaf of $\mathscr{C}^r_{\mathsf{M}}$-modules, then there exists a smooth or real analytic vector bundle $\pi\colon \mathsf{E} \to \mathsf{M}$, as required, and a regular generalised subbundle $\mathsf{F} \subseteq \mathsf{E}$ of class $C^r$ such that $\mathscr{F}$ is isomorphic to $\mathscr{G}^r_{\mathsf{F}}$.*

Proof: Note that regular generalised subbundles of class $C^r$ are vector bundles of class $C^r$. Thus we shall suppose, without loss of generality that $\mathsf{F} = \mathsf{E}$ and that $\mathsf{E}$ is not smooth or real analytic, but of class $C^r$.

First let $\pi\colon \mathsf{E} \to \mathsf{M}$ be a vector bundle of class $C^r$ and let $x_0 \in \mathsf{M}$. Let $(\mathcal{V}, \psi)$ be a vector bundle chart such that the corresponding chart $(\mathcal{U}, \phi)$ for $\mathsf{M}$ contains $x_0$. Suppose that $\psi(\mathcal{V}) = \phi(\mathcal{U}) \times \mathbb{R}^m$ and let $\eta_1, \dots, \eta_m \in \Gamma^r(\mathsf{E}|\mathcal{U})$ satisfy $\psi(\eta_j(x)) = (\phi(x), \boldsymbol{e}_j)$ for $x \in \mathcal{U}$ and $j \in \{1, \dots, m\}$. Let us arrange the components $\eta^k_j$, $j, k \in \{1, \dots, m\}$, of the sections $\eta_1, \dots, \eta_m$ in an $m \times m$ matrix:

$$\boldsymbol{\eta}(x) = \begin{bmatrix} \eta^1_1(x) & \cdots & \eta^1_m(x) \\ \vdots & \ddots & \vdots \\ \eta^m_1(x) & \cdots & \eta^m_m(x) \end{bmatrix}.$$

Now let $\xi \in \Gamma^r(\mathsf{E}|\mathcal{U})$, let the components of $\xi$ be $\xi^k$, $k \in \{1,\dots,k\}$, and arrange the components in a vector

$$\boldsymbol{\xi}(x) = \begin{bmatrix} \xi^1(x) \\ \vdots \\ \xi^m(x) \end{bmatrix}.$$

Now fix $x \in \mathcal{U}$. We wish to solve the equation

$$\xi(x) = f^1(x)\eta_1(x) + \dots + f^m(x)\eta_m(x)$$

for $f^1(x),\dots,f^m(x) \in \mathbb{R}$. Let us write

$$\boldsymbol{f}(x) = \begin{bmatrix} f^1(x) \\ \vdots \\ f^m(x) \end{bmatrix}.$$

Writing the equation we wish to solve as a matrix equation we have

$$\boldsymbol{\xi}(x) = \boldsymbol{\eta}(x)\boldsymbol{f}(x).$$

Therefore,

$$\boldsymbol{f}(x) = \boldsymbol{\eta}^{-1}(x)\boldsymbol{\xi}(x),$$

noting that $\boldsymbol{\eta}(x)$ is invertible since the vectors $\eta_1(x),\dots,\eta_m(x)$ are linearly independent. By Cramer's Rule, or some such, the components of $\boldsymbol{\eta}^{-1}$ are $C^r$-functions of $x \in \mathcal{U}$, and so $\xi$ is a $C^r(\mathcal{U})$-linear combination of $\eta_1,\dots,\eta_m$, showing that $\Gamma^r(\mathsf{E}|\mathcal{U})$ is finitely generated. To show that this module is free, it suffices to show that $(\eta_1,\dots,\eta_m)$ is linearly independent over $C^r(\mathcal{U})$. Suppose that there exists $f^1,\dots,f^m \in C^r(\mathcal{U})$ such that

$$f^1\eta_1 + \dots + f^m\eta_m = 0_{\Gamma^r(\mathsf{E})}.$$

Then, for every $x \in \mathcal{U}$,

$$f^1(x)\eta_1(x) + \dots + f^m(x)\eta_m(x) = 0_x \quad \Longrightarrow \quad f^1(x) = \dots = f^m(x) = 0,$$

giving the desired linear independence.

Next suppose that $\mathscr{F}$ is a locally free, locally finitely generated sheaf of $\mathscr{C}^r_{\mathsf{M}}$-modules. Let us first define the total space of our vector bundle. For $x \in \mathsf{M}$ define

$$\mathsf{E}_x = \mathscr{F}_x/\mathfrak{m}_x\mathscr{F}_x.$$

By Propositions 4.1 and 4.2, $\mathsf{E}_x$ is a $\mathbb{R}$-vector space. We take $\mathsf{E} = \mathring{\cup}_{x\in\mathsf{M}} \mathsf{E}_x$. Let $x \in \mathsf{M}$ and let $\mathcal{U}_x$ be a neighbourhood of $x$ such that $F(\mathcal{U}_x)$ is a free $C^r(\mathcal{U}_x)$-module. By shrinking $\mathcal{U}_x$ if necessary, we suppose that it is the domain of a coordinate chart $(\mathcal{U}_x,\phi_x)$. Let $s_1,\dots,s_m \in F(\mathcal{U}_x)$ be such that $(s_1,\dots,s_m)$ is a basis for $F(\mathcal{U}_x)$. Note that $([s_1]_y,\dots,[s_m]_x)$ is a basis for $\mathscr{F}_y$ for each $y \in \mathcal{U}_x$. It is straightforward to show that

$$([s_1]_y + \mathfrak{m}_y\mathscr{F}_y,\dots,[s_m]_y + \mathfrak{m}_y\mathscr{F}_y)$$

is then a basis for $\mathsf{E}_y$. For $y \in \mathcal{U}$ the map

$$a^1([s_1]_y + \mathfrak{m}_y) + \cdots + a^m([s_m]_y + \mathfrak{m}_y) \mapsto (a^1, \dots, a^m)$$

is clearly an isomorphism. Now define $\mathcal{V}_x = \mathring{\cup}_{y\in\mathcal{U}_x} \mathsf{E}_y$ and define $\psi_x \colon \mathcal{V}_x \to \phi_x(\mathcal{U}) \times \mathbb{R}^m$ by

$$\psi_x(a^1([s_1]_y + \mathfrak{m}_y) + \cdots + a^m([s_m]_y + \mathfrak{m}_y)) = (\phi_x(y), (a^1, \dots, a^m)).$$

This is clearly a vector bundle chart for $\mathsf{E}$. Moreover, this construction furnishes a covering of $\mathsf{E}$ by vector bundle charts.

Next we show that two overlapping vector bundle charts satisfy the appropriate overlap condition. Thus let $x, y \in \mathsf{M}$ be such that $\mathcal{U}_x \cap \mathcal{U}_y$ is nonempty. Let $(s_1, \dots, s_m)$ and $(t_1, \dots, t_m)$ be bases for $F(\mathcal{U}_x)$ and $F(\mathcal{U}_y)$, respectively. (Note that the cardinality of these bases agrees since, for $z \in \mathcal{U}_x \cap \mathcal{U}_y$, $([s_1]_z, \dots, [s_m]_z)$ and $([t_1]_z, \dots, [t_m]_z)$ are both bases for $\mathscr{F}_z$, cf. [Hungerford 1980, Corollary IV.2.12].) Note that

$$r_{\mathcal{U}_x, \mathcal{U}_x\cap\mathcal{U}_x}(s_j) = \sum_{k=1}^{m} f_j^k r_{\mathcal{U}_y, \mathcal{U}_x\cap\mathcal{U}_y}(t_k)$$

for $f_j^k \in C^r(\mathcal{U}_x \cap \mathcal{U}_y)$, $j, k \in \{1, \dots, m\}$. At the stalk level we have

$$[s_j]_z = \sum_{k=1}^{m} [f_j^k]_z [t_k]_z,$$

from which we conclude that

$$([s_j]_z + \mathfrak{m}_z\mathscr{F}_z) = \sum_{k=1}^{m} f_j^k(z)([t_k]_z + \mathfrak{m}_z\mathscr{F}_z),$$

From this we conclude that the matrix

$$\boldsymbol{f}(z) = \begin{bmatrix} f_1^1(z) & \cdots & f_m^1(z) \\ \vdots & \ddots & \vdots \\ f_1^m(z) & \cdots & f_m^m(z) \end{bmatrix}$$

is invertible, being the change of basis matrix for the two bases for $\mathsf{E}_z$. Moreover, the change of basis formula gives

$$\psi_y \circ \psi_x^{-1}(\boldsymbol{z}, (a^1, \dots, a^m)) = \Big(\phi_y \circ \phi_x^{-1}(\boldsymbol{z}), \Big(\sum_{j=1}^{m} a^j f_j^1(z), \dots, \sum_{j=1}^{m} a^j f_j^m(z)\Big)\Big)$$

for every $z \in \mathcal{U}_x \cap \mathcal{U}_y$, where $\boldsymbol{z} = \phi_x(z)$. Thus we see that the covering by vector bundle charts has the proper overlap condition to define a vector bundle structure for $\mathsf{E}$.

It remains to show that $\mathscr{G}^r_{\mathsf{E}}$ is isomorphic to $\mathscr{F}$. Let $\mathcal{U} \subseteq \mathsf{M}$ be open and define $\Phi_{\mathcal{U}} \colon F(\mathcal{U}) \to \Gamma^r(\mathsf{E}|\mathcal{U})$ by

$$\Phi_{\mathcal{U}}(s)(x) = [s]_x + \mathfrak{m}_x\mathscr{F}_x.$$

For this definition to make sense, we must show that $\Phi_{\mathcal{U}}(s)$ is of class $C^r$. Let $y \in \mathcal{U}$ and, using the above constructions, let $(s_1, \dots, s_m)$ be a basis for $F(\mathcal{U}_y)$. Let us abbreviate

$\mathcal{V} = \mathcal{U} \cap \mathcal{U}_y$. Note that $(r_{\mathcal{U},\mathcal{V}}(s_1), \dots, r_{\mathcal{U},\mathcal{V}}(s_m))$ is a basis for $F(\mathcal{V})$. (To see that this is so, one can identify $F(\mathcal{U})$ with a section of the étale space $\mathrm{Et}(\mathscr{F})$ over $\mathcal{U}$, cf. the discussion at the end of Section 2.2, and having done this the assertion is clear.) We thus write

$$r_{\mathcal{U},\mathcal{V}}(s) = f^1 r_{\mathcal{U},\mathcal{V}}(s_1) + \dots + f^m r_{\mathcal{U},\mathcal{V}}(s_m).$$

In terms of stalks we thus have

$$[s]_z = [f^1]_z [s_1]_z + \dots + [f^m]_z [s_m]_z$$

for each $z \in \mathcal{V}$. Therefore,

$$\Phi_{\mathcal{U}}(s)(z) = f^1(z)([s_1]_z + \mathfrak{m}_z \mathscr{F}_z) + \dots + f^m(z)([s_m]_z + \mathfrak{m}_z \mathscr{F}_z),$$

which (recalling that $\mathcal{U}_y$, and so also $\mathcal{V}$, is a chart domain) gives the local representative of $\Phi_{\mathcal{U}}(s)$ on $\mathcal{V}$ as

$$\boldsymbol{z} \mapsto (\boldsymbol{z}, (f^1 \circ \phi_y^{-1}(\boldsymbol{z}), \dots, f^m \circ \phi_y^{-1}(\boldsymbol{z}))).$$

Since this local representative is of class $C^r$ and since this construction can be made for any $y \in \mathcal{U}$, we conclude that $\Phi_{\mathcal{U}}(s)$ is of class $C^r$.

Now, to show that the family of mappings $(\Phi_{\mathcal{U}})_{\mathcal{U} \text{ open}}$ is an isomorphism, by [Hartshorne 1977, Proposition II.1.1] it suffices to show that the induced mapping on stalks is an isomorphism. Let us denote the mapping of stalks at $x$ by $\Phi_x$. We again use our constructions from the first part of this part of the proof and let $(s_1, \dots, s_m)$ be a basis for $F(\mathcal{U}_x)$. Let us show that $\Phi_x$ is surjective. Let $[\xi]_x \in \mathscr{G}^r_{x,\mathsf{M}}$, supposing that $\xi \in \Gamma^r(\mathsf{E}|\mathcal{U})$. Let $\mathcal{V} = \mathcal{U} \cap \mathcal{U}_x$. Let the local representative of $\xi$ on $\mathcal{V}$ in the chart $(\mathcal{V}_x, \psi_x)$ be given by

$$\boldsymbol{y} \mapsto (\boldsymbol{y}, (f^1 \circ \phi_x^{-1}(\boldsymbol{y}), \dots, f^m \circ \phi_x^{-1}(\boldsymbol{y})))$$

for $f^1, \dots, f^m \in C^r(\mathcal{V})$. Then, if

$$[s]_x = [f^1]_x [s_1]_x + \dots + [f^m]_x [s_m]_x,$$

we have $\Phi_x([s]_x) = [\xi]_x$. To prove injectivity of $\Phi_x$, suppose that $\Phi_x([s_x]) = 0$. This means that $\Phi_x([s]_x)$ is the germ of a section of $\mathsf{E}$ over some neighbourhood $\mathcal{U}$ of $x$ that is identically zero. We may without loss of generality assume that $\mathcal{U} \subseteq \mathcal{U}_x$. We also assume without loss of generality (by restriction of necessary) that $s \in F(\mathcal{U})$. We thus have

$$\Phi_{\mathcal{U}}(s)(y) = 0, \qquad y \in \mathcal{U}.$$

Since $(r_{\mathcal{U}_x,\mathcal{U}}(s_1), \dots, r_{\mathcal{U}_x,\mathcal{U}}(s_m))$ is a basis for $F(\mathcal{U})$ we write

$$s = f^1 r_{\mathcal{U}_x,\mathcal{U}}(s_1) + \dots + f^m r_{\mathcal{U}_x,\mathcal{U}}(s_m).$$

for some uniquely defined $f^1, \dots, f^m \in C^r(\mathcal{U})$. We have

$$\Phi_{\mathcal{U}}(s)(y) = f^1(y)([s_1]_y + \mathfrak{m}_y \mathscr{F}_y) + \dots + f^m(y)([s_m]_y + \mathfrak{m}_y \mathscr{F}_y)$$

for each $y \in \mathcal{U}$. Since

$$([s_1]_y + \mathfrak{m}_y \mathscr{F}_y, \dots, [s_m]_y + \mathfrak{m}_y \mathscr{F}_y)$$

is a basis for $\mathsf{E}_y$, we must have $f^1(y) = \dots = f^m(y) = 0$ for each $y \in \mathcal{U}$, giving $[s]_x = 0$. ■

If we relax the assumption of regularity, it becomes more difficult, in fact generally not possible, to establish any general results concerning the finite generatedness of the sheaf. However, in the patchy real analytic case, we have the following important result.

**4.10 Theorem: (Patchy real analytic subsheaves are locally finitely generated)** *Let $\pi\colon \mathsf{E} \to \mathsf{M}$ be a real analytic vector bundle and let $\mathscr{F}$ be a patchy subsheaf of $\mathscr{G}^{\omega}_{\mathsf{E}}$. Then, for any $x_0 \in \mathsf{M}$, there exists a neighbourhood $\mathcal{U}$ of $x_0$ and sections $\xi_1, \dots, \xi_k$ of $\mathscr{F}$ over $\mathcal{U}$ such that, if $\xi$ is a section of $\mathscr{F}$ over $\mathcal{U}$, there exists $f^1, \dots, f^k \in C^{\omega}(\mathcal{U})$ such that*

$$\xi = f^1\xi_1 + \dots + f^k\xi_k.$$

*In particular, $\mathscr{F}$ is locally finitely generated.*

Proof: A complete account of all that is needed to prove this result actually takes a lot of effort. We content ourselves by pointing the interested reader to the textual literature for detailed proofs.

We shall need the notion of a germ of a function, not at a point, but at a closed set $A \subseteq \mathsf{M}$. In this case, we consider pairs $(f, \mathcal{U})$ where $\mathcal{U} \subseteq \mathsf{M}$ is open with $A \subseteq \mathcal{U}$ and where $f \in C^{\omega}(\mathcal{U})$. Two such pairs $(f_1, \mathcal{U}_1)$ and $(f_2, \mathcal{U}_2)$ are equivalent if there exists $\mathcal{V} \subseteq \mathcal{U}_1 \cap \mathcal{U}_2$ with $A \subseteq \mathcal{V}$ and such that $f_1|\mathcal{V} = f_2|\mathcal{V}$. The set of equivalence classes we denote by $\mathscr{C}^{\omega}_{A,\mathsf{M}}$. If $A, B \subseteq \mathsf{M}$ are closed sets such that $B \subseteq A$ we have the map $r_{A,B}\colon \mathscr{C}^{\omega}_{A,\mathsf{M}} \to \mathscr{C}^{\omega}_{B,\mathsf{M}}$ given by restriction. If $\mathsf{A} \subseteq \mathscr{C}^{\omega}_{B,\mathsf{M}}$ we define

$$r^{-1}_{A,B}(\mathsf{A}) = \{[f]_A \in \mathscr{C}^{\omega}_{A,\mathsf{M}} \mid r_{A,B}([f]_A) \in \mathsf{A}\}.$$

We abbreviate $r_{A,x_0} = r_{A,\{x_0\}}$. If $\mathsf{A}$ is a subring (resp. ideal), one readily verifies that $r^{-1}_{A,B}(\mathsf{A})$ is also a subring (resp. ideal). Similar constructions hold for germs of sections of a vector bundle on closed sets.

The next lemma contains the hard technicalities needed to prove the theorem.

**1 Lemma:** *Let $\pi\colon \mathsf{E} \to \mathsf{M}$ be a real analytic vector bundle, let $x_0 \in \mathsf{M}$, and let $\mathsf{A}$ be a submodule of $\mathscr{G}^{\omega}_{x_0,\mathsf{E}}$. Let $\xi_1, \dots, \xi_k$ be sections of $\mathsf{E}$ defined on some neighbourhood $\mathcal{U}$ of $x_0$ and such that $[\xi_1]_{x_0}, \dots, [\xi_k]_{x_0}$ generate $\mathsf{A}$ (since $\mathscr{G}^{\omega}_{x_0,\mathsf{E}}$ is Noetherian). Then there exists a compact set $C \subseteq \mathsf{M}$ such that $x_0 \in \operatorname{int}(C)$ and such that $[\xi_1]_C, \dots, [\xi_k]_C$ generate the module $r^{-1}_{C,x_0}(\mathsf{A})$.*

Outline of proof: In the holomorphic case this result is proved as [Gunning 1990b, Theorem H.8]. The result, as many of these finite generation results, uses induction on $\dim(\mathsf{M})$ and the Weierstrass Preparation Theorem. The result in the real analytic case can be proved in a similar manner to the holomorphic case, using the real analytic Weierstrass Preparation Theorem [Krantz and Parks 2002, Theorem 6.1.3]. The result stated in [Gunning 1990b] includes an estimate that is not required for our purposes, and, moreover, does not hold in the real analytic case. However, the algebraic constructions we need can be extracted after chasing off the details of the proof of the estimate. ▼

Let $\mathscr{U} = (\mathcal{U}_a)_{a \in A}$ be an open cover of $\mathsf{M}$ and let $(\mathscr{F}_a)_{a \in A}$ be a family sheaves defining a patchy subsheaf $\mathscr{F}_{\mathscr{U}}$ as in Definition 3.22. Let $x_0 \in \mathsf{M}$ and let $a \in A$ be such that $x_0 \in \mathcal{U}_a$. Note that, for any compact set $C \subseteq \mathcal{U}_a$ and for any $\xi \in F_a(\mathcal{U}_a)$, we have $r_{C,x_0}([\xi]_C) = [\xi]_{x_0}$. Therefore, letting

$$\mathsf{A} = \{[\xi]_{x_0} \mid \xi \in F_a(\mathcal{U}_a)\},$$

we see that

$$r^{-1}_{C,x_0}(\mathsf{A}) = \{[\xi]_C \mid \xi \in F_a(\mathcal{U}_a)\}.$$

Suppose that $\mathsf{A}$ is generated by germs $[\xi_1]_{x_0}, \dots, [\xi_k]_{x_0}$, this by virtue of the fact that $\mathscr{G}^\omega_{x_0,\mathsf{E}}$ is Noetherian. By the lemma let $C \subseteq \mathcal{U}_a$ be a compact set such that $x_0 \in \operatorname{int}(C)$ and such that $[\xi_1]_C, \dots, [\xi_k]_C$ generate $r^{-1}_{C,x_0}(\mathsf{A})$. Now let $\mathcal{U}$ be a neighbourhood of $x_0$ such that $\mathcal{U} \subseteq C$. Let $\xi \in F_a(\mathcal{U})$ so $\xi$ is the restriction to $\mathcal{U}$ of a $\hat{\xi}$ section over $\mathcal{U}_a$, this because $\mathscr{F}$ is patchy. Note that $[\hat{\xi}]_C \in r^{-1}_{C,x_0}(\mathsf{A})$ and so there exists $[f^1]_C, \dots, [f^k]_C \in \mathscr{C}^\omega_{C,\mathsf{M}}$ such that

$$[\hat{\xi}]_C = [f^1]_C[\xi_1]_C + \dots + [f^k]_C[\xi_k]_C.$$

Therefore, we have

$$\xi = (f^1|\mathcal{U})\xi_1|\mathcal{U} + \dots + (f^k|\mathcal{U})\xi_k|\mathcal{U},$$

which is the first part of the result. The second assertion follows from Proposition 4.19(iv). ■

From the theorem, we have the following corollaries, the third of which is that with which we are presently concerned, but the first two of which are also extremely important.

**4.11 Corollary: (Patchy real analytic subsheaves are coherent)** *If $\pi\colon \mathsf{E} \to \mathsf{M}$ is a real analytic vector bundle and if $\mathscr{F}$ is a patchy subsheaf of $\mathscr{G}^\omega_{\mathsf{E}}$, then $\mathscr{F}$ is coherent.*

Proof: By Theorem 4.10 we know that $\mathscr{F}$ is locally finitely generated. By Oka's Theorem, $\mathscr{G}^\omega_{\mathsf{E}}$ is coherent. Since finitely generated subsheaves of coherent sheaves are coherent [Grauert and Remmert 1984, page 235], the result follows. ■

The following corollary is one that is often used.

**4.12 Corollary: (Submodules of analytic vector fields are locally finitely generated)** *If $\pi\colon \mathsf{E} \to \mathsf{M}$ is a real analytic vector bundle and if $\mathscr{M} \subseteq \Gamma^\omega(\mathsf{E})$ is a submodule of global sections, then $\mathscr{M}$ is locally finitely generated.*

Proof: This follows from Theorem 4.10 using the fact that the sheaf $\mathscr{F}_{\mathscr{M}}$ (see Definition 3.19 for the notation) is obviously patchy; indeed, it is a patchy sheaf defined by the trivial open covering. ■

The following result is also interesting and useful.

**4.13 Corollary: (The subsheaf of sections of a real analytic generalised subbundle is locally finitely generated)** *If $\pi\colon \mathsf{E} \to \mathsf{M}$ is a real analytic vector bundle and if $\mathsf{F}$ is an analytic generalised subbundle of $\mathsf{E}$, then $\mathscr{G}^\omega_{\mathsf{F}}$ is locally finitely generated.*

Proof: Let $x_0 \in \mathsf{M}$. There then exists a neighbourhood $\mathcal{U}$ of $x_0$ and local generators $(\xi_a)_{a\in A}$ for $\mathsf{F}$ defined on $\mathcal{U}$. Let $\mathscr{M}$ be the submodule of $\Gamma^\omega(\mathsf{E}|\mathcal{U})$ generated by the local generators. We then have the sheaf $\mathscr{F}_{\mathscr{M}}$ over $\mathcal{U}$, which is obviously patchy (it has a single patch). By Corollary 4.12 it follows that $\mathscr{M}$ is locally finitely generated. Thus, by shrinking $\mathcal{U}$ we arrive at a neighbourhood of $x_0$ with local sections $\eta_1, \dots, \eta_k \in \Gamma^\omega(\mathsf{E}|\mathcal{U})$ such that

$$\mathsf{F}_x = \operatorname{span}_{\mathbb{R}}(\xi_a(x)|\ a \in A) = \operatorname{span}_{\mathbb{R}}(\eta_j(x)|\ j \in \{1, \dots, k\}),$$

cf. Proposition 3.16. ■

Note that neither of the preceding two corollaries is a consequence of the Noetherian property of germs of real analytic sheaves, but a consequence of the deeper Theorem 4.10. This fact is routinely misunderstood [e.g., Agrachev and Sachkov 2004, Corollary 5.3]. Let us make this explicit by using one of our by now stock examples.

**4.14 Example: (Finite generation of germs does not imply locally finitely generated)** We take $\mathsf{M} = \mathbb{R}$ and consider the trivial vector bundle $\mathsf{E} = \mathbb{R} \times \mathbb{R}$ with projection $\pi(x, v) = x$. We take $S = \{0\} \cup \{\frac{1}{j} \mid j \in \mathbb{Z}_{>0}\}$ and define a subsheaf $\mathscr{F}_S$ of $\mathscr{G}^\omega_{\mathsf{E}}$ by

$$\mathscr{F}_S(\mathcal{U}) = \{\xi \in \Gamma^\omega(\mathsf{E}|\mathcal{U}) \mid \xi(x) = 0 \text{ for all } x \in \mathcal{U} \cap S\}.$$

The stalks $\mathscr{F}_{S,x}$, $x \in \mathbb{R}$, are finitely generated since they are Noetherian. However, as we saw in Example 3.24, $\mathscr{F}_S$ is not patchy, and, therefore, not locally finitely generated by Theorem 4.10. Specifically, $\mathscr{F}_S$ is not locally finitely generated about 0. •

In the smooth case, if we drop the assumption of regularity, then the sheaf of sections of a singular generalised subbundle may be locally finitely generated or not, depending on the precise example. Let us illustrate what can happen.

**4.15 Example: (A smooth singular subbundle whose subsheaf of sections is finitely generated)** We consider $\mathsf{M} = \mathbb{R}^2$ and the distribution $\mathsf{D}$ given by

$$\mathsf{D}_{(x_1,x_2)} = \begin{cases} \mathsf{T}_{(x_1,x_2)}\mathbb{R}^2, & x_1 \neq 0, \\ \operatorname{span}_{\mathbb{R}}(\frac{\partial}{\partial x_1}), & x_1 = 0. \end{cases}$$

This distribution is generated by any pair of vector fields

$$X_1(x_1, x_2) = \frac{\partial}{\partial x_1}, \quad X_2(x_1, x_2) = f(x_1)\frac{\partial}{\partial x_2},$$

where $f \in C^\infty(\mathbb{R})$ satisfies $f^{-1}(0) = \{0\}$. Thus $\mathsf{D}$ is a smooth distribution. Moreover, we claim that if $X$ is any section of $\mathsf{D}$ then $X = f^1 X_1 + f^2 X_2$ for some $f^1, f^2 \in C^\infty(\mathbb{R}^2)$, provided we take $f$ to be defined by $f(x) = x$. Indeed, let us write

$$X = g^1\frac{\partial}{\partial x_1} + g^2\frac{\partial}{\partial x_2},$$

noting that we must have $g^2(0, x_2) = 0$ for every $x_2 \in \mathbb{R}$. We write

$$g^2(x_1, x_2) = \int_0^{x_1} \frac{\partial g^2}{\partial x_1}(\xi, x_2)\,\mathrm{d}\xi = x_1 \int_0^1 \frac{\partial g^2}{\partial x_1}(x_1\eta, x_2)\,\mathrm{d}\eta.$$

Thus our claim follows by taking

$$f^1(x_1, x_2) = g^1(x_1, x_2), \quad f^2(x_1, x_2) = \int_0^1 \frac{\partial g^2}{\partial x_1}(x_1\eta, x_2)\,\mathrm{d}\eta.$$

This shows that, not only does $\mathsf{D}$ have a finite number of generators, but also that $\Gamma^\infty(\mathsf{D})$ is free and finitely generated. It follows that $\mathscr{G}^\infty_{\mathsf{D}}$ is free and finitely generated.

By choosing more pathological smooth generators, e.g., by taking

$$f(x) = \begin{cases} \mathrm{e}^{-1/x^2}, & x \neq 0, \\ 0, & x = 0, \end{cases}$$

one imagines that the algebraic properties of the distribution should deteriorate. However, this is not seen in this case until one looks at the Lie algebra generated by the generators. The reader can see this clearly in Example 6.10. •

Be careful to understand that the example does not contradict Theorem 4.9. For the previous example, the theorem merely says that the sheaf $\mathscr{G}_{\mathsf{D}}^{\infty}$ of the example is isomorphic to the sheaf of sections of some subbundle, in this case the trivial bundle with two-dimensional fibre.

There is actually a simpler example of a subbundle that illustrates the main point of the preceding example. Indeed, the distribution $\mathsf{D}$ on $\mathbb{R}$ defined by

$$\mathsf{D}_x = \begin{cases} \mathsf{T}_x\mathbb{R}, & x \neq 0, \\ \{0\}, & x = 0 \end{cases}$$

has the desired property of being a singular subbundle whose subsheaf of sections is free and finitely generated. This can be proved along the lines of the preceding example. However, we prefer the example we work out because we shall encounter it again in a different context, and it is illustrative to be able to compare the various properties of the distribution.

The next example we consider shows that it is possible to have a distribution whose submodule of sections is not finitely generated.

**4.16 Example: (A singular distribution whose submodule of sections is not locally finitely generated)** On $\mathsf{M} = \mathbb{R}^2$ we consider the distribution $\mathsf{D}$ generated by the two smooth vector fields

$$X_1(x_1, x_2) = \frac{\partial}{\partial x_1}, \quad X_2(x_1, x_2) = f(x_1)\frac{\partial}{\partial x_2},$$

where

$$f(x) = \begin{cases} \mathrm{e}^{-1/x^2}, & x \in \mathbb{R}_{>0}, \\ 0, & x \in \mathbb{R}_{\leq 0}. \end{cases}$$

We claim that $\Gamma^\infty(\mathsf{D})$ is not locally finitely generated. To see this, we refer ahead to Example 8.7 where we show that $\mathsf{D}$ is involutive but not integrable. By Frobenius's Theorem, Theorem 8.6, it follows that $\Gamma^\infty(\mathsf{D})$ is not locally finitely generated. •

Just as we saw after Example 4.15, it is possible for the preceding example to illustrate the point with a simpler distribution. Indeed, the distribution $\mathsf{D}$ on $\mathbb{R}$ defined by

$$\mathsf{D}_x = \begin{cases} \mathsf{T}_x\mathbb{R}, & x \in \mathbb{R}_{>0}, \\ \{0\}, & x \in \mathbb{R}_{\leq 0}, \end{cases}$$

has the property that it is involutive but not integrable. Thus, again according to Frobenius's Theorem, $\Gamma^\infty(\mathsf{D})$ is not locally finitely generated.

**4.4. Relationship between various notions of "locally finitely generated".** Note that we now have *three* versions of "locally finitely generated" floating about, and they do not apply to the same thing, and the relationships between them are not perfectly clear. To be precise, we have the notion of a locally finitely generated sheaf of $\mathscr{C}_{\mathsf{M}}^r$-modules from Definition 2.14, the notion of a locally finitely generated generalised subbundle from Definition 3.1, and the notion of a locally finitely generated submodule of sections from Definition 3.20.

The following property of locally finitely generated sheaves is of value.

**4.17 Lemma: (Local generators for locally finitely generated sheaves)** *Let* $\mathsf{M}$ *be a smooth or real analytic manifold, as required, let* $r \in \mathbb{Z}_{\geq 0} \cup \{\infty, \omega\}$*, and let* $\mathscr{F} = (F(\mathcal{U}))_{\mathcal{U}\,open}$ *be a locally finitely generated sheaf of* $\mathscr{C}^r_{\mathsf{M}}$*-modules. If, for* $x_0 \in \mathsf{M}$*,* $[s_1]_{x_0}, \dots, [s_k]_{x_0}$ *are generators for the* $\mathscr{C}^\omega_{x_0,\mathsf{M}}$*-module* $\mathscr{F}_{x_0}$*, then there exists a neighbourhood* $\mathcal{U}$ *of* $x_0$ *such that* $[s_1]_x, \dots, [s_k]_x$ *are generators for* $\mathscr{F}_x$ *for each* $x \in \mathcal{U}$*.*

Proof: By hypothesis, there exists a neighbourhood $\mathcal{V}$ of $x_0$ and sections $t_1, \dots, t_k \in F(\mathcal{V})$ such that $[t_1]_x, \dots, [t_m]_x$ generate $\mathscr{F}_x$ for all $x \in \mathcal{V}$. Since $[s_1]_{x_0}, \dots, [s_k]_{x_0}$ generate $\mathscr{F}_{x_0}$,

$$[t_l]_{x_0} = \sum_{j=1}^{k} [a^j_l]_{x_0} [s_j]_{x_0}, \qquad l \in \{1, \dots, m\},$$

for germs $[a^j_l]_{x_0} \in \mathscr{R}_{x_0}$. We can assume, possibly by shrinking $\mathcal{V}$, that $s_1, \dots, s_k \in F(\mathcal{V})$ and $a^j_l \in R(\mathcal{V})$, $j \in \{1, \dots, k\}$, $l \in \{1, \dots, m\}$. By definition of germ, there exists a neighbourhood $\mathcal{U} \subseteq \mathcal{V}$ of $x_0$ such that

$$r_{\mathcal{V},\mathcal{U}}(t_l) = \sum_{j=1}^{k} r_{\mathcal{V},\mathcal{U}}(a^j_l) r_{\mathcal{V},\mathcal{U}}(s_j), \qquad l \in \{1, \dots, m\}.$$

Taking germs at $x \in \mathcal{U}$ shows that the generators $[t_1]_x, \dots, [t]_m$ of $\mathscr{F}_x$ are linear combinations of $[s_1]_x, \dots, [s_k]_x$, as desired. ■

Note that the property of being locally finitely generated is one about stalks, not one about local sections. That is to say, if $[s_1]_{x_0}, \dots, [s_k]_{x_0}$ generate $\mathscr{F}_{x_0}$ and if $\mathcal{U}$ is a neighbourhood of $x_0$ for which $[s_1]_x, \dots, [s_k]_x$ generate $\mathscr{F}_x$ for each $x \in \mathcal{U}$, it is not clearly the case that $s_1, \dots, s_k$ generate $F(\mathcal{U})$. What we can state is the following result, which is a consequence of the vanishing of the cohomology groups of the sheaves in these cases.

**4.18 Lemma: (Local finite generation of modules of sections)** *Let* $r \in \mathbb{Z}_{\geq 0} \cup \{\infty, \omega\}$*, let* $\mathsf{M}$ *be a smooth or real analytic manifold, as required, and let* $\mathscr{F} = (F(\mathcal{U}))_{\mathcal{U}\,open}$ *be a sheaf of* $\mathscr{C}^r_{\mathsf{M}}$*-modules. Assume one of the two cases:*

*(i)* $r = \infty$*;*

*(ii)* $r = \omega$ *and* $\mathscr{F}$ *is coherent;*

*Let* $x_0 \in \mathsf{M}$*. If* $[(s_1, \mathcal{U})]_{x_0}, \dots, [(s_k, \mathcal{U})]_{x_0}$ *generate* $\mathscr{F}_{x_0}$*, then there exists a neighbourhood* $\mathcal{W} \subseteq \mathcal{U}$ *of* $x_0$ *such that* $r_{\mathcal{U},\mathcal{V}}(s_1), \dots, r_{\mathcal{U},\mathcal{V}}(s_k)$ *generate* $F(\mathcal{V})$ *for every open set* $\mathcal{V} \subseteq \mathcal{W}$*.*

Proof: From the proof of Lemma 4.17 we see that there exists a neighbourhood $\mathcal{W}$ of $x_0$ such that $([s_1]_x, \dots, [s_k]_x)$ generate $\mathscr{F}_x$ for every $x \in \mathcal{W}$. If $\mathcal{V} \subseteq \mathcal{W}$, we then have a presheaf morphism $\Phi = (\Phi_{\mathcal{V}'})_{\mathcal{V}' \subseteq \mathcal{V}\,\text{open}}$ from $(\mathscr{C}^r_{\mathcal{V}})^k$ to $\mathscr{F}|\mathcal{V}$ given by

$$\Phi_{\mathcal{V}'}(f^1, \dots, f^k) = f^1 r_{\mathcal{U},\mathcal{V}'}(s_1) + \dots + f^k r_{\mathcal{U},\mathcal{V}'}(s_k),$$

Note that the sequence

$$(\mathscr{C}^r_{\mathcal{V}})^k \xrightarrow{\Phi} \mathscr{F}|\mathcal{V} \longrightarrow 0 \tag{4.1}$$

is exact, by which we mean that it is exact on stalks. If $s \in F(\mathcal{V})$, exactness of (4.1) implies that, for $x \in \mathcal{V}$,

$$[s]_x = [g^1]_x [s_1]_x + \dots + [g^k]_x [s_k]_x$$

for $[g^1]_1, \dots, [g^k]_x \in \mathscr{C}^r_{x,\mathsf{M}}$. Since the preceding expression involves only a finite number of germs, there exists a neighbourhood $\mathcal{V}_x \subseteq \mathcal{V}$ of $x$ such that

$$r_{\mathcal{V},\mathcal{V}_x}(s) = g^1_x r_{\mathcal{U},\mathcal{V}_x}(s_1) + \dots + g^k_x r_{\mathcal{U},\mathcal{V}_x}(s_k)$$

for $g^1_x, \dots, g^k_x \in C^r(\mathcal{V}_x)$. Let $\mathscr{U} = (\mathcal{V}_x)_{x\in\mathcal{V}}$. If $\mathcal{V}_x \cap \mathcal{V}_y \neq \varnothing$, define $g^j_{xy} \in C^r(\mathcal{V}_x \cap \mathcal{V}_y)$ by

$$g^j_{xy} = g^j_x|\mathcal{V}_x \cap \mathcal{V}_y - g^j_y|\mathcal{V}_x \cap \mathcal{V}_y, \qquad j \in \{1,\dots,k\},$$

and note that $((g^1_{xy}, \dots, g^k_{xy}))_{x,y\in\mathcal{V}} \in \mathrm{Z}^1(\mathscr{U}, \ker(\Phi))$. We now note the following:

1. if $r = \infty$ then $\mathrm{H}^1(\mathscr{U}; \ker(\Phi)) = 0$ by Theorem 2.21;
2. if $r = \omega$ then $\ker(\Phi)$ is coherent [Grauert and Remmert 1984, page 237], and so $\mathrm{H}^1(\mathscr{U}; \ker(\Phi)) = 0$ by Cartan's Theorem B.

Since $\mathrm{H}^1(\mathscr{U}; \ker(\Phi)) = 0$, for each $x \in \mathcal{V}$ there exists $((h^1_x, \dots, h^k_x))_{x\in\mathcal{V}_x} \in \mathrm{C}^1(\mathscr{U}; \ker(\Phi))$ such that

$$h^j_y|\mathcal{V}_x \cap \mathcal{V}_y - h^j_x|\mathcal{V}_x \cap \mathcal{V}_y = g^j_{xy} = g^j_x|\mathcal{V}_x \cap \mathcal{V}_y - g^j_y|\mathcal{V}_x \cap \mathcal{V}_y, \qquad j \in \{1,\dots,k\}.$$

Define $f^j_x \in C^r(\mathcal{V}_x)$ by $f^j_x = g^j_x + h^j_x$, and note that

$$f^j_x|\mathcal{V}_x \cap \mathcal{V}_y = f^j_y|\mathcal{V}_x \cap \mathcal{V}_y, \qquad j \in \{1,\dots,k\}.$$

Thus there exists $f^j \in C^r(\mathcal{V})$ such that $f^j|\mathcal{V}_x = f^j_x$ for each $j \in \{1,\dots,k\}$ and $x \in \mathcal{V}$. Moreover, since

$$h^1_x r_{\mathcal{U},\mathcal{V}_x}(s_1) + \dots + h^k_x r_{\mathcal{U},\mathcal{V}_x}(s_k) = 0,$$

we have

$$f^1 r_{\mathcal{U},\mathcal{V}}(s_1) + \dots + f^k r_{\mathcal{U},\mathcal{V}}(s_k) = s,$$

as desired. ∎

We can now state the following result which connects the notions of "locally finitely generated."

