The International Journal of Nonlinear Mechanics
30(6), 793–815, 1995
DOI: 10.1016/0020-7462(95)00024-0

Variational principles for constrained systems: theory and experiment

Andrew D. Lewis* Richard M. Murray†

1994/10/07
Last updated: 1998/04/09

Abstract

In this paper we present two methods, the nonholonomic method and the vakonomic method, for deriving equations of motion for a mechanical system with constraints. The resulting equations are compared. Results are also presented from an experiment for a model system: a ball rolling without sliding on a rotating table. Both sets of equations of motion for the model system are compared with the experimental results. The effects of various forms of friction are considered in the nonholonomic equations. With appropriate friction terms, the nonholonomic equations of motion for the model system give reasonable agreement with the experimental observations.

Keywords. variational methods, mechanics, nonholonomic, vakonomic.

AMS Subject Classifications (2020). 70E99, 70H25, 70H26.

1. Introduction

Until recently there has been little attention paid to nonholonomic constraints in the geometric mechanics literature. There has been some recent effort to cast some of the ideas of nonholonomic mechanics in a more mathematical setting to make it consistent with the treatment received by unconstrained mechanics. For a survey of such efforts see [Bloch, Krishnaprasad, Marsden, and Murray 1996] and the references contained therein.

For deriving equations of motion for systems with constraints, there are at least two methods one may use. We call them the *nonholonomic* method and the *vakonomic* method. The nonholonomic method is the classical method for deriving equations of motion for constrained systems. A thorough exposition of this method, in classical language, may be found in [Pars 1965]. In this reference one will find various methods of determining equations of motion for systems with constraints. All of these equations of motion are equivalent and differ only in how the constraint forces are handled. The vakonomic method was originally proposed in [Kozlov 1983]. This method treats mechanical systems with constraints

*Professor, Department of Mathematics and Statistics, Queen's University, Kingston, ON K7L 3N6, Canada
Email: `andrew.lewis@queensu.ca`, URL: `http://www.mast.queensu.ca/~andrew/`
Work performed while a Graduate Research Assistant at the California Institute of Technology.
Research supported in part by a grant from the Powell Foundation.
†Professor, Control and Dynamical Systems, Mail Code 104–44 California Institute of Technology, Pasadena, CA 91125, U.S.A.
Email: `murray@indra.caltech.edu`, URL: `http://avalon.caltech.edu/~murray/`

as a standard constrained variational problem and the equations of motion are derivable using techniques from the calculus of variations with constraints. Kharlomov [1992] gives a critique of the vakonomic method which presents some "thought experiments" for certain systems, including a billiard ball. A counterpoint of this critique appears in [Kozlov 1992].

In this paper we present the nonholonomic and vakonomic methods for deriving equations of motion for systems with constraints and compare them with each other. We consider systems with what we shall call *affine* constraints. Pars [1965] refers to these systems as *acatastatic*. This separates our presentation slightly from the usual presentations of constrained mechanics where the affine part of the constraint is zero. We also define what it means for an affine constraint to be *holonomic*. This may be thought of as a modest generalisation of the Frobenius notion of integrability for distributions to affine constraints. In the case where the constraints are holonomic, the nonholonomic and vakonomic equations are shown to give the same physical motions for the system. These results are presented in Section 2.

In Section 3 we introduce our example of a ball rolling on a rotating table. We point out that this is a system with affine constraints. We derive the equations for this system using both the nonholonomic and vakonomic methods. In the nonholonomic approach an analytical solution is possible. With the vakonomic method we present some simulations to determine the behaviour of the system. We show that for the ball on the rotating table, it is not possible to obtain the solutions for the nonholonomic method as a subset of the solutions for the vakonomic method. In this section we also present some data from an experiment which was performed. We show that, with the addition of suitable friction terms to the nonholonomic model, it is possible to obtain reasonable agreement of the analytical and experimental data. This provides some justification for the adoption of the nonholonomic method as a legitimate way to model mechanical systems with constraints.

In Section 4 we present some questions which still need to be addressed regarding variational methods and their applicability for modelling physical systems. We also include four appendices in which we present various technical details. The reader may refer to these at appropriate times, but an understanding of the material should not be too severely jeopardised if a reading of the appendices is omitted.

2. Methods for modelling mechanical systems with constraints

In this section we present the nonholonomic and vakonomic methods for deriving the equations of motion of a mechanical system with constraints. We shall try to be somewhat precise without overly burdening the presentation with technicalities. If the reader is so inclined he may refer to appendices which give details.

We shall use the following notation:

Q	: a smooth configuration manifold which is n-dimensional, paracompact, and connected.
$\tau_M \colon TM \to M$	: the tangent bundle projection of a manifold M.
$\pi_M \colon T^*M \to M$	: the cotangent bundle projection of a manifold M.
L	: a Lagrangian which is simply a function on $TQ \times \mathbb{R}$.
$\dot{c}(t)$	: the derivative of a curve c. It is defined by $\dot{c}(t) = Tc(t, 1)$.
J	: the classical functional (see Appendix A).

d : the exterior derivative of a differential form.

All mappings shall be assumed to be smooth unless otherwise stated.

We now make clear the type of constraints we shall consider.