**4.19 Proposition: (Relationships between various notions of "locally finitely generated")** *Let $r \in \mathbb{Z}_{\geq 0} \cup \{\infty, \omega\}$ and let $\pi\colon \mathsf{E} \to \mathsf{M}$ be a smooth or analytic vector bundle, as required. Let $\mathscr{F}$ be a subsheaf of $\mathscr{G}^r_{\mathsf{E}}$ and let $\mathscr{M} \subseteq \Gamma^r(\mathsf{E})$ be a submodule. The following statements hold:*

*(i) if the subsheaf $\mathscr{F}$ is locally finitely generated, then the generalised subbundle $\mathsf{F}(\mathscr{F})$ is locally finitely generated;*

*(ii) if the subsheaf $\mathscr{F}$ is locally finitely generated, then, for each $x \in \mathsf{M}$, there exists a neighbourhood $\mathcal{U}$ of $x$ such that $F(\mathcal{U})$ is a finitely generated $C^r(\mathcal{U})$-module;*

*(iii) if the submodule $\mathscr{M}$ is locally finitely generated, then the generalised subbundle $\mathsf{F}(\mathscr{M})$ is locally finitely generated;*

*(iv) if the submodule $\mathscr{M}$ is locally finitely generated, then the subsheaf $\mathscr{F}_{\mathscr{M}}$ is locally finitely generated.*

Proof: (i) Let $x_0 \in \mathsf{M}$. Since $\mathscr{F}$ is locally finitely generated, there exists a neighbourhood $\mathcal{U}$ of $x_0$ and sections $\xi_1, \dots, \xi_k \in F(\mathcal{U})$ such that $([\xi_1]_x, \dots, [\xi_k]_x)$ generate $\mathscr{F}_x$ for each $x \in \mathcal{U}$. If $e_x \in \mathsf{F}(\mathscr{F})_x$ for $x \in \mathcal{U}$, by definition of $\mathsf{F}(\mathscr{F})$ we have $e_x = \xi(x)$ for $[\xi]_x \in \mathscr{F}_x$. It follows that $(\xi_1(x), \dots, \xi_k(x))$ span $\mathsf{F}(\mathscr{F})_x$, showing that $\mathsf{F}(\mathscr{F})$ is locally finitely generated.

(ii) This follows from Lemma 4.18 and the Oka Coherence Theorem, $\mathscr{F}$ being a finitely generated subsheaf of the coherent sheaf $\mathscr{G}^r_{\mathsf{E}}$, and so coherent [Grauert and Remmert 1984, page 235].

(iii) Let $x_0 \in \mathsf{M}$. Since $\mathscr{M}$ is locally finitely generated, there exists a neighbourhood $\mathcal{U}$ of $x_0$ and $\xi_1, \dots, \xi_k \in \mathscr{M}$ which generate $F_{\mathscr{M}}(\mathcal{U})$ as a $C^r(\mathcal{U})$-module. If $e_x \in \mathsf{F}(\mathscr{M})_x$ for $x \in \mathcal{U}$, by definition of $\mathsf{F}(\mathscr{M})$ we have $e_x = \xi(x)$ for $\xi \in \mathscr{M}$. It follows that $(\xi_1(x), \dots, \xi_k(x))$ span $\mathsf{F}(\mathscr{M})_x$, showing that $\mathsf{F}(\mathscr{M})$ is locally finitely generated.

(iv) Let $x_0 \in \mathsf{M}$. Since $\mathscr{M}$ is locally finitely generated, there exists a neighbourhood $\mathcal{U}$ of $x_0$ and $\xi_1, \dots, \xi_k \in \mathscr{M}$ which generate $F_{\mathscr{M}}(\mathcal{U})$ as a $C^r(\mathcal{U})$-module. Let $x \in \mathcal{U}$ and let $[(\xi, \mathcal{V})]_x \in \mathscr{F}_{\mathscr{M},x}$ for some $\mathcal{V} \subseteq \mathcal{U}$. Thus

$$\xi = f^1(\eta_1|\mathcal{V}) + \dots + f^m(\eta_m|\mathcal{V})$$

for $f^1, \dots, f^m \in C^r(\mathcal{V})$ and $\eta_1, \dots, \eta_m \in \mathscr{M}$. Let us write

$$\eta_a|\mathcal{U} = g^1_a\xi_1 + \dots + g^k_a\xi_k$$

for $g^j_a \in C^r(\mathcal{U})$, $a \in \{1, \dots, m\}$, $j \in \{1, \dots, k\}$. Thus

$$\xi = \sum_{j=1}^k \sum_{a=1}^m f^j(g^a_j|\mathcal{V})(\xi_a|\mathcal{V}),$$

giving

$$[\xi]_x = \sum_{j=1}^k \sum_{a=1}^m [f^j]_x[g^a_j]_x[\xi_a]_x,$$

so showing that $([\xi_1]_x, \dots, [\xi_k]_x)$ generate $\mathscr{F}_{\mathscr{M},x}$ for every $x \in \mathcal{U}$. ■

Note that the implications for a generalised subbundle being locally finitely generated are not present. This is because this is not interesting in the following sense. An analytic generalised subbundle $\mathsf{F}$ always has the property that $\mathscr{G}^\omega_{\mathsf{F}}$ is locally finitely generated; this is Corollary 4.13 above. On the other hand, a smooth generalised subbundle $\mathsf{F}$ is globally finitely generated by Theorem 5.1 below, but $\mathscr{G}^\infty_{\mathsf{F}}$ may not be locally finitely generated as can be seen from Example 4.16.

## 5. Global generators for generalised subbundles

One of the potential problems with our definition of a generalised subbundle is that it relies on generators that are only locally defined. In the smooth or finitely differentiable case, these local generators can very often (for example, when the neighbourhood on which the generators are defined is not dense) be extended using the Tietsze Extension Theorem [Abraham, Marsden, and Ratiu 1988, Theorem 5.5.9] to give global generators. However, in the real analytic case, matters are more complicated, as generally a local section simply *cannot* be extended to be globally defined. Here one must thus attenuate one's objectives, and merely wonder whether globally defined generators exist at all. In this section we address these questions, considering separately the smooth and finitely differentiable cases, and the real analytic case.

**5.1. Global generators for $C^r$-generalised subbundles.** Let us first look at the smooth and finitely differentiable case. The result here is due to Sussmann [2008], and we give the proof here since the result is an important one.

**5.1 Theorem: (Generalised subbundles of class $C^r$ are finitely generated)** *Let $r \in \mathbb{Z}_{\geq 0} \cup \{\infty\}$ and let $\pi\colon \mathsf{E} \to \mathsf{M}$ be a smooth vector bundle whose fibres have bounded dimension and for which $\mathsf{M}$ is a smooth paracompact Hausdorff manifold of bounded dimension. If $\mathsf{F} \subseteq \mathsf{E}$ is a generalised subbundle then the following statements are equivalent:*

*(i) $\mathsf{F}$ is of class $C^r$;*

*(ii) for each $x_0 \in \mathsf{M}$ and each $v_{x_0} \in \mathsf{F}_{x_0}$, there exists a neighbourhood $\mathcal{N}$ of $x_0$ and a $C^r$-section $\xi \in \Gamma^r(\mathsf{E})$ such that $\xi(x_0) = v_{x_0}$ and $\xi(x) \in \mathsf{F}_x$ for each $x \in \mathcal{N}$;*

*(iii) there exists a family $(\xi_1, \dots, \xi_k)$ of $C^r$-sections on $\mathsf{M}$ such that*

$$\mathsf{F}_x = \operatorname{span}_{\mathbb{R}}(\xi_1(x), \dots, \xi_k(x))$$

*for each $x \in \mathsf{M}$.*

Proof: (i) $\Longrightarrow$ (ii) Suppose that $\mathsf{F}$ is of class $C^r$, let $x_0 \in \mathsf{M}$, and let $v_{x_0} \in \mathsf{F}_{x_0}$. Let $\mathcal{N}$ be a neighbourhood of $x_0$ and let $(\xi_j)_{j\in J}$ be a family of sections on $\mathcal{N}$ of class $C^r$ such that

$$\mathsf{F}_x = \operatorname{span}_{\mathbb{R}}(\xi_j(x)|\ j \in J)$$

for $x \in \mathcal{N}$. Let $j_1, \dots, j_k \in J$ be such that $(\xi_{j_1}(x_0), \dots, \xi_{j_k}(x_0))$ is a basis for $\mathsf{F}_{x_0}$. Then

$$v_{x_0} = c_1\xi_{j_1}(x_0) + \cdots + c_k\xi_{j_k}(x_0)$$

for some uniquely defined $c_1, \dots, c_k \in \mathbb{R}$. The section $\xi = c_1\xi_{j_1} + \cdots + c_k\xi_{j_k}$ defined on $\mathcal{N}$ is then of class $C^r$, is $\mathsf{F}$-valued on $\mathcal{N}$, and satisfies $\xi(x_0) = v_{x_0}$.

(ii) $\Longrightarrow$ (iii) Since $\mathsf{E}$ has bounded fibre dimension there exists a least integer $n_0 \in \mathbb{Z}_{\geq 0}$ such that $\operatorname{rank}(\mathsf{F}_x) \leq n_0$ for every $x \in \mathsf{M}$. Use the notation $\operatorname{rank}(\mathsf{F}_x) = \operatorname{rank}_{\mathsf{F}}(x)$. For $k \in \{0, 1, \dots, n_0 + 1\}$ denote

$$\mathcal{U}_k = \{x \in \mathsf{M} \mid \ \operatorname{rank}_{\mathsf{F}}(x) \geq k\}.$$

By Proposition 3.6, $\operatorname{rank}_{\mathsf{F}}$ is lower semicontinuous, and so $\mathcal{U}_k$ is open for each $k \in \{0, 1, \dots, n_0 + 1\}$. Moreover,

$$\varnothing = \mathcal{U}_{n_0+1} \subseteq \mathcal{U}_{n_0} \subseteq \cdots \subseteq \mathcal{U}_1 \subseteq \mathcal{U}_0 = \mathsf{M}.$$

We wish to define a certain open cover of each of the open sets $\mathcal{U}_k$, $k \in \{0, 1, \dots, n_0 + 1\}$, with a certain property. We do this inductively.

The following two general lemmata will be key in our inductive construction. Note that the notation of the lemmata may or may nor correspond to the notation of the theorem and its proof. So beware.

**1 Lemma:** *Let* $\mathsf{M}$ *be a smooth, paracompact, Hausdorff manifold all of whose connected components have dimension bounded by* $n \in \mathbb{Z}_{\geq 0}$ *and let* $(\mathcal{W}_j)_{j\in J}$ *be an open cover of* $\mathsf{M}$*. Then there exists an open cover* $(\mathcal{V}_a)_{a\in A}$ *of* $\mathsf{M}$ *with the following properties:*

(i) $(\mathcal{V}_a)_{a\in A}$ *is a refinement of* $(\mathcal{W}_j)_{j\in J}$ *(i.e., for each* $j \in J$ *there exists* $a \in A$ *such that* $\mathcal{W}_j \subseteq \mathcal{V}_a$*);*

(ii) *there exist subsets* $A_1, \dots, A_{n+1} \subseteq A$ *such that* $A = \mathring{\cup}_{l=1}^{n+1} A_l$ *and such that, whenever* $a_1, a_2 \in A_l$ *for some* $l \in \{1, \dots, n+1\}$*, it holds that* $\mathcal{V}_{a_1} \cap \mathcal{V}_{a_2} = \varnothing$*.*

Proof: Since $\mathsf{M}$ is paracompact it possesses a Riemannian metric by Corollary 5.5.13 of [Abraham, Marsden, and Ratiu 1988]. Therefore, we may assume that $\mathsf{M}$ is a metric space, and we denote the metric by d. A smooth manifold can be triangulated by which we mean that there exists a homeomorphism $\Phi\colon S \to \mathsf{M}$ where $S$ is a union of simplices which intersect only at their boundaries [Munkres 1966, Theorem 8.4]. By successive barycentric subdivisions of the simplices of $S$ we can assume that all simplices $S_b$, $b \in B$, comprising $S$ are such that $\Phi(S_b) \subseteq \mathcal{W}_j$ for some $j \in J$. Let us adopt the usual slight abuse of terminology and say that $\Phi(S_b)$, $b \in B$, is a simplex. Let $m \in \{0, 1, \dots, n\}$ and define

$$\mathscr{F}_m = \{F \subseteq \mathsf{M} \mid\ F \text{ is an open } m\text{-dimensional face of some simplex}\}$$

and denote $|\mathscr{F}_m| = \cup_{F\in\mathscr{F}_m} F$. For $F \in \mathscr{F}_m$ denote

$$C(F) = \mathrm{cl}(\cup\{F' \in \mathscr{F}_m \mid\ F' \neq F\}).$$

Note that $F$ is both open and closed in the relative topology of $|\mathscr{F}_m|$. Also, $C(F)$ is closed in the relative topology of $|\mathscr{F}_m|$ since its intersection with any compact set is a finite union of closed sets, and so closed. This implies that $F \cap C(F) = \varnothing$ by virtue of $F$ being relatively open. Now let $x \in F$. Then $\{x\}$ and $C(F)$ are disjoint closed sets, and so

$$\mathrm{d}(x, C(F)) = \inf\{\mathrm{d}(x, y) \mid\ y \in C(F)\}$$

is positive. (Indeed, suppose otherwise. Then there exists a sequence $(y_k)_{k\in\mathbb{Z}_{>0}}$ in $C(F)$ converging to $x$. Thus $x \in C(F)$ since $C(F)$ is relatively closed. This contradicts the fact that $\{x\}$ and $C(F)$ are disjoint.) Denote by $\mathsf{B}(x, \frac{1}{2}\mathrm{d}(x, C(F)))$ the open ball in $\mathsf{M}$ of radius $\frac{1}{2}\mathrm{d}(x, C(F))$ and define $B(F) = \cup_{x\in F}\mathsf{B}(x, \frac{1}{2}\mathrm{d}(x, C(F)))$.

We claim that if $F_1, F_2 \in \mathscr{F}_m$ are disjoint, then $B(F_1)$ and $B(F_2)$ are disjoint. Suppose otherwise and let $x \in B(F_1) \cap B(F_2)$, Let $x_1 \in F_1$ and $x_2 \in F_2$ be such that $y \in \mathsf{B}(x_j, \frac{1}{2}\mathrm{d}(x_j, C(F_j)))$, $j \in \{1, 2\}$. Then

$$\begin{aligned}\mathrm{d}(x_1, x_2) &\leq \mathrm{d}(x_1, x) + \mathrm{d}(x_2, x) < \tfrac{1}{2}\mathrm{d}(x_1, C(F_1)) + \tfrac{1}{2}\mathrm{d}(x_2, C(F_2))\\ &\leq \max\{\mathrm{d}(x_1, C(F_1)), \tfrac{1}{2}\mathrm{d}(x_2, C(F_2))\}.\end{aligned}$$

Thus either

$$\mathrm{d}(x_1, x_2) < \mathrm{d}(x_1, C(F_1)) \quad \text{or} \quad \mathrm{d}(x_1, x_2) < \mathrm{d}(x_2, C(F_2)).$$

In the first case we have $x_2 \in F_1$, contradicting the fact that $x_2 \in C(F_1)$, and in the second case we have $x_1 \in F_2$, contradicting the fact that $x_1 \in C(F_2)$. Thus $B(F_1)$ and $B(F_2)$ are disjoint if $F_1, F_2 \in \mathscr{F}_m$ are disjoint.

For $F \in \mathscr{F}_m$ let $j_F \in J$ be such that $F \subseteq \mathcal{W}_{j_F}$, this being possible since our triangulation of $\mathsf{M}$ was chosen in precisely this manner. Define an open set $\mathcal{V}_F = B(F) \cap \mathcal{W}_{j_F}$. Note that

$$F \subseteq \mathcal{V}_F \subseteq \mathcal{W}_{j_F}.$$

Moreover, since $B(F_1)$ and $B(F_2)$ are disjoint for $F_1, F_2 \in \mathscr{F}_m$ disjoint, it follows that $\mathcal{V}_{F_1}$ and $\mathcal{V}_{F_2}$ are disjoint for $F_1, F_2 \in \mathscr{F}_m$ disjoint. If $x \in \mathsf{M}$ then $x$ belongs to some open $m$-dimensional face from the triangulation of $\mathsf{M}$ for some $m \in \{0, 1, \dots, n\}$. Therefore,

$$\mathsf{M} = \bigcup_{m=0}^{n} \bigcup_{F \in \mathscr{F}_m} \mathcal{V}_F.$$

Thus we have an open cover of $\mathsf{M}$ that refines $(\mathcal{W}_j)_{j \in J}$. The index set for the open cover is the set

$$A = \{F \mid F \in \mathscr{F}_m,\ m \in \{0, 1, \dots, n\}\}.$$

It remains to show that the index set for the open cover satisfies the second condition in the statement of the lemma. For $m \in \{1, \dots, n+1\}$ let $A_m = \mathscr{F}_{m-1}$ so that $A$ is the disjoint union of $A_1, \dots, A_{n+1}$. As we have shown above, for each $m \in \{1, \dots, n+1\}$, the family of open sets $(\mathcal{V}_F)_{F \in A_m}$ is a pairwise disjoint union of open sets, just as is asserted in the statement of the lemma. ▼

The following technical lemma will also be useful. The result is well-known, but we were not able to find a proof for it, so we provide one here.

**2 Lemma:** *If $\mathcal{U}$ is an open subset of a smooth, paracompact, Hausdorff manifold $\mathsf{M}$ then there exists $f \in C^\infty(\mathsf{M})$ such that $f(x) \in \mathbb{R}_{>0}$ for all $x \in \mathcal{U}$ and $f(x) = 0$ for all $x \in \mathsf{M} \setminus \mathcal{U}$.*

Proof: We shall construct $f$ as the limit of a sequence of smooth functions converging in the weak $C^\infty$-topology. We equip $\mathsf{M}$ with a Riemannian metric $\mathbb{G}$, this by paracompactness of $\mathsf{M}$ [Abraham, Marsden, and Ratiu 1988, Corollary 5.5.13]. We denote by $\nabla$ the Levi-Civita connection of $\mathbb{G}$. Let $g \in C^\infty(\mathsf{M})$. If $K \subseteq \mathsf{M}$ is compact and if $r \in \mathbb{Z}_{\geq 0}$, we define

$$\|g\|_{r,K} = \sup\{\|\nabla^j g(x)\| \mid x \in K,\ j \in \{0, 1, \dots, r\}\},$$

where $\|\cdot\|$ indicates the norm induced on tensors by the norm associated with the Riemannian metric. One readily sees that the family of seminorms $\|\cdot\|_{r,K}$, $r \in \mathbb{Z}_{\geq 0}$, $K \subseteq \mathsf{M}$ compact, defines a locally convex topology agreeing with other definitions of the weak topology. Thus, if a sequence $(g_j)_{j \in \mathbb{Z}_{>0}}$ satisfies

$$\lim_{j \to \infty} \|g - g_j\|_{r,K} = 0, \qquad r \in \mathbb{Z}_{\geq 0},\ K \subseteq \mathsf{M} \text{ compact},$$

then $g$ is infinitely differentiable [Michor 1980, §4.3].

We suppose that $\mathsf{M}$ is connected since, if it is not, we can construct $f$ for each connected component, which suffices to give $f$ on $\mathsf{M}$. If $\mathsf{M}$ is paracompact, connectedness allows us to conclude that $\mathsf{M}$ is second countable [Abraham, Marsden, and Ratiu 1988, Proposition 5.5.11]. Using Lemma 2.76 of [Aliprantis and Border 2006], we let $(K_j)_{j \in \mathbb{Z}_{>0}}$ be a sequence of compact subsets of $\mathcal{U}$ such that $K_j \subseteq \operatorname{int}(K_{j+1})$ for $j \in \mathbb{Z}_{>0}$ and such that $\cup_{j \in \mathbb{Z}_{>0}} K_j = \mathcal{U}$. For $j \in \mathbb{Z}_{>0}$ let $g_j \colon \mathsf{M} \to [0, 1]$ be a smooth function such that $g_j(x) = 1$ for

$x \in K_j$ and $g_j(x) = 0$ for $x \in \mathsf{M} \setminus K_{j+1}$; see [Abraham, Marsden, and Ratiu 1988, Proposition 5.5.8]. Let us define $\alpha_j = \|g_j\|_{j,K_{j+1}}$ and take $\epsilon_j \in \mathbb{R}_{>0}$ to satisfy $\epsilon_j < (\alpha_j 2^j)^{-1}$. We define $f$ by

$$f(x) = \sum_{j=1}^{\infty} \epsilon_j g_j(x),$$

and claim that $f$ as defined satisfies the conclusions of the lemma. First of all, since each of the functions $g_j$ takes values in $[0, 1]$ we have

$$|f(x)| \leq \sum_{j=1}^{\infty} |\epsilon_j g_j(x)| \leq \sum_{j=1}^{\infty} \epsilon_j \|g_j\|_{0,K_{j+1}} \leq \sum_{j=1}^{\infty} \epsilon_j \|g_j\|_{j,K_{j+1}} \leq \sum_{j=1}^{\infty} \frac{1}{2^j} \leq 1,$$

and so $f$ is well-defined. If $x \in \mathcal{U}$ then there exists $N \in \mathbb{Z}_{>0}$ such that $x \in K_N$. Thus $g_N(x) = 1$ and so $f(x) \in \mathbb{R}_{>0}$. If $x \in \mathsf{M} \setminus \mathcal{U}$ then $g_j(x) = 0$ for all $j \in \mathbb{Z}_{>0}$ and so $f(x) = 0$. All that remains to show is that $f$ is infinitely differentiable.

Let $x \in \mathsf{M}$, let $m \in \mathbb{Z}_{>0}$, and let $j \in \mathbb{Z}_{\geq 0}$ be such that $j \leq m$. If $x \notin K_{m+1}$ then $g_m$ is zero in a neighbourhood of $x$, and so $\|\nabla^j g_m(x)\| = 0$. If $x \in K_{m+1}$ then

$$\begin{aligned}\|\nabla^j g_m(x)\| &\leq \sup\{\|\nabla^j g_m(x')\| \mid x' \in K_{m+1}\} \\ &\leq \sup\{\|\nabla^j g_m(x')\| \mid x' \in K_{m+1},\ j \in \{0, 1, \ldots, m\}\} = \alpha_m.\end{aligned}$$

Thus, whenever $j \leq m$ we have $\|\nabla^j g_m(x)\| \leq \alpha_m$ for every $x \in \mathsf{N}$.

Let us define $f_m \in C^\infty(\mathsf{M})$ by

$$f_m(x) = \sum_{j=1}^{m} \epsilon_j g_j(x).$$

Let $K \subseteq \mathsf{M}$ be compact, let $r \in \mathbb{Z}_{\geq 0}$, and let $\epsilon \in \mathbb{R}_{>0}$. Take $N \in \mathbb{Z}_{>0}$ sufficiently large that

$$\sum_{m=m_1+1}^{m_2} \frac{1}{2^m} < \epsilon,$$

for $m_1, m_2 \geq N$ with $m_1 < m_2$, this being possible by convergence of $\sum_{j=1}^{\infty} \frac{1}{2^j}$. Then, for $m_1, m_2 \geq N$,

$$\begin{aligned}\|f_{m_1} - f_{m_2}\|_{r,K} &= \sup\{\|\nabla^j f_{m_1}(x) - \nabla^j f_{m_2}(x)\| \mid x \in K,\ j \in \{0, 1, \ldots, r\}\} \\ &= \sup\left\{\left\| \sum_{m=m_1+1}^{m_2} \epsilon_m \nabla^j g_m(x) \right\| \;\middle|\; x \in K,\ j \in \{0, 1, \ldots, r\}\right\} \\ &\leq \sup\left\{ \sum_{m=m_1+1}^{m_2} \epsilon_m \|\nabla^j g_m(x)\| \;\middle|\; x \in K,\ j \in \{0, 1, \ldots, r\}\right\} \leq \sum_{m_1+1}^{m_2} \frac{1}{2^m} < \epsilon.\end{aligned}$$

Thus, for every $r \in \mathbb{Z}_{\geq 0}$ and $K \subseteq \mathsf{M}$ compact, $(f_m)_{m \in \mathbb{Z}_{>0}}$ is a Cauchy sequence in the norm $\|\cdot\|_{r,K}$. Completeness of the weak $C^\infty$-topology implies that the sequence $(f_m)_{m \in \mathbb{Z}_{>0}}$ converges to a function that is infinitely differentiable. ▼

With the use of the preceding lemmata, we can prove the following, reverting now to the notation of the theorem and its proof.

**3 Lemma:** *There exist* $C^r$*-sections* $\xi^k_m$, $k \in \{0, 1, \dots, n_0\}$, $m \in \{1, \dots, n+1\}$, *on* $\mathsf{M}$, *taking values in* $\mathsf{F}$, *such that, for every* $k \in \{0, 1, \dots, n_0\}$ *and for every* $x \in \mathcal{U}_k$,

$$\dim(\mathrm{span}_{\mathbb{R}}(\{\xi^j_m(x') \mid j \in \{0, 1, \dots, k\},\ m \in \{1, \dots, n+1\}\})) \geq k.$$

Proof: We prove this by induction on $k$. For $k = 0$ the assertion is obvious. Indeed, one merely takes the sections $\xi^0_1, \dots, \xi^0_{n+1}$ to all be zero, and the conclusion holds in this case. Now assume that the conclusions hold for $k \in \{0, 1, \dots, s\}$. Let $x \in \mathcal{U}_{s+1}$ and define

$$\mathsf{U}^s_x = \mathrm{span}_{\mathbb{R}}(\{\xi^k_m(x) \mid k \in \{0, 1, \dots, s\},\ m \in \{1, \dots, n+1\}\}),$$

noting that $\dim(\mathsf{U}^s_x) \geq s$ by the induction hypothesis. Since $x \in \mathcal{U}_x$ there exists $v_x \in \mathsf{F}_x$ such that

$$\dim(\mathrm{span}_{\mathbb{R}}(\mathsf{U}^s_x \cup \{v_x\})) \geq s + 1.$$

By the hypotheses of part (ii) of the theorem, there exists a neighbourhood $\mathcal{W}^{s+1}_x$ of $x$ and a $C^r$-section $\xi_x$ on $\mathcal{W}^{s+1}_x$, taking values in $\mathsf{F}$, such that $\xi_x(x) = v_x$. By multiplying $\xi_x$ by a smooth function that is positive on $\mathcal{W}^{s+1}_x$ and with compact support [see Abraham, Marsden, and Ratiu 1988, Theorem 5.5.9] we can extend $\xi_x$ to a $\mathsf{F}$-valued $C^r$-section on all of $\mathsf{M}$. By shrinking $\mathcal{W}^{s+1}_x$ if necessary, we can suppose that

$$\dim(\mathrm{span}_{\mathbb{R}}(\mathsf{U}^s_{x'} \cup \{\xi_x(x')\})) \geq s + 1, \qquad x' \in \mathcal{W}^{s+1}_x,$$

by lower semicontinuity of the rank of a generalised subbundle (Proposition 3.6). With $\mathcal{W}^{s+1}_x$ so specified we have $\mathcal{W}^{s+1}_x \subseteq \mathcal{U}_{s+1}$. Note that $(\mathcal{W}^{s+1}_x)_{x \in \mathcal{U}_{s+1}}$ is then an open cover of $\mathcal{U}_k$.

By Lemma 1 let $(\mathcal{V}^{s+1}_a)_{a \in A}$ be a refinement of the open cover $(\mathcal{W}^{s+1}_x)_{x \in \mathcal{U}_{s+1}}$ of $\mathcal{U}_{s+1}$ such that the index set $A$ is a disjoint union of sets $A_1, \dots, A_{n+1}$ such that $\mathcal{V}^{s+1}_{a_1} \cap \mathcal{V}^{s+1}_{a_2} = \varnothing$ whenever $a_1, a_2 \in A_l$ are distinct for some $l \in \{1, \dots, n+1\}$. Let us denote $\mathcal{V}^{s+1}_l = \cup_{a \in A_l} \mathcal{V}^{s+1}_a$, $l \in \{1, \dots, n+1\}$. For $l \in \{1, \dots, m+1\}$ and $a \in A_l$, let $x_{l,a} \in \mathcal{U}_k$ be such that $\mathcal{V}^{s+1}_a \subseteq \mathcal{W}^{s+1}_{x_{l,a}}$. Define a $C^r$-section $\eta^{s+1}_l$ on $\mathcal{V}^{s+1}_l$ by asking that, if $x \in \mathcal{V}^{s+1}_a$ for some (necessarily unique) $a \in A_l$, then $\eta^{s+1}_l(x) = \xi_{x_{l,a}}(x)$. The section $\eta^{s+1}_l$ will then have the property that

$$\dim(\mathrm{span}_{\mathbb{R}}(\mathsf{U}^s_x \cup \{\eta^{s+1}_l(x)\})) \geq s + 1, \qquad x \in \mathcal{V}_l,\ l \in \{1, \dots, n+1\}. \tag{5.1}$$

For each $l \in \{1, \dots, n+1\}$, by Lemma 2 let $f_l \in C^\infty(\mathsf{M})$ be such that $f_l(x) \in \mathbb{R}_{>0}$ for $x \in \mathcal{V}_l$ and such that $f_l(x) = 0$ for $x \in \mathsf{M} \setminus \mathcal{V}_l$. Then define $\xi^{s+1}_l = f_l \eta^{s+1}_l$ so that

$$\dim(\mathrm{span}_{\mathbb{R}}(\mathsf{U}^s_x \cup \{\xi^{s+1}_l(x)\})) \geq s + 1, \qquad x \in \mathcal{V}_l,\ l \in \{1, \dots, n+1\}, \tag{5.2}$$

by (5.1), since $\xi^{s+1}_l(x)$ is a nonzero multiple of $\eta^{s+1}_l(x)$ for all $x \in \mathcal{V}_l$. To complete the proof of the lemma, let $x \in \mathcal{U}_{s+1}$ and let $a \in A$ be such that $x \in \mathcal{V}_a$. Then $a \in A_l$ for some unique $l \in \{1, \dots, n+1\}$ and so $x \in \mathcal{V}_l$. Therefore, by (5.2) and recalling the definition of $\mathsf{U}^s_x$,

$$\begin{aligned} &\mathrm{span}_{\mathbb{R}}(\mathsf{U}^s_x \cup \{\xi^{s+1}_l(x)\}) \subseteq \mathrm{span}_{\mathbb{R}}(\{\xi^j_m(x) \mid j \in \{0, 1, \dots, s+1\},\ m \in \{1, \dots, n+1\}\}) \\ \implies \quad &\dim(\mathrm{span}_{\mathbb{R}}(\{\xi^j_m(x) \mid j \in \{0, 1, \dots, s+1\},\ m \in \{1, \dots, n+1\}\})) \geq s + 1, \end{aligned}$$

as desired. ▼

We now conclude the proof of this part of the theorem by showing that the $\mathsf{F}$-valued $C^r$-sections

$$(\xi^j_m \mid\ j \in \{1,\dots,n_0\},\ m \in \{1,\dots,n+1\})$$

from Lemma 3 generate $\mathsf{F}$. Let $x \in \mathsf{M}$ and let $k = \operatorname{rank}_{\mathsf{F}}(x)$ so that $x \in \mathcal{U}_k$. By Lemma 3 we have

$$\begin{aligned}\dim(\operatorname{span}_{\mathbb{R}}(\{\xi^j_m(x) \mid\ j \in \{1,\dots,n_0\},\ m \in \{1,\dots,n+1\}\}))\\ \geq \dim(\operatorname{span}_{\mathbb{R}}(\{\xi^j_m(x) \mid\ j \in \{1,\dots,k\},\ m \in \{1,\dots,n+1\}\})) \geq k.\end{aligned}$$

However, since $\dim(\mathsf{F}_x) = k$ and since all sections $\xi^j_m$, $j \in \{1,\dots,n_0\}$, $m \in \{1,\dots,n+1\}$, are $\mathsf{F}$-valued, we conclude that

$$\operatorname{span}_{\mathbb{R}}(\{\xi^j_m(x) \mid\ j \in \{1,\dots,n_0\},\ m \in \{1,\dots,n+1\}\}) = \mathsf{F}_x,$$

as desired.

(iii) $\Longrightarrow$ (i) This is obvious. ■

Note that the theorem does not say that the sections $\xi_1,\dots,\xi_k$ from part (iii) generate the stalks of the sheaf $\mathscr{G}^r_{\mathsf{F}}$. Indeed, as in Example 4.16, the stalks of a subsheaf of $C^r$-sections, $r \in \mathbb{Z}_{\geq 0} \cup \{\infty\}$, are not generally finitely generated.

**5.2. Global generators for real analytic generalised subbundles.** In the real analytic case, we cannot prove that there are generally finitely many global generators. However, we can use Cartan's Theorem A, stated in Theorem 2.18, to give local generation by global sections. This is, of course, highly nontrivial since generally a local section of a real analytic vector bundle cannot be extended to a global section.

**5.2 Theorem: (Generalised real analytic subbundles are locally generated by global sections)** *Let $\pi\colon \mathsf{E} \to \mathsf{M}$ be a real analytic vector bundle for which $\mathsf{M}$ is paracompact and Hausdorff. If $\mathsf{F} \subseteq \mathsf{E}$ is a generalised subbundle, then the following statements are equivalent:*

*(i) $\mathsf{F}$ is real analytic;*

*(ii) for each $x_0 \in \mathsf{M}$ and each $v_{x_0} \in \mathsf{F}_{x_0}$, there exists a real analytic section $\xi \in \Gamma^\omega(\mathsf{F})$ over $\mathsf{M}$ such that $\xi(x_0) = v_{x_0}$;*

*(iii) for each $x_0 \in \mathsf{M}$ there exists a neighbourhood $\mathcal{N}$ of $x_0$ and a real analytic sections $\xi_1,\dots,\xi_k \in \Gamma^\omega(\mathsf{F})$ over $\mathsf{M}$ such that*

$$\mathsf{F}_x = \operatorname{span}_{\mathbb{R}}(\xi_1(x),\dots,\xi_k(x))$$

*for each $x \in \mathcal{N}$.*

Proof: (i) $\Longrightarrow$ (ii) By Proposition 4.5 let $V_{x_0} \in (\mathscr{C}^\omega_{x_0,\mathsf{M}}/\mathfrak{m}_{x_0}) \otimes_{\mathscr{C}^\omega_{x_0,\mathsf{M}}} \mathscr{G}^\omega_{x_0,\mathsf{F}}$ be such that $\tilde{\iota}_{x_0}(V_{x_0}) = v_{x_0}$, where $\tilde{\iota}_{x_0}$ is as in the proof of Proposition 4.5. We write

$$V_{x_0} = \sum_{j=1}^{k} ([f^j]_{x_0} + \mathfrak{m}_{x_0}) \otimes [\xi_j]_{x_0}$$

for some $[f^1]_{x_0},\dots,[f^k]_{x_0}\in\mathscr{C}^\omega_{x_0,\mathsf{M}}$ and $[\xi_1]_{x_0},\dots,[\xi_k]_{x_0}\in\mathscr{G}^\omega_{x_0,\mathsf{F}}$. By Corollary 4.11 we know that $\mathscr{G}^\omega_{\mathsf{F}}$ is coherent. Therefore, by Cartan's Theorem A (Theorem 2.18), there exist global sections $(\beta_a)_{a\in A}$ of $\mathsf{F}$ such that the germs $[\beta_a]$, $a\in A$, generate $\mathscr{G}^\omega_{x_0,\mathsf{F}}$ as a $\mathscr{C}^\omega_{x_0,\mathsf{M}}$-module. It follows that we can write

$$V_{x_0}=\sum_{a\in A}([g^a]_{x_0}+\mathfrak{m}_{x_0})\otimes[\beta_a]_{x_0},$$

where all but finitely many terms in the sum are zero. Now define

$$\xi=\sum_{a\in A}g^a(x_0)\xi_a,$$

again noting that all but finitely many terms in the sum are zero. Since

$$v_{x_0}=\tilde{\imath}_{x_0}(V_{x_0})=\tilde{\imath}_{x_0}\Big(\sum_{a\in A}([g^a]_{x_0}+\mathfrak{m}_{x_0})\otimes[\beta_a]_{x_0}\Big)=\sum_{a\in A}g^a(x_0)\xi_a(x_0)$$

(recalling from the proof of Proposition 4.5 the definition of $\tilde{\imath}_{x_0}$), it follows that $\xi(x_0)=v_{x_0}$, as desired.

(ii) $\Longrightarrow$ (iii) Let us take the index set $A=\mathsf{F}$ and for $a\in A$ let $\xi_a$ be a real analytic global section of $\mathsf{F}$ such that $\xi_a(x)=a$. Let $\mathsf{A}\subseteq\mathscr{G}^\omega_{x_0,\mathsf{F}}$ be the submodule generated by $[\xi_a]_{x_0}$, $a\in A$. By Lemma 1 from the proof of Theorem 4.10, there exists $a_1,\dots,a_k\in A$ and a compact set $C\subseteq\mathsf{M}$ such that $x_0\in\operatorname{int}(C)$ and such that $[\xi_{a_1}]_C,\dots,[\xi_{a_k}]_C$ generate $\rho^{-1}_{C,x_0}(\mathsf{A})$. Then, as we saw in the proof of Theorem 4.10, $[\xi_{a_1}]_x,\dots,[\xi_{a_k}]_x$ generate $\mathscr{G}^\omega_{x,\mathsf{F}}$ for each $x\in\mathcal{N}$ for any neighbourhood $\mathcal{N}\subseteq C$. It follows that

$$\mathsf{F}_x=\operatorname{span}_{\mathbb{R}}(\xi_{a_1}(x),\dots,\xi_{a_k}(x))$$

for each $x\in\mathcal{N}$.

(iii) $\Longrightarrow$ (i) This is obvious. ■

**5.3. Swan's Theorem for regular generalised subbundles.** In this section we consider a different sort of result for global generators when the subbundle is regular. In this case, regularity provides additional algebraic structure for the space of sections. The result was proved for $r=0$ and for a compact base by Swan [1962]. An analogous result for vector bundles over an algebraic variety over an algebraically closed field, e.g., in the case of a holomorphic algebraic variety, was proved by Serre [1955]. The result we give is in the $C^r$-case for $r\in\mathbb{Z}_{\geq 0}\cup\{\infty,\omega\}$.

Let us first recall some notions from commutative algebra. We let $\mathsf{R}$ be a commutative unit ring and let $\mathsf{A}$ be a unital $\mathsf{R}$-module. The module $\mathsf{A}$ is ***projective*** if there exists (1) a free module $\mathsf{B}=\oplus_{i\in I}\mathsf{R}$, (2) a submodule $\mathsf{C}\subseteq\mathsf{B}$, and (3) an injective homomorphism $\phi\colon\mathsf{A}\to\mathsf{B}$ for which $\mathsf{B}=\operatorname{image}(\phi)\oplus\mathsf{C}$. In brief, $\mathsf{A}$ is a direct summand of a free module.

With this algebraic definition, we have the following result.

**5.3 Theorem: (Swan's Theorem for regular generalised subbundles)** *Let $r\in\mathbb{Z}_{\geq 0}\cup\{\infty,\omega\}$ and let $\pi\colon\mathsf{E}\to\mathsf{M}$ be a smooth or real analytic vector bundle, as required, whose fibres have bounded dimension and for which $\mathsf{M}$ is a smooth paracompact Hausdorff manifold of bounded dimension. The following statements hold:*

*(i) if* $\mathsf{F}$ *is a regular generalised subbundle of* $\mathsf{E}$ *of class* $C^r$*, then* $\Gamma^r(\mathsf{F})$ *is a finitely generated projective module over* $C^r(\mathsf{M})$*; that is to say,* $\Gamma^r(\mathsf{F})$ *is a direct summand of a finitely generated free module over* $C^r(\mathsf{M})$*;*

*(ii) if* $\mathscr{M}$ *is a finitely generated projective module over* $C^r(\mathsf{M})$ *then* $\mathscr{M}$ *is isomorphic to the module* $\Gamma^r(\mathsf{F})$ *of* $C^r$*-sections of a* $C^r$*-generalised subbundle of* $\mathsf{E}$*.*

Proof: (i) Since $\mathsf{F}$ is a vector bundle of class $C^r$, we shall without loss of generality suppose that $\mathsf{F} = \mathsf{E}$ and that $\mathsf{E}$ is of class $C^r$. The proof of this part of the theorem breaks into two parts. Were we to have access to an embedding theorem for $C^0$-manifolds, this could be averted, but we are not aware of such a theorem. In any case, we give separate proofs for $r \in \mathbb{Z}_{\geq 0}$ and for $r \in \{\infty, \omega\}$. We comment that both proofs work for $r \in \mathbb{Z}_{>0} \cup \{\infty\}$, but the first part applies to $r = 0$ and the second part applies to $r = \omega$.

We first consider $r \in \mathbb{Z}_{\geq 0}$. By Theorem 5.1, let $\xi_1, \dots, \xi_k$ be globally defined generators for $\mathsf{E}$. Let $\mathbb{R}^k_{\mathsf{M}}$ denote the trivial vector bundle $\mathsf{M} \times \mathbb{R}^k$ and define a vector bundle map $\Psi \colon \mathbb{R}^k_{\mathsf{M}} \to \mathsf{E}$ by

$$\Psi(x, (v_1, \dots, v_k)) = v_1\xi_1(x) + \dots + v_k\xi_k(x).$$

Clearly $\Psi$ is surjective and $\ker(\Psi)$ is a $C^r$-subbundle of $\mathbb{R}^k_{\mathsf{M}}$ [Abraham, Marsden, and Ratiu 1988, Proposition 3.4.18]. Let $\langle \cdot, \cdot \rangle$ be the standard inner product on $\mathbb{R}^k$ which we think of as a vector bundle metric on $\mathbb{R}^k_{\mathsf{M}}$. Define $\mathsf{G}_x$ to be the orthogonal complement to $\ker(\Psi_x)$. Note that $\mathsf{G}$ is then a $C^r$-subbundle of $\mathbb{R}^k_{\mathsf{M}}$. We claim that $\Psi|\mathsf{G}$ is a $C^r$-vector bundle isomorphism onto $\mathsf{E}$. Certainly $\Psi$ is a $C^r$-vector bundle map. We claim that $\Psi|\mathsf{G}$ is onto $\mathsf{E}$. Indeed, let $x \in \mathsf{M}$ and let $v_x \in \mathsf{E}_x$. Then, since $\operatorname{image}(\Psi) = \mathsf{E}$, there exists $(x, \boldsymbol{v}) \in \mathbb{R}^k_{\mathsf{M}}$ such that $\Psi(x, \boldsymbol{v}) = v_x$. Write $(x, \boldsymbol{v}) = (x, \boldsymbol{u} + \boldsymbol{w})$ for $(x, \boldsymbol{u}) \in \ker(\Psi)$ and where $\boldsymbol{w}$ is orthogonal to $\boldsymbol{u}$, i.e., $(x, \boldsymbol{w}) \in \mathsf{G}_x$. Then

$$v_x = \Psi(x, \boldsymbol{v}) = \Psi(x, \boldsymbol{u}) + \Psi(x, \boldsymbol{w}) = \Psi(x, \boldsymbol{w}),$$

and so $\Psi|\mathsf{G}$ is onto $\mathsf{E}$. To show that $\Psi$ is injective, suppose that $\Psi(x, \boldsymbol{v}_1) = \Psi(x, \boldsymbol{v}_2)$ for $(x, \boldsymbol{v}_1), (x, \boldsymbol{v}_2) \in \mathsf{G}_x$. Then

$$\Psi(x, \boldsymbol{v}_2 - \boldsymbol{v}_1) = 0_x \quad \Longrightarrow \quad (x, \boldsymbol{v}_2 - \boldsymbol{v}_1) \in \ker(\Psi), \quad \Longrightarrow \quad \boldsymbol{v}_2 - \boldsymbol{v}_1 = \boldsymbol{0},$$

as desired. Now we recall that $C^r$-vector bundles over $\mathsf{M}$ are isomorphic if and only if their sets of sections are isomorphic as $C^r(\mathsf{M})$-modules, cf. [Nelson 1967, §6]. Thus $\Gamma^r(\mathsf{E})$ and $\Gamma^r(\mathsf{G})$ are isomorphic.

To complete this part of the proof, note that $\Gamma^r(\mathbb{R}^k_{\mathsf{M}})$ is a finitely generated free module over $C^r(\mathsf{M})$. To see this, one can easily show that the sections $x \mapsto (x, \boldsymbol{e}_j)$, $j \in \{1, \dots, k\}$, form a basis for all sections, where $\boldsymbol{e}_j \in \mathbb{R}^k$ is the $j$th standard basis vector. Moreover, any section $\xi$ of $\mathbb{R}^k_{\mathsf{M}}$ with $\xi(x) = (x, \boldsymbol{v}(x))$ can be written uniquely as

$$\xi(x) = (x, \boldsymbol{u}(x)) + (x, \boldsymbol{w}(x))$$

with $(x, \boldsymbol{u}(x)) \in \ker(\Psi_x)$ and where $\boldsymbol{w}(x)$ is orthogonal to $\boldsymbol{u}(x)$. That is, $\xi = \eta + \zeta$ for $\eta \in \Gamma^r(\ker(\Psi))$ and $\zeta \in \Gamma^r(\mathsf{E})$, and this decomposition is unique. That is to say, $\Gamma^r(\mathbb{R}^k_{\mathsf{M}}) = \Gamma^r(\ker(\Psi)) \oplus \Gamma^r(\mathsf{E})$, the direct sum being one of $C^r(\mathsf{M})$-modules. This gives this part of the theorem for $r \in \mathbb{Z}_{\geq 0}$.

For $r \in \{\infty, \omega\}$ we first note that there is a $C^r$-proper embedding $\Phi\colon \mathsf{E} \to \mathbb{R}^N$ into Euclidean space for some $N \in \mathbb{Z}_{>0}$. For $r = \infty$ this is the Whitney Embedding Theorem [Lee 2003, Theorem 6.9]. In the real analytic case, this is the Grauert Embedding Theorem [Grauert 1958]. Note that $T\Phi$ is an injective vector bundle morphism over $\Phi$. Now let $Z(\mathsf{E}) \subseteq \mathsf{E}$ be the zero section, and note that $Z(\mathsf{E})$ is canonically diffeomorphic to $\mathsf{M}$. Thus the restricted vector bundle $\mathsf{TE}|Z(\mathsf{E})$ is to be regarded as a vector bundle over $\mathsf{M}$. Since the fibres of $\mathsf{E}$ intersect $Z(\mathsf{E})$ transversally at points $0_x$ of $Z(\mathsf{E})$, we have

$$\mathsf{T}_{0_x}\mathsf{E} \simeq \mathsf{T}_x\mathsf{M} \oplus \mathsf{T}_{0_x}\mathsf{E}_x \simeq \mathsf{T}_x\mathsf{M} \oplus \mathsf{E}_x.$$

Thus $\mathsf{TE}|Z(\mathsf{E}) \simeq \mathsf{TM} \oplus \mathsf{E}$ and so $T\Phi$, when restricted to $\mathsf{TE}|Z(\mathsf{E})$, has image as a vector bundle over the submanifold $\Phi(\mathsf{M})$. This is, moreover, a subbundle of the trivial bundle $\Phi(\mathsf{M}) \times \mathbb{R}^N$. Thus we have $\mathsf{E}$ as a subbundle of the trivial bundle $\mathbb{R}^N_{\mathsf{M}} \triangleq \mathsf{M} \times \mathbb{R}^N$. Let $\langle\cdot,\cdot\rangle$ be the standard inner product on $\mathbb{R}^N$ which we think of as a vector bundle metric on $\mathbb{R}^N_{\mathsf{M}}$. Define $\mathsf{G}_x$ to be the orthogonal complement to $\mathsf{E}_x$, noting that $\mathsf{G}$ is then a $C^r$-subbundle of $\mathbb{R}^N_{\mathsf{M}}$ and that $\mathbb{R}^N_{\mathsf{M}} = \mathsf{E} \oplus \mathsf{G}$. Let $\pi_1\colon \mathbb{R}^N_{\mathsf{M}} \to \mathsf{E}$ and $\pi_2\colon \mathbb{R}^N_{\mathsf{M}} \to \mathsf{G}$ be the projections, thought of as vector bundle morphisms. Note that $\Gamma^r(\mathbb{R}^N_{\mathsf{M}})$ is isomorphic, as a $C^r(\mathsf{M})$-module, to $C^r(\mathsf{M})^N$. Moreover, the map from $\Gamma^r(\mathbb{R}^N_{\mathsf{M}})$ to $\Gamma^r(\mathsf{E}) \oplus \Gamma^r(\mathsf{G})$ given by

$$\boldsymbol{\xi} \mapsto (\pi_1 \circ \boldsymbol{\xi}) \oplus (\pi_2 \circ \boldsymbol{\xi})$$

can be directly verified to be an isomorphism of $C^r(\mathsf{M})$-modules. In particular, $\Gamma^r(\mathsf{E})$ is a summand of the free, finitely generated $C^r(\mathsf{M})$-module $\Gamma^r(\mathbb{R}^N_{\mathsf{M}})$.

(ii) By definition, there exists a module $\mathscr{N}$ over $C^r(\mathsf{M})$ such that

$$\mathscr{M} \oplus \mathscr{N} \simeq \underbrace{C^r(\mathsf{M}) \oplus \cdots \oplus C^r(\mathsf{M})}_{k \text{ factors}}.$$

The direct sum on the right is naturally isomorphic to the set of sections of the trivial vector bundle $\mathbb{R}^k_{\mathsf{M}} = \mathsf{M} \times \mathbb{R}^k$. Thus we can write $\mathscr{M} \oplus \mathscr{N} = \Gamma^r(\mathbb{R}^k_{\mathsf{M}})$. For $a \in \{1, 2\}$, let $\Pi_a\colon \Gamma^r(\mathbb{R}^k_{\mathsf{M}}) \to \Gamma^r(\mathbb{R}^k_{\mathsf{M}})$ be the projection onto the $a$th factor. As per [Nelson 1967, §6] (essentially), associated with $\Pi_a$ is a vector bundle map $\pi_a\colon \mathbb{R}^k_{\mathsf{M}} \to \mathbb{R}^k_{\mathsf{M}}$. Since $\Pi_a \circ \Pi_a = \Pi_a$ (by virtue of $\Pi_a$ being a projection), $\pi_a \circ \pi_a = \pi_a$. To show that $\mathscr{M}$ is the set of sections of a vector subbundle of $\mathbb{R}^k_{\mathsf{M}}$ it suffices to show that $\pi_1$ has locally constant rank. Following along the lines of the proof of Proposition 3.6 one can show that $x \mapsto \operatorname{rank}(\pi_{a,x})$ is lower semicontinuous for $a \in \{1, 2\}$. However, since $\operatorname{rank}(\pi_{1,x}) = \operatorname{rank}(\pi_{2,x}) = k$ for all $x \in \mathsf{M}$, if $x \mapsto \operatorname{rank}(\pi_{1,x})$ is lower semicontinuous at $x_0$, then $x \mapsto \operatorname{rank}(\pi_{2,x})$ is upper semicontinuous at $x_0$. Thus we conclude that both of these functions must be continuous at $x_0$. Since $x \mapsto \operatorname{rank}(\pi_{1,x})$ is integer-valued, it must therefore be locally constant. ■

## 6. Differential constructions associated to distributions

In this section we turn, for the first time, exclusively to distributions, i.e., generalised subbundles of the tangent bundle $\mathsf{TM}$ of a manifold $\mathsf{M}$. Here one has some particular features of the tangent bundle that come into play, mainly associated with the fact that sections of the tangent bundle are vector fields, which have associated to them a great deal of additional and important structure. It is these features that we focus on in this section. We begin by indicating how standard vector field operations interact with our sheaf formalism for distributions. Then we look at two structures associated to distributions: invariant distributions and distributions arising from Lie algebras of vector fields.

**6.1. Local diffeomorphisms and flows of vector fields.** A standard assumption made in the differential geometry literature when dealing with vector fields is that they are complete, i.e., if $X$ is a vector field on $\mathsf{M}$, the flow $x \mapsto \Phi_t^X(x)$ is defined for every $(t,x) \in \mathbb{R} \times \mathsf{M}$. In order for the results in the paper to be as general as possible, we shall not make this assumption. Thus we have to introduce some technicalities to deal with this, following Sussmann [1973]. The first is the notion of a local diffeomorphism, a notion which we will encounter again in our treatment of the Orbit Theorem in Section 7.

**6.1 Definition: (Local diffeomorphisms, groups of local diffeomorphisms)** Let $r \in \{\infty, \omega\}$ and let $\mathsf{M}$ be a $C^r$-manifold.

(i) A ***$C^r$-local diffeomorphism*** on $\mathsf{M}$ is a pair $(\Phi, \mathcal{U})$ where $\mathcal{U} \subseteq \mathsf{M}$ is (a possibly empty) open subset called the ***domain*** and where $\Phi \colon \mathcal{U} \to \Phi(\mathcal{U})$ is a $C^r$-diffeomorphism. The ***image*** of $(\Phi, \mathcal{U})$ is the open set $\Phi(\mathcal{U})$.

(ii) If $(\Phi, \mathcal{U})$ and $(\Psi, \mathcal{V})$ are $C^r$-local diffeomorphisms, their ***composition*** $(\Psi, \mathcal{V}) \circ (\Phi, \mathcal{U})$ is the $C^r$-local diffeomorphism $(\Psi \circ \Phi|\Phi^{-1}(\mathcal{V}), \Phi^{-1}(\mathcal{V}))$.

(iii) If $(\Phi, \mathcal{U})$ is a $C^r$-local diffeomorphism, its ***inverse*** $(\Phi, \mathcal{U})^{-1}$ is the $C^r$-local diffeomorphism $(\Phi^{-1}, \Phi(\mathcal{U}))$.

(iv) A ***group of $C^r$-local diffeomorphisms*** is a family $\mathscr{G}$ of $C^r$-local diffeomorphisms such that, if $(\Phi, \mathcal{U}), (\Psi, \mathcal{V}) \in \mathscr{G}$ then $(\Psi, \mathcal{V}) \circ (\Psi, \mathcal{U}) \in \mathscr{G}$ and $(\Phi, \mathcal{U})^{-1} \in \mathscr{G}$. •

The notion of a local diffeomorphism with an empty domain is possibly confusing, but is a technical convenience. We shall often make a slight abuse of notation by denoting a local diffeomorphism by $\Phi$ rather than $(\Phi, \mathcal{U})$. In cases when we do this, we believe there will be no loss in clarity.

Next we consider local diffeomorphisms generated by flows of vector fields. We recall that if $X \in \Gamma^1(\mathsf{TM})$ then the flow $\Phi_t^X(x)$ is defined for $(t,x)$ in an open subset of $\mathbb{R} \times \mathsf{M}$ that we denote by $D(X)$. For a vector field $X$, we denote by $I(X, x_0) \subseteq \mathbb{R}$ the domain of the maximal integral curve of $X$ through $x_0$. For $t \in \mathbb{R}$ we denote by $\mathcal{U}(X,t)$ the largest (possibly empty) open subset of $\mathsf{M}$ such that $(\Phi_t^X, \mathcal{U}(X,t))$ is a local diffeomorphism. We note that, given $(t_0, x_0) \in \mathbb{R} \times \mathsf{M}$ there exists a neighbourhood $\mathcal{U}$ of $x_0$ such that $\Phi_t^X(x)$ is defined for $t \in (-t_0, t_0)$.

**6.2. Lie brackets, (local) diffeomorphisms, and sheaves.** In this section we indicate how two standard constructions normally defined for vector fields can be applied to sheaves. This has the advantage of systematically handling situations where one wishes to deal with objects that are not globally defined.

First let us consider the Lie bracket. For $r \in \{\infty, \omega\}$ and for a $C^r$-manifold $\mathsf{M}$, we note that $\Gamma^r(\mathsf{TM})$ has the structure of a Lie algebra. This Lie algebra structure can be defined most conveniently by recalling that the set of vector fields is in one-to-one correspondence with the set of derivations of $C^r(\mathsf{M})$ (see [Abraham, Marsden, and Ratiu 1988, Theorem 4.2.16] for the classical $r = \infty$ case and [Grabowski 1981] for the $r = \omega$

case). The derivation associated to a vector field $X$, denoted by $\mathscr{L}_X$, is defined simply by $\mathscr{L}_X f(x) = \langle \boldsymbol{d}f(x); X(x)\rangle$. For $X, Y \in \Gamma^r(\mathsf{TM})$ we note that

$$f \mapsto \mathscr{L}_X\mathscr{L}_Y f - \mathscr{L}_Y\mathscr{L}_X f$$

is a derivation (one can just check this). Thus, associated to this derivation is a unique vector field that we denote by $[X, Y]$ which is the ***Lie bracket*** of $X$ and $Y$. This Lie bracket is easily verified to have the properties that render $\Gamma^r(\mathsf{TM})$ a $\mathbb{R}$-Lie algebra.

Let us now extend the above construction to sheaves. Let $X \in \Gamma^r(\mathsf{TM})$ be a smooth or real analytic vector field. Let $\mathcal{U} \subseteq \mathsf{M}$ be open. We denote by $\mathscr{L}_{X,\mathcal{U}}$ the derivation on $C^r(\mathcal{U})$ defined by $X|\mathcal{U}$. Note that the diagram

$$\begin{tikzcd} C^r(\mathcal{U}) \arrow[r, "\mathscr{L}_{X,\mathcal{U}}"] \arrow[d, "r_{\mathcal{U},\mathcal{V}}"'] & C^r(\mathcal{U}) \arrow[d, "r_{\mathcal{U},\mathcal{V}}"] \\ C^r(\mathcal{V}) \arrow[r, "\mathscr{L}_{X,\mathcal{V}}"'] & C^r(\mathcal{V}) \end{tikzcd}$$

commutes for open sets $\mathcal{U}, \mathcal{V} \subseteq \mathsf{M}$ with $\mathcal{V} \subseteq \mathcal{U}$. Thus we have a well-defined map $\mathscr{L}_{X,x} \colon \mathscr{C}^r_{x,\mathsf{M}} \to \mathscr{C}^r_{x,\mathsf{M}}$ which is a derivation of the $\mathbb{R}$-algebras. Thus we can think of $\mathscr{L}_X$ as being a morphism of sheaves that is a derivation. The set of such derivations then forms a $\mathbb{R}$-Lie algebra with bracket

$$[\mathscr{L}_{X,x}, \mathscr{L}_{Y,x}]([f]_x) = \mathscr{L}_{X,x}(\mathscr{L}_{Y,x}([f]_x)) - \mathscr{L}_{Y,x}(\mathscr{L}_{X,x}([f]_x)).$$

Note that $\mathscr{L}_{X,x}$ depends only on the germ of $X$ at $x$.

Now we consider diffeomorphisms acting on functions and vector fields. Thus we let $r \in \{\infty, \omega\}$, let $\Phi \in C^r(\mathsf{M}; \mathsf{M})$ be a diffeomorphism, let $f \in C^r(\mathsf{M})$ be a function, and let $X \in \Gamma^r(\mathsf{TM})$. The ***pull-back*** (resp. ***push-forward***) of $f$ by $\Phi$ is defined by $\Phi^* f = f \circ \Phi$ (resp. $\Phi_* f = f \circ \Phi^{-1}$) and the ***pull-back*** (resp. ***push-forward*** of $X$ by $\Phi$ is defined by $\Phi_* X = T\Phi \circ X \circ \Phi^{-1}$ (resp. $\Phi_* X = T\Phi^{-1} \circ X \circ \Phi$).

Let us formulate this in sheaf language, allowing for local diffeomorphisms. Thus we let $r \in \{\infty, \omega\}$, let $\mathsf{M}$ be a $C^r$-manifold, and let $(\Phi, \mathcal{U})$ be a local diffeomorphism with $\mathcal{U} \neq \varnothing$. Let us first consider the action of $(\Phi, \mathcal{U})$ on functions. We recall from Definition 2.12 that $\Phi$ induces the direct image sheaf $\Phi_*\mathscr{C}^r_{\mathcal{U}}$ and the inverse image sheaf $\Phi^{-1}\mathscr{C}^r_{\mathcal{U}}$ defined by

$$\Phi_*\mathscr{C}^r_{\mathcal{U}}(\mathcal{W}) = C^r(\Phi^{-1}(\mathcal{W})), \qquad \mathcal{W} \in \Phi(\mathcal{U}),$$

and

$$\Phi^{-1}\mathscr{C}^r_{\mathcal{U}}(\mathcal{V}) = C^r(\Phi(\mathcal{V})), \qquad \mathcal{V} \subseteq \mathcal{U}.$$

The pull-back morphism is the sheaf morphism $\Phi^*$ from $\Phi^{-1}\mathscr{C}^r_{\Phi(\mathcal{U})}$ to $\mathscr{C}^r_{\mathcal{U}}$ defined by asking that $\Phi^*_{\mathcal{V}} \colon C^r(\Phi(\mathcal{V})) \to C^r(\mathcal{V})$ be given by $\Phi^*_{\mathcal{V}}(g) = g \circ (\Phi|\mathcal{V})$. Note that the diagram

$$\begin{tikzcd} C^r(\Phi(\mathcal{V})) \arrow[r, "\Phi^*_{\mathcal{V}}"] \arrow[d, "r_{\Phi(\mathcal{V}),\Phi(\mathcal{W})}"'] & C^r(\mathcal{V}) \arrow[d, "r_{\mathcal{V},\mathcal{W}}"] \\ C^r(\Phi(\mathcal{W})) \arrow[r, "\Phi^*_{\mathcal{W}}"'] & C^r(\mathcal{W}) \end{tikzcd}$$

commutes for each pair $\mathcal{V}, \mathcal{W} \subseteq \mathcal{U}$ of open sets for which $\mathcal{W} \subseteq \mathcal{V}$. Thus we have an induced mapping $\Phi_x^*$ from the stalk $\Phi^{-1}\mathscr{C}^r_{x,\mathcal{U}}$ to the stalk $\mathscr{C}^r_{x,\mathcal{U}}$ given explicitly by

$$\Phi_x^*([g]_{\Phi(x)}) = [g \circ \Phi]_x$$

for $x \in \mathcal{U}$.

The push-forward morphism is the sheaf morphism $\Phi_*$ from $\mathscr{C}^r_{\mathcal{U}}$ to $\Phi^{-1}\mathscr{C}^r_{\mathcal{U}}$ defined by asking that $\Phi_{*,\mathcal{V}}\colon C^r(\mathcal{V}) \to C^r(\Phi(\mathcal{V}))$ be given by $\Phi_{*,\mathcal{V}}(f) = f \circ (\Phi^{-1}|\Phi(\mathcal{V}))$. Note that the diagram

$$\begin{tikzcd} C^r(\mathcal{V}) \arrow[r, "\Phi_{*,\mathcal{V}}"] \arrow[d, "r_{\mathcal{W},\mathcal{V}}"'] & C^r(\Phi(\mathcal{V})) \arrow[d, "r_{\Phi(\mathcal{V}),(\mathcal{W})}"] \\ C^r(\mathcal{W}) \arrow[r, "\Phi_{*,\mathcal{W}}"'] & C^r(\Phi(\mathcal{W})) \end{tikzcd}$$

commutes for every pair $\mathcal{W}, \mathcal{V} \subseteq \mathcal{U}$ of open sets for which $\mathcal{V} \subseteq \mathcal{W}$. Thus we have an induced mapping $\Phi_{*,x}$ from the stalk $\mathscr{C}^r_{x,\mathcal{U}}$ to the stalk $\Phi^{-1}\mathscr{C}^r_{x,\mathcal{U}}$ given explicitly by

$$\Phi_{*,x}([f]_x) = [f \circ \Phi^{-1}]_{\Phi(x)}$$

for $x \in \mathcal{U}$.

Now let us see how this can be used to give the sheaf version of the pull-back of a vector field by a local diffeomorphism. Let $\mathsf{M}$ and $(\Phi, \mathcal{U})$ be as above, let $\mathcal{V} \subseteq \mathcal{U}$ be open, and let $f \in C^r(\mathcal{V})$ be a local section of $\mathscr{C}^r_{\mathcal{U}}$ over $\mathcal{V}$. Let $Y$ be a local section of $\mathscr{G}^r_{\mathsf{TM}}$ over $\Phi(\mathcal{V})$, i.e., an element of the inverse image sheaf $\Phi^{-1}\mathscr{G}^\infty_{\mathsf{TM}|\mathcal{U}}(\mathcal{V})$. Then, for $y \in \Phi(\mathcal{V})$, the tangent vector $T_y\Phi^{-1}(Y(y))$ can be used to derive the function $f$ at $\Phi^{-1}(y)$. Thus we can define a morphism $\mathscr{L}^{\Phi^*}_{Y,\mathcal{V}}\colon C^r(\mathcal{V}) \to C^r(\Phi^{-1}(\mathcal{V}))$ by

$$\mathscr{L}^{\Phi^*}_{Y,\mathcal{V}} f(y) = \langle \boldsymbol{d}f(\Phi^{-1}(y)); T_y\Phi^{-1}(Y(y))\rangle, \qquad y \in \Phi(\mathcal{V}).$$

Alternatively, we can regard $\mathscr{L}^{\Phi^*}_{Y,\mathcal{V}}$ as a morphism from $\mathscr{C}^r_{\mathcal{U}}(\mathcal{V}) = C^r(\mathcal{V})$ to itself by defining

$$\mathscr{L}^{\Phi^*}_{Y,\mathcal{V}} f(x) = \langle \boldsymbol{d}f(x); T_{\Phi(x)}\Phi^{-1}(Y(\Phi(x)))\rangle, \qquad x \in \mathcal{V}.$$

This map is a derivation of $\mathbb{R}$-algebras, i.e., it is $\mathbb{R}$-linear and satisfies

$$\mathscr{L}^{\Phi^*}_{Y,\mathcal{V}} fg = f(\mathscr{L}^{\Phi^*}_{Y,\mathcal{V}} g) + (\mathscr{L}^{\Phi^*}_{Y,\mathcal{V}} f)g$$

for every $f, g \in C^r(\mathcal{V})$. Thus there exists a unique vector field on $\mathcal{V}$, which we denote by $\Phi^*_{\mathcal{V}}Y$, such that

$$\mathscr{L}_{\Phi^*_{\mathcal{V}}Y} f = \mathscr{L}^{\Phi^*}_{Y,\mathcal{V}} f$$

for every $f \in C^r(\mathcal{V})$. Thus $\Phi^*_{\mathcal{V}}Y \in \mathscr{G}^r_{\mathsf{TM}|\mathcal{U}}(\mathcal{V})$. Clearly the diagram

$$\begin{tikzcd} C^r(\mathcal{V}) \arrow[r, "\mathscr{L}^{\Phi^*}_{Y,\mathcal{V}}"] \arrow[d, "r_{\mathcal{V},\mathcal{W}}"'] & C^r(\mathcal{V}) \arrow[d, "r_{\mathcal{V},\mathcal{W}}"] \\ C^r(\mathcal{W}) \arrow[r, "\mathscr{L}^{\Phi^*}_{Y,\mathcal{W}}"'] & C^r(\mathcal{W}) \end{tikzcd}$$

commutes for open sets $\mathcal{V}, \mathcal{W} \subseteq \mathcal{U}$ satisfying $\mathcal{W} \subseteq \mathcal{V}$. Thus, for each $x \in \mathcal{U}$ we have a well-defined morphism of stalks:

$$\mathscr{L}^{\Phi^*}_{Y,x} \colon \mathscr{C}^r_{x,\mathcal{U}} \to \mathscr{C}^r_{x,\mathcal{U}}.$$

Moreover, this map is a derivation. That is, it is $\mathbb{R}$-linear and satisfies

$$\mathscr{L}^{\Phi^*}_{Y,x}[f]_x[g]_x = [f]_x(\mathscr{L}^{\Phi^*}_{Y,x}[g]_x) + (\mathscr{L}^{\Phi^*}_{Y,x}[f]_x)[g]_x$$

for $[f]_x, [g]_x \in \mathscr{C}^r_{x,\mathcal{U}}$. Thus there exists a unique vector field germ, which we denote by $\Phi^*_x[Y]_{\Phi(x)}$, such that

$$\mathscr{L}_{\Phi^*_x[Y]_{\Phi(x)}}[f]_x = \mathscr{L}^{\Phi^*}_{Y,x}[f]_x$$

for every $[f]_x \in \mathscr{C}^r_{x,\mathcal{U}}$. The upshot of the preceding development is that, associated with the local diffeomorphism $(\Phi, \mathcal{U})$, is a sheaf morphism $\Phi^*$ from $\Phi^{-1}\mathscr{G}^r_{\mathsf{TM}|\mathcal{U}}$ to $\mathscr{G}^r_{\mathcal{U}}$.

Finally, we define the sheaf version of the push-forward of a vector field by a local diffeomorphism. Let $\mathsf{M}$ and $(\Phi, \mathcal{U})$ be as above, let $\mathcal{V} \subseteq \mathcal{U}$ be open, and let $g \in C^r(\Phi(\mathcal{V}))$ be a local section of $\Phi^{-1}\mathscr{C}^r_{\mathcal{U}}$ over $\mathcal{V}$. Let $X \in \Gamma^r(\mathcal{V})$ be a local section of $\mathscr{G}^r_{\mathsf{TM}}$ over $\mathcal{V}$, i.e., an element of $\mathscr{G}^r_{\mathsf{TM}|\mathcal{U}}(\mathcal{V})$. Then, for $x \in \mathcal{V}$, the tangent vector $T_x\Phi(X(x)) \in \mathsf{T}_{\Phi(x)}\mathsf{M}$ can be used to derive the function $g$ at $\Phi(x)$. Thus, we can define a morphism $\mathscr{L}^{\Phi_*}_{X,\mathcal{V}} \colon C^r(\Phi(\mathcal{V})) \to C^r(\mathcal{V})$ by

$$\mathscr{L}^{\Phi_*}_{X,\mathcal{V}} g(x) = \langle \boldsymbol{d}g(\Phi(x)); T_x\Phi(X(x))\rangle, \qquad x \in \mathcal{V}.$$

Alternatively, we can regard $\mathscr{L}^{\Phi_*}_{X,\mathcal{V}}$ as a morphism from $\Phi^{-1}\mathscr{C}^r_{\mathcal{U}}(\mathcal{V}) = C^r(\Phi(\mathcal{V}))$ to itself by defining

$$\mathscr{L}^{\Phi_*}_{X,\mathcal{V}} g(y) = \langle \boldsymbol{d}g(y); T_{\Phi^{-1}(y)}\Phi(X(\Phi^{-1}(y)))\rangle, \qquad y \in \Phi(\mathcal{V}).$$

This map is a derivation of $\mathbb{R}$-algebras, i.e., it is $\mathbb{R}$-linear and satisfies

$$\mathscr{L}^{\Phi_*}_{X,\mathcal{V}} gh = g(\mathscr{L}^{\Phi_*}_{X,\mathcal{V}} h) + (\mathscr{L}^{\Phi_*}_{X,\mathcal{V}} g)h$$

for every $g, h \in C^r(\Phi(\mathcal{V}))$. Thus there exists a unique vector field on $\Phi(\mathcal{V})$, which we denote by $\Phi_{*,\mathcal{V}}X$, such that

$$\mathscr{L}_{\Phi_{*,\mathcal{V}}X} g = \mathscr{L}^{\Phi_*}_{X,\mathcal{V}} g$$

for every $g \in C^r(\Phi(\mathcal{V}))$. Thus $\Phi_{*,\mathcal{V}}X \in \Phi^{-1}\mathscr{G}^r_{\mathsf{TM}|\mathcal{U}}(\mathcal{V})$. Clearly the diagram

$$\begin{tikzcd}
C^r(\Phi(\mathcal{V})) \arrow[r, "\mathscr{L}^{\Phi_*}_{X,\mathcal{V}}"] \arrow[d, "r_{\Phi(\mathcal{V}),\Phi(\mathcal{W})}"'] & C^r(\Phi(\mathcal{V})) \arrow[d, "r_{\Phi(\mathcal{V}),\Phi(\mathcal{W})}"] \\
C^r(\Phi(\mathcal{W})) \arrow[r, "\mathscr{L}^{\Phi_*}_{X,\mathcal{W}}"'] & C^r(\Phi(\mathcal{W}))
\end{tikzcd}$$

commutes for open sets $\mathcal{V}, \mathcal{W} \subseteq \mathcal{U}$ satisfying $\mathcal{W} \subseteq \mathcal{V}$. Thus, for each $y \in \Phi(\mathcal{U})$ we have a well-defined morphism of stalks:

$$\mathscr{L}^{\Phi_*}_{X,y} \colon \Phi^{-1}\mathscr{C}^r_{y,\mathcal{U}} \to \Phi^{-1}\mathscr{C}^r_{y,\mathcal{U}}.$$

Moreover, this map is a derivation. That is, it is $\mathbb{R}$-linear and satisfies

$$\mathscr{L}^{\Phi_*}_{X,y}[g]_y[h]_y = [g]_y(\mathscr{L}^{\Phi_*}_{X,y}[h]_y) + (\mathscr{L}^{\Phi_*}_{X,y}[g]_y)[h]_y$$

for $[g]_y, [h]_y \in \Phi^{-1}\mathscr{C}^r_{y,\mathcal{U}}$. Thus there exists a unique vector field germ, which we denote by $\Phi_{*,x}[X]_{\Phi^{-1}(y)}$, such that

$$\mathscr{L}_{\Phi_{*,x}[X]_{\Phi^{-1}(y)}}[g]_y = \mathscr{L}^{\Phi_*}_{X,y}[g]_y$$

for every $[g]_y \in \Phi^{-1}\mathscr{C}^r_{y,\mathcal{U}}$. The upshot of the preceding development is that, associated with the local diffeomorphism $(\Phi, \mathcal{U})$, is a sheaf morphism $\Phi_*$ from $\mathscr{G}^r_{\mathsf{TM}|\mathcal{U}}$ to $\Phi^{-1}\mathscr{G}^r_{\mathsf{TM}|\mathcal{U}}$.

The preceding discussion is lengthy and notation-laden. Therefore, it is worth summarising the constructions we have made. We let $(\Phi, \mathcal{U})$ be a local diffeomorphism, and note that our preceding developments define the following sheaf morphisms:

1. $\Phi^*\colon \Phi^{-1}\mathscr{C}^r_{\mathcal{U}} \to \mathscr{C}^r_{\mathcal{U}}$;
2. $\Phi_*\colon \mathscr{C}^r_{\mathcal{U}} \to \Phi^{-1}\mathscr{C}^r_{\mathcal{U}}$;
3. $\Phi^*\colon \Phi^{-1}\mathscr{G}^r_{\mathsf{TM}|\mathcal{U}} \to \mathscr{G}^r_{\mathsf{TM}|\mathcal{U}}$;
4. $\Phi_*\colon \mathscr{G}^r_{\mathsf{TM}|\mathcal{U}} \to \Phi^{-1}\mathscr{G}^r_{\mathsf{TM}|\mathcal{U}}$.

**6.3. Lie subalgebras of vector fields and subsheaves.** In this section we consider Lie subalgebras of the Lie algebras $\Gamma^r(\mathsf{TM})$ and $\mathscr{G}^r_{\mathsf{TM}}$ for $r \in \{\infty, \omega\}$. Let us first define the objects of interest.

**6.2 Definition: (Lie subalgebras)** Let $r \in \{\infty, \omega\}$, let $\mathsf{M}$ be a manifold of class $C^r$, and let $x \in \mathsf{M}$.

(i) A ***Lie subalgebra*** of vector fields is a Lie subalgebra of the $\mathbb{R}$-Lie algebra $\Gamma^r(\mathsf{TM})$.

(ii) A ***Lie subalgebra*** of $\mathscr{G}^r_{\mathsf{TM}}$ is an assignment to each open set $\mathcal{U} \subseteq \mathsf{M}$ a Lie subalgebra $L(\mathcal{U})$ of vector fields on $\mathcal{U}$ with the property that $L(\mathcal{V}) = r_{\mathcal{U},\mathcal{V}}(L(\mathcal{U}))$ for every pair $\mathcal{U}, \mathcal{V}$ of open sets for which $\mathcal{V} \subseteq \mathcal{U}$.

(iii) A ***Lie subalgebra*** of germs of vector fields at $x$ is a Lie subalgebra of the $\mathbb{R}$-Lie algebra $\mathscr{G}^r_{x,\mathsf{TM}}$.

(iv) If $\mathscr{X} \subseteq \Gamma^r(\mathsf{TM})$ then the ***Lie subalgebra generated by $\mathscr{X}$*** is the smallest Lie subalgebra of vector fields containing $\mathscr{X}$. The Lie subalgebra is denoted by $\mathscr{L}^{(\infty)}(\mathscr{X})$.

(v) If $\mathscr{X}$ is a subsheaf of sets of the sheaf $\mathscr{G}^r_{\mathsf{TM}}$—i.e., an assignment to each open set $\mathcal{U} \subseteq \mathsf{M}$ a subset $X(\mathcal{U}) \subseteq \Gamma^r(\mathsf{TM}|\mathcal{U})$ with the assignment satisfying $X(\mathcal{V}) = r_{\mathcal{U},\mathcal{V}}(X(\mathcal{U}))$ for every pair of open sets $\mathcal{U}, \mathcal{V}$ for which $\mathcal{V} \subseteq \mathcal{V}$—the ***Lie subalgebra of generated by $\mathscr{X}$*** is the Lie subalgebra of $\mathscr{G}^r_{\mathsf{TM}}$ defined by assigning to the open set $\mathcal{U}$ the Lie subalgebra of vector fields on $\mathcal{U}$ generated by $X(\mathcal{U})$. This Lie subalgebra is denoted by $\mathscr{L}^{(\infty)}(\mathscr{X}) = (\mathscr{L}^{(\infty)}(X(\mathcal{U})))_{\mathcal{U}\ \mathrm{open}}$.

(vi) If $\mathscr{X}_x \subseteq \mathscr{G}^r_{x,\mathsf{TM}}$, the ***Lie subalgebra generated by $\mathscr{X}_x$*** is the smallest Lie subalgebra of germs of vector fields at $x$ containing $\mathscr{X}_x$. This Lie subalgebra is denoted by $\mathscr{L}^{(\infty)}(\mathscr{X}_x)$. •

One can easily give a precise characterisation of a Lie subalgebra generated by a set in the three cases of the preceding definition.

**6.3 Proposition: (Generated Lie algebras)** *Let $r \in \{\infty, \omega\}$ and let $\mathsf{M}$ be a $C^r$-manifold. Then the following statements hold:*

*(i) if $\mathscr{X} \subseteq \Gamma^r(\mathsf{TM})$, the Lie subalgebra generated by $\mathscr{X}$ is generated by finite $\mathbb{R}$-linear combinations of vector fields of the form*

$$[X_k, [X_{k-1}, \dots, [X_2, X_1] \cdots]], \qquad k \in \mathbb{Z}_{>0},\ X_1, \dots, X_k \in \mathscr{X};$$

*(ii) if $\mathscr{X} = (X(\mathcal{U}))_{\mathcal{U}, open}$ is a subsheaf of sets of $\mathscr{G}^r_{\mathsf{TM}}$, then the Lie subalgebra of $\mathscr{G}^r_{\mathsf{TM}}$ generated by $\mathscr{X}$ is such that $\mathscr{L}^{(\infty)}(X(\mathcal{U}))$ is generated by finite $\mathbb{R}$-linear combinations of vector fields of the form*

$$[X_k, [X_{k-1}, \dots, [X_2, X_1] \cdots]], \qquad k \in \mathbb{Z}_{>0},\ X_1, \dots, X_k \in X(\mathcal{U});$$

*(iii) if $x \in \mathsf{M}$ and if $\mathscr{X}_x \subseteq \mathscr{G}^r_{x,\mathsf{TM}}$, the Lie subalgebra generated by $\mathscr{X}_x$ is generated by finite $\mathbb{R}$-linear combinations of germs of vector fields of the form*

$$[[X_k]_x, [[X_{k-1}]_x, \dots, [[X_2]_x, [X_1]_x] \cdots]], \qquad k \in \mathbb{Z}_{>0},\ [X_1]_x, \dots, [X_k]_x \in \mathscr{X}.$$

Proof: We prove the first statement only. The second follows from this and the third follows via an entirely similar computation. For vector fields $X_1, \dots, X_k \in \mathscr{X}$, since $\mathscr{L}^{(\infty)}(\mathscr{X})$ is a Lie subalgebra of $\Gamma^r(\mathsf{TM})$, it follows by induction that $[X_k, [X_{k-1}, \dots, [X_2, X_1] \cdots]] \in \mathscr{L}^{(\infty)}(\mathscr{X})$. Since $\mathscr{L}^{(\infty)}(\mathscr{X})$ is a subspace of the $\mathbb{R}$-vector space $\Gamma^r(\mathsf{TM})$, it also follows that all $\mathbb{R}$-linear combinations of such vector fields are in $\mathscr{L}^{(\infty)}(\mathscr{X})$.