2.1 Definition: An ***affine constraint*** on Q is a pair, (D, γ) where D is a distribution on Q and γ is a vector field on Q. A curve $c\colon [a, b] \to Q$ will be said to ***satisfy*** the affine constraint (D, γ) if $\dot{c}(t) - \gamma(c(t)) \in D(c(t))$ for all $t \in [a, b]$. •

We shall assume that D has a constant rank k for simplicity. We will use this fact to suppose, at least locally, the existence of $n - k$ linearly independent one-forms, $\omega^1, \ldots, \omega^{n-k}$, which annihilate the distribution. That is to say we have

$$D(q) = \ker\{\omega^1(q), \ldots, \omega^{n-k}(q)\}.$$

All solutions of the constrained system are required to satisfy the condition

$$\omega^a(\dot{c}(t)) = \omega^a(\gamma(c(t))), \qquad a = 1, \ldots, n - k.$$

At this time, readers unfamiliar with techniques in the calculus of variations as applied to mechanics may wish to refer to Appendix A. Here they will find the notions of a ***variation*** and an ***infinitesimal variation*** defined.

2.1. The nonholonomic method. In this variational method, one applies the constraints *after* making the functional J stationary. Let us formulate this problem more precisely. Let (D, γ) be an affine constraint on Q. As in Appendix A, for $q_1, q_2 \in Q$ we define the set of twice differentiable curves which connect q_1 to q_2 and satisfy the constraints as

$$\begin{aligned} C^2(q_1, q_2, [a, b], D, \gamma) = \{c\colon [a, b] \to Q \mid \ & c \text{ is a } C^2 \text{ curve}, \\ & c(a) = q_1,\ c(b) = q_2, \text{ and } \dot{c}(t) - \gamma(c(t)) \in D(c(t)) \text{ for } t \in [a, b]\}. \end{aligned}$$

From now on we shall tacitly assume that $C^2(q_1, q_2, [a, b], D, \gamma)$ is not empty. That is to say, we suppose that there are C^2 curves which connect q_1 and q_2 and which satisfy the affine constraint. We shall regard $C^2(q_1, q_2, [a, b], D, \gamma)$ as a subset of $C^2(q_1, q_2, [a, b])$. At a point $c \in C^2(q_1, q_2, [a, b], D, \gamma)$, we define a subset of the tangent space $T_c C^2(q_1, q_2, [a, b])$ by

$$\begin{aligned} X_c(q_1, q_2, [a, b], D) = \{u\colon [a, b] \to TQ \mid \ & u \text{ is } C^2,\ \tau_Q \circ u = c, \\ & u(a) = 0,\ u(b) = 0, \text{ and } u(t) \in D(c(t)) \text{ for } t \in [a, b]\}. \end{aligned}$$

For a discussion of the meaning of $X_c(q_1, q_2, [a, b], D)$ see Appendix A. We define the functional J by

$$\begin{aligned} J\colon C^2(q_1, q_2, [a, b], D, \gamma) &\to \mathbb{R} \\ c &\mapsto \int_a^b L(\dot{c}(t), t)\,\mathrm{d}t \end{aligned}.$$

The variational problem is stated as a definition.

2.2 Definition: A curve, $c \in C^2(q_1, q_2, [a, b], D, \gamma)$, will be called a ***solution to the nonholonomic constrained variational problem*** if $\boldsymbol{d}J(c) \cdot u = 0$ for every $u \in X_c(q_1, q_2, [a, b], D)$. •

The following result is natural given our definition of the problem. Recall that a Lagrangian is said to be ***regular*** if the corresponding Legendre transformation is a local diffeomorphism (see [Abraham and Marsden 1978]).

2.3 Proposition: *Let L be a regular Lagrangian on Q, and let (D, γ) be an affine constraint on Q. Then $c \in C^2(q_1, q_2, [a, b], D, \gamma)$ is a solution of the nonholonomic constrained variational problem if and only if*

$$\left[\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{q}^i}\right) - \frac{\partial L}{\partial q^i}\right] u^i(t) = 0$$

for every $u \in X_c(q_1, q_2, [a, b], D)$.

Proof: Let c_s be a variation whose infinitesimal variation is $u \in X_c(q_1, q_2, [a, b], D)$. Then, as in the proof of Proposition A.1, we have

$$\boldsymbol{d}J(c) \cdot u = \int_a^b \left(\frac{\partial L}{\partial q^i}\frac{\partial q^i}{\partial s} + \frac{\partial L}{\partial \dot{q}^i}\frac{\partial \dot{q}^i}{\partial s}\right)\bigg|_{s=0} \mathrm{d}t.$$

In this case we simply have

$$\left.\frac{\partial q^i(t,s)}{\partial s}\right|_{s=0} = u^i(t), \quad \text{and} \quad \left.\frac{\partial \dot{q}^i(t,s)}{\partial s}\right|_{s=0} = \dot{u}^i(t).$$

If we do the usual integration by parts we have

$$\boldsymbol{d}J(c) \cdot u = \int_a^b \left(\frac{\partial L}{\partial q^i} - \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{q}^i}\right) u^i \, \mathrm{d}t$$

from which the proposition follows. ■

2.4 Remarks:

1. Note that we do not require c_s to be in $C^2(q_1, q_2, [a, b], D, \gamma)$ for $s \neq 0$. Thus we do not require our variations to satisfy the constraints. We only require the infinitesimal variations to satisfy the (non-affine) constraints. For a discussion of this see Appendix A. The fact that the variations do not necessarily satisfy the constraints allows us to interchange the order of differentiation with respect to s and t in determining $\partial \dot{q}^i / \partial s$. In classical terms, this allows us to interchange the "operators" δ and $\mathrm{d}/\mathrm{d}t$.