To prove the opposite inclusion, it suffices to show—since $\mathscr{L}^{(\infty)}(\mathscr{X})$ is the smallest Lie subalgebra containing $\mathscr{X}$—that the set of all $\mathbb{R}$-linear combinations in the statement of the proposition forms a Lie algebra. If we have two $\mathbb{R}$-linear combinations of vector fields of the form stated in the proposition, their Lie bracket will be in $\mathscr{L}^{(\infty)}(\mathscr{X})$ if and only if the Lie bracket of each of the summands is in $\mathscr{L}^{(\infty)}(\mathscr{X})$ (by linearity of the Lie bracket). Consider two vector fields of the form stated in the proposition:

$$\begin{aligned} X &= [X_k, [X_{k-1}, \dots, [X_2, X_1] \cdots]] \\ Y &= [Y_l, [Y_{l-1}, \dots, [Y_2, Y_1] \cdots]]. \end{aligned}$$

We shall prove by induction that $[X, Y] \in \mathscr{L}^{(\infty)}(\mathscr{X})$ for any $k$ and $l$. Note that $[X, Y] \in \mathscr{L}^{(\infty)}(\mathscr{X})$ for any $Y$ and $l$, and for $k = 1$. Now suppose this is true for $k = 1, \dots, m$. Then, taking $k = m + 1$, we have

$$[X, Y] = [[X_{m+1}, X^1], Y]$$

where $X^1 = [X_m, \dots, [X_2, X_1] \cdots]$. By the Jacobi identity we have

$$[[X_{m+1}, X^1], Y] + [[Y, X_{m+1}], X^1] + [[X^1, Y], X_{m+1}] = 0_{\Gamma^r(\mathsf{TM})}.$$

This gives

$$[X, Y] = [X^1, [Y, X_{m+1}]] + [X_{m+1}, [X^1, Y]].$$

By the induction hypothesis, $[X^1, [X_{m+1}, Y]] \in \mathscr{L}^{(\infty)}(\mathscr{X})$ since $X^1$ is a bracket of length $m$. Also $[X^1, Y] \in \mathscr{L}^{(\infty)}(\mathscr{X})$ so the second term on the right is in $\mathscr{L}^{(\infty)}(\mathscr{X})$. Thus the set of linear combinations of the form stated in the proposition forms a Lie subalgebra, giving the result. ■

Let $\mathscr{X} \subseteq \Gamma^r(\mathsf{TM})$ be a family of vector fields. Note that since $\mathscr{L}^{(\infty)}(\mathscr{X})$ is a subspace, one immediately has

$$\mathscr{L}^{(\infty)}(\mathscr{X}) = \mathscr{L}^{(\infty)}(\operatorname{span}_{\mathbb{R}}(\mathscr{X})).$$

However, it is not generally the case that $\mathscr{L}^{(\infty)}(\mathscr{X})$ is a submodule. Similarly flavoured statements hold for $\mathscr{X}$ a subsheaf of sets of $\mathscr{G}^r_{\mathsf{TM}}$ or for $\mathscr{X}_x$ a subset of germs from $\mathscr{G}^r_{x,\mathsf{TM}}$. In this respect, however, the following result is useful.

**6.4 Proposition: (The Lie algebra generated by a submodule is a submodule)** *If $r \in \{\infty, \omega\}$, if $\mathsf{M}$ is a $C^r$-manifold, and if $x \in \mathsf{M}$, then the following statements hold:*

*(i) if $\mathscr{M} \subseteq \Gamma^r(\mathsf{TM})$ is a submodule of vector fields, then $\mathscr{L}^{(\infty)}(\mathscr{M})$ is also a submodule of vector fields;*

*(ii) if $\mathscr{F}$ is a subsheaf $\mathscr{C}^r_{\mathsf{M}}$-modules of $\mathscr{G}^r_{\mathsf{TM}}$, then $\mathscr{L}^{(\infty)}(\mathscr{F})$ is also a subsheaf $\mathscr{C}^r_{\mathsf{M}}$-modules of $\mathscr{G}^r_{\mathsf{TM}}$.*

*(iii) if $\mathscr{F}_x \subseteq \mathscr{G}^r_{x,\mathsf{TM}}$ is a submodule of germs of vector fields, then $\mathscr{L}^{(\infty)}(\mathscr{F}_x)$ is also a submodule of germs of vector fields.*

Proof: We prove the first statement only; the second follows directly from this and the third is proved in exactly the same way. By Proposition 6.3 it suffices to show that, for any $f \in C^r(\mathsf{M})$ and for any $X_1, \dots, X_k \in \mathscr{M}$,

$$f[X_k, [X_{k-1}, \dots, [X_2, X_1] \cdots]] \in \mathscr{L}^{(\infty)}(\mathscr{M}).$$

We prove this by induction on $k$, it clearly being true for $k = 1$. Assume now that the statement holds for $k \in \{1, \dots, m+1\}$. Then

$$f[X_{m+1}, [X_m, \dots, [X_2, X_1] \cdots]] = [f X_{m+1}, [X_m, \dots, [X_2, X_1] \cdots]] + (\mathscr{L}_{[X_m, \dots, [X_2, X_1] \cdots]} f) X_k.$$

By the induction hypothesis, $(\mathscr{L}_{[X_m, \dots, [X_2, X_1] \cdots]} f) X_k \in \mathscr{L}^{(\infty)}(\mathscr{M})$. Since $f X_{m+1} \in \mathscr{M}$, by Proposition 6.3 it follows that $[f X_{m+1}, [X_m, \dots, [X_2, X_1] \cdots]] \in \mathscr{L}^{(\infty)}(\mathscr{M})$, giving the result. ■

Let $\mathscr{X}$ be a family of smooth or real analytic vector fields on a smooth or real analytic manifold $\mathsf{M}$. Following the notation of Definition 3.15, associated with the family of vector fields $\mathscr{L}^{(\infty)}(\mathscr{X})$ is the distribution $\mathsf{D}(\mathscr{L}^{(\infty)}(\mathscr{X}))$ which we abbreviate by $\mathsf{L}^{(\infty)}(\mathscr{X})$. The following result simplifies some parts of the subsequent discussion. For the statement, we refer to Definition 3.15 for the notation for the submodule of vector fields generated by a family of vector fields.

**6.5 Proposition: (Characterisation of $\mathsf{L}^{(\infty)}(\mathscr{X})$ for families of vector fields)** *Let $r \in \{\infty, \omega\}$, let $\mathsf{M}$ be a $C^r$-manifold, and let $\mathscr{X} \subseteq \Gamma^r(\mathsf{TM})$. Then the distributions*

*(i) $\mathsf{L}^{(\infty)}(\mathscr{X})$,*

*(ii) $\mathsf{L}^{(\infty)}(\operatorname{span}_{\mathbb{R}}(\mathscr{X}))$,*

*(iii) $\mathsf{L}^{(\infty)}(\langle \mathscr{X} \rangle)$, and*

*(iv) $\mathsf{D}(\langle \mathscr{L}^{(\infty)}(\mathscr{X}) \rangle)$*

*agree.*

Proof: It is clear that

$$\mathsf{L}^{(\infty)}(\mathscr{X}) \subseteq \mathsf{L}^{(\infty)}(\mathrm{span}_{\mathbb{R}}(\mathscr{X})) \subseteq \mathsf{L}^{(\infty)}(\langle\mathscr{X}\rangle).$$

We will show that $\mathsf{L}^{(\infty)}(\langle\mathscr{X}\rangle) \subseteq \mathsf{L}^{(\infty)}(\mathscr{X})$. By Proposition 6.3 it suffices to show that

$$[Y_k, [Y_{k-1}, \dots, [Y_2, Y_1] \cdots ]](x) \in \mathsf{L}^{(\infty)}(\mathscr{X})_x$$

for every $Y_1, \dots, Y_k \in \langle\mathscr{X}\rangle$ and for every $x \in \mathsf{M}$.

We prove this by first showing that

$$[Y_k, [Y_{k-1}, \dots, [Y_2, Y_1] \cdots ]] = f^1 Z_1 + \cdots + f^s Z_s$$

for every $Y_1, \dots, Y_k \in \langle\mathscr{X}\rangle$, and where $f^1, \dots, f^s \in C^r(\mathsf{M})$ and

$$Z_j = [X_{j,l}, [X_{j,l-1}, \dots, [X_{j,2}, X_{j,1}] \cdots ]]$$

for $X_{j,1}, \dots, X_{j,l} \in \mathscr{X}$ with $l \in \{1, \dots, k\}$. This we prove by induction on $k$. It is clearly true for $k = 1$, so suppose it holds for $k \in \{1, \dots, m\}$ and let $Y_1, \dots, Y_m, Y_{m+1} \in \langle\mathscr{X}\rangle$. Write

$$Y_{m+1} = f^1 X_1 + \cdots + f^s X_s, \qquad f^1, \dots, f^s \in C^r(\mathsf{M}),\ X_1, \dots, X_s \in \mathscr{X}.$$

Then

$$[Y_{m+1}, [Y_m, \dots, [Y_2, Y_1] \cdots ]] = \sum_{j=1}^{s} (f^j [X_j, [Y_m, \dots, [Y_2, Y_1]]] - (\mathscr{L}_{[Y_m, \dots, [Y_2, Y_1] \cdots ]} f^j) X_j).$$

By the induction hypothesis,

$$[Y_m, \dots, [Y_2, Y_1] \cdots ] = g^1 Z_1 + \cdots + g^d Z_d$$

for $g^1, \dots, g^d \in C^r(\mathsf{M})$ and where

$$Z_a = [X_{a,l_a}, [X_{a,l_a-1}, \dots, [X_{a,2}, X_{a,1}] \cdots ]]$$

for $X_{a,1}, \dots, X_{a,l_a} \in \mathscr{X}$ and where $l_a \in \{1, \dots, m\}$ for each $a \in \{1, \dots, d\}$. Then

$$\begin{aligned} [X_j, [Y_m, \dots, [Y_2, Y_1] \cdots ]] &= \sum_{a=1}^{d} [X_j, g^a [X_{a,l_a}, [X_{a,l_a-1}, \dots, [X_{a,2}, X_{a,1}] \cdots ]]] \\ &= \sum_{a=1}^{d} (g^a [X_j, [X_{a,l_a}, [X_{a,l_a-1}, \dots, [X_{a,2}, X_{a,1}] \cdots ]]] \\ &\quad + (\mathscr{L}_{X_j} g^a)[X_{a,l_a}, [X_{a,l_a-1}, \dots, [X_{a,2}, X_{a,1}] \cdots ]]). \end{aligned}$$

This proves that $[Y_{m+1}, [Y_m, \dots, [Y_2, Y_1] \cdots ]]$ has the desired form.

From this it immediately follows from Proposition 6.3 that

$$[Y_k, [Y_{k-1}, \dots, [Y_2, Y_1] \cdots ]](x) \in \mathsf{L}^{(\infty)}(\mathscr{X})_x$$

for every $Y_1, \dots, Y_k \in \langle\mathscr{X}\rangle$ and for every $x \in \mathsf{M}$, and so the first three distributions in the statement of the proposition are equal. The equality of these distributions with the fourth distribution in the statement of the proposition follows from Proposition 3.16. ■

There is a corresponding sheaf version of the preceding result which we state for completeness; it follows immediately from the result above. Following the notation of Definition 3.17, we denote $\mathsf{L}^{(\infty)}(\mathscr{X}) = \mathsf{D}(\mathscr{L}^{(\infty)}(\mathscr{X}))$ for a subsheaf of sets $\mathscr{X}$ of $\mathscr{G}^r_{\mathsf{TM}}$.

**6.6 Proposition: (Characterisation of $\mathsf{L}^{(\infty)}(\mathscr{X})$ for subsheaves of sets of vector fields)** *Let $r \in \{\infty, \omega\}$, let $\mathsf{M}$ be a $C^r$-manifold, and let $\mathscr{X} = (X(\mathcal{U}))_{\mathcal{U}\ \mathrm{open}}$ be a subsheaf of sets of $\mathscr{G}^r_{\mathsf{TM}}$. Then the distributions*

*(i)* $\mathsf{L}^{(\infty)}(\mathscr{X})$,

*(ii)* $\mathsf{L}^{(\infty)}(\langle\mathscr{X}\rangle)$, *and*

*(iii)* $\mathsf{D}(\langle\mathscr{L}^{(\infty)}(\mathscr{X})\rangle)$

*agree.*

The developments with Lie algebras thus far in this section have had to do with families of vector fields and subsheaves of sets of $\mathscr{G}^r_{\mathsf{TM}}$. We now turn to Lie algebraic constructions in the case when $\mathscr{X} = \Gamma^r(\mathsf{D})$ is the submodule of sections or where $\mathscr{X} = \mathscr{G}^r_{\mathsf{D}}$ is the sheaf of submodules of sections of a distribution $\mathsf{D}$.

Let us first consider the submodule case. Thus we let $\mathscr{M} \subseteq \Gamma^r(\mathsf{TM})$ be a submodule of vector fields, and we recall that $\mathscr{M} \subseteq \Gamma^r(\mathsf{D}(\mathscr{M}))$, but that the inclusion is, in general, strict, cf. Example 3.14. Moreover, unlike some of the other anomalies we have encountered and will encounter in the paper, this one is not a result of a lack of analyticity. However, there are interesting conclusions that hold in the analytic case, and indeed more generally in the locally finitely generated case.

**6.7 Theorem: (Sometimes $\mathsf{L}^{(\infty)}(\mathscr{X}) = \mathsf{L}^{(\infty)}(\Gamma^r(\mathsf{D}(\mathscr{X})))$)** *Let $r \in \{\infty, \omega\}$, let $\mathsf{M}$ be a $C^r$-manifold, and let $\mathscr{X} \subseteq \Gamma^r(\mathsf{TM})$ be a family of vector fields such that*

*(i) $\langle\mathscr{X}\rangle$ is a locally finitely generated submodule of $\Gamma^r(\mathsf{TM})$ and*

*(ii) the module (by Proposition 6.4) $\mathscr{L}^{(\infty)}(\langle\mathscr{X}\rangle)$ is locally finitely generated.*

*Then $\mathsf{L}^{(\infty)}(\mathscr{X}) = \mathsf{L}^{(\infty)}(\Gamma^r(\mathsf{D}(\mathscr{X})))$.*

Proof: Our proof will rely on the Orbit Theorem and various constructions and results from the theory of control systems and differential inclusions.

Let $x_0 \in \mathsf{M}$. Denote $\mathscr{M} = \langle\mathscr{X}\rangle$. By hypothesis, there exists a neighbourhood $\mathcal{U}$ of $x_0$, a finite subset, say $\mathscr{X}' = (X_1, \dots, X_k)$, of $\mathscr{X}$, and $m \in \mathbb{Z}_{>0}$ such that $F_{\mathscr{X}'}(\mathcal{U}) = F_{\mathscr{M}}(\mathcal{U})$ (see Definition 3.19 for the notation) and such that the vector fields

$$[X_{a_1}, [X_{a_2}, \dots, [X_{a_{l-1}}, X_{a_l}] \cdots ]]|\mathcal{U}, \qquad l \in \{1, \dots, m\},\ a_1, \dots, a_l \in \{1, \dots, k\},$$

generate $\mathscr{L}^{(\infty)}(F_{\mathscr{M}}(\mathcal{U}))$, using Proposition 6.3. Thus both $\mathscr{L}^{(\infty)}(F_{\mathscr{X}'})$ and $\mathscr{L}^{(\infty)}(F_{\mathscr{M}}(\mathcal{U}))$ generate locally finitely generated modules and, moreover, these modules agree. Thus, by Theorem 7.21,

$$\mathrm{Orb}(x_0, \mathscr{X}'|\mathcal{U}) = \mathrm{Orb}(x_0, \mathscr{M}|\mathcal{U}),$$

Consider now two control systems defined on $\mathcal{U}$:

$$\Sigma\colon \quad \xi'(t) = \sum_{j=1}^{k} \mu^j(t) X_j(\xi(t)), \qquad \boldsymbol{\mu}(t) \in U \triangleq \{\boldsymbol{e}_1, \dots, \boldsymbol{e}_k, -\boldsymbol{e}_1, \dots, -\boldsymbol{e}_k\},$$

$$\mathrm{conv}(\Sigma)\colon \quad \xi'(t) = \sum_{j=1}^{k} \mu^j(t) X_j(\xi(t)), \qquad \boldsymbol{\mu}(t) \in \mathrm{conv}(U).$$

In each case, we consider controls to be locally integrable functions taking values in the control set. We have the associated differential inclusions defined by the set-valued right-hand sides

$$F(x) = \Big\{ \sum_{j=1}^{k} u^j X_j(x) \;\Big|\; \boldsymbol{u} \in U \Big\},$$
$$\operatorname{conv}(F)(x) = \Big\{ \sum_{j=1}^{k} u^j X_j(x) \;\Big|\; \boldsymbol{u} \in \operatorname{conv}(U) \Big\},$$

respectively.

We now state a lemma which is often used, but for which we were unable to locate a proof.

**1 Lemma:** *Let $(\mathcal{M}, \mathrm{d}_{\mathcal{M}})$ and $(\mathcal{N}, \mathrm{d}_{\mathcal{N}})$ be metric spaces, let $\mathcal{U}$ be a compact topological space, and let $f \colon \mathcal{M} \times \mathcal{U} \to \mathcal{N}$ be a continuous map for which the map $x \mapsto f(x, u)$ is locally Lipschitz for each $u \in \mathcal{U}$. Then the set-valued map*

$$x \mapsto F(x) \triangleq \{ f(x, u) \mid u \in \mathcal{U} \}$$

*is locally Lipschitz, i.e., for each $x \in \mathcal{M}$ there exists $L \in \mathbb{R}_{>0}$ and a neighbourhood $\mathcal{X}$ of $x$ such that*

$$F(x_1) \subseteq \bigcup_{y \in F(x_2)} \mathsf{B}_{\mathcal{N}}(y, L\mathrm{d}_{\mathcal{M}}(x_1, x_2))$$

*for each $x_1, x_2 \in \mathcal{X}$, where $\mathsf{B}_{\mathcal{N}}(y, r)$ is the ball of radius $r$ centred at $y \in \mathcal{N}$.*

Proof: Since $f$ is locally Lipschitz in its first argument, for each $x \in \mathcal{M}$ and $u \in \mathcal{U}$, there exists $L_u \in \mathbb{R}_{>0}$ and a neighbourhood $\mathcal{X}_u$ of $x$ such that

$$\mathrm{d}_{\mathcal{N}}(f(x_1, u), f(x_2, u)) \le 2L_u \mathrm{d}_{\mathcal{M}}(x_1, x_2)$$

for every $x_1, x_2 \in \mathcal{X}_u$. Continuity of $f$ and the metric ensures that there exists a neighbourhood $\mathcal{Z}_u \subseteq \mathcal{U}$ of $u$ such that

$$\mathrm{d}_{\mathcal{N}}(f(x_1, v_1), f(x_2, v_2)) < L_u \mathrm{d}_{\mathcal{M}}(x_1, x_2)$$

for every $x_1, x_2 \in \mathcal{X}_u$ and every $v_1, v_2 \in \mathcal{Z}_u$, possibly by also shrinking $\mathcal{X}_u$. Let

$$\mathcal{X} = \cap_{j=1}^{k} \mathcal{X}_{u_j}, \quad L = \max\{L_{u_1}, \dots, L_{u_k}\}.$$

Now let $x_1, x_2 \in \mathcal{X}$ and let $f(x_1, u) \in F(x_1)$ for some $u \in \mathcal{U}$. Then $u \in \mathcal{Z}_j$ for some $j \in \{1, \dots, k\}$ and so

$$\mathrm{d}_{\mathcal{N}}(f(x_1, u), f(x_2, u)) < L_{u_j} \mathrm{d}_{\mathcal{M}}(x_1, x_2) \le L\mathrm{d}_{\mathcal{M}}(x_1, x_2),$$

giving $f(x_1, u) \in \mathsf{B}(, \mathcal{N})(f(x_2, u), L\mathrm{d}_{\mathcal{M}}(x_1, x_2))$. We thus conclude that

$$F(x_1) \subseteq \bigcup_{y \in F(x_2)} \mathsf{B}(, \mathcal{N})(y, L\mathrm{d}_{\mathcal{M}}(x_1, x_2)),$$

as desired. ▼

Let us define the family of vector fields

$$\operatorname{conv}(\mathscr{X}') = \Big\{ \sum_{j=1}^{k} u^j X_j \;\Big|\; \boldsymbol{u} \in \operatorname{conv}(U) \Big\}.$$

Since the vector fields $X_1, \dots, X_k$ are locally Lipschitz, by the lemma the differential inclusion defined by $F$ is locally Lipschitz. It is also clearly compact-valued. By the relaxation theorem of Filippov [1967] and Ważewski [1962], it follows that the reachable set for $\Sigma$ is dense in the reachable set for $\operatorname{conv}(\Sigma)$. Since both $\Sigma$ and $\operatorname{conv}(\Sigma)$ are symmetric, i.e., if $v_x \in F(x)$ then $-v_x \in F(x)$ and similarly for $\operatorname{conv}(F)$, it follows that the reachable set and the orbit agree (see Proposition 4.3 in [Jakubczyk 2002], for example). Thus $\operatorname{Orb}(x_0, \mathscr{X}')$ is dense in $\operatorname{Orb}(x_0, \operatorname{conv}(\mathscr{X}'))$. By the Orbit Theorem, $\operatorname{Orb}(x_0, \mathscr{X}')$ is an immersed submanifold of the immersed submanifold $\operatorname{Orb}(x_0, \operatorname{conv}(\mathscr{X}'))$. Since $x_0 \in \operatorname{int}(\operatorname{Orb}(x_0, \mathscr{X}'))$ and $x_0 \in \operatorname{int}(\operatorname{Orb}(x_0, \operatorname{conv}(\mathscr{X}')))$ (interior being taken in the orbit topology in each case), this implies that the tangent spaces to the two orbits agree at $x_0$. By shrinking $\mathcal{U}$ we can ensure that $\operatorname{Orb}(x_0, \mathscr{X}') = \operatorname{Orb}(x_0, \operatorname{conv}(\mathscr{X}'))$.

Now note that, if $X \in \Gamma^r(\mathsf{D}(\mathscr{X})|\mathcal{U})$, then $X(x) \in \operatorname{span}_{\mathbb{R}}(X_1(x), \dots, X_k(x))$ for each $x \in \mathcal{U}$ since $(X_1, \dots, X_k)$ generate $F_{\mathscr{M}}(\mathcal{U})$. Therefore, points in $\operatorname{Orb}(x_0, \Gamma^r(\mathsf{D}(\mathscr{X})|\mathcal{U}))$ are endpoints of concatenations of curves whose tangent vectors are positive multiples of tangent vectors in $\operatorname{conv}(F)$. Thus points in $\operatorname{Orb}(x_0, \Gamma^r(\mathsf{D}(\mathscr{X})|\mathcal{U})))$ are endpoints of concatenations of curves tangent to $\operatorname{Orb}(x_0, \operatorname{conv}(\mathscr{X}'))$. Thus $\operatorname{Orb}(x_0, \Gamma^r(\mathsf{D}(\mathscr{X})|\mathcal{U})) \subseteq \operatorname{Orb}(x_0, \operatorname{conv}(\mathscr{X}'))$. Since the opposite inclusion is obvious, we have $\operatorname{Orb}(x_0, \Gamma^r(\mathsf{D}(\mathscr{X})|\mathcal{U})) = \operatorname{Orb}(x_0, \operatorname{conv}(\mathscr{X}'))$.

Putting the above arguments together, the tangent spaces at $x_0$ of $\operatorname{Orb}(\Gamma^r(\mathsf{D}(\mathscr{X})|\mathcal{U}))$ and $\operatorname{Orb}(x_0, \mathscr{X}|\mathcal{U})$ agree. By Theorems 7.18 and 7.21 we have

$$\mathsf{L}^{(\infty)}(\Gamma^r(\mathsf{D}(\mathscr{X})))_{x_0} \subseteq \mathsf{T}_{x_0} \operatorname{Orb}(x_0, \Gamma^r(\mathsf{D}(\mathscr{X})|\mathcal{U})) = \mathsf{T}_{x_0} \operatorname{Orb}(x_0, \mathscr{X}|\mathcal{U}) = \mathsf{L}^{(\infty)}(\mathscr{X})_{x_0}.$$

Since the inclusion $\mathsf{L}^{(\infty)}(\mathscr{X})_{x_0} \subseteq \mathsf{L}^{(\infty)}(\Gamma^r(\mathsf{D}(\mathscr{X})))_{x_0}$ is clear and since $x_0$ is arbitrary, the theorem follows. ■

The result has two interesting and often applicable corollaries.

**6.8 Corollary: (In the smooth constant rank case, $\mathsf{L}^{(\infty)}(\mathscr{X}) = \mathsf{L}^{(\infty)}(\Gamma^\infty(\mathsf{D}(\mathscr{X})))$)** *Let $\mathsf{M}$ be a $C^\infty$-manifold, let $\mathscr{X} \subseteq \Gamma^\infty(\mathsf{TM})$, and let $x$ be a regular point of $\mathsf{D}(\mathscr{X})$ and of $\mathsf{L}^{(\infty)}(\mathscr{X})$. Then $\mathsf{L}^{(\infty)}(\mathscr{X})_x = \mathsf{L}^{(\infty)}(\Gamma^\infty(\mathsf{D}(\mathscr{X})))_x$ for every $x \in \mathsf{M}$.*

Proof: This follows from Theorem 6.7, along with Theorem 4.9. ■

**6.9 Corollary: (In the analytic case, $\mathsf{L}^{(\infty)}(\mathscr{X}) = \mathsf{L}^{(\infty)}(\Gamma^\omega(\mathsf{D}(\mathscr{X})))$)** *Let $\mathsf{M}$ be a $C^\omega$-manifold and let $\mathscr{X} \subseteq \Gamma^\omega(\mathsf{TM})$. Then $\mathsf{L}^{(\infty)}(\mathscr{X})_x = \mathsf{L}^{(\infty)}(\Gamma^\omega(\mathsf{D}(\mathscr{X})))_x$.*

Proof: This follows from Theorem 6.7, along with Theorem 4.10 (noting that $\mathscr{F}_{\mathscr{X}}$ is obviously patchy). ■

Let us consider a few examples that illustrate the subtlety of the preceding results.

**6.10 Examples: (The relationship between $\mathsf{L}^{(\infty)}(\mathscr{X})$ and $\mathsf{L}^{(\infty)}(\Gamma^r(\mathsf{D}(\mathscr{X})))$)** We consider $\mathsf{M} = \mathbb{R}^2$ and the distribution $\mathsf{D}$ given by

$$\mathsf{D}_{(x_1,x_2)} = \begin{cases} \mathsf{T}_{(x_1,x_2)}\mathbb{R}^2, & x_1 \neq 0, \\ \operatorname{span}_{\mathbb{R}}(\frac{\partial}{\partial x_1}), & x_1 = 0. \end{cases}$$

This distribution is generated by any pair of vector fields

$$X_1(x_1,x_2) = \frac{\partial}{\partial x_1}, \quad X_2(x_1,x_2) = f(x_1)\frac{\partial}{\partial x_2},$$

where $f \in C^\infty(\mathbb{R})$ satisfies $f^{-1}(0) = \{0\}$. Thus $\mathsf{D}$ is a smooth distribution. Moreover, we claim that if $X$ is any section of $\mathsf{D}$ then $X = f^1 X_1 + f^2 X_2$ for some $f^1, f^2 \in C^\infty(\mathbb{R}^2)$, provided we take $f$ defined by $f(x) = x$. Indeed, let us write

$$X = g^1\frac{\partial}{\partial x_1} + g^2\frac{\partial}{\partial x_2},$$

noting that we must have $g^2(0,x_2) = 0$ for every $x_2 \in \mathbb{R}$. We write

$$g^2(x_1,x_2) = \int_0^{x_1} \frac{\partial g^2}{\partial x_1}(\xi, x_2)\,\mathrm{d}\xi = x_1\int_0^1 \frac{\partial g^2}{\partial x_1}(x_1\eta, x_2)\,\mathrm{d}\eta.$$

Thus our claim follows by taking

$$f^1(x_1,x_2) = g^1(x_1,x_2), \quad f^2(x_1,x_2) = \int_0^1 \frac{\partial g^2}{\partial x_1}(x_1\eta, x_2)\,\mathrm{d}\eta.$$

This shows that, not only does $\mathsf{D}$ have a finite number of generators (as per Theorem 5.1), but also that $\Gamma^\infty(\mathsf{D})$ is finitely generated. Note that it is possible to choose generators for $\mathsf{D}$ that do not generate $\Gamma^\infty(\mathsf{D})$, e.g., by taking $f(x) = x^2$ we see that $X_1$ and $X_2$ as above have this property, cf. Example 3.10. By choosing more pathological generators, e.g., by taking

$$f(x) = \begin{cases} \mathrm{e}^{-1/x^2}, & x \neq 0, \\ 0, & x = 0, \end{cases}$$

one imagines that the algebraic properties of the distribution should deteriorate. We shall see now that this is true as concerns the Lie algebra generated by the generators.

We take $\mathscr{X} = (X_1, X_2)$ and consider a few $f$'s.

1. First let us consider $f(x) = x$. We compute

$$[X_1, X_2](x_1,x_2) = \frac{\partial}{\partial x_2}.$$

Therefore, $\mathsf{L}^{(\infty)}(\mathscr{X}) = \mathsf{T}\mathbb{R}^2$. Thus we must have $\mathsf{L}^{(\infty)}(\mathscr{X}) = \mathsf{L}^{(\infty)}(\Gamma^r(\mathsf{D}(\mathscr{X})))$. Since $f$ is analytic, this is in agreement with Corollary 6.9.

2. Next consider $f(x) = x^2$. In this case we have

$$[X_1, X_2](x_1,x_2) = 2x_1\frac{\partial}{\partial x_2}, \quad [X_1,[X_1,X_2]](x_1,x_2) = 2\frac{\partial}{\partial x_2}.$$

Thus we can again conclude that $\mathsf{L}^{(\infty)}(\mathscr{X}) = \mathsf{T}\mathbb{R}^2$, implying that $\mathsf{L}^{(\infty)}(\mathscr{X}) = \mathsf{L}^{(\infty)}(\Gamma^r(\mathsf{D}(\mathscr{X})))$. This again is consistent with Corollary 6.9. Note, however, the distribution $\mathsf{L}^{(\infty)}(\mathscr{X})$ is generated by different brackets than was the case when we took $f(x) = x$. Thus the fact that $\mathsf{L}^{(\infty)}(\mathscr{X}) = \mathsf{L}^{(\infty)}(\Gamma^r(\mathsf{D}(\mathscr{X})))$ is less obvious in this case.

3. The final case we consider is

$$f(x) = \begin{cases} \mathrm{e}^{-1/x^2}, & x \neq 0, \\ 0, & x = 0. \end{cases}$$

We note that

$$\mathsf{D}(\mathscr{X})_{x_1,x_2} = \begin{cases} \mathsf{T}_{(x_1,x_2)}\mathbb{R}^2, & x_1 \neq 0, \\ \operatorname{span}_{\mathbb{R}}(\frac{\partial}{\partial x_1}), & x_1 = 0. \end{cases}$$

Thus, for example, the vector fields

$$X_1'(x_1,x_2) = \frac{\partial}{\partial x_1}, \quad X_2'(x_1,x_2) = x_1\frac{\partial}{\partial x_2}$$

generate $\mathsf{D}(\mathscr{X}')$. As above, one can compute $[X_1, X_2] = \frac{\partial}{\partial x_2}$, and so we have $\mathsf{L}^{(\infty)}(\Gamma^r(\mathsf{D}(\mathscr{X}))) = \mathsf{T}\mathbb{R}^2$. However, one can easily show that $\mathsf{L}^{(\infty)}(\mathscr{X}) = \mathsf{D}(\mathscr{X})$ and so $\mathsf{L}^{(\infty)}(\mathscr{X}) \subset \mathsf{L}^{(\infty)}(\Gamma^r(\mathsf{D}(\mathscr{X}))q)$. By Theorem 6.7 we conclude that $\mathscr{L}^{(\infty)}(\langle\mathscr{X}\rangle)$ is not locally finitely generated. This is a problem with the generators $\mathscr{X}$ for $\mathsf{D}(\mathscr{X})$ being smooth but not analytic. •

Theorem 6.7 and its corollaries can also be adapted to subsheaves instead of submodules.

**6.11 Theorem: (Sometimes $\mathsf{L}^{(\infty)}(\mathscr{X}) = \mathsf{L}^{(\infty)}(\mathscr{G}^r_{\mathsf{D}(\mathscr{X})})$)** *Let $r \in \{\infty, \omega\}$, let $\mathsf{M}$ be a $C^r$-manifold, and let $\mathscr{X} = (X(\mathcal{U}))_{\mathcal{U}\ open}$ be a subsheaf of sets of the sheaf $\mathscr{G}^r_{\mathsf{TM}}$ such that*

*(i) $\langle\mathscr{X}\rangle$ is a locally finitely generated subsheaf of $\mathscr{G}^r_{\mathsf{TM}}$ and*

*(ii) the subsheaf (by Proposition 6.4) $\mathscr{L}^{(\infty)}(\langle\mathscr{X}\rangle)$ is locally finitely generated.*

*Then $\mathsf{L}^{(\infty)}(\mathscr{X}) = \mathsf{L}^{(\infty)}(\mathscr{G}^r_{\mathsf{D}(\mathscr{X})})$.*

Proof: This follows from Theorem 6.7. ■

**6.12 Corollary: (In the smooth constant rank case, $\mathsf{L}^{(\infty)}(\mathscr{X}) = \mathsf{L}^{(\infty)}(\mathscr{G}^r_{\mathsf{D}(\mathscr{X})})$)** *let $\mathsf{M}$ be a $C^\infty$-manifold, let $\mathscr{X} = (X(\mathcal{U}))_{\mathcal{U}\ open}$ be a subsheaf of sets of the sheaf $\mathscr{G}^r_{\mathsf{TM}}$, and let $x$ be a regular point of $\mathsf{D}(\mathscr{X})$ and of $\mathsf{L}^{(\infty)}(\mathscr{X})$. Then $\mathsf{L}^{(\infty)}(\mathscr{X}) = \mathsf{L}^{(\infty)}(\mathscr{G}^r_{\mathsf{D}(\mathscr{X})})$.*

**6.13 Corollary: (In the patchy analytic case, $\mathsf{L}^{(\infty)}(\mathscr{X}) = \mathsf{L}^{(\infty)}(\mathscr{G}^r_{\mathsf{D}(\mathscr{X})})$)** *Let $\mathsf{M}$ be a $C^\omega$-manifold and let $\mathscr{X} = (X(\mathcal{U}))_{\mathcal{U}\ open}$ be a subsheaf of sets of the sheaf $\mathscr{G}^r_{\mathsf{TM}}$ for which $\langle\mathscr{X}\rangle$ is patchy. Then $\mathsf{L}^{(\infty)}(\mathscr{X}) = \mathsf{L}^{(\infty)}(\mathscr{G}^r_{\mathsf{D}(\mathscr{X})})$.*

**6.4. Distributions and subsheaves invariant under vector fields and diffeomorphisms.** Using the constructions of the preceding section, in this section we introduce the notion of distributions and subsheaves that are invariant under vector fields and diffeomorphisms. In this section, when we say "subsheaf" of $\mathscr{G}^r_{\mathsf{TM}}$ we mean a subsheaf of $\mathscr{C}^r_{\mathsf{M}}$-modules.

**6.14 Definition: (Distributions and subsheaves invariant under vector fields and diffeomorphisms)** Let $r \in \{\infty, \omega\}$, let $\mathsf{M}$ be a manifold of class $C^r$, let $\mathsf{D}$ be a distribution of class $C^r$, let $\mathscr{F}$ be a subsheaf of $\mathscr{G}^r_{\mathsf{TM}}$, let $X$ be a $C^r$-vector field, and let $(\Phi, \mathcal{U})$ be a $C^r$-local diffeomorphism. The distribution $\mathsf{D}$

(i) is ***invariant*** under $X$ if $[X, Y] \in \Gamma^r(\mathsf{D})$ for every $Y \in \Gamma^r(\mathsf{D})$ and

(ii) is ***invariant*** under $(\Phi, \mathcal{U})$ if $\Phi^* Y \in \Gamma^r(\mathsf{D}|\Phi(\mathcal{U}))$ for every $Y \in \Gamma^r(\mathsf{D}|\mathcal{U})$.

The subsheaf $\mathscr{F}$

(iii) is ***invariant*** under $X$ at $x$ if $[[X, Y]]_x \in \mathscr{F}_x$ for every $[Y]_x \in \mathscr{F}_x$,

(iv) is ***invariant*** under $(\Phi, \mathcal{U})$ at $x \in \mathcal{U}$ if $\Phi^*_x [X]_x \in \mathscr{F}_{\Phi(x)}$ for every $[X]_x \in \mathscr{F}_x$,

(v) is ***invariant*** under $X$ if it is invariant under $X$ at $x$ for each $x \in \mathsf{M}$, and

(vi) is ***invariant*** under $(\Phi, \mathcal{U})$ if it is invariant under $\Phi$ at $x$ for each $x \in \mathcal{U}$. •

One would like to think that invariance under a vector field and invariance under its flow are equivalent. This is true under suitable hypotheses. Let us first look at the case of invariant sheaves. Here the statement requires us to exploit the topologies on stalks of subsheaves of vector bundles from Section 2.8.

**6.15 Theorem: (Invariant sheaves)** *Let $r \in \{\infty, \omega\}$, let $\mathsf{M}$ be a $C^r$-manifold, let $\mathscr{F} = (F(\mathcal{U}))_{\mathcal{U}\ \mathrm{open}}$ be a subsheaf of $\mathscr{G}^r_{\mathsf{TM}}$, let $X \in \Gamma^r(\mathsf{TM})$, and let $x \in \mathsf{M}$. Consider the following two statements:*

*(i) $\mathscr{F}$ is invariant under $X$ at $x$;*

*(ii) for each $T \in \mathbb{R}_{>0}$ there exists a neighbourhood $\mathcal{U}$ of $x$ such that $\mathscr{F}$ is invariant under $(\Phi^X_t, \mathcal{U})$ at $x$ for each $t \in [-T, T]$.*

*Then (ii) $\implies$ (i) if $\mathscr{F}_x$ is a closed submodule of $\mathscr{G}^r_{x,\mathsf{TM}}$, and (i) $\implies$ (ii) if there exists a neighbourhood $\mathcal{V}$ of $x$ such that $F(\mathcal{V})$ is a finitely generated $C^r(\mathcal{V})$-module.*

Proof: (i) $\implies$ (ii) Let $x \in \mathsf{M}$ and let $X \in \Gamma^r(\mathsf{TM})$ satisfy

$$\{[X, Y]_x \mid [Y]_x \in \mathscr{F}_x\} \subseteq \mathscr{F}_x.$$

Suppose that $Y_1, \dots, Y_k$ generate $F(\mathcal{V})$ for some neighbourhood $\mathcal{V}$ of $x$. By hypothesis,

$$[X, Y_j]_x = \sum_{i=1}^k [f^i_j]_x [Y_i]_x, \qquad j \in \{1, \dots, k\},$$

for some $[f^i_j]_x \in \mathscr{C}^r_{x,\mathsf{M}}$, $i, j \in \{1, \dots, k\}$. As this expression involves only finitely many germs, we may assume $\mathcal{V}$ sufficiently small that

$$[X, Y_j](y) = \sum_{i=1}^k f^i_j(y) Y_i(y), \qquad y \in \mathcal{V}.$$

Let $\mathcal{U} \subseteq \mathcal{V}$ be sufficiently small that $\Phi^X_t(y) \in \mathcal{V}$ for every $t \in [-T, T]$ and every $y \in \mathcal{U}$. For $t \in [-T, T]$ and $y \in \mathcal{U}$ define $v_j(t, y) = (\Phi^X_t)^* Y_j(y)$, $j \in \{1, \dots, k\}$, so that $t \mapsto v_j(t, y)$ is a curve in $\mathsf{T}_y\mathsf{M}$. By [Abraham, Marsden, and Ratiu 1988, Theorem 4.2.19]

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}v_j(t,y) &= \frac{\mathrm{d}}{\mathrm{d}t}(\Phi^X_t)^*Y_j(y) = (\Phi^X_t)^*[X,Y_j](y) = (\Phi^X_t)^*\Big(\sum_{i=1}^k f^i_jY_i\Big)(y)\\ &= \sum_{i=1}^k(\Phi^X_t)^*f^i_j(y)(\Phi^X_t)^*Y_i(y) = \sum_{i=1}^k(\Phi^X_t)^*f^i_j(y)v_i(t).\end{aligned}$$

Define $\boldsymbol{A}_y(t)\in\mathbb{R}^{k\times k}$ by

$$A^i_{y,j}(t) = (\Phi^X_t)^*f^i_j(y)$$

and let $\boldsymbol{\Psi}_y\colon\mathbb{R}\to\mathbb{R}^{k\times k}$ be the solution to the matrix initial value problem

$$\frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{\Psi}_y(t) = \boldsymbol{A}_y(t)\boldsymbol{\Psi}_y(t), \qquad \boldsymbol{\Psi}_y(0) = \boldsymbol{I}_k.$$

We claim that

$$v_j(t,y) = \sum_{i=1}^k\Psi^i_{y,j}(t)Y_i(y).$$

Indeed,

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\Big(\sum_{i=1}^k\Psi^i_{y,j}(t)Y_i(y)\Big) &= \sum_{i=1}^k\frac{\mathrm{d}}{\mathrm{d}t}\Psi^i_{y,j}(t)Y_i(y) = \sum_{i,l=1}^kA^l_{y,j}(t)\Psi^i_{y,l}(t)Y_i(y)\\ &= \sum_{l=1}^k(\Phi^X_t)^*f^l_j(y)\Big(\sum_{i=1}^k\Psi^i_{y,l}(t)Y_i(y)\Big).\end{aligned}$$

Moreover,

$$\sum_{i=1}^k\Psi^i_{y,j}(0)Y_i(y) = Y_j(y), \qquad v_j(0,y) = Y_j(y).$$

Thus

$$t\mapsto v_j(t,y)\ \text{ and }\ t\mapsto\sum_{i=1}^k\Psi^i_{y,j}(t)Y_i(y)$$

satisfy the same differential equation with the same initial condition. Thus they are equal. This gives

$$(\Phi^X_t)^*Y_j(y) = \sum_{i=1}^k\Psi^i_{y,j}(t)Y_i(y)$$

for every $t\in[-T,T]$ and $y\in\mathcal{U}$. Now let $[Y]_x\in\mathscr{F}_x$ and suppose that $Y$ is a local section over $\mathcal{W}\subseteq\mathcal{U}$. Following the proof of Proposition 4.19(iv), we can write

$$[Y]_x = \sum_{j=1}^k[\eta^j]_x[Y_j]_x$$

for $[\eta^j]_x \in \mathscr{C}^r_{x,\mathsf{M}}$, $j \in \{1, \dots, k\}$. Therefore, possibly after shrinking $\mathcal{W}$, we can write

$$Y = \eta^1(Y_j|\mathcal{W}) + \dots + \eta^k(Y_k|\mathcal{W})$$

for some $\eta^1, \dots, \eta^k \in C^r(\mathcal{W})$. Therefore, for $y \in \mathcal{W}$ and $t \in [-T, T]$,

$$(\Phi^X_t)^* Y(y) = \sum_{j=1}^k \eta^j(y)(\Phi^X_t)^* Y_j(y) = \sum_{j=1}^k \eta^j(y) \sum_{i=1}^k \Psi^i_{y,j}(t) Y_i(y), \tag{6.1}$$

and so $[(\Phi^X_t)^* Y]_x \in \mathscr{F}_x$. This gives this part of the theorem.

(ii) $\Longrightarrow$ (i) This part of the proof we carry out separately for $r = \infty$ and $r = \omega$.

Let us first consider the case $r = \infty$. Let $[Y]_x \in \mathscr{F}_x$, let $\mathcal{V}$ be a sufficiently small neighbourhood that $Y$ can be taken as being defined on $\mathcal{V}$, and let $\mathcal{U} \subseteq \mathcal{V}$ and $T \in \mathbb{R}_{>0}$ be sufficiently small that $\Phi^X_t(y) \in \mathcal{V}$ for every $y \in \mathcal{U}$ and $t \in [-T, T]$. We can without loss of generality assume that $\mathcal{V}$ is the domain of an admissible chart, and so deal with local representatives. Let us state a lemma that will be helpful here and in the second part of the proof where we treat the real analytic case.

**1 Lemma:** *Let $\mathcal{U}_1 \subseteq \mathbb{R}^{n_1}$ and $\mathcal{U}_2 \subseteq \mathbb{R}^{n_2}$ be open sets, let $\mathsf{V}$ be a finite-dimensional normed vector space, and let $f \colon \mathcal{U}_1 \times \mathcal{U}_2 \to \mathsf{V}$ be continuously differentiable. Then, for any compact set $K \subseteq \mathcal{U}_2$, any $\boldsymbol{x}_1 \in \mathcal{U}_1$, and any $\boldsymbol{v} \in \mathbb{R}^{n_1}$,*

$$\lim_{t\to 0}\Big(\sup\Big\{\Big\|\frac{1}{t}(f(\boldsymbol{x}_1 + t\boldsymbol{v}, \boldsymbol{x}_2) - f(\boldsymbol{x}_1, \boldsymbol{x}_2)) - \boldsymbol{D}_1 f(\boldsymbol{x}_1, \boldsymbol{x}_2) \cdot \boldsymbol{v}\Big\| \;\Big|\; \boldsymbol{x}_2 \in K\Big\}\Big) = 0.$$

Proof: Define

$$g_{\boldsymbol{x}_1,\boldsymbol{v}}(t, \boldsymbol{x}_2) = f(\boldsymbol{x}_1 + t\boldsymbol{v}, \boldsymbol{x}_2) - f(\boldsymbol{x}_1, \boldsymbol{x}_2) - t\boldsymbol{D}_1 f(\boldsymbol{x}_1, \boldsymbol{x}_2) \cdot \boldsymbol{v}.$$

Note that $(t, \boldsymbol{x}_2) \mapsto \frac{1}{t} g_{\boldsymbol{x}_1,\boldsymbol{v}}(t, \boldsymbol{x}_2)$ is continuous for $t$ sufficiently near zero (taking its value when $t = 0$ to be zero). This implies that, for every $\epsilon \in \mathbb{R}_{>0}$, there exists $\delta \in \mathbb{R}_{>0}$ such that $\|\frac{1}{t} g_{\boldsymbol{x}_1,\boldsymbol{v}}(t, \boldsymbol{x}_2)\| < \epsilon$ for every $t \in [-\delta, \delta]$ and $\boldsymbol{x}_2 \in K$. From this the result follows. ▼

Applying the lemma successively to all derivatives, and recalling the notation from Section 2.8, we have

$$\lim_{t\to 0}\Big\|\frac{1}{t}((\Phi^X_t)^* Y - Y) - [X, Y]\Big\|_{r,K} = 0$$

for every compact $K \subseteq \mathcal{U}$ and every $r \in \mathbb{Z}_{\geq 0}$, using [Abraham, Marsden, and Ratiu 1988, Theorem 4.2.19]. This shows that

$$\lim_{t\to 0} \frac{1}{t}((\Phi^X_t)^* Y - Y) = [X, Y]$$

in $\Gamma^\infty(\mathsf{TM}|\mathcal{U})$ in the weak $C^\infty$-topology described in Section 2.8. Since we are assuming that $\mathscr{F}_x$ is closed, $r^{-1}_{\mathcal{U},x}(\mathscr{F}_x)$ is closed (since $r_{\mathcal{U},x}$ is continuous as in [Köthe 1969, §19.5]). By hypothesis, $\frac{1}{t}((\Phi^X_t)^* Y - Y) \in r^{-1}_{\mathcal{U},x}(\mathscr{F}_x)$ for every sufficiently small $t$, it follows that $[X.Y] \in r^{-1}_{\mathcal{U},x}(\mathscr{F}_x)$. Thus $r_{\mathcal{U},x}([X, Y]) \in \mathscr{F}_x$ for every sufficiently small neighbourhood $\mathcal{U}$ of $x$ and so $[[X, Y]]_x \in \mathscr{F}_x$, which is the result in the smooth case.

Now we consider the real analytic case. We let $[Y]_x \in \mathscr{F}_x$ and suppose that $Y$ is defined in a neighbourhood $\mathcal{V}$ of $x$. As above, let $\mathcal{U} \subseteq \mathcal{V}$ be a neighbourhood of $x$ and let $T \in \mathbb{R}_{>0}$ be such that $\Phi^X_t(y) \in \mathcal{V}$ for every $y \in \mathcal{V}$ and $t \in [-T, T]$. We suppose that $\mathcal{V}$ is the domain of a coordinate chart and work locally in Euclidean space. We let $\overline{\mathcal{Z}}$ be a neighbourhood in the complexification of $(-T, T) \times \mathcal{V}$. Let $\overline{x}$ be the image of $x$ in this complexification. By shrinking $\overline{\mathcal{Z}}$ we extend the vector fields $X$ and $Y$ to holomorphic vector fields $\overline{X}$ and $\overline{Y}$, respectively, on $\overline{\mathcal{Z}}$. We can extend the real analytic map

$$(-T, T) \times \mathcal{U} \ni (t, y) \mapsto \Phi^X_t(y) \in \mathcal{V} \tag{6.2}$$

to a holomorphic map defined in a neighbourhood of $(-T, T) \times \mathcal{V}$ in $\overline{\mathcal{Z}}$. Let us denote the variables in the complexification by $(\tau, z)$. Let us write the complexification of the map (6.2) as $(\tau, z) \mapsto \overline{\Phi}^{\overline{X}}_\tau(z)$. Note that $\frac{\mathrm{d}}{\mathrm{d}\tau}\big|_{\tau=0}(\overline{\Phi}^{\overline{X}}_\tau)^*\overline{Y}(z)$ is the holomorphic extension of $[X, Y]$, since it agrees with $[X, Y]$ at points in $\mathcal{V}$ using [Abraham, Marsden, and Ratiu 1988, Theorem 4.2.19]. Let us denote this extension by $[\overline{X}, \overline{Y}]$. Using the lemma above (noting that complex differentiation with respect to $\tau = t + \mathrm{i}s$ is represented by the Jacobian of the corresponding real mapping) and recalling the notation from Section 2.8,

$$\lim_{\tau\to 0}\left\|\frac{1}{\tau}((\overline{\Phi}^{\overline{X}}_\tau)^*\overline{Y} - \overline{Y}) - [\overline{X}, \overline{Y}]\right\|_K = 0$$

for every compact set $K \subseteq \mathcal{V}$. We now recall the notation introduced preceding the proof of Theorem 2.26. Since every submodule of $\mathscr{G}^\omega_{x,\mathsf{E}}$ is closed by Theorem 2.26, the definition of the topology on $\mathscr{G}^\omega_{x,\mathsf{E}}$ ensures that $r^{-1}_{\overline{\mathcal{Z}},\overline{x}}(\rho^{-1}_{\overline{x},x}(\mathscr{F}_x))$ is closed, cf. [Köthe 1969, §19.5]. Thus $[\overline{X}, \overline{Y}] \in r^{-1}_{\overline{\mathcal{Z}},\overline{x}}(\rho^{-1}_{\overline{x},x}(\mathscr{F}_x))$ since, by hypothesis, $\frac{1}{\tau}((\overline{\Phi}^{\overline{X}}_\tau)^*\overline{Y} - \overline{Y}) \in r^{-1}_{\overline{\mathcal{Z}},\overline{x}}(\rho^{-1}_{\overline{x},x}(\mathscr{F}_x))$ for $\tau$ sufficiently near zero. It follows, therefore, that $[[X, Y]]_x \in \mathscr{F}_x$, giving the theorem. ■

The theorem then has the following immediate corollary, when combined with Theorem 2.26 and Proposition 4.6.

**6.16 Corollary: (Real analytic invariant sheaves)** *Let $\mathsf{M}$ be a real analytic manifold, let $\mathscr{F}$ be a subsheaf of $\mathscr{G}^\omega_{\mathsf{TM}}$, let $X \in \Gamma^r(\mathsf{TM})$, and let $x \in \mathsf{M}$. Then the following two statements are equivalent:*

*(i) $\mathscr{F}$ is invariant under $X$ at $x$;*

*(ii) for each $T \in \mathbb{R}_{>0}$ there exists a neighbourhood $\mathcal{U}$ of $x$ such that $\mathscr{F}$ is invariant under $(\Phi^X_t, \mathcal{U})$ at $x$ for each $t \in [-T, T]$.*

Next we consider the relationship between invariance of distributions under vector fields and their flows.

**6.17 Theorem: (Invariant distributions)** *Let $r \in \{\infty, \omega\}$, let $\mathsf{M}$ be a $C^r$-manifold, let $\mathsf{D}$ be a $C^r$-distribution, and let $X \in \Gamma^r(\mathsf{TM})$. Consider the following two statements:*

*(i) $\mathsf{D}$ is invariant under $X$;*

*(ii) for each $T \in \mathbb{R}_{>0}$ and each $x \in \mathsf{M}$ there exists a neighbourhood $\mathcal{U}$ of $x$ such that $\mathsf{D}|\mathcal{U}$ is invariant under $(\Phi^X_t, \mathcal{U})$ for each $t \in [-T, T]$.*

*Then (ii) $\Longrightarrow$ (i) always, and (i) $\Longrightarrow$ (ii) if $\mathscr{G}^r_{\mathsf{D}}$ is locally finitely generated.*

Proof: (i) $\Longrightarrow$ (ii) This follows from the corresponding part of Theorem 6.15, taking particular note that in the proof of the theorem the equation (6.1) holds in a neighbourhood of $x$.

(ii) $\Longrightarrow$ (i) Let $x \in \mathsf{M}$ and let $\epsilon \in \mathbb{R}_{>0}$ be such that $\Phi^X_t(x)$ exists for $t \in (-\epsilon, \epsilon)$. Then, for each $t \in (-\epsilon, \epsilon)$, we have

$$(\Phi^X_t)_* Y(x) \in \mathsf{D}_x.$$

Therefore, since $\mathsf{D}_x$ is a subspace, we use [Abraham, Marsden, and Ratiu 1988, Theorem 4.2.19] to compute

$$[Y, X](x) = \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} (\Phi^X_t)_* Y(x) \in \mathsf{D}_x,$$

as desired. ■

Of course, the preceding theorem then has the following immediate corollary, when combined with Proposition 4.6.

**6.18 Corollary: (Real analytic invariant distributions)** *Let* $\mathsf{M}$ *be a real analytic manifold, let* $\mathsf{D}$ *be an analytic distribution on* $\mathsf{M}$*, and let* $X \in \Gamma^r(\mathsf{TM})$*. Then the following two statements are equivalent:*

- *(i)* $\mathsf{D}$ *is invariant under* $X$ *at* $x$*;*
- *(ii) for each* $T \in \mathbb{R}_{>0}$ *and each* $x \in \mathsf{M}$ *there exists a neighbourhood* $\mathcal{U}$ *of* $x$ *such that* $\mathsf{D}|\mathcal{U}$ *is invariant under* $(\Phi^X_t, \mathcal{U})$ *for each* $t \in [-T, T]$*.*

## 7. The Orbit Theorem

In this section we consider an important theorem that is not so immediately connected to the theory of distributions, but, as we shall see, leads to important theorems regarding special classes of distributions. Contributions to the Orbit Theorem have been made by Hermann [1962], Krener [1974], Lobry [1970], Stefan [1974a], Stefan [1974b], and Sussmann [1973].

### 7.1. Partially defined vector fields.

In our development of the Orbit Theorem, we follow Sussmann [1973] and consider the quite general setting where vector fields are not only not necessarily complete, but only defined on an open subset of the manifold. In this section we develop the notation and rules for dealing with these sorts of vector fields.

**7.1 Definition: (Partially defined vector field, everywhere defined family of partially defined vector fields)** Let $r \in \{\infty, \omega\}$ and let $\mathsf{M}$ be a manifold of class $C^r$.

- (i) A ***partially defined vector field of class $C^r$*** is a pair $(X, \mathcal{U})$ where $\mathcal{U} \subseteq \mathsf{M}$ is open and where $X$ is a $C^r$-vector field on $\mathcal{U}$.
- (ii) A family $\mathscr{X} = ((X_j, \mathcal{U}_j))_{j \in J}$ of partially defined vector fields of class $C^r$ is ***everywhere defined*** if, for every $x \in \mathsf{M}$, there exists $j \in J$ such that $x \in \mathcal{U}_j$. •

Let us outline, for clarity, how some of the standard vector field constructions apply to partially defined vector fields. The ***sum*** of two partially defined vector fields $(X, \mathcal{U})$ and $(Y, \mathcal{V})$ is the partially defined vector field

$$(X, \mathcal{U}) + (Y, \mathcal{V}) = (X|\mathcal{U} \cap \mathcal{V} + Y|\mathcal{U} \cap \mathcal{V}, \mathcal{U} \cap \mathcal{V}).$$

Given partially defined vector fields $(X, \mathcal{U})$ and $(Y, \mathcal{V})$, their ***Lie bracket*** is the partially defined vector field

$$[(X, \mathcal{U}), (Y, \mathcal{V})] = ([X|\mathcal{U} \cap \mathcal{V}, Y|\mathcal{U} \cap \mathcal{V}], \mathcal{U} \cap \mathcal{V}).$$

Motivated by the constructions of Section 6.3, let us say that a family $\mathscr{X} = ((X_j, \mathcal{U}_j))_{j \in J}$ of partially defined vector fields is a ***Lie algebra*** of partially defined vector fields if

$$(X_{j_1}, \mathcal{U}_{j_1}) + (X_{j_2}, \mathcal{U}_{j_2}), [(X_{j_1}, \mathcal{U}_{j_1}), (X_{j_2}, \mathcal{U}_{j_2})] \in \mathscr{X}$$

for every $j_1, j_2 \in J$. For a family $\mathscr{X} = ((X_j, \mathcal{U}_j))_{j \in J}$ of partially defined vector fields on $\mathsf{M}$, we denote by $\mathscr{L}^{(\infty)}(\mathscr{X})$ the smallest Lie algebra of partially defined vector fields containing $\mathscr{X}$.

Associated in a natural way to a partially defined family of vector fields is a distribution.

**7.2 Definition: (Distribution associated to a partially defined family of vector fields)** Let $r \in \{\infty, \omega\}$ and let $\mathsf{M}$ be a manifold of class $C^r$. Given a family $\mathscr{X} = ((X_j, \mathcal{U}_j))_{j \in J}$ of partially defined vector fields of class $C^r$, we can define a distribution $\mathsf{D}(\mathscr{X})$ by

$$\mathsf{D}(\mathscr{X})_x = \operatorname{span}_{\mathbb{R}}(X_j(x) |\ x \in \mathcal{U}_j). \qquad \bullet$$

In the definition we did not assert anything about whether the distribution $\mathsf{D}(\mathscr{X})$ is of class $C^r$ if $\mathscr{X}$ is a family of partially defined $C^r$-vector fields. In the smooth case, the resulting distribution *is* smooth, provided that all vector fields are bounded.

**7.3 Proposition: (Distributions defined by families of partially defined smooth vector fields are smooth)** *Let $\mathsf{M}$ be a smooth paracompact Hausdorff manifold and let $\mathbb{G}$ be a smooth Riemannian metric on $\mathsf{M}$. If $\mathscr{X} = ((X_j, \mathcal{U}_j))_{j \in J}$ is a family of smooth partially defined vector fields $\mathsf{M}$ such that*

$$\sup\{\|X_j(x)\|_{\mathbb{G}} \mid x \in \mathcal{U}_j\} < \infty$$

*for each $j \in J$, then the distribution $\mathsf{D}(\mathscr{X})$ is smooth. (Here $\|\cdot\|_{\mathbb{G}}$ denotes the norm on the tangent spaces induced by the Riemannian metric $\mathbb{G}$.)*

Proof: For each $j \in J$, we use Lemma 2 from the proof of Theorem 5.1 to assert the existence of $f_j \in C^\infty(\mathsf{M})$ such that $f_j(x) \in \mathbb{R}_{>0}$ for $x \in \mathcal{U}_j$ and $f_j(x) = 0$ for $x \in \mathsf{M} \setminus \mathcal{U}_j$. We can then define a smooth vector field $\hat{X}_j$ on $\mathsf{M}$ by

$$\hat{X}_j(x) = \begin{cases} f_j(x) X_j(x), & x \in \mathcal{U}_j, \\ 0, & x \in \mathsf{M} \setminus \mathcal{U}_j. \end{cases}$$

Define the family $\hat{\mathscr{X}} = (\hat{X}_j)_{j \in J}$ of vector fields on $\mathsf{M}$. It is clear that $\mathsf{D}(\mathscr{X}) = \mathsf{D}(\hat{\mathscr{X}})$, giving the result. ■

For families of partially defined analytic vector fields, the corresponding assertion does not generally hold.

**7.4 Example: (Distributions defined by families of analytic partially defined vector fields are not generally analytic)** Note that Example 7.7–4 below shows that families of partially defined $C^\omega$-vector fields do not define $C^\omega$-distributions. Indeed, in this example, the family of analytic partially vector fields $\mathscr{X} = ((X_j, \mathcal{U}_j))_{j\in\{1,2\}}$ defines the distribution

$$\mathsf{D}(\mathscr{X})_{(x_1,x_2)} = \begin{cases} \operatorname{span}_{\mathbb{R}}(\frac{\partial}{\partial x_1}), & x_1 \leq -1, \\ \operatorname{span}_{\mathbb{R}}(\frac{\partial}{\partial x_2}), & x_1 \geq -1, \\ \mathsf{T}_{(x_1,x_2)}\mathbb{R}^2, & x_1 \in (-1,1). \end{cases}$$

By Proposition 3.7, this is not an analytic distribution. It *is*, however, a smooth distribution. •

Let $\mathscr{X}$ be a family of partially defined vector fields and, following our notation of Section 6.3, let us abbreviate $\mathsf{D}(\mathscr{L}^{(\infty)}(\mathscr{X}))$ by $\mathsf{L}^{(\infty)}(\mathscr{X})$. By Proposition 6.3, $\mathsf{L}^{(\infty)}(\mathscr{X})_x$ is generated by tangent vectors of the form

$$[(X_{j_k}, \mathcal{U}_{j_k}), [(X_{j_{k-1}}, \mathcal{U}_{j_{k-1}}), \dots, [(X_{j_2}, \mathcal{U}_{j_2}), (X_{j_1}, \mathcal{U}_{j_1})] \cdots]](x),$$

where $k \in \mathbb{Z}_{>0}$ and $(X_{j_1}, \mathcal{U}_{j_1}), \dots, (X_{j_k}, \mathcal{U}_{j_k}) \in \mathscr{X}$ are such that $x \in \mathcal{U}_{j_1} \cap \cdots \cap \mathcal{U}_{j_k}$.

It is also convenient to introduce some notation regarding germs of partially defined vector fields. Thus we let $\mathscr{X} = ((X_j, \mathcal{U}_j))_{j\in J}$ be a family of partially defined vector fields and we denote by

$$\mathscr{X}_x = \{r_{\mathcal{U}_j,x}(X_j) \mid x \in \mathcal{U}_j\}$$

the set of germs at $x$ of vector fields from $\mathscr{X}$. Note that

$$\mathsf{D}(\mathscr{X})_x = \operatorname{span}_{\mathbb{R}}(X(x) \mid [X]_x \in \mathscr{X}_x). \tag{7.1}$$

**7.2. Orbits.** Now we turn to defining orbits for partially defined families of vector fields. The first step is to study a "group" of diffeomorphisms associated with a partially defined family of vector fields. For a partially defined vector field $(X, \mathcal{U})$ and for $t \in \mathbb{R}$, we recall from Section 6.1 that $\mathcal{U}(X,t) \subseteq \mathcal{U}$ is the open set such that $\Phi^X_t(x)$ is defined for each $x \in \mathcal{U}(X,t)$.

**7.5 Definition: (Local group of diffeomorphisms generated by a family of vector fields)** Let $r \in \{\infty, \omega\}$, let $\mathsf{M}$ be a $C^r$-manifold, let $\mathscr{D}$ be a family of $C^r$-local diffeomorphisms, and let $\mathscr{X}$ be a family of partially defined vector fields of class $C^r$.

(i) A ***group of $C^r$-local diffeomorphisms*** is a family $\mathscr{G} = ((\Phi_j, \mathcal{U}_j))_{j\in J}$ of $C^r$-local diffeomorphisms such that $(\Phi_{j_1}, \mathcal{U}_{j_1}) \circ (\Phi_{j_2}, \mathcal{U}_{j_2}) \in \mathscr{G}$ and $(\Phi_j, \mathcal{U}_j)^{-1} \in \mathscr{G}$ for every $j, j_1, j_2 \in J$.

(ii) A group $\mathscr{G} = ((\Phi_j, \mathcal{U}_j))_{j\in J}$ of $C^r$-local diffeomorphisms is ***everywhere defined*** if, for each $x \in \mathsf{M}$, there exists $j \in J$ such that $x \in \mathcal{U}_j$.

(iii) The ***group of local diffeomorphisms*** generated by $\mathscr{D}$ is the smallest group of $C^r$-local diffeomorphisms containing $\mathscr{D}$ and which is closed under the operations of composition and inverse of local diffeomorphisms.

(iv) The ***group of local diffeomorphisms*** generated by $\mathscr{X}$ is the group of $C^r$-local diffeomorphisms generated by those local diffeomorphisms of the form $(\Phi^X_t, \mathcal{U}(X,t))$ for $(X, \mathcal{U}) \in \mathscr{X}$ and for $t \in \mathbb{R}$. •

Note that a group of local diffeomorphisms is not a actually a group in the usual sense of the word since, for example, local diffeomorphisms with empty domain do not have unique inverses.

Let us obtain a concrete description of $\mathrm{Diff}(\mathscr{X})$. First of all, to simplify notation, since the open set $\mathcal{U}(X,t)$ associated with the mapping $\Phi^X_t$ is implicit, we shall write $\Phi^X_t$ for the local diffeomorphism $(\Phi^X_t, \mathcal{U}(X,t))$. Then, if $\boldsymbol{X} = (X_1, \dots, X_k)$ is a family of vector fields from $\mathscr{X}$ and if $\boldsymbol{t} = (t_1, \dots, t_k) \in \mathbb{R}^k$, then we denote

$$\Phi^{\boldsymbol{X}}_{\boldsymbol{t}} = \Phi^{X_k}_{t_k} \circ \cdots \circ \Phi^{X_1}_{t_1},$$

which we think of as a composition of local diffeomorphisms and so a local diffeomorphism. For $x \in \mathsf{M}$ we shall also write

$$\Phi^{\boldsymbol{X}}_{\boldsymbol{t}}(x) = \Phi^{X_k}_{t_k} \circ \cdots \circ \Phi^{X_1}_{t_1}(x),$$

with the understanding that this is defined if $x$ is in the domain of the local diffeomorphism $\Phi^{\boldsymbol{X}}_{\boldsymbol{t}}$. The set of such $x$'s we denote by $\mathcal{U}(\boldsymbol{X}, \boldsymbol{t})$, noting that this is an open subset of $\mathsf{M}$. One can then directly verify that

$$\mathrm{Diff}(\mathscr{X}) = \left\{\Phi^{\boldsymbol{X}}_{\boldsymbol{t}} \;\middle|\; \boldsymbol{X} \in \mathscr{X}^k,\ \boldsymbol{t} \in \mathbb{R}^k,\ k \in \mathbb{Z}_{>0}\right\}. \tag{7.2}$$

The preceding discussion is greatly complicated by the fact that we allow vector fields from $\mathscr{X}$ to possibly not be complete and/or not globally defined. If all vector fields from $\mathscr{X}$ are complete and globally defined, then one easily sees that

$$\mathrm{Diff}(\mathscr{X}) = \left\{\Phi^{X_k}_{t_k} \circ \cdots \circ \Phi^{X_1}_{t_1} \;\middle|\; X_1, \dots, X_k \in \mathscr{X},\ t_1, \dots, t_k \in \mathbb{R}\right\}.$$

The reader would benefit by keeping this special case in mind.

We can now define what we mean by an orbit for a family of partially defined vector fields.

**7.6 Definition: (Orbit)** Let $r \in \{\infty, \omega\}$, let $\mathsf{M}$ be a $C^r$-manifold, and let $\mathscr{X} = ((X_j, \mathcal{U}_j))_{j\in J}$ be an everywhere defined family of partially defined $C^r$-vector fields. The ***orbit*** of $\mathscr{X}$ through $x_0 \in \mathsf{M}$ is the set

$$\mathrm{Orb}(x_0, \mathscr{X}) = \left\{\Phi^{\boldsymbol{X}}_{\boldsymbol{t}}(x_0) \;\middle|\; \boldsymbol{X} \in \mathscr{X}^k,\ \boldsymbol{t} \in \mathbb{R}^k,\ k \in \mathbb{Z}_{>0}\right\}. \qquad \bullet$$

Note that two distinct orbits are disjoint. Thus the set of orbits defines a partition of $\mathsf{M}$.

Let us understand the concept of an orbit by using some examples. Most of our examples involve complete vector fields, so obviating some of the complications of the constructions above.

**7.7 Examples: (Orbits)**

1. We take $\mathsf{M} = \mathbb{R}^2$ and define

$$X_1 = x_1 \frac{\partial}{\partial x_1}, \quad X_2 = x_2 \frac{\partial}{\partial x_2}.$$

The flows of $X_1$ and $X_2$ are easily determined explicitly. Using these flows one can readily determine the orbits for $\mathscr{X} = (X_1, X_2)$. Let us illustrate how to do this in two cases; the other cases follow in the same manner.

(a) $\boldsymbol{x}_0 = (x_{01}, x_{02})$ *with* $x_{01} \in \mathbb{R}_{>0}$ *and* $x_{20} = 0$*:* Since $X_2 = 0$ on the $x_1$-axis and since $X_1$ is tangent to the $x_1$-axis, $\mathrm{Orb}(\boldsymbol{x}_0, \mathscr{X})$ will be contained in the $x_1$-axis. Moreover, if $x_1 \in \mathbb{R}_{>0}$ and if we define $t_1 = \log \frac{x_1}{x_{01}}$, then $\Phi^{X_1}_{t_1}(\boldsymbol{x}_0) = (x_1, 0)$. Moreover, for any $t \in \mathbb{R}_{>0}$,

$$\Phi^{X_1}_{t}(\boldsymbol{x}_0) \in \{(x_1, 0) \mid x_1 \in \mathbb{R}_{>0}\}.$$

Thus we must have

$$\mathrm{Orb}(\boldsymbol{x}_0, \mathscr{X}) = \{(x_1, 0) \mid x_1 \in \mathbb{R}_{>0}\}.$$

(b) $\boldsymbol{x}_0 = (x_{01}, -x_{02})$ *with* $x_{01}, x_{02} \in \mathbb{R}_{>0}$*:* Here we let $(x_1, -x_2) \in \mathbb{R}^2$ with $x_1, x_2 \in \mathbb{R}_{>0}$. We then define $t_1 = \log \frac{x_1}{x_{01}}$ and $t_2 = \log \frac{x_2}{x_{02}}$ and note that $\Phi^{X_2}_{t_2} \circ \Phi^{X_1}_{t_1}(\boldsymbol{x}_0) = (x_1, x_2)$. Moreover, for $t_1, \dots, t_k \in \mathbb{R}$ and for $j_1, \dots, j_k \in \{1, 2\}$,

$$\Phi^{X_{j_k}}_{t_k} \circ \dots \circ \Phi^{X_{j_1}}_{t_1}(\boldsymbol{x}_0) \in \{(x_1, -x_2) \mid x_1, x_2 \in \mathbb{R}_{>0}\}.$$

This shows that

$$\mathrm{Orb}(\boldsymbol{x}_0, \mathscr{X}) = \{(x_1, -x_2) \mid x_1, x_2 \in \mathbb{R}_{>0}\}.$$

In any case, it is easy to see that there are nine distinct orbits for the family of vector fields $\mathscr{X} = (X_1, X_2)$, and these are determined to be

$$\begin{aligned}
\mathrm{Orb}_1((0,0), \mathscr{X}) &= \{(0,0)\},\\
\mathrm{Orb}_2((1,0), \mathscr{X}) &= \{(x_1, 0) \mid x_1 \in \mathbb{R}_{>0}\},\\
\mathrm{Orb}_3((-1,0), \mathscr{X}) &= \{(-x_1, 0) \mid x_1 \in \mathbb{R}_{>0}\},\\
\mathrm{Orb}_4((0,1), \mathscr{X}) &= \{(0, x_2) \mid x_2 \in \mathbb{R}_{>0}\},\\
\mathrm{Orb}_5((0,-1), \mathscr{X}) &= \{(0, -x_2) \mid x_2 \in \mathbb{R}_{>0}\},\\
\mathrm{Orb}_6((1,1), \mathscr{X}) &= \{(x_1, x_2) \mid x_1, x_2 \in \mathbb{R}_{>0}\},\\
\mathrm{Orb}_7((-1,1), \mathscr{X}) &= \{(-x_1, x_2) \mid x_1, x_2 \in \mathbb{R}_{>0}\},\\
\mathrm{Orb}_8((1,-1), \mathscr{X}) &= \{(x_1, -x_2) \mid x_1, x_2 \in \mathbb{R}_{>0}\},\\
\mathrm{Orb}_9((-1,-1), \mathscr{X}) &= \{(-x_1, -x_2) \mid x_1, x_2 \in \mathbb{R}_{>0}\}.
\end{aligned}$$

We depict these orbits in Figure 3.

2. We let $\mathsf{M} = \mathbb{R}^2$ and define

$$X_1 = \frac{\partial}{\partial x_1}, \quad X_2 = f(x_1)\frac{\partial}{\partial x_2},$$

where

$$f(x) = \begin{cases} \mathrm{e}^{-1/x^2}, & x \in \mathbb{R}_{>0},\\ 0, & x \in \mathbb{R}_{\leq 0}. \end{cases}$$

We take $\mathscr{X} = (X_1, X_2)$ and claim that $\mathrm{Orb}(\boldsymbol{x}, \mathscr{X}) = \mathbb{R}^2$ for every $\boldsymbol{x} \in \mathbb{R}^2$. It suffices to show that, for example, $\mathrm{Orb}(\boldsymbol{0}, \mathscr{X}) = \mathbb{R}^2$. For $(x_1, x_2) \in \mathbb{R}^2$ with $x_1 > 0$ we define $t_1 = x_1$ and $t_2 = \frac{x_2}{f(x_1)}$, and directly compute

$$\Phi^{X_2}_{t_2} \circ \Phi^{X_1}_{t_1}(\boldsymbol{0}) = (x_1, x_2).$$

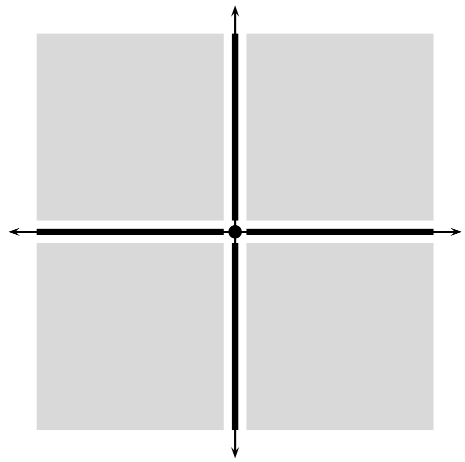

**Figure 3.** Orbits

If $(x_1, x_2) \in \mathbb{R}^2$ with $x_1 \leq 0$ we define $t_1 = 1$, $t_2 = \frac{x_2}{f(x_1)}$, and $t_3 = -1 + x_1$, and directly compute

$$\Phi^{X_3}_{t_3} \circ \Phi^{X_2}_{t_2} \circ \Phi^{X_1}_{t_1}(\mathbf{0}) = (x_1, x_2).$$

This shows that $\mathrm{Orb}(\mathbf{0}, \mathscr{X}) = \mathbb{R}^2$ as desired.

3. For $u \in \mathbb{R}$ define a vector field $X_u$ on $\mathbb{R}^2$ by

$$X_u(\boldsymbol{x}) = (\boldsymbol{x}, \boldsymbol{A}\boldsymbol{x} + \boldsymbol{b}u),$$

where

$$\boldsymbol{A} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad \boldsymbol{b} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

We then consider the family of vector fields on $\mathbb{R}^2$ given by $\mathscr{X} = (X_u)_{u\in\mathbb{R}}$. Let $(x_{11}, x_{12}), (x_{21}, x_{22}) \in \mathbb{R}^2$ and let $T \in \mathbb{R} \setminus \{0\}$. Define

$$u_1 = \frac{-3Tx_{12} - Tx_{22} - 4x_{11} + 4x_{21}}{T^2},$$
$$u_2 = \frac{Tx_{12} + 3Tx_{22} + 4x_{11} - 4x_{21}}{T^2}.$$

One can then directly verify (by solving the linear differential equations defining the flow) that

$$(x_{21}, x_{22}) = \Phi^{X_{u_2}}_{T/2} \circ \Phi^{X_{u_1}}_{T/2}(x_{11}, x_{12}), \tag{7.3}$$

showing that $\mathrm{Orb}(\boldsymbol{x}, \mathscr{X}) = \mathbb{R}^2$ for every $\boldsymbol{x} \in \mathbb{R}^2$.

4. The final example we consider is one where the vector fields are partially defined, but not globally defined. We take $\mathsf{M} = \mathbb{R}^2$ and define a family $\mathscr{X} = ((X_j, \mathcal{U}_j))_{j\in\{1,2\}}$ of partially defined vector fields by

$$\mathcal{U}_1 = \{(x_1, x_2) \mid x_1 < 1\}, \quad \mathcal{U}_2 = \{(x_1, x_2) \mid x_1 > -1\}$$

and

$$X_1 = \frac{\partial}{\partial x_1}, \quad X_2 = \frac{\partial}{\partial x_2}.$$

It is easy to see that

$$\operatorname{Orb}((x_{01}, x_{02}), \mathscr{X}) = \begin{cases} \{(x_1, x_2) \mid x_1 < 1\}, & x_{01} < 1, \\ \{(x_{01}, x_2) \mid x_2 \in \mathbb{R}\}, & x_{01} \geq 1. \end{cases}$$

Note that the orbits are analytic submanifolds. •

**7.3. Fixed-time orbits.** In this section we consider a modification of the notion of an orbit as defined in the previous section. Let $r \in \{\infty, \omega\}$, let $\mathsf{M}$ be a $C^r$-manifold, and let $\mathscr{X}$ be a family of partially vector fields of class $C^r$. Above we defined $\operatorname{Diff}(\mathscr{X})$ as the subgroup of local diffeomorphisms defined by flows of vector fields from $\mathscr{X}$. In this section we modify this construction slightly to give the orbit corresponding to flows whose “total time” is fixed. To make this construction, we first consider flows whose “total time” is zero.

For convenience and to reestablish notation, we recall the explicit characterisation of $\operatorname{Diff}(\mathscr{X})$ from above. As above, for a vector field $X$ we shall often denote the local diffeomorphism $(\Phi^X_t, \mathcal{U}(X, t))$ simply by $\Phi^X_t$. Let $\boldsymbol{X} = (X_1, \dots, X_k)$ be a finite family of vector fields from $\mathscr{X}$ and let $\boldsymbol{t} = (t_1, \dots, t_k) \in \mathbb{R}^k$. Then we define a local diffeomorphism

$$\Phi^{\boldsymbol{X}}_{\boldsymbol{t}} = \Phi^{X_k}_{t_k} \circ \cdots \circ \Phi^{X_1}_{t_1},$$

understanding implicitly that this is only interesting when the resulting composition has nonempty domain. With this notation,

$$\operatorname{Diff}(\mathscr{X}) = \{\Phi^{\boldsymbol{X}}_{\boldsymbol{t}} \mid \boldsymbol{X} \in \mathscr{X}^k,\ \boldsymbol{t} \in \mathbb{R}^k,\ k \in \mathbb{Z}_{>0}\}.$$

Now let $T \in \mathbb{R}$. Define

$$\operatorname{Diff}_T(\mathscr{X}) = \Big\{\Phi^{\boldsymbol{X}}_{\boldsymbol{t}} \;\Big|\; \boldsymbol{X} \in \mathscr{X}^k,\ \boldsymbol{t} \in \mathbb{R}^k,\ \sum_{j=1}^k t_k = T,\ k \in \mathbb{Z}_{>0}\Big\}.$$

The case where $T = 0$ is particularly interesting, as we shall see below. The following properties of $\operatorname{Diff}_0(\mathscr{X})$ are useful in understanding some of the subsequent constructions.