2. Observe that, unlike Hamilton's Principle, the nonholonomic constrained variational problem does not immediately give the equations of motion. This task is taken up when we discuss the Principle of Virtual Work in Appendix B. There we will show that the equations of motion for the nonholonomic method are

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{q}^i}\right) - \frac{\partial L}{\partial q^i} = \lambda_a \omega_i^a, \qquad i = 1, \ldots, n \tag{2.1}$$

along with the constraint equations

$$\omega_i^a \dot{q}^i = \omega_i^a \gamma^i, \qquad a = 1, \ldots, n-k.$$

There are other forms of the equations of motion for the nonholonomic method. An example of another form is the so-called ***Lagrange-d'Alembert*** equations. See [Bloch, Krishnaprasad, Marsden, and Murray 1996] for a discussion of this along with other forms of the equations of motion using Ehresmann connections on fibre bundles.

3. See Figure 1 for a visual representation of the nonholonomic constrained variational problem. Observe how it differs from the representation of the vakonomic problem next to it. •

2.2. The vakonomic method. In this variational technique one makes the functional J stationary *after* asking that the solutions satisfy the constraints. Thus this is a classical constrained minimisation problem, and may be solved with techniques from the calculus of variations with constraints. To make this method precise we must introduce some involved notation. Therefore, we postpone the technical proofs to Appendix C.

We begin with the definition of the solution to the vakonomic problem.

2.5 Definition: A curve, $c \in C^2(q_1, q_2, [a,b], D, \gamma)$, will be called a ***solution to the vakonomic constrained variational problem*** if c is a critical point of $J \mid C^2(q_1, q_2, [a,b], D, \gamma)$. •

In Appendix C we show that the equations of motion for a vakonomic system may derived as Lagrange's equations for the appended Lagrangian

$$\mathscr{L}(\dot{c}(t), t) = L(\dot{c}(t), t) - \lambda_a(t)[\omega^a(\dot{c}(t)) - \omega^a(\gamma(c(t)))]$$

defined on $Q \times \mathbb{R}^{n-k}$. Here $(\lambda_1, \ldots, \lambda_{n-k})$ are to be regarded as generalised coordinates for $\mathbb{R}^{n-k}$.

Let us further examine the equations of motion for the vakonomic problem. In coordinates we have

$$\mathscr{L}(q, \dot{q}, t) = L(q, \dot{q}, t) - \lambda_a \omega_i^a \dot{q}^i + \lambda_a \omega_i^a \gamma^i.$$

Lagrange's equations for the Lagrangian $\mathscr{L}$ then read

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial \mathscr{L}}{\partial \dot{q}^i}\right) - \frac{\partial \mathscr{L}}{\partial q^i} &= \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{q}^i} - \lambda_a \omega_i^a\right) - \frac{\partial L}{\partial q^i} + \\ &\qquad \lambda_a \frac{\partial \omega_j^a}{\partial q^i} \dot{q}^j - \lambda_a \frac{\partial \omega_j^a}{\partial q^i} \gamma^j - \lambda_a \omega_j^a \frac{\partial \gamma^j}{\partial q^i} \\ &= \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{q}^i}\right) - \frac{\partial L}{\partial q^i} - \dot{\lambda}_a \omega_i^a - \lambda_a \frac{\partial \omega_j^a}{\partial q^i} \gamma^j - \lambda_a \omega_j^a \frac{\partial \gamma^j}{\partial q^i} = 0, \qquad (2.2) \\ &\qquad i = 1, \ldots, n. \end{aligned}$$

Appended to these are the constraint equations which are simply the "λ-part" of Lagrange's equations:

$$\omega_i^a \dot{q}^i = \omega_i^a \gamma^i, \qquad a = 1, \ldots, n-k. \qquad (2.3)$$

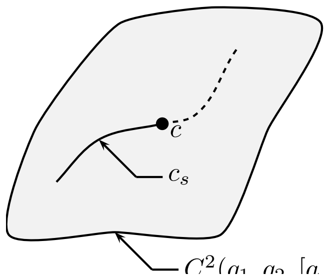

Figure 1. A representation of the nonholonomic constrained variational problem

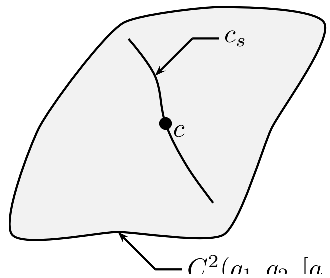

Figure 2. A representation of the vakonomic constrained variational problem

2.6 Remarks: 1. Observe that, in practice, the equations (2.2) and (2.3) constitute a set of implicit first order ordinary differential equations in the variables $(q, \dot{q}, \lambda)$. This means that one must specify initial conditions for the Lagrange multipliers for the vakonomic problem.

2. In the case when $\gamma = 0$, the equations of motion for the vakonomic problem look like the equations of motion for the nonholonomic problem except there is now a $\dot{\lambda}_a$ in place of λ_a.

3. See Figure 2 for a visual representation of the vakonomic constrained variational problem. •

2.3. The nonholonomic and vakonomic methods compared. Generally, the nonholonomic and vakonomic methods yield different equations of motion. This is readily seen by observing that the vakonomic equations have $\dot{\lambda}_a$'s in them which are not present in the nonholonomic equations. For certain systems, however, it is possible to choose the initial conditions for the Lagrange multipliers in the vakonomic equations in such a way that the resulting solution is exactly that determined by the nonholonomic method. This occurs, for example, in the example of a penny rolling upright on a planar surface (see [Bloch and Crouch 1995]). This is not the case in general, however, as we will show when we discuss the ball rolling on the rotating table.

It also turns out that when the constraints are holonomic, the nonholonomic and vakonomic problems are equivalent. We shall say that an affine constraint (D, γ) is ***holonomic*** if D is integrable and if γ is a section in D. Notice that this is a modest generalisation of what we would denote as an holonomic constraint for systems with no affine part. In that case the constraint is simply the distribution, D, and is holonomic if D is integrable.

2.7 Remark: Note that if (D, γ) is an holonomic affine constraint, then $C^2(q_1, q_2, [a, b], D, \gamma)$ is non-empty if and only if q_1 and q_2 lie in the same connected component of a leaf of the foliation defined by D. Also, any curve that is in a leaf of the foliation defined by D will automatically satisfy the constraints. Thus the definition is only a mild generalisation of the usual notion of integrability of a distribution. •

Let Λ be a leaf of the foliation defined by D. Given a Lagrangian on Q we may define a Lagrangian L_Λ on Λ by restriction of L to $T\Lambda \times \mathbb{R} \subset TQ \times \mathbb{R}$. With this Lagrangian we may define a function on $C^2(q_1, q_2, [a, b], D, \gamma)$ by

$$\begin{aligned} J_\Lambda \colon C^2(q_1, q_2, [a, b], D, \gamma) &\to \mathbb{R} \\ c &\mapsto \int_a^b L_\Lambda(\dot{c}(t), t)\, \mathrm{d}t. \end{aligned} \tag{2.4}$$

The result is thus stated.

2.8 Proposition: *Let L be a Lagrangian on Q, and let (D, γ) be an integrable affine constraint on Q. Let $c \in C^2(q_1, q_2, [a, b], D, \gamma)$ where q_1 and q_2 lie in a leaf Λ of the foliation determined by D. Let J_Λ be the function defined by (2.4). Then the following are equivalent:*

(i) c is a solution of the nonholonomic constrained variational problem.

(ii) c is a solution of the vakonomic constrained variational problem.

(iii) c is a critical point of J_Λ.

(iv) c is a solution of Lagrange's equations on Λ with Lagrangian L_Λ.

Proof: By Frobenius' theorem, we may choose coordinates, $(x^1, \dots, x^k, y^1, \dots, y^{n-k})$, around any point $q \in \Lambda$ which have the properties:

1. $(x^1, \dots, x^k)$ are coordinates for Λ,
2. the injection of Λ into Q looks like $(x^1, \dots, x^k) \mapsto (x^1, \dots, x^k, 0, \dots, 0)$, and
3. $D = \ker\{dy^1, \dots, dy^{n-k}\} = \langle \frac{\partial}{\partial x^1}, \dots, \frac{\partial}{\partial x^k} \rangle$.

We first look at the equations of motion for the nonholonomic problem. By (2.1) we know that $c \in C^2(q_1, q_2, [a, b], D, \gamma)$ is a solution of the nonholonomic constrained variational problem if and only if

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{q}^i}\right) - \frac{\partial L}{\partial q^i} = \lambda_a \omega_i^a, \qquad i = 1, \dots, n \tag{2.5}$$

for some $\lambda_1, \dots, \lambda_{n-k}$ defined on $[a, b]$. In the coordinates $(x^1, \dots, x^k, y^1, \dots, y^{n-k})$ above, the curve c looks like

$$t \mapsto (x^1(t), \dots, x^k(t), 0, \dots, 0).$$

The equations (2.5) in the coordinates $(x^1, \dots, x^k, y^1, \dots, y^{n-k})$ are thus

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{x}^\sigma}\right) - \frac{\partial L}{\partial x^\sigma} = 0, \qquad \sigma = 1, \dots, k, \tag{2.6}$$

$$\frac{\partial^2 L}{\partial \dot{y}^a \partial t} - \frac{\partial L}{\partial y^a} = \lambda_a, \qquad a = 1, \dots, n-k. \tag{2.7}$$

Note that (2.7) simply specifies the Lagrange multipliers and has no effect on the solution in Q since all the time evolution there is specified by (2.6).

Now we turn to the vakonomic problem. The appended Lagrangian to be used in the coordinates coordinates $(x^1, \dots, x^k, y^1, \dots, y^{n-k})$ is

$$\mathscr{L} = L - \lambda_a \dot{y}^a.$$

We may easily determine that the equations (2.2) appear in these coordinates as

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{x}^\sigma}\right) - \frac{\partial L}{\partial x^\sigma} = 0, \qquad \sigma = 1,\dots,k \tag{2.8}$$

$$\frac{\partial^2 L}{\partial \dot{y}^a \partial t} - \frac{\partial L}{\partial y^a} = \dot{\lambda}_a, \qquad a = 1,\dots,n-k. \tag{2.9}$$

Here again we have used the fact that $y^1 = \cdots = y^{n-k} = 0$ along c. As with the nonholonomic equations, (2.9) serves to determine the Lagrange multipliers and does not affect the time evolution of the coordinates $(x^1,\dots,x^k)$.

In both the nonholonomic and vakonomic equations, the constraint equations are null since γ is a section of D.