**7.8 Proposition: ($\operatorname{Diff}_0(\mathscr{X})$ is a “normal subgroup” of $\operatorname{Diff}(\mathscr{X})$)** *Let $r \in \{\infty, \omega\}$, let $\mathsf{M}$ be a $C^r$-manifold, and let $\mathscr{X} \subseteq \Gamma^r(\mathsf{TM})$. Then the following statements hold:*

*(i) $\operatorname{Diff}_0(\mathscr{X})$ is a subgroup of the group $\operatorname{Diff}(\mathscr{X})$ of local diffeomorphisms; that is, if $(\Phi, \mathcal{U}), (\Psi, \mathcal{V}) \in \operatorname{Diff}_0(\mathscr{X})$, then $(\Phi, \mathcal{U})^{-1} \in \operatorname{Diff}_0(\mathscr{X})$ and $(\Phi, \mathcal{U}) \circ (\Psi, \mathcal{V}) \in \operatorname{Diff}_0(\mathscr{X})$;*

*(ii) $\operatorname{Diff}_0(\mathscr{X})$ is a normal subgroup; that is, if $(\Phi, \mathcal{U}) \in \operatorname{Diff}_0(\mathscr{X})$ and if $(\Psi, \mathcal{V}) \in \operatorname{Diff}(\mathscr{X})$, then $(\Psi, \mathcal{V}) \circ (\Phi, \mathcal{U}) \circ (\Psi, \mathcal{V})^{-1} \in \operatorname{Diff}_0(\mathscr{X})$.*

Proof: (i) Let $\boldsymbol{X} = (X_1, \dots, X_k)$ and $\boldsymbol{Y} = (Y_1, \dots, Y_m)$ be families of vector fields and $\boldsymbol{t} = (t_1, \dots, t_k)$ and $\boldsymbol{s} = (s_1, \dots, s_m)$ be families of real numbers such that $\Phi = \Phi^{\boldsymbol{X}}_{\boldsymbol{t}}$ and $\Psi = \Phi^{\boldsymbol{Y}}_{\boldsymbol{s}}$. Thus

$$\sum_{j=1}^k t_j = \sum_{l=1}^m s_l = 0.$$

Then $(\Phi, \mathcal{U})^{-1}$ is defined by
$$\Phi = \Phi^{X_1}_{-t_1} \circ \cdots \circ \Phi^{X_k}_{-t_k},$$
and so $(\Phi, \mathcal{U})^{-1} \in \mathrm{Diff}_0(\mathscr{X})$. Similarly, $(\Phi, \mathcal{U}) \circ (\Psi, \mathcal{V})$ is defined by
$$\Phi^{X_k}_{t_k} \circ \cdots \circ \Phi^{X_1}_{t_1} \circ \Phi^{Y_m}_{t_m} \circ \cdots \Phi^{Y_1}_{t_1},$$
and so $(\Phi, \mathcal{U}) \circ (\Psi, \mathcal{V}) \in \mathrm{Diff}_0(\mathscr{X})$.

(ii) Let $\boldsymbol{X} = (X_1, \dots, X_k)$ and $\boldsymbol{Y} = (Y_1, \dots, Y_m)$ be families of vector fields and $\boldsymbol{t} = (t_1, \dots, t_k)$ and $\boldsymbol{s} = (s_1, \dots, s_m)$ be families of real numbers such that $\Phi = \Phi^{\boldsymbol{X}}_{\boldsymbol{t}}$ and $\Psi = \Phi^{\boldsymbol{Y}}_{\boldsymbol{s}}$. Thus
$$\sum_{j=1}^{k} t_j = 0.$$
Note that $(\Psi, \mathcal{V}) \circ (\Phi, \mathcal{U}) \circ (\Psi, \mathcal{V})^{-1}$ is defined by
$$\Phi^{Y_m}_{s_m} \circ \cdots \circ \Phi^{Y_1}_{s_1} \circ \Phi^{X_k}_{t_k} \circ \cdots \circ \Phi^{X_1}_{t_1} \circ \Phi^{Y_1}_{-s_1} \circ \cdots \circ \Phi^{Y_m}_{-s_m},$$
and so $(\Psi, \mathcal{V}) \circ (\Phi, \mathcal{U}) \circ (\Psi, \mathcal{V})^{-1} \in \mathrm{Diff}_0(\mathscr{X})$, as desired. ■

We can now make the following definition.

**7.9 Definition: (Fixed-time orbit)** Let $r \in \{\infty, \omega\}$, let $\mathsf{M}$ be a $C^r$-manifold, let $\mathscr{X} \subseteq \Gamma^r(\mathsf{TM})$ be a family of $C^r$-vector fields, and let $T \in \mathbb{R}$. The ***T-orbit*** of $\mathscr{X}$ through $x_0 \in \mathsf{M}$ is the set
$$\mathrm{Orb}_T(x_0, \mathscr{X}) = \Big\{\Phi^{\boldsymbol{X}}_{\boldsymbol{t}}(x_0) \;\Big|\; \boldsymbol{X} \in \mathscr{X}^k,\ \boldsymbol{t} \in \mathbb{R}^k,\ \sum_{j=1}^{k} t_j = T,\ k \in \mathbb{Z}_{>0}\Big\}.$$
A ***fixed-time orbit*** of $\mathscr{X}$ through $x_0 \in \mathsf{M}$ is a set of the form $\mathrm{Orb}_T(x_0, \mathscr{X})$ for some $T \in \mathbb{R}$. •

Let us give some examples of fixed-time orbits.

**7.10 Examples: (Fixed-time orbits)** We resume some of the examples whose orbits we studied in Example 7.7.

1. Let $\mathsf{M} = \mathbb{R}^2$ and take $X_1 = x_1 \frac{\partial}{\partial x_1}$ and $X_2 = x_2 \frac{\partial}{\partial x_2}$. We determined the orbits of $\mathscr{X} = (X_1, X_2)$ in Example 7.7–1. One readily computes
$$\mathrm{Orb}_T((x_1, 0), \mathscr{X}) = \{(x_1 \mathrm{e}^T, 0)\}$$
and
$$\mathrm{Orb}_T((0, x_2), \mathscr{X}) = \{(0, x_2 \mathrm{e}^T)\}.$$
Thus the $T$-orbits for points on the coordinate axes are singletons. With a little more effort, or better, an application of Theorem 7.21 below, we can determine the $T$-orbits for the other points. The $T$-orbits are the submanifolds
$$\mathsf{S}^+_c = \{(x, cx^{-1}) \mid x \in \mathbb{R}_{>0}\}, \quad \mathsf{S}^-_c = \{(x, cx^{-1}) \mid x \in \mathbb{R}_{>0}\},$$
parameterised by the nonzero real number $c$. In Figure 4 we show the collection of these fixed-time orbits. To determine which of the submanifolds is a $T$-orbit through a point $(x_1, x_2)$ that is not on a coordinate axis, one follows the integral curve of either $X_1$ or $X_2$ (it matters not which) for time $T$ and the submanifold upon which one finds oneself is the $T$-orbit for $(x_1, x_2)$.

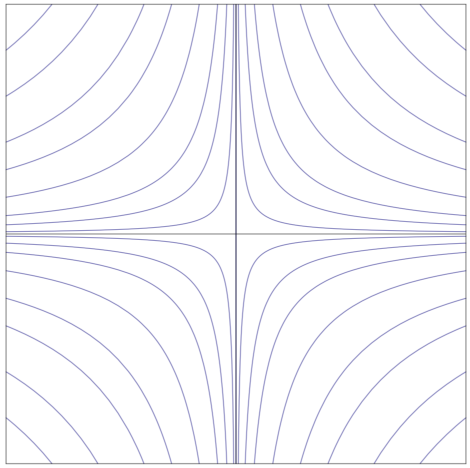

**Figure 4.** Fixed-time orbits

2. We consider here the example first considered in Example 7.7–3. Thus we consider the family $(X_u)_{u\in\mathbb{R}}$ of vector fields with

$$X_u(\boldsymbol{x}) = (\boldsymbol{x}, \boldsymbol{A}\boldsymbol{x} + \boldsymbol{b}u),$$

where

$$\boldsymbol{A} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad \boldsymbol{b} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

We saw in (7.3) that $\mathrm{Orb}_T(\boldsymbol{x}, \mathscr{X}) = \mathbb{R}^2$ for every $\boldsymbol{x} \in \mathbb{R}^2$ and $T \in \mathbb{R} \setminus \{0\}$.

3. Let us now consider the partially defined vector fields $\mathscr{X} = ((X_1, \mathcal{U}_1), (X_2, \mathcal{U}_2))$ from Example 7.7–4. As with Example 1 above and by Theorem 7.21, it suffices to determine the 0-orbit through each point. The $T$-orbit for a point is then determined by following a curve that is a concatenation of integral curves for $X_1$ and $X_2$ for a total time $T$. The $T$-orbit with be the 0-orbit through the resulting point. A few moments of thought will lead one to conclude that

$$\mathrm{Orb}_0((x_1, x_2), \mathscr{X}) = \begin{cases} \{(x_1 + s, x_2 - s) \mid s \in (-\infty, 1 - x_1)\}, & x_1 < 1, \\ \{(x_1, x_2)\}, & x_1 \geq 1. \end{cases}$$

The 0-orbits are depicted in Figure 5. Note that the dimensions of the 0-orbits differ on disjoint open sets. •

The following result gives a useful characterisation of $\mathrm{Diff}_T(\mathscr{X})$.

**7.11 Proposition: (Characterisation of $\mathrm{Orb}_T(x_0, \mathscr{X})$)** *Let $r \in \{\infty, \omega\}$, let $\mathsf{M}$ be a $C^r$-manifold, and let $\mathscr{X}$ be a family of partially defined vector fields of class $C^r$. For $x_0 \in \mathsf{M}$ and $T \in \mathbb{R}$ suppose that $\mathrm{Orb}_T(x_0, \mathscr{X}) \neq \varnothing$. Then, for every $x \in \mathrm{Orb}_T(x_0, \mathscr{X})$, we have*

$$\mathrm{Orb}_T(x_0, \mathscr{X}) = \{\Phi(x) \mid \Phi \in \mathrm{Diff}_0(\mathscr{X})\}.$$

Proof: Write $x = \Phi^{\boldsymbol{X}}_{\boldsymbol{t}}(x_0)$ for $\boldsymbol{X} \in \mathscr{X}^k$ and $\boldsymbol{t} \in \mathbb{R}^k$ with $\sum_{j=1}^k t_j = T$. If $\Phi \in \mathrm{Diff}_0(\mathscr{X})$ then we have $\Phi = \Phi^{\boldsymbol{Y}}_{\boldsymbol{s}}$ for $\boldsymbol{Y} \in \mathscr{X}^m$ and $\boldsymbol{s} \in \mathbb{R}^m$ with $\sum_{l=1}^m s_l = 0$. Then,

$$\Phi(x) = \Phi^{\boldsymbol{Y}}_{\boldsymbol{s}} \circ \Phi^{\boldsymbol{X}}_{\boldsymbol{t}}(x_0),$$

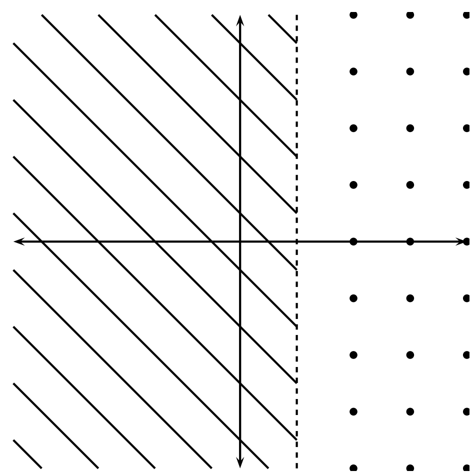

**Figure 5.** The 0-orbits for a family of partially defined vector fields

and so it is evident that $\Phi(x) \in \mathrm{Orb}_T(x_0, \mathscr{X})$. Thus

$$\{\Phi(x) \mid \Phi \in \mathrm{Diff}_0(\mathscr{X})\} \subseteq \mathrm{Orb}_T(x_0, \mathscr{X}).$$

Conversely, let $y \in \mathrm{Orb}_T(x_0, \mathscr{X})$ and write $y = \Phi^{\boldsymbol{Y}}_{\boldsymbol{s}}(x_0)$ for $\boldsymbol{Y} \in \mathscr{X}^m$ and $\boldsymbol{s} \in \mathbb{R}^m$ with $\sum_{l=1}^m s_l = T$. If we write $\hat{\boldsymbol{t}} = (t_k, \dots, t_1)$, then

$$y = \Phi^{\boldsymbol{Y}}_{\boldsymbol{s}}(x_0) = \Phi^{\boldsymbol{Y}}_{\boldsymbol{s}} \circ \Phi^{\boldsymbol{X}}_{-\hat{\boldsymbol{t}}}(x),$$

from which we deduce that

$$\mathrm{Orb}_T(x_0, \mathscr{X}) \subseteq \{\Phi(x) \mid \Phi \in \mathrm{Diff}_0(\mathscr{X})\},$$

as desired. ■

The following corollary gives a particular simple form for the fixed-time orbit in some cases.

**7.12 Corollary: (Characterisation of $\mathbf{Orb}_{\boldsymbol{T}}(\boldsymbol{x_0}, \mathscr{X})$ in nice cases)** *Let $r \in \{\infty, \omega\}$, let $\mathsf{M}$ be a $C^r$-manifold, and let $\mathscr{X}$ be a family of partially defined vector fields of class $C^r$. Let $X \in \mathscr{X}$, $x_0 \in \mathsf{M}$, and $T \in \mathbb{R}_{>0}$ satisfy $x \in \mathcal{U}(X, T)$. Then*

$$\mathrm{Orb}_T(x_0, \mathscr{X}) = \{\Phi \circ \Phi^X_T(x_0) \mid \Phi \in \mathrm{Diff}_0(\mathscr{X})\}.$$

**7.4. The Orbit Theorem.** Before we state the Orbit Theorem we need some terminology and notation.

Let $r \in \{\infty, \omega\}$ and let $\mathsf{M}$ be a $C^r$-manifold. For a $C^r$-local diffeomorphism $(\Phi, \mathcal{U})$ and for a partially defined vector field $(X, \mathcal{V})$, denote by $\Phi_* X$ the partially defined vector field with domain $\Phi(\mathcal{U} \cap \mathcal{V})$ and defined by $\Phi_* X(x) = T_{\Phi^{-1}(x)}\Phi \circ X \circ \Phi^{-1}(x)$.

With this terminology, we are ready to state the Orbit Theorem.

**7.13 Theorem: (Orbit Theorem)** *Let $r \in \{\infty, \omega\}$, let $\mathsf{M}$ be a $C^r$-manifold, and let $\mathscr{X}$ be an everywhere defined family of partially defined vector fields of class $C^r$. Then, for each $x_0 \in \mathsf{M}$,*

*(i)* $\mathrm{Orb}(x_0, \mathscr{X})$ *is a connected immersed $C^r$-submanifold of $\mathsf{M}$ and*

*(ii) for each $x \in \mathrm{Orb}(x_0, \mathscr{X})$,*

$$\mathsf{T}_x \mathrm{Orb}(x_0, \mathscr{X}) = \mathrm{span}_{\mathbb{R}}(\Phi_* X(x) \mid \; \Phi \in \mathrm{Diff}(\mathscr{X}),\ X \in \mathscr{X}).$$

*Moreover, $\mathsf{M}$ is the disjoint union of the set of orbits.*

Proof: Let us denote a family of partially defined vector fields by

$$\mathscr{O}(\mathscr{X}) = \{\Phi_* X \mid \Phi \in \mathrm{Diff}(\mathscr{X}),\ X \in \mathscr{X}\}.$$

We also define a distribution $\mathsf{O}$ by

$$\mathsf{O}_x = \mathrm{span}_{\mathbb{R}}(\Phi_* X(x) | \; \Phi \in \mathrm{Diff}(\mathscr{X}),\ X \in \mathscr{X}).$$

The following lemma gives a useful property of these subspaces.

**1 Lemma:** *For $x_0 \in \mathsf{M}$ and for $x \in \mathrm{Orb}(x_0, \mathscr{X})$,* $\dim(\mathsf{O}_x) = \dim(\mathsf{O}_{x_0})$.

Proof: If $x \in \mathrm{Orb}(x_0, \mathscr{X})$ then there exists $\Psi \in \mathrm{Diff}(\mathscr{X})$ such that $x = \Psi(x_0)$. Let $\Phi \in \mathrm{Diff}(\mathscr{X})$ be such that $x_0$ is in the image of $\Phi$ and let $X \in \mathscr{X}$ so that $\Phi_* X(x_0) \in \mathsf{O}_{x_0}$. Then

$$\begin{aligned} T_{x_0}\Psi(\Phi_* X(x_0)) &= T_{x_0}\Psi \circ T_{\Phi^{-1}(x_0)}\Phi \circ X \circ \Phi^{-1}(x_0) \\ &= T_{x_0}\Psi \circ T_{\Phi^{-1}(x_0)}\Phi \circ X \circ \Phi^{-1} \circ \Psi^{-1} \circ \Psi(x_0) \\ &= T_{(\Psi\circ\Phi)^{-1}(x)}(\Psi \circ \Phi) \circ X \circ (\Psi \circ \Phi)^{-1}(x) \\ &= (\Psi \circ \Phi)_* X(x) \in \mathsf{O}_x \end{aligned}$$

since $\Psi \circ \Phi \in \mathrm{Diff}(\mathscr{X})$. Since $T_{x_0}\Psi$ is an isomorphism, this shows that $\dim(\mathsf{O}_x) \geq \dim(\mathsf{O}_{x_0})$. This argument can be reversed to give the opposite inequality. ▼

For $x \in \mathsf{M}$ suppose that $\dim(\mathsf{O}_x) = m_x$. Let $\mathscr{Y}_x = (Y_1, \dots, Y_{m_x}) \subseteq \mathscr{O}(\mathscr{X})$ be such that $(Y_1(x), \dots, Y_{m_x}(x))$ is a basis for $\mathsf{O}_x$. Define $\phi_x \colon \mathbb{R}^{m_x} \to \mathsf{M}$ by

$$\phi_x(t_1, \dots, t_{m_x}) = \Phi^{Y_{m_x}}_{t_{m_x}} \circ \dots \circ \Phi^{Y_1}_{t_1}(x).$$

Since $\frac{\partial \phi_x}{\partial t_j}$ is equal to $Y_j(x)$ when evaluated to $t_1 = \dots = t_{m_x} = 0$, it follows that $\phi_x$ is an embedding in a neighbourhood of $\mathbf{0} \in \mathbb{R}^{m_x}$; let us denote the image of this neighbourhood under $\phi_x$ by $\mathcal{U}(\mathscr{Y}_x)$. Thus $\mathcal{U}(\mathscr{Y}_x)$ is a $C^r$-submanifold.

**2 Lemma:** $\mathcal{U}(\mathscr{Y}_x) \subseteq \mathrm{Orb}(x, \mathscr{X})$.

Proof: By definition we can write $Y_j = \Phi_{j*} X_j$ for $\Phi_j \in \mathrm{Diff}(\mathscr{X})$ and $X_j \in \mathscr{X}$, $j \in \{1, \dots, m_x\}$. We recall from [Abraham, Marsden, and Ratiu 1988, Proposition 4.2.4] the formula $\Phi^Y_t = \Phi \circ \Phi^X_t \circ \Phi^{-1}$ which holds if $Y = \Phi_* X$ for vector fields $X$ and $Y$ and a diffeomorphism $\Phi$. Using this formula we have, for $\boldsymbol{t} \in \phi_x^{-1}(\mathcal{U}(\mathscr{Y}_x))$,

$$\phi_x(\boldsymbol{t}) = \Phi^{Y_{m_x}}_{t_{m_x}} \circ \dots \circ \Phi^{Y_1}_{t_1}(x) = \Phi_{m_x} \circ \Phi^{X_{m_x}}_{t_{m_x}} \circ \Phi^{-1}_{m_x} \circ \dots \circ \Phi_1 \circ \Phi^{X_1}_{t_1} \circ \Phi^{-1}_1(x),$$

showing that $\mathcal{U}(\mathscr{Y}_x) \subseteq \mathrm{Orb}(x, \mathscr{X})$, as desired, since all mappings in the above composition are in $\mathrm{Diff}(\mathscr{X})$. ▼

**3 Lemma:** $\mathsf{T}_y\mathcal{U}(\mathscr{Y}_x) = \mathsf{O}_y$ *for every* $y \in \mathcal{U}(\mathscr{Y}_x)$.

Proof: Let $j \in \{1, \dots, m_x\}$ and let

$$\Phi_{\boldsymbol{t}} = \Phi^{Y_{m_x}}_{t_{m_x}} \circ \dots \circ \Phi^{Y_{j+1}}_{t_{j+1}}.$$

Then

$$\begin{aligned}\frac{\partial \phi_x}{\partial t_j} &= \frac{\partial}{\partial t_j}\Phi^{Y_{m_x}}_{t_{m_x}} \circ \dots \circ \Phi^{Y_1}_{t_1}(x)\\ &= \frac{\partial}{\partial t_j}\Phi_{\boldsymbol{t}} \circ \Phi^{Y_j}_{t_j} \circ \Phi^{-1}_{\boldsymbol{t}} \circ \Phi^{Y_{m_x}}_{t_{m_x}} \circ \dots \circ \Phi^{Y_1}_{t_1}(x)\\ &= T\Phi_{\boldsymbol{t}} \circ Y_j \circ \Phi^{-1}_{\boldsymbol{t}} \circ \phi_x(\boldsymbol{t}) = \Phi_{\boldsymbol{t}*}Y_j(\phi_x(\boldsymbol{t})) \in \mathsf{O}_{\phi_x(\boldsymbol{t})}.\end{aligned}$$

Since this holds for every $j \in \{1, \dots, m_x\}$, it follows that $\mathrm{image}(T_{\boldsymbol{t}}\phi_x) \subseteq \mathsf{O}_{\phi_x(\boldsymbol{t})}$. By Lemma 1 we have

$$\dim(\mathsf{O}_{\phi_x(\boldsymbol{t})}) = \dim(\mathsf{O}_x) = \dim(\mathrm{image}(T_{\boldsymbol{t}}\phi_x)).$$

Therefore,

$$\mathsf{O}_{\phi_x(\boldsymbol{t})} = \mathrm{image}(T_{\boldsymbol{t}}\phi_x) = \mathsf{T}_{\phi_x(\boldsymbol{t})}\mathcal{U}(\mathscr{Y}_x),$$

as desired. ▼

**4 Lemma:** *The subsets* $(\mathcal{U}(\mathscr{Y}_x))_{x\in\mathsf{M}}$ *for* $\mathscr{Y}_x$ *as defined above, form a basis for a topology on* $\mathsf{M}$.

Proof: By, e.g., Theorem 5.3 of [Willard 1970], it suffices to show that for any pair $\mathcal{U}(\mathscr{Y}_{x_1})$ and $\mathcal{U}(\mathscr{Y}_{x_2})$ of such subsets there exists $\mathcal{U}(\mathscr{Y}_x)$ such that

$$\mathcal{U}(\mathscr{Y}_x) \subseteq \mathcal{U}(\mathscr{Y}_{x_1}) \cap \mathcal{U}(\mathscr{Y}_{x_2}).$$

Let $x \in \mathcal{U}(\mathscr{Y}_{x_1}) \cap \mathcal{U}(\mathscr{Y}_{x_2})$ and let $Y_1, \dots, Y_{m_x} \in \mathscr{O}(\mathscr{X})$ be such that

$$\mathsf{O}_x = \mathrm{span}_{\mathbb{R}}(Y_1(x), \dots, Y_{m_x}(x)).$$

Let $\phi_x\colon \mathbb{R}^{m_x} \to \mathsf{M}$ be the map defined above. By Lemma 3 it follows that

$$Y_1(y), \dots, Y_{m_x}(y) \in \mathsf{T}_y\mathcal{U}(\mathscr{Y}_{x_1}), \quad Y_1(y), \dots, Y_{m_x}(y) \in \mathsf{T}_y\mathcal{U}(\mathscr{Y}_{x_2})$$

for every $y$ in a neighbourhood of $x$. Therefore, the integral curves of the vector fields $Y_1, \dots, Y_{m_x}$ with initial conditions in $\mathcal{U}(\mathscr{Y}_{x_1})$ (resp. $\mathcal{U}(\mathscr{Y}_{x_2})$) nearby $x$ remain in $\mathcal{U}(\mathscr{Y}_{x_1})$ (resp. $\mathcal{U}(\mathscr{Y}_{x_2})$). Therefore, concatenations of these integral curves nearby $x$ will also remain in $\mathcal{U}(\mathscr{Y}_{x_1})$ (resp. $\mathcal{U}(\mathscr{Y}_{x_2})$). In short, for $t_1, \dots, t_{m_x}$ sufficiently near zero,

$$\Phi^{Y_{m_k}}_{t_{m_k}} \circ \dots \circ \Phi^{Y_1}_{t_1}(x) \in \mathcal{U}(\mathscr{Y}_{x_1}) \quad (\text{resp. } \Phi^{Y_{m_k}}_{t_{m_k}} \circ \dots \circ \Phi^{Y_1}_{t_1}(x) \in \mathcal{U}(\mathscr{Y}_{x_2})).$$

Thus, by restricting $\phi_x$ to a small enough neighbourhood $\mathcal{N}$ of $\boldsymbol{0}$, if we define $\mathcal{U}(\mathscr{Y}_x) = \phi_x(\mathcal{N})$, we have

$$\mathcal{U}(\mathscr{Y}_x) \subseteq \mathcal{U}(\mathscr{Y}_{x_1}) \cap \mathcal{U}(\mathscr{Y}_{x_2}).$$ ▼

Let us call the topology on $\mathsf{M}$ generated by the sets $(\mathcal{U}(\mathscr{Y}_x))_{x\in\mathsf{M}}$ the ***orbit topology***.

**5 Lemma:** *In the orbit topology, the orbits are connected, open, and closed.*

Proof: Let $X \in \mathscr{X}$. Since the integral curve $t \mapsto \Phi_t^X(x)$ is a continuous curve of $\mathsf{M}$ and since it is tangent to $\mathcal{U}(\mathscr{Y}_x)$, it follows that the curve is continuous in the relative topology on $\mathcal{U}(\mathscr{Y}_x)$. Since $\mathcal{U}(\mathscr{Y}_x)$ is a submanifold, the relative topology is the same as the topology induced by the immersion $\phi_x$. Thus the integral curve $t \mapsto \Phi_t^X(x)$ is continuous in the orbit topology. Therefore, by definition of orbits, orbits are path connected and so connected.

If $x \in \mathrm{Orb}(x_0, \mathscr{X})$ then every set $\mathcal{U}(\mathscr{Y}_x)$ is a subset of $\mathrm{Orb}(x_0, \mathscr{X})$. Since $\mathcal{U}(\mathscr{Y}_x)$ is open, it follows that $\mathrm{Orb}(x_0, \mathscr{X})$ is open.

Note that $\mathsf{M}$ is a disjoint union of orbits. Therefore, the complement of an orbit is a union of orbits. Thus the complement of an orbit is a union of open sets and so open. Thus an orbit is closed. ▼

This last lemma shows that the orbits are the connected components of $\mathsf{M}$ in the orbit topology. This, in particular, gives the final assertion of the theorem.

**6 Lemma:** *For* $x_0 \in \mathsf{M}$,

$$\mathrm{Orb}(x_0, \mathscr{X}) = \bigcup_{x \in \mathrm{Orb}(x_0, \mathscr{X})} \mathcal{U}(\mathscr{Y}_x),$$

*the union being over the neighbourhoods* $\mathcal{U}(\mathscr{Y}_x)$ *constructed above.*

Proof: It is trivial that $\mathrm{Orb}(x_0, \mathscr{X}) \subseteq \bigcup_{x \in \mathrm{Orb}(x_0, \mathscr{X})} \mathcal{U}(\mathscr{Y}_x)$. The converse inclusion follows from Lemma 2. ▼

Since each of the sets $\mathcal{U}(\mathscr{Y}_x)$ is diffeomorphic to an open subset of $\mathbb{R}^{m_x}$ by $\phi_x^{-1}$, it follows that $(\mathcal{U}(\mathscr{Y}_x), \phi_x^{-1})$ is a chart for $\mathrm{Orb}(x_0, \mathscr{X})$ for every $x \in \mathrm{Orb}(x_0, \mathscr{X})$. The overlap map between intersection charts $(\mathcal{U}(\mathscr{Y}_{x_1}), \phi_{x_1}^{-1})$ and $(\mathcal{U}(\mathscr{Y}_{x_2}), \phi_{x_2}^{-1})$ are obtained by concatenations of flows of vector fields from $\mathscr{X}$, and so are diffeomorphisms. This shows that $\mathrm{Orb}(x_0, \mathscr{X})$ is an immersed submanifold as in (i). Assertion (ii) follows from the Lemma 3 and the definition of the differentiable structure on the orbits. ■

**7.14 Remark: (The orbit topology)** In the proof of the Orbit Theorem we prescribed a topology on $\mathsf{M}$ that we will, on occasion, make reference to. This topology is called the ***orbit topology***, and is defined as follows, using the notation from the proof of the Orbit Theorem. For a family $\mathscr{X}$ of partially defined $C^r$-vector fields, $r \in \{\infty, \omega\}$, and for $x \in \mathsf{M}$, let $\mathscr{Y}_x = (Y_1, \dots, Y_{m_x})$ be vector fields from the family

$$\mathscr{O}(\mathscr{X}) = \{\Phi_* X \mid \Phi \in \mathrm{Diff}(\mathscr{X}),\ X \in \mathscr{X}\}$$

for which $(Y_1(x), \dots, Y_{m_x}(x))$ are a basis for $\mathsf{T}_x \mathrm{Orb}(x, \mathscr{X})$. Define $\phi_x \colon \mathbb{R}^{m_x} \to \mathsf{M}$ by

$$\phi_x(t_1, \dots, t_{m_x}) = \Phi_{t_{m_x}}^{Y_{m_x}} \circ \cdots \circ \Phi_{t_1}^{Y_1}(x),$$

and note that this map, restricted to a neighbourhood of $\mathbf{0} \in \mathbb{R}^{m_x}$, is an embedding. The image of this neighbourhood under $\phi_x$ is denoted by $\mathcal{U}(\mathscr{Y}_x)$. The sets $\mathcal{U}(\mathscr{Y}_x)$ form a basis for a topology, and this topology is the orbit topology. In the proof of the Orbit Theorem it was shown that the orbits are the connected components of $\mathsf{M}$ in the orbit topology.

Sometimes the following characterisation of the orbit topology is useful. Let $x \in \mathsf{M}$, let $k \in \mathbb{Z}_{>0}$, let $\boldsymbol{X} = (X_1, \dots, X_k) \subseteq \mathscr{X}$, and let $\mathcal{U} \subseteq \mathbb{R}^k$ be a neighbourhood of $\mathbf{0}$ be such that the map

$$\mathcal{U} \ni \boldsymbol{t} \mapsto \Phi_{\boldsymbol{t}}^{\boldsymbol{X}}(x) \in \mathsf{M}$$

is defined. One may then define the orbit topology as the final topology induced by the above family of mappings, i.e., the finest topology for which all of these maps are continuous. •

It is interesting to consider whether there are stronger conclusions that can be drawn from the Orbit Theorem when the vector fields are globally defined. In the smooth case when the partially defined vector fields are bounded, by Proposition 7.3, there is no extra structure present when the vector fields from $\mathscr{X}$ are globally defined. In the analytic case, however, if the vector fields are globally defined, there are additional conclusions that can be drawn. To state this clearly, the following definition is convenient.

**7.15 Definition: (Regular and singular orbits)** Let $r \in \{\infty, \omega\}$, let $\mathsf{M}$ be a $C^r$-manifold, and let $\mathscr{X}$ be a family of partially defined vector fields of class $C^r$. An orbit $\mathsf{O} \subseteq \mathsf{M}$ for $\mathscr{X}$ is

(i) ***regular*** if, for each $x_0 \in \mathsf{O}$ there exists a neighbourhood $\mathcal{U}$ of $x_0$ such that $\dim(\mathrm{Orb}(x, \mathscr{X})) = \dim(\mathsf{O})$ for each $x \in \mathcal{U}$ and is

(ii) ***singular*** if it is not regular. •

With this terminology, we have the following result, first for the smooth case.

**7.16 Proposition: (Regular orbits for families of smooth vector fields)** *Let* $\mathsf{M}$ *be a smooth manifold and let* $\mathscr{X}$ *be a family of smooth partially defined vector fields. Then the union of the regular orbits for* $\mathscr{X}$ *is an open dense subset of* $\mathsf{M}$.

Proof: By the Orbit Theorem, the distribution whose subspace at $x$ is $\mathsf{T}_x \mathrm{Orb}(x, \mathscr{X})$ is the distribution associated to a family of smooth partially defined vector fields. Thus, by Proposition 7.3, this is a smooth distribution. The result follows from Proposition 3.6. ■

**7.17 Proposition: (Regular orbits for families of analytic vector fields)** *Let* $\mathsf{M}$ *be a real analytic manifold and let* $\mathscr{X} \subseteq \Gamma^\omega(\mathsf{TM})$ *be a family of real analytic vector fields. Then the following statements hold:*

*(i) the set of singular orbits for* $\mathscr{X}$ *is a locally analytic subset of* $\mathsf{M}$*;*

*(ii) if* $\mathsf{M}$ *is connected, then all regular orbits of* $\mathscr{X}$ *have the same dimension.*

Proof: By Theorem 7.21 below it follows that the distribution whose subspace at $x$ is $\mathsf{T}_x \mathrm{Orb}(x, \mathscr{X})$ is analytic. The result then follows from Proposition 3.7. ■

Note that in the analytic case, we really do require that the vector fields be globally defined; Example 7.7–4 suffices to illustrate this.

The Orbit Theorem gives us an insightful description of the tangent spaces to $\mathscr{X}$-orbits. However, computationally the description is not the most useful since it requires that we know something about the group $\mathrm{Diff}(\mathscr{X})$. One can wonder whether there is a simpler "infinitesimal" description. If one has some intuition about things analytic, one might imagine that such an infinitesimal description is possible for the analytic version of the

Orbit Theorem. We shall show that this is true. We begin by describing a subspace of the tangent spaces to the $\mathscr{X}$-orbits.

The proof of the next theorem is adapted from that of Jakubczyk [2002, Proposition 4.15].

**7.18 Theorem: (A subspace of the tangent space of an orbit)** *Let $r \in \{\infty, \omega\}$, let $\mathsf{M}$ be a $C^r$-manifold, and let $\mathscr{X}$ be a family of $C^r$-partially defined vector fields on $\mathsf{M}$. Then*

$$\mathsf{L}^{(\infty)}(\mathscr{X})_{x_0} \subseteq \mathsf{T}_{x_0} \operatorname{Orb}(x_0, \mathscr{X})$$

*for every $x_0 \in \mathsf{M}$.*

Proof: We abbreviate a local diffeomorphism $(\Phi, \mathcal{U})$ by $\Phi$ and a partially defined vector field $(X, \mathcal{U})$ by $X$.

Let $x_0 \in \mathsf{M}$. By Proposition 6.3 it suffices to show that

$$[X_k, [X_{k-1}, \dots, [X_2, X_1] \cdots]](x_0) \in \mathsf{T}_{x_0} \operatorname{Orb}(x_0, \mathscr{X}) \tag{7.4}$$

for any vector fields $X_1, \dots, X_k \in \mathscr{X}$ whose domains contain $x_0$. We prove this using some notation and three lemmata. First the notation. For local diffeomorphisms $\Phi$ and $\Psi$ of $\mathsf{M}$ denote

$$[\Phi, \Psi] = \Phi^{-1} \circ \Psi^{-1} \circ \Phi \circ \Psi.$$

With this notation we have the following lemma.

**1 Lemma:** *Let $X_1, \dots, X_k \in \Gamma^r(\mathsf{TM})$ and recursively define*

$$\begin{aligned}
\Psi_1(t_1) &= \Phi^{X_1}_{t_1}, \\
\Psi_2(t_1, t_2) &= [\Phi^{X_1}_{t_1}, \Phi^{X_2}_{t_2}], \\
\Psi_3(t_1, t_2, t_3) &= [[\Phi^{X_1}_{t_1}, \Phi^{X_2}_{t_2}], \Phi^{X_3}_{t_3}], \\
&\vdots \\
\Psi_k(t_1, t_2, \dots, t_k) &= [\cdots [\Phi^{X_1}_{t_1}, \Phi^{X_2}_{t_2}], \dots, \Phi^{X_k}_{t_k}],
\end{aligned}$$

*for $t_1, \dots, t_k \in \mathbb{R}$ such that all flows are defined. Then*

$$\frac{\partial}{\partial t_1}\Big|_{t_1=0} \Psi_k(t_1, \dots, t_k)(x_0) = -X_1(x_0) + \frac{\partial}{\partial t_1}\Big|_{t_1=0} \Phi^{X_k}_{t_k} \circ \Psi_{k-1}(t_1, \dots, t_{k-1}) \circ \Phi^{X_2}_{t_2}(x_0).$$

Proof: It is easy to see by induction that if any of the numbers $t_1, \dots, t_k$ are zero, then $\Psi_k = \mathrm{id}_{\mathsf{M}}$. We shall use this fact frequently.

First note that differentiation of the relation

$$\Psi_{k-1}(t_1, \dots, t_{k-1}) \circ \Psi_{k-1}(t_1, \dots, t_{k-1})^{-1}(x) = x$$

gives

$$\begin{aligned}
&\frac{\partial}{\partial t_1}(\Psi_{k-1}(t_1, \dots, t_{k-1})^{-1}(x)) \\
&\qquad = -T\Psi_{k-1}(t_1, \dots, t_{k-1})^{-1} \cdot \Big(\Big(\frac{\partial}{\partial t_1}\Psi_{k-1}(t_1, \dots, t_{k-1})\Big) \circ \Psi_{k-1}(t_1, \dots, t_{k-1})^{-1}(x)\Big).
\end{aligned}$$

Evaluating at $t_1 = 0$ and using the fact stated at the beginning of the proof then gives

$$\frac{\partial}{\partial t_1}\Big|_{t_1=0}(\Psi_{k-1}(t_1,\dots,t_{k-1})^{-1}(x)) = -X_1(x).$$

Using this fact along with the statement made at the beginning of the lemma, we calculate

$$\begin{aligned}\frac{\partial}{\partial t_1}\Big|_{t_1=0}\Psi_k(t_1,\dots,t_k)(x_0) &= \frac{\partial}{\partial t_1}\Big|_{t_1=0}[\Psi_{k-1}(t_1,\dots,t_{k-1}),\Phi^{X_k}_{t_k}](x_0)\\ &= \frac{\partial}{\partial t_1}\Big|_{t_1=0}\Psi_{k-1}(t_1,\dots,t_{k-1})^{-1}\circ\Phi^{X_2}_{-t_2}\circ\Psi_{k-1}(t_1,\dots,t_{k-1})\circ\Phi^{X_k}_{t_k}(x_0)\\ &= -X_1(x_0) + \frac{\partial}{\partial t_1}\Big|_{t_1=0}\Phi^{X_k}_{t_k}\circ\Psi_{k-1}(t_1,\dots,t_{k-1})\circ\Phi^{X_2}_{t_2}(x_0),\end{aligned}$$

giving the result. ▼

We also recall the definition of the pull-back of a vector field $X$ by a diffeomorphism $\Phi$: $\Phi^*X = T\Phi^{-1}\circ X\circ\Phi$. With this notation we have the following lemma.

**2 Lemma:** *With the notation from Lemma 1,*

$$\frac{\partial}{\partial t_k}\cdots\frac{\partial}{\partial t_1}\Big|_{t_1=\cdots=t_k=0}\Psi_k(t_1,\dots,t_k)(x_0) = \frac{\partial}{\partial t_k}\cdots\frac{\partial}{\partial t_2}\Big|_{t_2=\cdots=t_k=0}((\Phi^{X_k}_{t_k})^*\cdots(\Phi^{X_2}_{t_2})^*X_1)(x_0).$$

Proof: We prove this by induction on $k$. For $k = 2$ we use Lemma 1 to determine that

$$\begin{aligned}\frac{\partial}{\partial t_2}\frac{\partial}{\partial t_1}\Big|_{t_1=t_2=0}\Psi_2(t_1,t_2)(x_0) &= \frac{\partial}{\partial t_2}\frac{\partial}{\partial t_1}\Big|_{t_1=t_2=0}\Phi^{X_2}_{-t_2}\circ\Phi^{X_1}_{t_1}\circ\Phi^{X_2}_{t_2}(x_0)\\ &= \frac{\partial}{\partial t_2}\Big|_{t_2=0}T\Phi^{X_2}_{-t_2}\circ X_1(\Phi^{X_2}_{t_2}(x_0)) = \frac{\partial}{\partial t_2}\Big|_{t_2=0}((\Phi^{X_s}_{t_2})^*X_1)(x_0),\end{aligned}$$

giving the lemma for $k = 2$.