Lagrange's equations on Λ for the Lagrangian L_Λ are

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L_\Lambda}{\partial \dot{x}^\sigma}\right) - \frac{\partial L_\Lambda}{\partial x^\sigma} = 0, \qquad \sigma = 1,\dots,k. \tag{2.10}$$

Note that since $y^1 = \cdots = y^{n-k} = 0$ along c we have

$$\frac{\partial L_\Lambda}{\partial \dot{x}^\sigma} = \frac{\partial L}{\partial \dot{x}^\sigma}, \quad \text{and} \quad \frac{\partial L_\Lambda}{\partial x^\sigma} = \frac{\partial L}{\partial x^\sigma}, \qquad \sigma = 1,\dots,k. \tag{2.11}$$

From (2.6) and (2.8) we see that the components $(x^1,\dots,x^k)$ evolve according to the same equations of motion in the nonholonomic and vakonomic problems. This proves that (i) is equivalent to (ii). Using (2.10) and (2.11) we also see that (iv) is equivalent to both (i) and (ii). Hamilton's Principle implies that (iii) is equivalent to (iv). This completes the proof. ■

2.4. Realising constraints. As a final word in our presentation of the nonholonomic and vakonomic methods, we say a few things about "realising constraints". One may think of constraints as being a limiting process where certain dynamic properties become large and so limit the motion to the constrained directions. This may be made precise in the vakonomic and nonholonomic models. These notions are given in their precise forms by Arnol′d [1988], but we shall give rough descriptions of these limits here.

The vakonomic solutions may be regarded as a limit as an inertial term becomes large. The inertial term is a degenerate one which supplies no inertial forces to motions allowed by the constraints. When this term goes to infinity, the solutions of Lagrange's equations approach a solution for the vakonomic problem.

The nonholonomic solutions may be regarded as a limit as viscosity becomes large. To be more precise, we add Rayleigh dissipation to the mechanical system which does no work on motions allowed by the constraints (thus the dissipation function is degenerate). Then, as we make the magnitude of the dissipation function go to infinity, the corresponding solutions to Lagrange's equations approach the solutions to the nonholonomic equations.

As a simple example of using these limits to obtain constraints, consider the system in Figure 3. We wish to impose the (holonomic, non-affine) constraint $x = 0$. There are several ways to do this. One way would be to let the mass M get large. This would correspond to the vakonomic limit. Another way to impose the constraint $x = 0$ would be to let the damping coefficient c tend to infinity. This would correspond to the nonholonomic limit.

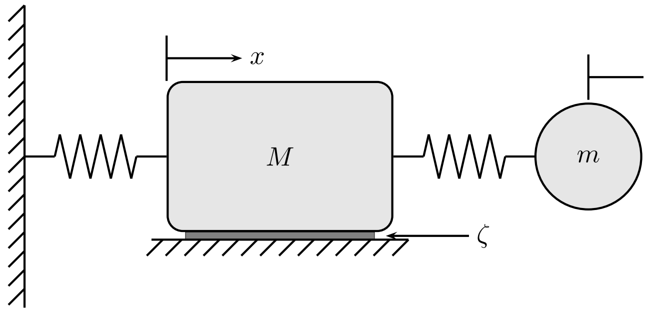

Figure 3. An example of realising constraints

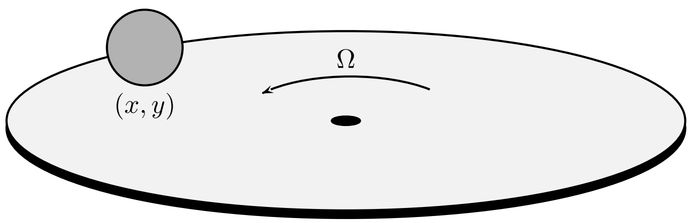

Figure 4. The rolling ball

In each case care must be taken in the limit, and the convergence to the vakonomic and nonholonomic solutions in each case is not uniform in time. Note that in this case, since the constraint is holonomic, the limiting processes should produce the same motions by Proposition 2.8.

3. The ball on a rotating table

In this section we present the mechanical system which we study analytically, numerically, and experimentally. The system is a ball rolling on a uniformly rotating table with no sliding (see Figure 4). Here (x, y) denotes the position of the point of contact of the ball with respect to the centre of rotation of the table. The z-axis will be perpendicular to the plane of the table. The ball is assumed to be spherical and to have uniform mass density. The parameters in the problem are:

m	: mass of the ball
r	: radius of the ball
I	: moment of inertia of the ball
Ω	: rotational velocity of the table

The configuration space for the system is $Q = \mathbb{R}^2 \times SO(3)$. We shall use (x, y, R) to represent a typical point in Q. The constraints for the system are given by

$$\begin{aligned}\dot{x} - r\boldsymbol{e}_1^T \dot{R}R^T \boldsymbol{e}_3 &= -\Omega y \\ \dot{y} + r\boldsymbol{e}_3^T \dot{R}R^T \boldsymbol{e}_2 &= \Omega x\end{aligned}$$

where $\{\boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3\}$ is the standard basis for $\mathbb{R}^3$. Since the matrix $\dot{R}R^T$ is skew symmetric (it represents the angular velocity of the ball in spatial coordinates), we may write

$$\dot{R}R^T = \begin{bmatrix} 0 & -\xi^3 & \xi^2 \\ \xi^3 & 0 & -\xi^1 \\ -\xi^2 & \xi^1 & 0 \end{bmatrix}$$

and ξ^1, ξ^2, ξ^3 are the rotational velocities about the x, y, z axes, respectively. With this notation, the constraints assume a more recognisable form:

$$\begin{aligned}\dot{x} - r\xi^2 &= -\Omega y \\ \dot{y} + r\xi^1 &= \Omega x.\end{aligned}$$

The Lagrangian for the rolling ball is

$$L = -\frac{1}{4} I \operatorname{tr}(\dot{R}R^T \dot{R}R^T) + \frac{1}{2} m(\dot{x}^2 + \dot{y}^2).$$

3.1. The nonholonomic ball. First we concentrate on the nonholonomic model for the rolling ball. This model turns out to be equivalent to using Newton's equations, so we will simply balance forces and torques to obtain the equations of motion. As above, ξ^1, ξ^2, ξ^3 are the components of the spatial angular velocity of the ball. The forces acting on the ball satisfy

$$m\ddot{x} = R_x + F_x \tag{3.1a}$$
$$m\ddot{y} = R_y + F_y \tag{3.1b}$$
$$I\dot{\xi}^1 = rR_y + T_1 \tag{3.1c}$$
$$I\dot{\xi}^2 = -rR_x + T_2 \tag{3.1d}$$
$$I\dot{\xi}^3 = T_3. \tag{3.1e}$$

Here R_x, R_y are the constraint forces which are to be determined from the constraint equations, and F_x, F_y, T_1, T_2, T_3 are external forces in the appropriate directions. We shall include external forces which arise from dissipative effects in Section 3.2.

Setting the external forces to zero for the moment, it is possible to explicitly derive equations of motion for the variables x, y which are independent of the rotational velocities ξ^1, ξ^2, ξ^3. To do this we determine from the constraints that

$$\xi^1 = \frac{\Omega}{r} x - \frac{1}{r}\dot{y} \tag{3.2a}$$
$$\xi^2 = \frac{\Omega}{r} y + \frac{1}{r}\dot{x} \tag{3.2b}$$

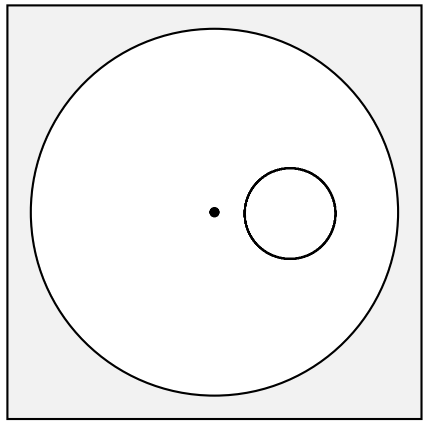

Figure 5. An orbit for the nonholonomic equations of motion with no external forces

and, differentiating this, that

$$\dot{\xi}^1 = \frac{\Omega}{r}\dot{x} - \frac{1}{r}\ddot{y} \tag{3.3a}$$

$$\dot{\xi}^2 = \frac{\Omega}{r}\dot{y} + \frac{1}{r}\ddot{x}. \tag{3.3b}$$

With external forces F_x, F_y, T_1, T_2, T_3 set to zero, we may solve for R_x, R_y in terms of $\dot{\xi}_1, \dot{\xi}_2$ from equations (3.1c) and (3.1d). These may be put in terms of $x, y, \dot{x}, \dot{y}, \ddot{x}, \ddot{y}$ from (3.2a), (3.2b), (3.3a), and (3.3b). These expressions for R_x, R_y in terms of $x, y, \dot{x}, \dot{y}, \ddot{x}, \ddot{y}$ are then substituted into (3.1a) and (3.1b) to get the following equations for x, y:

$$\begin{bmatrix} I + mr^2 & 0 \\ 0 & I + mr^2 \end{bmatrix} \begin{pmatrix} \ddot{x} \\ \ddot{y} \end{pmatrix} + \begin{bmatrix} 0 & I\Omega \\ -I\Omega & 0 \end{bmatrix} \begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}. \tag{3.4}$$

Once a solution to these equations has been found, it is possible to construct the solutions for ξ^1, ξ^2 directly from the constraint equations.

Note that these equations are, in fact, linear first order equations in $\dot{x}, \dot{y}$. We may readily determine the solution as

$$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = \frac{I + mr^2}{I\Omega} \begin{bmatrix} \sin\left(\frac{I\Omega}{I+mr^2}t\right) & \cos\left(\frac{I\Omega}{I+mr^2}t\right) \\ -\cos\left(\frac{I\Omega}{I+mr^2}t\right) & \sin\left(\frac{I\Omega}{I+mr^2}t\right) \end{bmatrix} \begin{pmatrix} \dot{x}(0) \\ \dot{y}(0) \end{pmatrix} + \begin{pmatrix} x(0) - \frac{I+mr^2}{I\Omega}\dot{y}(0) \\ y(0) + \frac{I+mr^2}{I\Omega}\dot{x}(0) \end{pmatrix}.$$

Thus, in the presence of no external forces, the nonholonomic model predicts that the point of contact of the ball will sweep out a circle on the table. See Figure 5.

3.2. Friction effects in the nonholonomic model. Here we consider the effects of adding dissipation to the equations of motion derived via the nonholonomic method. The types of dissipation we consider are:

1. *Translational viscous friction:* We are uncertain how to motivate adding this type of friction to the model, but we add it since it is easy to do, and does not make the analysis any more difficult. To add translational viscous friction we add terms of the form

$$F_x = -\nu \dot{x}$$
$$F_y = -\nu \dot{y}$$

to the equations of motion. Here ν is a strictly positive real number.