Now suppose the lemma holds for $k \in \{1,\dots,m-1\}$. An application of Lemma 1 and the induction hypothesis gives

$$\begin{aligned}&\frac{\partial}{\partial t_m}\cdots\frac{\partial}{\partial t_1}\Big|_{t_1=\cdots=t_m=0}\Psi_m(t_1,\dots,t_m)(x_0)\\ &\quad= \frac{\partial}{\partial t_m}\cdots\frac{\partial}{\partial t_1}\Big|_{t_1=\cdots=t_m=0}\Phi^{X_m}_{-t_m}\circ\Psi_{m-1}(t_1,\dots,t_{m-1})\circ\Phi^{X_m}_{t_m}(x_0)\\ &\quad= \frac{\partial}{\partial t_m}\cdots\frac{\partial}{\partial t_2}\Big|_{t_2=\cdots=t_m=0}T\Phi^{X_m}_{-t_m}\cdot\Big(\frac{\partial}{\partial t_1}\Big|_{t_1=0}\Psi_{m-1}(t_1,\dots,t_{m-1})\circ\Phi^{X_m}_{t_m}(x_0)\Big)\\ &\quad= \frac{\partial}{\partial t_m}\cdots\frac{\partial}{\partial t_2}\Big|_{t_2=\cdots=t_m=0}(\Phi^{X_m}_{t_m})^*\frac{\partial}{\partial t_1}\Big|_{t_1=0}\Psi_{m-1}(t_1,\dots,t_{m-1})\\ &\quad= \frac{\partial}{\partial t_m}\Big|_{t_m=0}(\Phi^{X_m}_{t_m})^*\frac{\partial}{\partial t_{m-1}}\cdots\frac{\partial}{\partial t_2}\frac{\partial}{\partial t_1}\Big|_{t_1=\cdots=t_{m-1}=0}\Psi_{m-1}(t_1,\dots,t_{m-1})\\ &\quad= \frac{\partial}{\partial t_m}\Big|_{t_m=0}(\Phi^{X_m}_{t_m})^*\frac{\partial}{\partial t_{m-1}}\cdots\frac{\partial}{\partial t_2}\Big|_{t_2=\cdots=t_{m-1}=0}((\Phi^{X_{m-1}}_{t_{m-1}})^*\cdots(\Phi^{X_2}_{t_2})^*X_1)(x_0)\\ &\quad= \frac{\partial}{\partial t_m}\cdots\frac{\partial}{\partial t_2}\Big|_{t_2=\cdots=t_m=0}(\Phi^{X_m}_{t_m})^*\cdots(\Phi^{X_2}_{t_2})^*X_1)(x_0),\end{aligned}$$

which is the result. (Note that our freely swapping partial derivatives with pull-backs is justified since we are differentiating the pull-back with respect to its argument, and the pull-back is linear in its argument.) ▼

Now we prove the key fact.

**3 Lemma:** *We use the notation from Lemma 2. For $x_0 \in \mathsf{M}$, if we define $\Psi_{x_0}(s) = \Psi_k(s,\dots,s)(x_0)$ for all $s \in \mathbb{R}$ such that the expression makes sense, then*

$$\frac{\mathrm{d}^j}{\mathrm{d}s^j}\Big|_{s=0} \Psi_{x_0}(s) = 0, \qquad j \in \{0,1,\dots,k-1\},$$

*and*

$$\frac{\mathrm{d}^k}{\mathrm{d}s^k}\Big|_{s=0} \Psi_{x_0}(s) = k![X_k,\dots,[X_2,X_1]\cdots](x_0).$$

Proof: Now let $j \in \{0,1,\dots,k-1\}$. By the Chain Rule for high-order derivatives, [Abraham, Marsden, and Ratiu 1988, Supplement 2.4A],

$$\frac{\mathrm{d}^j}{\mathrm{d}s^j}\Big|_{s^j=0} \Psi_{x_0}(s) = \sum_{\substack{j_1,\dots,j_k\in\{0,1,\dots,j\}\\ j_1+\dots+j_k=j}} \frac{j!}{j_1!\cdots j_k!}\frac{\partial^{j_1}}{\partial t_1^{j_1}}\cdots\frac{\partial^{j_k}}{\partial t_k^{j_k}}\Big|_{t_1=\dots=t_k=0}\Phi_k(t_1,\dots,t_k)(x_0). \quad (7.5)$$

Note that each term in the above will have $j_a = 0$ for some $a \in \{1,\dots,k\}$. The partial derivatives in (7.5), when evaluated at $t_1 = \dots = t_k = 0$, will then necessarily be taken with $t_a = 0$ for some $a \in \{1,\dots,k\}$. By our comment at the beginning of the proof of Lemma 1, it follows that all such partial derivatives will be zero.

By the same reasoning, in the expression

$$\frac{\mathrm{d}^k}{\mathrm{d}s^k}\Big|_{s^k=0} \Psi_{x_0}(s) = \sum_{\substack{j_1,\dots,j_k\in\{0,1,\dots,k\}\\ j_1+\dots+j_k=k}} \frac{k!}{j_1!\cdots j_k!}\frac{\partial^{j_1}}{\partial t_1^{j_1}}\cdots\frac{\partial^{j_k}}{\partial t_k^{j_k}}\Big|_{t_1=\dots=t_k=0}\Phi_k(t_1,\dots,t_k)(x_0)$$

for the $k$th derivative, the only nonzero term in the sum occurs when $j_1 = \dots = j_k = 1$, since otherwise at least one of the numbers $j_1,\dots,j_k$ will be zero. That is to say,

$$\frac{\mathrm{d}^k}{\mathrm{d}s^k}\Big|_{s^k=0} \Psi_{x_0}(s) = k!\frac{\partial}{\partial t_1}\cdots\frac{\partial}{\partial t_k}\Big|_{t_1=\dots=t_k=0}\Phi_k(t_1,\dots,t_k)(x_0).$$

Let us now turn to the proof of the fact that this expression is the iterated Lie bracket in the statement of the lemma.

We prove this by showing that, for any $j \in \{2,\dots,k\}$,

$$\begin{aligned}\frac{\partial}{\partial t_j}\Big|_{t_j=0}((\Phi^{X_k}_{t_k})^*\cdots(\Phi^{X_j}_{t_j})^*[X_{j-1},\dots,[X_2,X_1]\cdots])(x_0)&\\ = ((\Phi^{X_k}_{t_k})^*\cdots(\Phi^{X_{j+1}}_{t_{j+1}})^*[X_j,\dots,[X_2,X_1]\cdots])(x_0).& \quad (7.6)\end{aligned}$$

This we prove by induction on $j$. For $j = 2$ we have

$$\begin{aligned}\frac{\partial}{\partial t_2}\Big|_{t_2=0}((\Phi^{X_k}_{t_k})^*\cdots(\Phi^{X_2}_{t_2})^*X_1)(x_0) &= ((\Phi^{X_k}_{t_k})^*\cdots(\Phi^{X_3}_{t_3})^*\frac{\partial}{\partial t_2}\Big|_{t_2=0}\Phi^{X_2}_{t_2}X_1)(x_0)\\ &= ((\Phi^{X_k}_{t_k})^*\cdots(\Phi^{X_3}_{t_3})^*[X_2,X_1])(x_0),\end{aligned}$$

using the well-known characterisation of the Lie bracket by

$$[X,Y](x) = \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0}(\Phi^X_t)^*Y(x)$$

[see Abraham, Marsden, and Ratiu 1988, Theorem 4.2.19]. Now suppose that (7.6) holds for $j \in \{2, \dots, m-1\}$. Then we use the induction hypotheses to get

$$\begin{aligned}
&\frac{\partial}{\partial t_m}\Big|_{t_m=0}((\Phi^{X_k}_{t_k})^* \cdots (\Phi^{X_m}_{t_m})^*[X_{m-1}, \dots, [X_2, X_1]\cdots])(x_0) \\
&\qquad = ((\Phi^{X_k}_{t_k})^* \cdots (\Phi^{X_{m+1}}_{t_{m+1}})^* \frac{\partial}{\partial t_m}\Big|_{t_m=0}\Phi^{X_m}_{t_m}[X_{m-1}, \dots, [X_2, X_1]\cdots])(x_0) \\
&\qquad = ((\Phi^{X_k}_{t_k})^* \cdots (\Phi^{X_{m+1}}_{t_{m+1}})^*[X_m, \dots, [X_2, X_1]\cdots])(x_0),
\end{aligned}$$

giving (7.6).

Now we use Lemma 2 and recursively apply (7.6) to give

$$\begin{aligned}
\frac{\partial}{\partial t_k}\cdots\frac{\partial}{\partial t_1}\Big|_{t_1=\cdots=t_k=0}\Psi_k(t_1, \dots, t_k)(x_0) &= \frac{\partial}{\partial t_k}\cdots\frac{\partial}{\partial t_2}\Big|_{t_2=\cdots=t_k=0}((\Phi^{X_k}_{t_k})^* \cdots (\Phi^{X_2}_{t_2})^* X_1)(x_0) \\
&= [X_k, [X_{k-1}, \dots, [X_2, X_1]\cdots]](x_0)
\end{aligned}$$

as desired. ▼

We can now complete the proof of the theorem by verifying (7.4). Recalling the distribution $\mathsf{O}$ from the proof of the Orbit Theorem, the vector fields $X_1, \dots, X_k$ are $\mathsf{O}$-valued and so tangent to $\mathrm{Orb}(x_0, \mathscr{X})$. Therefore, with the notation of Lemma 3, $\Psi_{x_0}(s) \in \mathrm{Orb}(x_0, \mathscr{X})$ for every $s \in \mathbb{R}$. Therefore, since the first $k-1$ derivatives of $\Psi_{x_0}$ at $s = 0$ vanish by Lemma 3, the $k$th derivative of $\Psi_{x_0}$ is tangent to $\mathrm{Orb}(x_0, \mathscr{X})$. That is to say, by Lemma 3,

$$\frac{\mathrm{d}^k}{\mathrm{d}s^k}\Big|_{s=0}\Psi_{x_0}(s) = k![X_k, \dots, [X_2, X_1]\cdots](x_0) \in \mathsf{T}_{x_0}\mathrm{Orb}(x_0, \mathscr{X}),$$

showing that (7.4) holds. ■

An immediate consequence of Theorems 7.13 and 7.18 is the following well-known result of Rashevsky [1938] and Chow [1940/1941]. The proof of Chow is given in the Cartan-like framework of differential forms, whereas the proof of Rashevsky is, like our proof, a vector field proof.

**7.19 Corollary: (The Rashevsky–Chow Theorem)** *Let $\mathsf{M}$ be a connected $C^\infty$-manifold and let $\mathscr{X}$ be a family of smooth partially defined vector fields. If $\mathsf{L}^{(\infty)}(\mathscr{X}) = \mathsf{TM}$ then, for $x_1, x_2 \in \mathsf{M}$, there exists $X_1, \dots, X_k \in \mathscr{X}$ and $t_1, \dots, t_k \in \mathbb{R}$ such that*

$$x_2 = \Phi^{X_k}_{t_k} \circ \cdots \circ \Phi^{X_1}_{t_1}(x_1).$$

Proof: Let $x_1 \in \mathsf{M}$. By Theorem 7.18 we have $\mathsf{L}^{(\infty)}(\mathscr{X})_x \subseteq \mathsf{T}_x\mathrm{Orb}(x_1, \mathscr{X})$ for every $x \in \mathrm{Orb}(x_1, \mathscr{X})$ and so

$$\mathsf{L}^{(\infty)}(\mathscr{X})_x \subseteq \mathsf{T}_x\mathrm{Orb}(x_1, \mathscr{X}) \subseteq \mathsf{T}_x\mathsf{M} = \mathsf{L}^{(\infty)}(\mathscr{X})_x,$$

giving $\mathsf{T}_x\mathrm{Orb}(x_1, \mathscr{X}) = \mathsf{T}_x\mathsf{M}$ for every $x \in \mathrm{Orb}(x_1, \mathscr{X})$. Thus $\mathrm{Orb}(x_1, \mathscr{X})$ is an open submanifold of $\mathsf{M}$. Thus, recalling the basis for the orbit topology from the proof of the Theorem 7.13, open subsets of $\mathrm{Orb}(x_1, \mathscr{X})$ in the orbit topology are open subsets in the relative topology on $\mathsf{M}$. Since $\mathsf{M}$ is a disjoint union of its orbits, it is a disjoint union of open sets. Each component in this disjoint union is necessarily closed since its complement is open, being a union of open sets. Thus each orbit is a connected component of $\mathsf{M}$. Since $\mathsf{M}$ is assumed connected it follows that $\mathrm{Orb}(x_1, \mathscr{X}) = \mathsf{M}$, which is the result. ■

The converse of the Rashevsky–Chow Theorem is generally false.

**7.20 Example: (Failure of the converse of the Rashevsky–Chow Theorem)** Recall Example 7.7–2 where $\mathsf{M} = \mathbb{R}^2$ and where $\mathscr{X} = (X_1, X_2)$ is defined by

$$X_1 = \frac{\partial}{\partial x_1}, \quad X_2 = f(x_1)\frac{\partial}{\partial x_2},$$

where

$$f(x) = \begin{cases} \mathrm{e}^{-1/x^2}, & x \in \mathbb{R}_{>0}, \\ 0, & x \in \mathbb{R}_{\leq 0}. \end{cases}$$

In Example 7.7–2 we explicitly showed that $\mathsf{M} = \operatorname{Orb}(\mathbf{0}, \mathscr{X})$. However, one can also directly show that

$$\mathsf{L}^{(\infty)}(\mathscr{X})_{(x_1,x_2)} = \begin{cases} \mathsf{T}_{(x_1,x_2)}\mathbb{R}^2, & x_1 > 0, \\ \operatorname{span}_{\mathbb{R}}(\frac{\partial}{\partial x_1}), & x_1 \leq 0. \end{cases}$$

Thus $\mathsf{L}^{(\infty)}(\mathscr{X}) \subset \mathsf{TM}$. •

**7.5. The finitely generated Orbit Theorem.** Now we turn to characterising situations where the tangent spaces to the orbits are *exactly* the subspaces $\mathsf{L}^{(\infty)}(\mathscr{X})$.

**7.21 Theorem: (The Orbit Theorem in the finitely generated case)** *Let $r \in \{\infty, \omega\}$, let $\mathsf{M}$ be a $C^r$-manifold, and let $\mathscr{X}$ be a family of partially defined vector fields of class $C^r$ such that $\mathscr{L}^{(\infty)}(\mathscr{X})_x$ is a finitely generated submodule of $\mathscr{G}^r_{x,\mathsf{TM}}$ for each $x \in \mathsf{M}$. Then, for each $x_0 \in \mathsf{M}$,*

*(i) $\operatorname{Orb}(x_0, \mathscr{X})$ is a connected immersed $C^r$-submanifold of $\mathsf{M}$ and*

*(ii) for each $x \in \operatorname{Orb}(x_0, \mathscr{X})$, $\mathsf{T}_x \operatorname{Orb}(x_0, \mathscr{X}) = \mathsf{L}^{(\infty)}(\mathscr{X})_x$.*

*Moreover, $\mathsf{M}$ is the disjoint union of the set of orbits.*

Proof: From the Orbit Theorem and Theorem 7.18 it only remains to show that $\mathsf{T}_x \operatorname{Orb}(x_0, \mathscr{X}) \subseteq \mathsf{L}^{(\infty)}(\mathscr{X})_x$. For $X \in \mathscr{X}$ we have $[X, Y] \in \mathscr{L}^{(\infty)}(\mathscr{X})$ for every $Y \in \mathscr{L}^{(\infty)}(\mathscr{X})$, this since $\mathscr{L}^{(\infty)}(\mathscr{X})$ is a Lie subalgebra. Since $\mathscr{L}^{(\infty)}(\mathscr{X})_x$ is assumed to be finitely generated, Theorem 6.15 and (7.1) gives $(\Phi^X_t)_* Y(x) \in \mathsf{L}^{(\infty)}(\mathscr{X})_x$ for every $X \in \mathscr{X}$ and $t \in \mathbb{R}$ such that $x \in \Phi^X_t(\mathcal{U}(X, t))$. A trivial induction then gives

$$(\Phi^{X_k}_{t_k})_* \cdots (\Phi^{X_1}_{t_1})_* Y(x) = (\Phi^{X_k}_{t_k} \circ \cdots \circ \Phi^{X_1}_{t_1})_* Y(x) \in \mathsf{L}^{(\infty)}(\mathscr{X})_x \tag{7.7}$$

for every suitable $X_1, \dots, X_k \in \mathscr{X}$ and $t_1, \dots, t_k \in \mathbb{R}$, where we use the fact that push-forward commutes with composition [Abraham, Marsden, and Ratiu 1988, Proposition 4.2.3]. However, by Theorem 7.13,

$$\mathsf{T}_x \operatorname{Orb}(x_0, \mathscr{X}) = \{\Phi_* X(x) \mid \Phi \in \operatorname{Diff}(\mathscr{X}),\ X \in \mathscr{X}\},$$

and so (7.7) implies that $\mathsf{T}_x \operatorname{Orb}(x_0, \mathscr{X}) \subseteq \mathsf{L}^{(\infty)}(\mathscr{X})$. ■

**7.22 Remark: (The "submodule" assumption in the finitely generated Orbit Theorem)** In the statement of the preceding theorem we asked that $\mathscr{L}^{(\infty)}(\mathscr{X})_x$ be a submodule of $\mathscr{G}^r_{x,\mathsf{TM}}$ for each $x \in \mathsf{M}$. For a general family $\mathscr{X}$ of vector fields, it will not be the case that $\mathscr{L}^{(\infty)}(\mathscr{X})_x$ is a submodule. However, if all one is interested in is the tangent spaces to orbits, then, by Propositions 6.4 and 6.5, one can replace with $\mathscr{X}_x$ with the module $\langle \mathscr{X}_x \rangle$ generated by $\mathscr{X}_x$. Equivalently, also by Proposition 6.5, one can replace $\mathscr{L}^{(\infty)}(\mathscr{X})_x$ with the module $\langle \mathscr{L}^{(\infty)}(\mathscr{X})_x \rangle$ generated by $\mathscr{L}^{(\infty)}(\mathscr{X})_x$. As long as the module $\mathscr{L}^{(\infty)}(\langle \mathscr{X}_x \rangle)$ or $\langle \mathscr{L}^{(\infty)}(\mathscr{X})_x \rangle$ is locally finitely generated, it will hold that $\mathsf{T}_x \operatorname{Orb}(x_0, \mathscr{X}) = \mathsf{L}^{(\infty)}(\mathscr{X})_x$ for all $x_0 \in \mathsf{M}$ and $x \in \operatorname{Orb}(x_0, \mathscr{X})$. •

Again, it is important to distinguish between a distribution generated by a family of vector fields being finitely generated and the module generated by a family of vector fields being finitely generated.

This gives the following important results for families of analytic vector fields and certain families of smooth vector fields.

**7.23 Corollary: (The Orbit Theorem when $\mathsf{L}^{(\infty)}(\mathscr{X})$ has constant rank)** *Let* $\mathsf{M}$ *be a* $C^\infty$*-manifold and let* $\mathscr{X}$ *be a family of partially defined smooth vector fields such that the distribution* $\mathsf{L}^{(\infty)}(\mathscr{X})$ *is regular. Then, for each* $x_0 \in \mathsf{M}$*,*

*(i)* $\operatorname{Orb}(x_0, \mathscr{X})$ *is a connected immersed smooth submanifold of* $\mathsf{M}$ *and*

*(ii) for each* $x \in \operatorname{Orb}(x_0, \mathscr{X})$*,* $\mathsf{T}_x \operatorname{Orb}(x_0, \mathscr{X}) = \mathsf{L}^{(\infty)}(\mathscr{X})$*.*

*Moreover,* $\mathsf{M}$ *is the disjoint union of the set of orbits.*

Proof: This follows from Theorem 7.21, along with Theorem 4.9. ■

**7.24 Corollary: (The Orbit Theorem in the analytic case)** *Let* $\mathsf{M}$ *be an analytic manifold and let* $\mathscr{X}$ *be a family of partially defined analytic vector fields. Then, for each* $x_0 \in \mathsf{M}$*,*

*(i)* $\operatorname{Orb}(x_0, \mathscr{X})$ *is a connected immersed analytic submanifold of* $\mathsf{M}$ *and*

*(ii) for each* $x \in \operatorname{Orb}(x_0, \mathscr{X})$*,* $\mathsf{T}_x \operatorname{Orb}(x_0, \mathscr{X}) = \mathsf{L}^{(\infty)}(\mathscr{X})$*.*

*Moreover,* $\mathsf{M}$ *is the disjoint union of the set of orbits.*

Proof: This follows from Theorem 7.21, along with Theorem 4.10. ■

The hypothesis of finite generation is necessary, and Example 7.20 serves to demonstrate this necessity.

**7.6. The fixed-time Orbit Theorem.** In this section we give the version of the Orbit Theorem corresponding to the fixed-time orbits considered in Section 7.3. In order to understand the tangent spaces to the fixed-time orbits for a family $\mathscr{X} = ((X_j, \mathcal{U}_j))_{j \in J}$ of partially defined vector fields, we introduce a family of partially defined vector fields by

$$\mathscr{X}_0 = \Big\{ \sum_{l=1}^k \lambda_l (X_{j_l}, \mathcal{U}_{j_l}) \Bigm| j_1, \dots, j_k \in J,\ \lambda_1, \dots, \lambda_k \in \mathbb{R},\ \sum_{l=1}^k \lambda_j = 0,\ k \in \mathbb{Z}_{>0} \Big\}.$$

For brevity, in the preceding expression we have suppressed the domain of the partially defined vector fields.

With the family of vector fields $\mathscr{X}_0$ at our disposal, we can state the fixed-time Orbit Theorem.

**7.25 Theorem: (Fixed-time Orbit Theorem)** *Let $r \in \{\infty, \omega\}$, let $\mathsf{M}$ be a $C^r$-manifold, and let $\mathscr{X}$ be a family of partially defined vector fields of class $C^r$. For $x_0 \in \mathsf{M}$ and $T \in \mathbb{R}_{>0}$ satisfying $\mathrm{Orb}_T(x_0, \mathscr{X}) \neq \varnothing$, it holds that*

*(i)* $\mathrm{Orb}_T(x_0, \mathscr{X})$ *is a connected immersed $C^r$-submanifold of $\mathsf{M}$ and*

*(ii) for each $x \in \mathrm{Orb}_T(x_0, \mathscr{X})$,*

$$\mathsf{T}_x \mathrm{Orb}_T(x_0, \mathscr{X}) = L(\mathrm{aff}(\{\Phi_* X(x) \mid \Phi \in \mathrm{Diff}(\mathscr{X}),\ X \in \mathscr{X}\})),$$

*where the right-hand side of this expression denotes the linear subspace associated with the affine hull.*

*As a result,* $\dim(\mathrm{Orb}(x_0, \mathscr{X})) - \dim(\mathrm{Orb}_T(x_0, \mathscr{X})) \in \{0, 1\}$.

Proof: We shall consider the manifold $\mathbb{R} \times \mathsf{M}$ and so let us introduce some convenient notation for using this manifold. For $\mathcal{U} \subseteq \mathsf{M}$ open, note that $\mathsf{T}(\mathbb{R} \times \mathcal{U}) = \mathsf{T}\mathbb{R} \times \mathsf{T}\mathcal{U} \simeq \mathbb{R} \times \mathbb{R} \times \mathsf{T}\mathcal{U}$. For a partially defined vector field $(X, \mathcal{U})$ define a partially vector field $(\hat{X}, \mathbb{R} \times \mathcal{U})$ by

$$\hat{X}(s, x) = (s, 1, X(x)) \in \mathbb{R} \times \mathbb{R} \times \mathsf{T}\mathcal{U}.$$

The following simple lemma gives some of the useful properties—the last of which we shall not use until the proof of Theorem 7.30 below—of this natural extension of vector fields from $\mathsf{M}$ to $\mathbb{R} \times \mathcal{U}$.

**1 Lemma:** *For partially defined vector fields $(X, \mathcal{U})$ and $(Y, \mathcal{V})$, the following statements hold when they make sense:*

*(i)* $\Phi_t^{\hat{X}}(s, x) = (s + t, \Phi_t^X(x))$;

*(ii)* $(\Phi_t^{\hat{X}})_* \hat{Y} = \widehat{(\Phi_t^X)_* Y}$;

*(iii)* $[\hat{X}, \hat{Y}](s, x) = (s, 0, [X, Y](x))$.

Proof: (i) We have

$$\frac{\mathrm{d}}{\mathrm{d}t}(s + t, \Phi_t^X(x)) = (s + t, 1, X(\Phi_t^X(x))) = \hat{X}(s + t, \Phi_t^X(x)),$$

from which the desired conclusion follows by definition of integral curves.

(ii) Using (i) we compute

$$\begin{aligned}(\Phi_t^{\hat{X}})_* \hat{Y}(s, x) &= T\Phi_t^{\hat{X}} \circ \hat{Y} \circ \Phi_{-t}^{\hat{X}}(s, x) = T\Phi_t^{\hat{X}}(s - t, 1, Y(\Phi_{-t}^X(x))) \\ &= (s, 1, T\Phi_t^X \circ Y \circ \Phi_{-t}^X(x)) = \widehat{(\Phi_t^X)_* Y}(s, x),\end{aligned}$$

as desired.

(iii) Let $f \in C^r(\mathbb{R} \times \mathcal{U})$, denote by $f_s \colon \mathcal{U} \to \mathbb{R}$ and $f^x \colon \mathbb{R} \to \mathbb{R}$ the functions $f_s(x) = f^x(s) = f(s, x)$. Then compute

$$\hat{X} f(s, x) = \frac{\mathrm{d}f^x}{\mathrm{d}s} + X f_s(x), \quad \hat{Y} f(s, x) = \frac{\mathrm{d}f^x}{\mathrm{d}s} + Y f_s(x),$$

which gives

$$(\hat{X}\hat{Y} - \hat{Y}\hat{X})f(s,x) = \frac{\mathrm{d}^2 f^x}{\mathrm{d}x^2} + Y\frac{\partial f}{\partial s}(s,x) + X\frac{\partial f}{\partial s}(s,x) + XYf_s(x) \\ - \frac{\mathrm{d}^2 f^x}{\mathrm{d}x^2} - X\frac{\partial f}{\partial s}(s,x) - Y\frac{\partial f}{\partial s}(s,x) - YXf_s(x) = [X,Y]f_s(x),$$

and our result follows from this. ▼

Now let us define a family $\hat{\mathscr{X}}$ of partially defined $C^r$-vector fields on $\mathbb{R}\times\mathsf{M}$ by

$$\hat{\mathscr{X}} = \{(\hat{X}, \mathbb{R}\times\mathcal{U}) \mid (X,\mathcal{U}) \in \mathscr{X}\}.$$

As usual, when it is convenient we will suppress the domain $\mathbb{R}\times\mathcal{U}$ when we write the partially defined vector field $(\hat{X}, \mathbb{R}\times\mathcal{U})$. We can describe the fixed-time orbit in terms of the orbits of $\hat{\mathscr{X}}$.

**2 Lemma:** *For each* $s_0 \in \mathbb{R}$*,*

$$\mathrm{Orb}_T(x_0, \mathscr{X}) = \{x \in \mathsf{M} \mid (s_0 + T, x) \in \mathrm{Orb}((s_0, x_0), \hat{\mathscr{X}})\}.$$

Proof: Let $x \in \mathrm{Orb}_T(x_0, \mathscr{X})$ so that $x = \Phi^{\boldsymbol{X}}_{\boldsymbol{t}}(x_0)$ where $\boldsymbol{X} \in \mathscr{X}^k$ and $\boldsymbol{t} \in \mathbb{R}^k$ satisfies $\sum_{j=1}^k t_j = T$. If $\boldsymbol{X} = (X_1, \dots, X_k)$, let us denote $\hat{\boldsymbol{X}} = (\hat{X}_1, \dots, \hat{X}_k)$. Then, by Lemma 1, note that

$$\Phi^{\hat{\boldsymbol{X}}}_{\boldsymbol{t}}(s_0, x_0) = (s_0 + T, \Phi^{\boldsymbol{X}}_{\boldsymbol{t}}(x_0)) = (s_0 + T, x).$$

This shows that

$$\mathrm{Orb}_T(x_0, \mathscr{X}) \subseteq \{x \in \mathsf{M} \mid (s_0 + T, x) \in \mathrm{Orb}((s_0, x_0), \hat{\mathscr{X}})\}.$$

Conversely, suppose that $(s_0+T, x) \in \mathrm{Orb}((s_0, x_0), \hat{\mathscr{X}})$. Thus there exists $\hat{\boldsymbol{X}} \in \hat{\mathscr{X}}^k$ and $\boldsymbol{t} \in \mathbb{R}^k$ such that $(s_0 + T, x) = \Phi^{\hat{\boldsymbol{X}}}_{\boldsymbol{t}}(s_0, x_0)$. By the form of the flow of vector fields from $\hat{\mathscr{X}}$ as given in Lemma 1 we must have $\sum_{j=1}^k t_j = T$ and $x = \Phi^{\boldsymbol{X}}_{\boldsymbol{t}}(x_0)$, giving

$$\{x \in \mathsf{M} \mid (s_0 + T, x) \in \mathrm{Orb}((s_0, x_0), \hat{\mathscr{X}})\} \subseteq \mathrm{Orb}_T(x_0, \mathscr{X}),$$

as desired. ▼

(i) In what follows, we make the identification

$$\mathrm{Orb}_T(x_0, \mathscr{X}) = \{(T, x) \in \mathbb{R}\times\mathsf{M} \mid (T, x) \in \mathrm{Orb}((0, x_0), \hat{\mathscr{X}})\}, \tag{7.8}$$

according to the lemma. We can now prove the first part of the theorem. Let $\mathsf{M}_T = \{T\}\times\mathsf{M}$, noting that $\mathsf{M}_T$ is obviously a submanifold of $\mathbb{R}\times\mathsf{M}$. We claim that

$$\mathsf{T}_{(T,x)}\,\mathrm{Orb}((0, x_0), \hat{\mathscr{X}}) + \mathsf{T}_{(T,x)}\mathsf{M}_T = \mathsf{T}_{(T,x)}(\mathbb{R}\times\mathsf{M}).$$

This follows since $\mathrm{codim}(\mathsf{M}_T) = 1$ and since, for each $X \in \mathscr{X}$, the vector field $\hat{X}$ is tangent to $\mathrm{Orb}((0, x_0), \hat{\mathscr{X}})$ and is not tangent to $\mathsf{M}_T$. Thus $\mathrm{Orb}_T(x_0, \mathscr{X})$ and $\mathsf{M}_T$ intersect transversely, and so it follows from Abraham, Marsden, and Ratiu [1988, Corollary 3.5.13] that

$$\mathrm{Orb}_T(x_0, \mathscr{X}) = \mathrm{Orb}((0, x_0), \hat{\mathscr{X}}) \cap \mathsf{M}_T$$

is an immersed submanifold of class $C^r$ by the Orbit Theorem. To prove the first part of the theorem, it remains to show that $\mathrm{Orb}_T(x_0,\mathscr{X})$ is connected. Let $x\in\mathrm{Orb}_T(x_0,\mathscr{X})$, let $X_1,\dots,X_k\in\mathscr{X}$, and let $t_1,\dots,t_k\in\mathbb{R}$ be such that

$$\Phi^{X_1}_{t_1}\circ\cdots\circ\Phi^{X_k}_{t_k}(x_0)=x,\quad \sum_{j=1}^k t_j=0,$$

this being possible by Proposition 7.11. Then the curve

$$[0,1]\ni t\mapsto \Phi^{X_1}_{t_1t}\circ\cdots\circ\Phi^{X_k}_{t_kt}(x_0)$$

is a curve in $\mathrm{Orb}_T(x_0,\mathscr{X})$ (again by Proposition 7.11) connecting $x_0$ and $x$, giving path connectedness and so connectedness of $\mathrm{Orb}_T(x_0,\mathscr{X})$.

(ii) Given our identification of $\mathrm{Orb}_T(x_0,\mathscr{X})$ as an immersed submanifold of $\mathbb{R}\times\mathsf{M}$, we can describe its tangent space accordingly:

$$\mathsf{T}_{(T,x)}\mathrm{Orb}_T(x_0,\mathscr{X})=\{(a,v_x)\in\mathsf{T}_{(T,x)}(\mathbb{R}\times\mathsf{M})\mid\ a=0,\ (0,v_x)\in\mathsf{T}_{(T,x)}\mathrm{Orb}((0,x_0),\hat{\mathscr{X}})\}. \tag{7.9}$$

By the Orbit Theorem,

$$\mathsf{T}_{(T,x)}\mathrm{Orb}((0,x_0),\hat{\mathscr{X}})=\mathrm{span}_{\mathbb{R}}(\hat{\Phi}_*\hat{X}(T,x)\mid\ \hat{\Phi}\in\mathrm{Diff}(\hat{\mathscr{X}}),\ \hat{X}\in\hat{\mathscr{X}}).$$

By Lemma 1, the definition of $\mathrm{Diff}(\hat{\mathscr{X}})$ from (7.2), and an elementary induction, for $\hat{\Phi}\in\mathrm{Diff}(\hat{\mathscr{X}})$ and $\hat{X}\in\hat{\mathscr{X}}$ we have

$$\hat{\Phi}_*\hat{X}(T,x)=\widehat{\Phi_*X}(T,x)=(T,1,\Phi_*X(x)) \tag{7.10}$$

for $\Phi\in\mathrm{Diff}(\mathscr{X})$. Let $\Phi_1,\dots,\Phi_k\in\mathrm{Diff}(\hat{\mathscr{X}})$, let $\hat{X}_1,\dots,\hat{X}_k\in\hat{\mathscr{X}}$, and let $\lambda_1,\dots,\lambda_k\in\mathbb{R}$. Then

$$\begin{aligned}&\sum_{j=1}^k\lambda_j\hat{\Phi}_{j*}\hat{X}_j(T,x)\in\mathsf{T}_{(T,x)}\mathrm{Orb}_T(x_0,\mathscr{X})\\ \iff\quad&\Big(T,\sum_{j=1}^k\lambda_j,\sum_{j=1}^k\lambda_j\Phi_{j*}X_j(x)\Big)\in\mathsf{T}_{(T,x)}\mathrm{Orb}_T(x_0,\mathscr{X})\end{aligned}$$

using (7.10). We immediately conclude from (7.9) that $\sum_{j=1}^k\lambda_j=0$, and so

$$\mathsf{T}_x\mathrm{Orb}_T(x_0,\mathscr{X})=L(\mathrm{aff}(\{\Phi_*X(x)\mid\ \Phi\in\mathrm{Diff}(\mathscr{X}),\ X\in\mathscr{X}\})),$$

recalling that the linear part of the affine hull of a subset $S$ of a vector space $\mathsf{V}$ is given by

$$\sum_{j=1}^k\lambda_jv_j,\qquad k\in\mathbb{Z}_{>0},\ v_1,\dots,v_k\in S,\ \lambda_1,\dots,\lambda_k\in\mathbb{R},\ \sum_{j=1}^k\lambda_j=0$$

(this follows directly from the definition of the affine hull, cf. Theorem 1.2.5 of [Webster 1994] and Theorem 1.2 of [Rockafellar 1970]).

For the final assertion of the theorem, let us abbreviate

$$\mathsf{V}_x = \mathrm{span}_{\mathbb{R}}(\{\Phi_* X(x) \mid \Phi \in \mathrm{Diff}(\mathscr{X}),\ X \in \mathscr{X}\}) \subseteq \mathsf{T}_x\mathsf{M}$$

and

$$\mathsf{A}_x = \mathrm{aff}(\{\Phi_* X(x) \mid \Phi \in \mathrm{Diff}(\mathscr{X}),\ X \in \mathscr{X}\}) \subseteq \mathsf{T}_x\mathsf{M}.$$

We have two cases, cf. the discussion on the bottom of page 8 of [Webster 1994].

1. $\mathsf{A}_x = \mathsf{V}_x$: In this case we have $\mathsf{V}_x = \mathsf{A}_x = L(\mathsf{A}_x)$ and so

$$\dim(\mathrm{Orb}(x_0, \mathscr{X})) = \dim(\mathrm{Orb}_T(x_0, \mathscr{X}))$$

since the tangent spaces to the orbit and the fixed-time orbits agree.

2. $\mathsf{A}_x \subset \mathsf{V}_x$: In this case there exists $v_0 \in \mathsf{V}_x \setminus \mathsf{A}_x$ and

$$\mathsf{V}_x = \mathrm{span}_{\mathbb{R}}(\{v_0\} \cup \mathsf{A}_x),$$

which gives

$$\dim(\mathrm{Orb}(x_0, \mathscr{X})) = \dim(\mathrm{Orb}_T(x_0, \mathscr{X})) + 1$$

as desired. ■

**7.26 Remark: (The fixed-time orbit topology)** As with the proof of the Orbit Theorem, the proof of the fixed-time Orbit Theorem prescribes a topology on $\mathsf{M}$. Let us extract this ***fixed-time orbit topology*** here. We will use the notation from the proof of the fixed-time Orbit Theorem. In the proof we saw that $\mathrm{Orb}_T(x_0, \mathscr{X})$ was naturally identified with an immersed submanifold of $\mathbb{R} \times \mathsf{M}$ and was also a subset of the (non-fixed-time) orbit $\mathrm{Orb}((0, x_0), \hat{\mathscr{X}})$. Since $\mathrm{Orb}((0, x_0), \hat{\mathscr{X}})$ has the orbit topology, the natural topology on $\mathrm{Orb}_T(x_0, \mathscr{X})$ is that induced by this orbit topology.

As with the orbit topology, one can describe the fixed-time orbit topology as a final topology. We do this as follows. Denote

$$\mathbb{R}^k_0 = \{\boldsymbol{t} \in \mathbb{R}^k \mid t_1 + \cdots + t_k = 0\},$$

noting that $\mathbb{R}^k_0$ is a $(k-1)$-dimensional subspace. Now let $x \in \mathsf{M}$, let $k \in \mathbb{Z}_{>0}$, let $\boldsymbol{X} = (X_1, \dots, X_k) \subseteq \mathscr{X}$, and let $\mathcal{U} \subseteq \mathbb{R}^k$ be a neighbourhood of $\boldsymbol{0}$ be such that the map

$$\mathbb{R}^k_0 \cap \mathcal{U} \ni \boldsymbol{t} \mapsto \Phi^{\boldsymbol{X}}_{\boldsymbol{t}}(x) \in \mathsf{M}$$

is defined. One may then define the orbit topology as the final topology induced by the above family of mappings. •

As with Theorem 7.18 for orbits, there is an easily described subspace of the tangent spaces to the fixed-time orbits. Let us describe this here. We let $\mathscr{D}(\mathscr{X})$ be the derived algebra of $\mathscr{L}^{(\infty)}(\mathscr{X})$. That is to say, $\mathscr{D}(\mathscr{X})$ is the subspace of $\mathscr{L}^{(\infty)}(\mathscr{X})$ generated by vector fields of the form $[Y_1, Y_2]$ where $Y_1, Y_2 \in \mathscr{L}^{(\infty)}(\mathscr{X})$. (Here again we are suppressing the domain of partially defined vector fields.) As with Lie algebras of globally defined vector fields, we have the notion of an ideal in $\mathscr{L}^{(\infty)}(\mathscr{X})$. Indeed, a subset $\mathscr{I} \subseteq \mathscr{L}^{(\infty)}(\mathscr{X})$ is an ***ideal*** if $[X, Y] \in \mathscr{I}$ for every $X \in \mathscr{I}$ and $Y \in \mathscr{L}^{(\infty)}(\mathscr{X})$. Note that $\mathscr{I}_x$ is then an ideal of $\mathscr{L}^{\infty)}(\mathscr{X})_x$ in the usual sense.

With the preceding comments, we have the following explicit characterisation of the derived algebra.