2. *Rolling friction:* This type of friction arises from the resistance that the ball encounters as it rolls over the surface of the table. One may think of the situation as depicted in Figure 6. As the ball rolls over the surface, there is an elastic deformation of the surface which creates resistance to the ball's motion. The force resulting from rolling friction turns out to be proportional to the weight of the ball, and is in the direction opposite the direction the ball is moving relative to the table. Thus the rolling friction force has the form

$$F_x = -\delta m g \frac{v_x}{\|v\|}$$
$$F_y = -\delta m g \frac{v_y}{\|v\|}$$

where g is the acceleration due to gravity and $v = (v_x, v_y)$ is the relative velocity of the ball with respect to the table. Thus

$$v_x = \dot{x} + \Omega y$$
$$v_y = \dot{y} - \Omega x.$$

See [Bidwell 1962, Flom 1962] and [Koizumi, Shibazaki, Nishio, and Nishiwaki 1983] for a discussion of how this form of the rolling friction force arises. The coefficient δ may be determined experimentally for the materials involved. An experimental study of rolling contact may be found in [Flom 1962]. Unfortunately none of the experimental results presented in this work apply directly to the parameters in our experiment so it is difficult to extrapolate an appropriate value for δ without further study of rolling friction itself.

3. *Rotational viscous friction:* In this type of friction, air resistance is modelled. The effects are assumed to be viscous so the forces added to the equations of motion are of the form

$$T_1 = -\mu \xi^1 \tag{3.5a}$$
$$T_2 = -\mu \xi^2. \tag{3.5b}$$

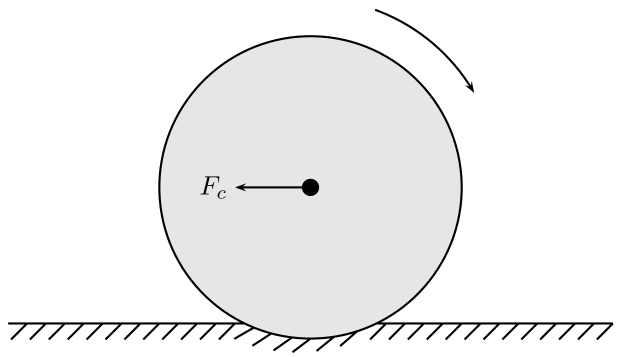

Figure 6. Rolling friction

With these types of friction it is still possible to reduce the equations of motion to equations involving only the x, y variables. Proceeding much as we did when there were no external forces we may arrive at the equations

$$\begin{bmatrix} I+mr^2 & 0 \\ 0 & I+mr^2 \end{bmatrix} \begin{pmatrix} \ddot{x} \\ \ddot{y} \end{pmatrix} + \begin{bmatrix} \mu+\nu r^2 & I\Omega \\ -I\Omega & \mu+\nu r^2 \end{bmatrix} \begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} + \begin{bmatrix} 0 & \mu\Omega \\ -\mu\Omega & 0 \end{bmatrix} \begin{pmatrix} x \\ y \end{pmatrix} - \frac{\delta m g}{\|v\|} \begin{pmatrix} v_x \\ v_y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \qquad (3.6)$$

which describe the motion of the point of contact of the ball on the table.

3.3. Analysis and simulation of the nonholonomic ball. Here we perform some analysis for (3.6) when such analysis is possible. When it is not, we perform simulations to give some idea of the behaviour.

Note on presentation of simulations: In our discussion of the various models for the ball rolling on a rotating table, we present some simulations. These simulations were done for a 2.54 cm diameter steel (density = 7.8 gm/cm^3) ball rolling on a table rotating at 45 rpm. These parameters were selected to match one of the experiments discussed below. So that one may visualise the experimental setup, the simulations are presented on a background which represents the experiment setup. In Figure 7 is a schematic of a plan view of the experiment. The surface on which the ball rolled was 30 cm in diameter. The arrow indicates the direction of motion in the simulations. The initial conditions for (x, y) for all simulations was ($x = 10$ cm, $y = 0$ cm, $\dot{x} = 0$ cm/sec, $\dot{y} = 5$ cm/sec). Initial conditions for ξ_1, ξ_2 are then determined by the constraints. For the nonholonomic method, the initial value of ξ_3 is inconsequential since its dynamics are decoupled from the rest of the dynamics. This, along with the initial conditions for the Lagrange multipliers, is given more consideration below. •

When $\delta = 0$ in (3.6), the equation is linear, and so it is possible to do some analysis on the equations directly.

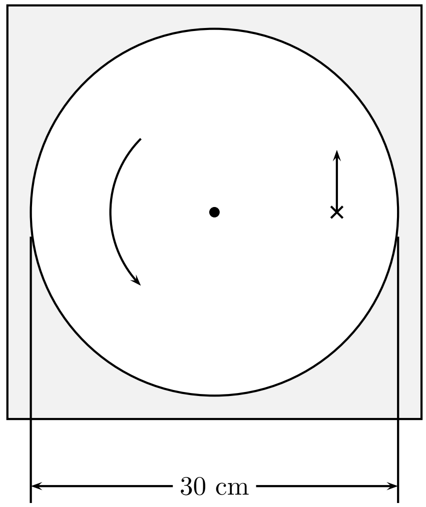

Figure 7. Schematic of the experimental setup which is a background for the simulation results.

1. $\delta = \mu = \nu = 0$*:* This is the situation dealt with in Section 3.1. Here we note that the eigenvalues of the linear system are

$$\lambda \in \left\{ 0, 0, i\frac{I\Omega}{I + mr^2}, -i\frac{I\Omega}{I + mr^2} \right\}.$$

Again, refer to Figure 5 for a typical orbit.