**7.27 Proposition: (Characterisation of $\mathscr{D}(\mathscr{X})$)** *Let $r \in \{\infty, \omega\}$, let $\mathsf{M}$ be a $C^r$-manifold, and let $\mathscr{X}$ be a family of partially defined $C^r$-vector fields. Then the derived algebra $\mathscr{L}^{(\infty)}(\mathscr{X})$ is comprised of finite $\mathbb{R}$-linear combinations of vector fields of the form*

$$[X_k, [X_{k-1}, \dots, [X_2, X_1] \cdots]], \qquad X_1, \dots, X_k \in \mathscr{X}, \; k \geq 2.$$

Proof: We first claim that the derived algebra is an ideal of $\mathscr{L}^{(\infty)}(\mathscr{X})$, indeed the ideal generated by elements of the form $[X_1, X_2]$ for $X_1, X_2 \in \mathscr{X}$. First, it is clear from the definition that $\mathscr{D}(\mathscr{X})$ is an ideal. It is also clear, since $\mathscr{X} \subseteq \mathscr{L}^{(\infty)}(\mathscr{X})$, that $[X_1, X_2] \in \mathscr{D}(\mathscr{X})$ for every $X_1, X_2 \in \mathscr{X}$. Thus $\mathscr{D}(\mathscr{X})$ contains the ideal generated by brackets from $\mathscr{X}$. Now consider an element from $\mathscr{D}(\mathscr{X})$ of the form $[Y_1, Y_2]$ for $Y_1, Y_2 \in \mathscr{L}^{(\infty)}(\mathscr{X})$, noting that all elements of $\mathscr{D}(\mathscr{X})$ are finite linear combinations of such elements. By Proposition 6.3 (more properly, its adaptation to partially defined vector fields), $[Y_1, Y_2]$ is a finite linear combination of brackets of the type in the statement of the result. Moreover, if we consider the proof of Proposition 6.3, we can see that the brackets involved will be of the form in the statement of the proposition with $k \geq 2$. From this we conclude our claim that $\mathscr{D}(\mathscr{X})$ is the ideal of $\mathscr{L}^{(\infty)}(\mathscr{X})$ generated by brackets $[X_1, X_2]$ for $X_1, X_2 \in \mathscr{X}$.

From the fact that the derived algebra is an ideal and that it contains all vector fields of the form $[X_1, X_2]$ for $X_1, X_2 \in \mathscr{X}$, it follows that

$$[X_k, [X_{k-1}, \dots, [X_2, X_1] \cdots]] \in \mathscr{D}(\mathscr{X})$$

for every $X_1, \dots, X_k \in \mathscr{X}$, $k \geq 2$. Conversely, the set of finite $\mathbb{R}$-linear combinations of the form in the statement of the result is easily shown to be an ideal of $\mathscr{L}^{(\infty)}(\mathscr{X})$ and it clearly contains the brackets $[X_1, X_2]$ for $X_1, X_2 \in \mathscr{X}$. Thus $\mathscr{D}(\mathscr{X})$ is contained in the this set of linear combinations, which completes the proof. ■

We then define

$$\mathscr{I}(\mathscr{X}) = \operatorname{span}_{\mathbb{R}}(X + Y | \; X \in \mathscr{X}_0, \; Y \in \mathscr{D}(\mathscr{X})).$$

The following characterisation of $\mathscr{I}(\mathscr{X})$ is useful.

**7.28 Proposition: (Characterisation of $\mathscr{I}(\mathscr{X})$)** *Let $r \in \{\infty, \omega\}$, let $\mathsf{M}$ be a $C^r$-manifold, and let $\mathscr{X}$ be a family of partially defined $C^r$-vector fields. Then the following statements hold:*

*(i) $\mathscr{I}(\mathscr{X})$ is an ideal of $\mathscr{L}^{(\infty)}(\mathscr{X})$;*

*(ii) the codimension of $\mathscr{I}(\mathscr{X})_x$ in $\mathscr{L}^{(\infty)}(\mathscr{X})_x$ is zero if $\mathscr{I}(\mathscr{X})_x \cap \mathscr{X}_x \neq \varnothing$ and is one otherwise.*

Proof: (i) If $Y \in \mathscr{L}^{(\infty)}(\mathscr{X})$ and if $X \in \mathscr{I}(\mathscr{X})$, then $[Y, X]$ is obviously in the derived algebra of $\mathscr{L}^{(\infty)}(\mathscr{X})$, by definition of the derived algebra. Since $\mathscr{D}(\mathscr{X}) \subseteq \mathscr{I}(\mathscr{X})$, this part of the result follows.

(ii) From Proposition 6.3 and Proposition 7.27 we have that any element of $\mathscr{L}^{(\infty)}(\mathscr{X})$ can be written as

$$\sum_{j=1}^{k} \lambda_j X_j + Y, \qquad X_1, \dots, X_k \in \mathscr{X}, \; \lambda_1, \dots, \lambda_k \in \mathbb{R}, \; Y \in \mathscr{D}(\mathscr{X})$$

(suppressing the domains of partially defined vector fields, as usual). Thus $\mathscr{L}^{(\infty)}(\mathscr{X})_x$ is the sum of the subspaces $\mathscr{X}_x$ and $\mathscr{D}(\mathscr{X})_x$. Referring to Proposition 7.27, $\mathscr{I}(\mathscr{X})_x$ is the sum of the subspaces $L(\mathrm{aff}(\mathscr{X}_x))$ and $\mathscr{D}(\mathscr{X})_x$. Note that $L(\mathrm{aff}(\mathscr{X}_x))$ is a subspace of $\mathscr{X}_x$. Moreover, as in the last step in the proof of Theorem 7.25, $L(\mathrm{aff}(\mathscr{X}_x)) = \mathscr{X}_x$ if and only if $\mathscr{X}_x \cap L(\mathrm{aff}(\mathscr{X}_x)) \neq \varnothing$. Also as in the last step in the proof of Theorem 7.25, if $\mathscr{X}_x \cap L(\mathrm{aff}(\mathscr{X}_x)) = \varnothing$, then the codimension of $L(\mathrm{aff}(\mathscr{X}_x))$ in $\mathscr{X}_x$ is one. This gives this part of the result. ■

We define

$$\mathsf{I}(\mathscr{X})_x = \{X(x) \mid X \in \mathscr{I}(\mathscr{X})\}$$

so that $\mathsf{I}(\mathscr{X})$ is a distribution on $\mathsf{M}$. Since $\mathscr{I}(\mathscr{X})$ is an ideal of $\mathscr{L}^{(\infty)}(\mathscr{X})$, it is also a Lie subalgebra, and so is a Lie subalgebra of $\Gamma^r(\mathsf{TM})$. The picture one should have in mind is that $\mathscr{I}(\mathscr{X})$ is to $\mathrm{Orb}_T(x, \mathscr{X})$ what $\mathscr{L}^{(\infty)}(\mathscr{X})$ is to $\mathrm{Orb}(x, \mathscr{X})$. For example, one should think of $\mathscr{L}^{(\infty)}(\mathscr{X})$ as being the "Lie algebra" of the "Lie group" $\mathrm{Diff}(\mathscr{X})$. Upon doing so, one should think of $\mathscr{I}(\mathscr{X})$ as the Lie subalgebra (actually ideal) of $\mathscr{L}^{(\infty)}(\mathscr{X})$ corresponding to the "subgroup" (actually, "normal subgroup") $\mathrm{Diff}_0(\mathscr{X})$ of $\mathrm{Diff}(\mathscr{X})$. Moreover, we have the following theorem.

**7.29 Theorem: (A subspace of the tangent space of a fixed-time orbit)** *Let $r \in \{\infty, \omega\}$, let $\mathsf{M}$ be a $C^r$-manifold, and let $\mathscr{X}$ be a family of partially defined $C^r$-vector fields on $\mathsf{M}$. For $x_0 \in \mathsf{M}$ and $T \in \mathbb{R}$ satisfying $\mathrm{Orb}_T(x_0, \mathscr{X}) \neq \varnothing$, it holds that*

$$\mathsf{I}(\mathscr{X})_x \subseteq \mathsf{T}_x \mathrm{Orb}_T(x_0, \mathscr{X})$$

*for every $x \in \mathrm{Orb}_T(x_0, \mathscr{X})$.*

Proof: Let $x \in \mathrm{Orb}_T(x_0, \mathscr{X})$. Let $X_1, \dots, X_k \in \mathscr{X}$ and let $\lambda_1, \dots, \lambda_k \in \mathbb{R}$ satisfy $\sum_{j=1}^k \lambda_j = 0$. Then take $X = \sum_{j=1}^k \lambda_j X_j$ (we remind the reader once again that we suppress the domains of partially defined vector fields). We have

$$\begin{aligned} X(x) = \lambda_1 X_1(x) + \dots + \lambda_k X_k(x) &= \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} \Phi_t^X(x) = \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} \Phi_t^{\lambda_1 X_1 + \dots + \lambda_k X_k}(x) \\ &= \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} \Phi_t^{\lambda_1 X_1} \circ \dots \circ \Phi_t^{\lambda_k X_k}(x) = \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} \Phi_{\lambda_1 t}^{X_1} \circ \dots \circ \Phi_{\lambda_k t}^{X_k}(x). \end{aligned}$$

Since

$$\Phi_{\lambda_1 t}^{X_1} \circ \dots \circ \Phi_{\lambda_k t}^{X_k} \in \mathrm{Diff}_0(\mathscr{X}),$$

it follows from Proposition 7.11 that $X(x) \in \mathsf{T}_x \mathrm{Orb}_T(x_0, \mathscr{X})$.

Recalling the notation $[\Phi, \Psi]$ for local diffeomorphisms $\Phi$ and $\Psi$ from the proof of Theorem 7.18, note that for $t_1, t_2 \in \mathbb{R}$ we have $[\Phi_{t_1}^{X_1}, \Phi_{t_2}^{X_2}] \in \mathrm{Diff}_0(\mathscr{X})$. An induction then gives

$$[\cdots[\Phi_{t_1}^{X_1}, \Phi_{t_2}^{X_2}], \dots, \Phi_{t_k}^{X_k}] \in \mathrm{Diff}_0(\mathscr{X})$$

for vector fields $X_1, \dots, X_k \in \mathscr{X}$ and $t_1, \dots, t_k \in \mathbb{R}$. Thus the curve

$$s \mapsto [\cdots[\Phi_s^{X_1}, \Phi_s^{X_2}], \dots, \Phi_s^{X_k}](x)$$

is a curve in $\mathrm{Orb}_T(x_0, \mathscr{X})$ by Proposition 7.11. From Lemma 3 from the proof of Theorem 7.18 we then have

$$[X_k, \ldots, [X_2, X_1] \cdots](x) \in \mathsf{T}_x \mathrm{Orb}_T(x_0, \mathscr{X})$$

for any $X_1, \ldots, X_k \in \mathscr{X}$ and for $x \in \mathrm{Orb}_T(x_0, \mathscr{X})$. Combining this conclusion with that from the preceding paragraph, and using Proposition 6.3, we have that $\mathsf{I}(\mathscr{X})_x \subseteq \mathsf{T}_x \mathrm{Orb}_T(x_0, \mathscr{X})$, as desired. ■

**7.7. The finitely generated fixed-time Orbit Theorem.** Our final version of the Orbit Theorem considers the fixed-time orbits, but under the assumption that $\mathscr{I}(\mathscr{X})$ is a locally finitely generated module.

**7.30 Theorem: (The fixed-time Orbit Theorem in the finitely generated case)** *Let $r \in \{\infty, \omega\}$, let $\mathsf{M}$ be a $C^r$-manifold, and let $\mathscr{X}$ be a family of partially defined vector fields of class $C^r$ such that $\mathscr{L}^{(\infty)}(\mathscr{X})_x$ is a finitely generated submodule of $\mathscr{G}^r_{x,\mathsf{TM}}$ for each $x \in \mathsf{M}$. Then, for each $x_0 \in \mathsf{M}$ and $T \in \mathbb{R}$ such that $\mathrm{Orb}_T(x_0, \mathscr{X}) \neq \varnothing$,*

*(i) $\mathrm{Orb}_T(x_0, \mathscr{X})$ is a connected immersed $C^r$-submanifold of $\mathsf{M}$ and*

*(ii) for each $x \in \mathrm{Orb}_T(x_0, \mathscr{X})$, $\mathsf{T}_x \mathrm{Orb}_T(x_0, \mathscr{X}) = \mathsf{I}(\mathscr{X})_x$.*

Proof: From Proposition 7.11, the fixed-time Orbit Theorem, and Theorem 7.29, it only remains to show that $\mathsf{T}_x \mathrm{Orb}_T(x_0, \mathscr{X}) \subseteq \mathsf{I}^{(\infty)}(\mathscr{X})_x$.

We shall adopt the notation and assume the setting of Theorem 7.25. In particular, we recall the identifications (7.8) and (7.9). Note that since $\mathscr{L}^{(\infty)}(\mathscr{X})_x$ is finitely generated, it follows from Lemma 1 from the proof of Theorem 7.25 that $\mathscr{L}^{(\infty)}(\hat{\mathscr{X}})_x$ is also locally finitely generated. Let $x \in \mathrm{Orb}_T(x_0, \mathscr{X})$. From the finitely generated Orbit Theorem we have

$$\mathsf{T}_{(T,x)} \mathrm{Orb}((0, x_0), \hat{\mathscr{X}}) = \mathsf{L}^{(\infty)}(\hat{\mathscr{X}})_{(T,x)}.$$

From Lemma 1 from the proof of Theorem 7.25 we have

$$\mathsf{L}^{(\infty)}(\hat{\mathscr{X}})_{(T,x)} = \{(T, 0, v_x) \in \mathbb{R} \times \mathbb{R} \times \mathsf{TM} \mid \ v_x \in \mathsf{L}^{(\infty)}(\mathscr{X})\}.$$

By Proposition 6.3, if $\hat{Y} \in \mathscr{L}^{(\infty)}(\hat{\mathscr{X}})$ then we can write

$$\hat{Y} = \sum_{j=1}^{k} \lambda_j \hat{X}_j + X$$

for $X_1, \ldots, X_k \in \mathscr{X}$, for $\lambda_1, \ldots, \lambda_k \in \mathbb{R}$ and for $X \in \mathscr{D}(\hat{\mathscr{X}})$. If $\hat{Y}$ further has the property that $\hat{Y}(T, x) \in \mathsf{T}_{(T,x)} \mathrm{Orb}_T(x_0, \mathscr{X})$ (under the identification (7.9)), then it follows that $\sum_{j=1}^{k} \lambda_j = 0$, using the fact that $X(T, x) = (T, 0, X'(x))$ for some $X' \in \mathscr{D}(\mathscr{X})$. Thus $\mathsf{T}_{(T,x)} \mathrm{Orb}_T(x_0, \mathscr{X}) \subseteq \mathsf{I}(\mathscr{X})_x$, as desired. ■

As with the finitely generated Orbit Theorem, the assumption that $\mathscr{L}^{(\infty)}(\mathscr{X})_x$ is finitely generated is necessary, and Example 7.20 serves to demonstrate this necessity.

As is usual with these notions of finite generatedness, one has the following two cases where finite generation is guaranteed.

**7.31 Corollary: (The fixed-time Orbit Theorem when $\mathsf{L}^{(\infty)}(\mathscr{X})$ has constant rank)** *Let* $\mathsf{M}$ *be a* $C^\infty$*-manifold and let* $\mathscr{X}$ *be a family of partially defined smooth vector fields such that the distribution* $\mathsf{L}^{(\infty)}(\mathscr{X})$ *is regular. Then, for each* $x_0 \in \mathsf{M}$ *and* $T \in \mathbb{R}$ *such that* $\mathrm{Orb}_T(x_0, \mathscr{X}) \neq \varnothing$*,*

*(i)* $\mathrm{Orb}_T(x_0, \mathscr{X})$ *is a connected immersed smooth submanifold of* $\mathsf{M}$ *and*

*(ii) for each* $x \in \mathrm{Orb}_T(x_0, \mathscr{X})$*,* $\mathsf{T}_x \mathrm{Orb}_T(x_0, \mathscr{X}) = \mathsf{I}(\mathscr{X})_x$*.*

Proof: This follows from Theorem 7.30, along with Theorem 4.9. ■

**7.32 Corollary: (The fixed-time Orbit Theorem in the analytic case)** *Let* $\mathsf{M}$ *be an analytic manifold and let* $\mathscr{X}$ *be a family of partially defined analytic vector fields. Then, for each* $x_0 \in \mathsf{M}$ *and* $T \in \mathbb{R}$ *such that* $\mathrm{Orb}_T(x_0, \mathscr{X}) \neq \varnothing$*,*

*(i)* $\mathrm{Orb}_T(x_0, \mathscr{X})$ *is a connected immersed analytic submanifold of* $\mathsf{M}$ *and*

*(ii) for each* $x \in \mathrm{Orb}_T(x_0, \mathscr{X})$*,* $\mathsf{T}_x \mathrm{Orb}_T(x_0, \mathscr{X}) = \mathsf{I}(\mathscr{X})_x$*.*

Proof: This follows from Theorem 7.30, along with Theorem 4.10. ■

## 8. Frobenius's Theorem for subsheaves and distributions

We next use the Orbit Theorem to prove Frobenius's Theorem. We give the statement in terms of both distributions and subsheaves.

### 8.1. Involutive distributions and subsheaves.

Frobenius's Theorem connects two concepts: integrability and involutivity. Let us first consider involutivity.

**8.1 Definition: (Involutive distributions and subsheaves)** Let $r \in \{\infty, \omega\}$, let $\mathsf{M}$ be a manifold of class $C^r$, let $\mathsf{D}$ be a distribution of class $C^r$, and let $\mathscr{F} = (F(\mathcal{U}))_{\mathcal{U}\,\mathrm{open}}$ be a subsheaf of $\mathscr{G}^r_{\mathsf{TM}}$.

(i) The subsheaf $\mathscr{F}$ is ***involutive*** if $[X, Y] \in F(\mathcal{U})$ for every $X, Y \in F(\mathcal{U})$ and every open $\mathcal{U} \subseteq \mathsf{M}$, i.e., if $\mathscr{F}$ is a Lie subalgebra of $\mathscr{G}^r_{\mathsf{TM}}$.

(ii) The distribution $\mathsf{D}$ is ***involutive*** if $\mathsf{L}^{(\infty)}(\mathscr{G}^r_{\mathsf{D}}) = \mathsf{D}$. •

Let us explore the relationship between the two notions of involutivity. We start with two examples that show that, in general, there will be no exact correspondence between involutivity of sheaves and involutivity of the distributions they generate.

**8.2 Example: (The distribution generated by an involutive subsheaf may not be involutive)** We consider one of the cases of Example 6.10. Specifically, we take $\mathsf{M} = \mathbb{R}^2$ and take the two vector fields

$$X_1 = \frac{\partial}{\partial x_1}, \quad X_2 = f(x_1)\frac{\partial}{\partial x_2},$$

where

$$f(x) = \begin{cases} \mathrm{e}^{-1/x^2}, & x \neq 0, \\ 0, & x = 0, \end{cases}$$

We let $\mathscr{X} = \{X_1, X_2\}$ and let $\mathscr{F} = (F(\mathcal{U}))_{\mathcal{U}\,\mathrm{open}}$ be the subsheaf defined by $F(\mathcal{U}) = \mathscr{L}^{(\infty)}(\langle\mathscr{X}|\mathcal{U}\rangle)$. By Proposition 6.6 we have

$$\mathsf{D}(\mathscr{F}) = \mathsf{L}^{(\infty)}(\mathscr{X}).$$

A simple inductive argument shows that the only nonzero Lie brackets of the form in Proposition 6.3 are given by

$$\underbrace{[X_1, [X_1, \cdots, [X_1}_{k\ \text{times}}, X_2]\cdots]] = f^{(k)}\frac{\partial}{\partial x_2}.$$

Thus we conclude from Proposition 6.3 that

$$\mathsf{D}(\mathscr{F})_{(x_1,x_2)} = \begin{cases} \mathsf{T}_{(x_1,x_2)}\mathbb{R}^2, & x_1 \neq 0, \\ \operatorname{span}_{\mathbb{R}}(\frac{\partial}{\partial x_1}), & x_1 = 0. \end{cases}$$

From Example 6.10 we see that $\mathsf{D}(\mathscr{F})$ is not involutive, although $\mathscr{F}$ clearly is. •

**8.3 Example: (A non-involutive subsheaf may generate an involutive distribution)** Let us consider the vector fields

$$X_1(x_1,x_2) = (x_1^2 + x_2^2)\frac{\partial}{\partial x_1}, \quad X_2(x_1,x_2) = (x_1^2 + x_2^2)\frac{\partial}{\partial x_2}$$

on $\mathbb{R}^2$. We let $\mathscr{X} = \{X_1, X_2\}$ and take $\mathscr{F} = \langle\mathscr{X}\rangle$. By Proposition 3.16 the distribution generated by these vector fields is

$$\mathsf{D}(\mathscr{F})_{(x_1,x_2)} = \begin{cases} \mathsf{T}_{(x_1,x_2)}\mathbb{R}^2, & (x_1,x_2) \neq (0,0), \\ 0, & (x_1,x_2) = (0,0). \end{cases}$$

By Frobenius's Theorem below, $\mathsf{D}(\mathscr{F})$ is involutive since it is integrable (integrability is discussed in the next section).

We claim that $\mathscr{F}$ is also involutive. To see this, let $X, Y \in \langle X_1, X_2\rangle$ and write

$$X = \alpha_X X_1 + \beta_X X_2, \quad Y = \alpha_Y X_1 + \beta_X X_2$$

for smooth or analytic functions $\alpha_X$, $\alpha_Y$, $\beta_X$, and $\beta_Y$. A direct calculation then shows that $[X, Y] \in \langle X_1, X_2\rangle$. Thus, not only is $\mathsf{D}(\mathscr{F})$, the generators $(X_1, X_2)$ generate a subsheaf of vector fields that is involutive.

Next consider the generators

$$X_1'(x_1,x_2) = (x_1^2 + x_2^2)\frac{\partial}{\partial x_1}, \quad X_2'(x_1,x_2) = (x_1^4 + x_2^4)\frac{\partial}{\partial x_2}$$

for $\mathsf{D}(\mathscr{F})$. We let $\mathscr{F}'$ be the subsheaf generated by these vector fields. In this case we calculate

$$[X_1', X_2'](x_1,x_2) = \underbrace{-\frac{2x_2(x_1^4 + x_2^4)}{x_1^2 + x_2^2}}_{\alpha} X_1' + \underbrace{\frac{4x_1^3(x_1^2 + x_2^2)}{x_1^4 + x_2^4}}_{\beta} X_2'.$$

We calculate
$$\frac{\partial\beta}{\partial x_1} = \frac{4(x_1^8 - x_1^6x_2^2 + 5x_1^4 + x_2^4 + 3x_1^2x_2^6)}{(x_1^4 + x_2^4)^2},$$
which is not continuous since
$$\frac{\partial\beta}{\partial x_1}(x_1, 0) = 4, \quad \frac{\partial\beta}{\partial x_1}(0, x_2) = 0.$$
Thus, while $\mathsf{D}(\mathscr{F}') = \mathsf{D}(\mathscr{F})$ is involutive, the subsheaf $\mathscr{F}'$ is not. •

Note that the second of these examples applies to the analytic case. The first example, however, is smooth but not real analytic. Indeed, by Theorem 6.7, if $\mathscr{F}$ is an involutive analytic subsheaf of vector fields, then $\mathsf{D}(\mathscr{F})$ is necessarily involutive.

**8.2. Integral manifolds.** A related notion to an orbit is the following.

**8.4 Definition: (Integral manifold, integrable distribution, foliation)** Let $r \in \{\infty, \omega\}$, let $\mathsf{M}$ be a $C^r$-manifold, let $\mathscr{F}$ be a subsheaf of $\mathscr{G}^r_{\mathsf{TM}}$, and let $\mathsf{D}$ be a $C^r$-distribution.

(i) An ***integral manifold*** of $\mathsf{D}$ is a $C^r$-immersed submanifold $\mathsf{S}$ of $\mathsf{M}$ such that $\mathsf{T}_x\mathsf{S} = \mathsf{D}_x$ for every $x \in \mathsf{S}$.

(ii) An integral manifold $\mathsf{S}$ for $\mathsf{D}$ is ***maximal*** if it is connected, and if every connected integral manifold $\mathsf{S}'$ for $\mathsf{D}$ such that $\mathsf{S}' \cap \mathsf{S} \neq \varnothing$ is an open submanifold of $\mathsf{S}$.

(iii) The distribution $\mathsf{D}$ is ***integrable*** if, for each $x \in \mathsf{M}$, there exists an integral manifold of $\mathsf{D}$ containing $x$.

(iv) The subsheaf $\mathscr{F}$ is ***integrable*** if the distribution $\mathsf{D}(\mathscr{F})$ is integrable.

(v) A ***$C^r$-foliation*** of $\mathsf{M}$ is a family $(\mathsf{S}_a)_{a\in A}$ of pairwise disjoint immersed $C^r$-submanifolds such that

  (a) $\mathsf{M} = \cup_{a\in A}\mathsf{S}_a$ and

  (b) for each $x_0 \in \mathsf{M}$, there exists a neighbourhood $\mathcal{N}$ of $x$ and a family $(X_b)_{b\in B}$ of $C^r$-vector fields for which $\mathsf{T}_x\mathsf{S}_a = \operatorname{span}_{\mathbb{R}}(X_b(x) |\ b \in B)$. •

Let us illustrate these definitions with examples.

**8.5 Examples: (Integral manifolds)**

1. In Example 7.7–1 we considered an example with $\mathsf{M} = \mathbb{R}^2$ and define
$$X_1 = x_1\frac{\partial}{\partial x_1}, \quad X_2 = x_2\frac{\partial}{\partial x_2}.$$
By $\mathsf{D}$ we denote the distribution generated by the vector fields $\mathscr{X} = (X_1, X_2)$. The orbits for $\mathscr{X}$ are shown in Figure 3, and we note that these are also the maximal integral manifolds for $\mathsf{D}$. Note that the dimension of the integral manifolds passing through distinct points may have different dimensions. Moreover, the family of maximal integral manifolds comprises a foliation.

2. Let $\mathsf{M} = \mathbb{R}^3$ and define
$$X_1 = \frac{\partial}{\partial x_2}, \quad X_2 = \frac{\partial}{\partial x_1} + x_2\frac{\partial}{\partial x_3}.$$

We let $\mathsf{D}$ be the distribution generated by these vector fields. We shall see that Frobenius's Theorem provides an easy means of verifying that this distribution does not possess integral manifolds. However, let us verify this "by hand" to possibly get some insight. Let us fix $t \in \mathbb{R}$ and compute

$$\begin{aligned}
\Phi_t^{X_1}(0,0,0) &= (0,t,0),\\
\Phi_t^{X_2} \circ \Phi_t^{X_1}(0,0,0) &= (t,t,t^2),\\
\Phi_{-t}^{X_1} \circ \Phi_t^{X_2} \circ \Phi_t^{X_1}(0,0,0) &= (t,0,t^2),\\
\Phi_{-t}^{X_2} \circ \Phi_{-t}^{X_1} \circ \Phi_t^{X_2} \circ \Phi_t^{X_1}(0,0,0) &= (0,0,t^2).
\end{aligned}$$

Now suppose that $\mathsf{S}$ is an integral manifold for $\mathsf{D}$ containing $\mathbf{0} = (0,0,0)$. Thus $\mathsf{S}$ must be two-dimensional. Since $X_1, X_2 \in \Gamma^\infty(\mathsf{D})$ and since $\mathsf{TS} \subseteq \mathsf{D}$, it must be the case the integral curves, and therefore concatenations of integral curves, of $X_1$ and $X_2$ with initial conditions in $\mathsf{S}$ must remain in $\mathsf{S}$. Therefore, for each $t \in \mathbb{R}$, we must have $(0,0,t^2) \in \mathsf{S}$. Therefore, $\frac{\partial}{\partial x_3} \in \mathsf{T}_{\mathbf{0}}\mathsf{S}$. However, one readily checks that $(X_1(\mathbf{0}), X_2(\mathbf{0}), \frac{\partial}{\partial x_3})$ is linearly independent, prohibiting $\mathsf{S}$ from being two-dimensional. Thus $\mathsf{D}$ has no integral manifold passing through $\mathbf{0}$. One can show, in fact, that $\mathsf{D}$ possesses no integral manifolds passing through *any* point.

3. We next consider the example from Example 7.7–2. We take $\mathsf{M} = \mathbb{R}^2$ and define

$$X_1(x_1,x_2) = \frac{\partial}{\partial x_1}, \quad X_2(x_1,x_2) = f(x_1)\frac{\partial}{\partial x_2},$$

where

$$f(x) = \begin{cases} \mathrm{e}^{-1/x^2}, & x \in \mathbb{R}_{>0},\\ 0, & x \in \mathbb{R}_{\le 0}. \end{cases}$$

In Example 7.7–2 we showed that there was one orbit, and this was all of $\mathbb{R}^2$. If $(x_{01}, x_{02}) \in \mathbb{R}^2$ with $x_{01} > 0$ we can see that

$$\{(x_1,x_2) \in \mathbb{R}^2 \mid\ x_1 \in \mathbb{R}_{>0}\}$$

is the unique maximal integral manifold of $\mathsf{D}$ through $(x_{01}, x_{02})$. For $(x_{01}, x_{02}) \in \mathbb{R}^2$ with $x_{01} < 0$, the maximal integral manifold of $\mathsf{D}$ through $(x_{01}, x_{02})$ is

$$\{(-x_1, x_{02}) \mid\ x_1 \in \mathbb{R}_{>0}\}.$$

Note that there are no integral manifolds through points on the $x_2$-axis.

4. Our next example shows that integral manifolds can be isolated. We consider the vector fields

$$X_1(x_1,x_2,x_3) = x_1x_3\frac{\partial}{\partial x_1} + \frac{\partial}{\partial x_2}, \quad X_3(x_1,x_2,x_2) = \frac{\partial}{\partial x_3}$$

on $\mathbb{R}^3$, and let $\mathsf{D}$ be the distribution generated by these vector fields. Note that

$$\mathsf{S} = \{(x_1,x_2,x_3) \mid\ x_1 = 0\}$$

is an integral manifold for $\mathsf{D}$. However, $\mathsf{D}$ is not integrable. Let us verify this, using Frobenius's Theorem below. We calculate

$$[X_1, X_2](x_1, x_2, x_3) = -x_1 \frac{\partial}{\partial x_1},$$

and note that $[X_1, X_2](x_1, x_2, x_3) \in \mathsf{D}_{(x_1,x_2,x_3)}$ if and only if $x_1 = 0$. Thus $\mathsf{D}|(\mathbb{R}^3 \setminus \mathsf{S})$ is not involutive and so $\mathsf{D}$ possesses no integral manifolds other than $\mathsf{S}$.

5. Let us now consider the family $\mathscr{X} = ((X_j, \mathcal{U}_j))_{j \in \{1,2\}}$ of partially defined vector fields from Example 7.7–4. In this example, the orbits are also integral manifolds for $\mathsf{D}(\mathscr{X})$. The resulting family of immersed submanifolds defines a smooth foliation (cf. Proposition 7.3) but not an analytic foliation (cf. Example 7.4). •

**8.3. Frobenius's Theorem, examples, and counterexamples.** The smooth part of the following theorem was proved by Frobenius [1877] and the analytic part was proved by Nagano [1966]. Contributions also come from [Hermann 1960].

**8.6 Theorem: (Frobenius's Theorem)** *Let $r \in \{\infty, \omega\}$, let $\mathsf{M}$ be a $C^r$-manifold, let $\mathscr{F}$ be a subsheaf of $\mathscr{G}^r_{\mathsf{TM}}$, and let $\mathsf{D}$ be a $C^r$-distribution on $\mathsf{M}$. Then the following statements hold:*

*(i) if $\mathsf{D}$ is integrable then it is involutive;*

*(ii) if $\mathscr{G}^r_{\mathsf{D}}$ is locally finitely generated and if $\mathsf{D}$ is involutive, then it is integrable;*

*(iii) if $\mathscr{F}$ is locally finitely generated and involutive, then it is integrable.*

*In particular,*

*(iv) if $r = \infty$ and if $\mathrm{rank}_{\mathsf{D}}$ is locally constant, then $\mathsf{D}$ is integrable if and only if it is involutive and*

*(v) if $r = \omega$ then $\mathsf{D}$ is integrable if and only if it is involutive.*

*Moreover, in case the hypotheses are satisfied in either of the above cases, the set of maximal integral manifolds forms a foliation of $\mathsf{M}$.*

Proof: In the proof of the theorem we shall use the Orbit Theorem for a certain class of partially defined vector fields. Specifically, given a distribution $\mathsf{D}$, we shall consider the partially defined vector fields $(X, \mathcal{U})$ where $\mathcal{U} \subseteq \mathsf{M}$ is open and $X \in \Gamma^r(\mathsf{D}|\mathcal{U})$. Thus the class of partially defined vector fields is exactly the collection of local sections of the subsheaf $\mathscr{G}^r_{\mathsf{D}}$. For this reason, we this family of partially defined vector fields simply by $\mathscr{G}^r_{\mathsf{D}}$.

(i) First suppose that $\mathsf{D}$ is integrable. Let $x_0 \in \mathsf{M}$ and let $\mathsf{S}$ be the maximal integral manifold through $x_0$. Since every $\mathsf{D}$-valued vector field is tangent to $\mathsf{S}$ since $\mathsf{TS} \subseteq \mathsf{D}$, it follows that $\mathcal{U} \subseteq \mathsf{S}$ for some neighbourhood $\mathcal{U}$ of $x_0$ in the orbit topology. Thus $\mathsf{T}\mathcal{U} \subseteq \mathsf{TS}$. Moreover, if $x \in \mathcal{U}$ and if $v_x \in \mathsf{T}_x\mathsf{S} = \mathsf{D}_x$, then let $X \in \mathscr{G}^r_{\mathsf{D}}(\mathcal{U})$ be such that $v_x = X(x)$ (recalling Theorem 5.1 if $r = \infty$ or Theorem 5.2 if $r = \omega$). Then, since $\Phi^X_t(x) \in \mathcal{U} \subseteq \mathrm{Orb}(x_0, \mathscr{G}^r_{\mathsf{D}})$ for $t$ sufficiently small, it follows that $X(x) = v_x \in \mathsf{T}_x\mathcal{U}$. That is to say, for $x \in \mathcal{U}$, $\mathsf{T}_x\mathsf{S} \subseteq \mathsf{T}_x \mathrm{Orb}(x_0, \mathscr{G}^r_{\mathsf{D}})$. Thus $\mathsf{T}_x\mathsf{S} = \mathsf{T}_x \mathrm{Orb}(x_0, \mathscr{G}^r_{\mathsf{D}})$ since we obviously have $\mathsf{T}_x \mathrm{Orb}(x_0, \mathscr{G}^r_{\mathsf{D}}) \subseteq \mathsf{T}_x\mathsf{S}$. Since $\mathsf{L}^{(\infty)}(\mathscr{G}^r_{\mathsf{D}})_x \subseteq \mathsf{T}_x \mathrm{Orb}(x_0, \mathscr{G}^r_{\mathsf{D}})$ by Theorem 7.18, it follows that $\mathsf{L}^{(\infty)}(\mathscr{G}^r_{\mathsf{D}})_x = \mathsf{D}_x$, so $\mathsf{D}$ is involutive.

(ii) Conversely, suppose that $\mathsf{D}$ is involutive and that $\mathscr{G}^r_{\mathsf{D}}$ is locally finitely generated. Involutivity of $\mathsf{D}$ implies that if $X, Y \in \Gamma^r(\mathsf{D}|\mathcal{U})$ for some open set $\mathcal{U} \subseteq \mathsf{M}$, then $[X, Y](x) \in$

$\mathsf{D}_x$ for every $x \in \mathcal{U}$. Thus $[X, Y] \in \Gamma^r(\mathsf{D}|\mathcal{U})$ and so $\mathscr{G}^r_{\mathsf{D}}$ is an involutive subsheaf. Since $\mathscr{L}^{(\infty)}(\mathscr{G}^r_{\mathsf{D}}) = \mathscr{G}^r_{\mathsf{D}}$, it follows that $\mathscr{L}^{(\infty)}(\mathscr{G}^r_{\mathsf{D}})$ is locally finitely generated. Therefore, by Theorem 7.21, $\mathsf{T}_x \operatorname{Orb}(x, \mathscr{G}^r_{\mathsf{D}}) = \mathsf{D}_x$ for every $x \in \mathsf{M}$. Thus $\operatorname{Orb}(x, \mathscr{G}^r_{\mathsf{D}})$ is an integral manifold for $\mathsf{D}$ through $x$.

(iii) By Theorem 6.7 and involutivity of $\mathscr{F}$, $\mathsf{L}^{(\infty)}(\mathscr{F}) = \mathsf{L}^{(\infty)}(\mathscr{G}^r_{\mathsf{D}})$. This part of the result then follows from part (ii).

Parts (iv) and (v) follow from Theorem 4.9 and Theorem 4.10, respectively.

Now we verify the final assertion of the theorem. Disjointness of the orbits ensures that the orbits are maximal integral manifolds, and moreover shows that the maximal integral manifolds form a partition of $\mathsf{M}$. That this partition is a foliation follows since local generators for $\mathsf{D}$ will satisfy part (b) in the definition of a foliation. ■

Example 8.3 shows that part (i) of the proceeding theorem is not generally true for subsheaves; that is, it may be the case that an integrable locally finitely generated subsheaf may not be involutive. Note that Example 8.5–3 shows that any attempt to relax the constant rank condition in the $C^\infty$-case will be met with failure in general. Let us clarify this.

**8.7 Example: (Failure of the converse of Frobenius's Theorem in the smooth case)** We consider the vector fields

$$X_1(x_1, x_2) = \frac{\partial}{\partial x_1}, \quad X_2(x_1, x_2) = f(x_1)\frac{\partial}{\partial x_2}$$

on $\mathbb{R}^2$, where

$$f(x) = \begin{cases} \mathrm{e}^{-1/x^2}, & x \in \mathbb{R}_{>0}, \\ 0, & x \in \mathbb{R}_{\leq 0}. \end{cases}$$

As we have seen in Example 8.5–3, the distribution $\mathsf{D}$ generated by $(X_1, X_2)$ is not integrable since there is no integral manifold for $\mathsf{D}$ passing through the points of the form $(0, x_2)$, $x_2 \in \mathbb{R}$. We claim that $\mathsf{D}$ is involutive. To see this, suppose that $X, Y \in \Gamma^\infty(\mathsf{D})$ and write

$$X = \alpha_X \frac{\partial}{\partial x_1} + \beta_X \frac{\partial}{\partial x_2}, \quad Y = \alpha_Y \frac{\partial}{\partial x_1} + \beta_Y \frac{\partial}{\partial x_2}$$

for smooth functions $\alpha_X$, $\alpha_Y$, $\beta_X$, and $\beta_Y$. Then compute

$$\begin{aligned} [X, Y](x_1, x_2) = &\left(\frac{\partial \alpha_X}{\partial x_1}\alpha_Y + \frac{\partial \alpha_X}{\partial x_2}\beta_Y - \alpha_X \frac{\partial \alpha_Y}{\partial x_1} - \frac{\partial \alpha_Y}{\partial x_2}\beta_X\right)\frac{\partial}{\partial x_1} \\ &- \left(\alpha_X \frac{\partial \beta_Y}{\partial x_1} - \alpha_Y \frac{\partial \beta_X}{\partial x_1} - \frac{\partial \beta_X}{\partial x_2}\beta_Y + \beta_X \frac{\partial \beta_Y}{\partial x_2}\right)\frac{\partial}{\partial x_2}. \end{aligned}$$

We consider three cases.

1. $x_1 \in \mathbb{R}_{<0}$: Here, in some neighbourhood of $(x_1, x_2)$, $\beta_X$ and $\beta_Y$ are zero. In this case, in this neighbourhood $[X, Y]$ is collinear with $\frac{\partial}{\partial x_1}$ and so $[X, Y](x_1, x_2) \in \mathsf{D}_{(x_1, x_2)}$.

2. $x_1 = 0$: In this case, the requirement that $X$ and $Y$ are $\mathsf{D}$-valued implies that

$$\beta_X(x_1, x_2) = \beta_Y(x_1, x_2) = 0, \quad \frac{\partial \beta_X}{\partial x_1}(x_1, x_2) = \frac{\partial \beta_Y}{\partial x_1}(x_1, x_2) = 0.$$

Thus, in this case we again have $[X, Y](x_1, x_2)$ collinear with $\frac{\partial}{\partial x_1}$, and so $[X, Y](x_1, x_2) \in \mathsf{D}_{(x_1,x_2)}$.

3. $x_1 \in \mathbb{R}_{>0}$: Here we obviously have $[X, Y](x_1, x_2) \in \mathsf{D}_{(x_1,x_2)}$ since $\mathsf{D}_{(x_1,x_2)} = \mathsf{T}_{(x_1,x_2)}\mathbb{R}^2$ when $x_1 \in \mathbb{R}_{>0}$.

This gives the desired involutivity of $\mathsf{D}$. •

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