2. $\delta = \mu = 0$*:* In this case we have only translational viscous friction. It is still comparatively easy to determine the closed form of the solution as

$$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = \frac{I + mr^2}{I\Omega} e^{-\nu r^2 t} \begin{bmatrix} \sin\left(\frac{I\Omega}{I+mr^2}t\right) & \cos\left(\frac{I\Omega}{I+mr^2}t\right) \\ -\cos\left(\frac{I\Omega}{I+mr^2}t\right) & \sin\left(\frac{I\Omega}{I+mr^2}t\right) \end{bmatrix} \begin{pmatrix} \dot{x}(0) \\ \dot{y}(0) \end{pmatrix} + \begin{pmatrix} x(0) - \frac{I+mr^2}{I\Omega}\dot{y}(0) \\ y(0) + \frac{I+mr^2}{I\Omega}\dot{x}(0) \end{pmatrix}.$$

Note that the eigenvalues for the system are

$$\lambda \in \left\{ 0, 0, -\frac{\nu r^2}{I + mr^2} + i\frac{I\Omega}{I + mr^2}, -\frac{\nu r^2}{I + mr^2} - i\frac{I\Omega}{I + mr^2} \right\}.$$

Since $\nu > 0$, the orbits will be stable spirals. Since the two zero eigenvalues from the undamped case persist, the spirals will be asymptotic to a point on the table which depends on the initial conditions. See Figure 8 for a typical orbit in this case.

3. $\delta = 0$*:* In this case we eliminate only rolling friction so the equations are still linear. However, an explicit determination of the eigenvalues proves to be complicated. It

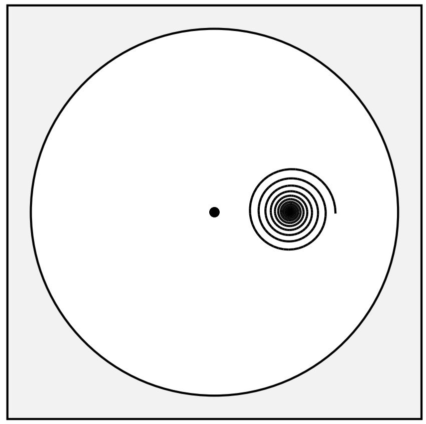

Figure 8. An orbit when $\delta = \mu = 0$ and $\nu = 5.0$.

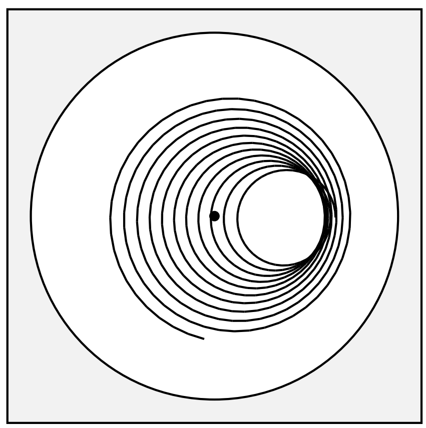

Figure 9. An orbit when $\delta = \nu = 0$ and $\mu = 6.0$.

is possible to obtain some stability boundaries using the Routh-Hurwitz method, but the resulting expressions are too bulky to allow any analysis. Therefore, a numerical investigation of the eigenvalues was performed, and the eigenvalues were typically found to be of the form

$$\lambda \in \{\alpha_1 \pm i\beta_1, \alpha_2 \pm i\beta_2\}$$

where $\alpha_1 < 0$ and the sign of α_2 was undetermined. When $\nu = 0$, (i.e., when only rotational viscous friction was present), α_2 was always observed to be positive. These numerical studies should not be regarded as conclusive, however. See Figure 9 for a typical orbit when only rotational friction is present.

4. $\delta \neq 0$*:* In this case the equations are nonlinear so numerical simulation was performed to obtain some trajectories. A typical trajectory is shown in Figure 10.

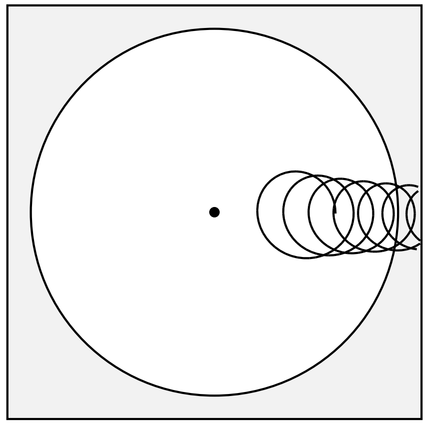

Figure 10. An orbit when $\mu = \nu = 0$ and $\delta = 0.005$.

3.4. The vakonomic ball. Now we analyse the rolling ball on a spinning table using the vakonomic approach. The derivation of the equations of motion does not proceed as easily in this case since we must begin from the Lagrangian formulation on $T(\mathbb{R}^2 \times SO(3))$. This analysis is performed by Bloch and Crouch [1995] in the case when $\Omega = 0$ (i.e., the stationary table). We follow their lead in our computations. In particular, we work on the configuration manifold $\mathbb{R}^2 \times \mathbb{R}^3 \times \mathbb{R}^{3\times 3}$ using (x, y) as coordinates for the $\mathbb{R}^2$ portion, (ξ^1, ξ^2, ξ^3) as coordinates for the $\mathbb{R}^3$ portion, and $\{R_{ij} \mid \; i, j = 1, 2, 3\}$ as coordinates for the $\mathbb{R}^{3\times 3}$ portion. Here $\boldsymbol{\xi}$ represents the angular velocity in inertial coordinates. To make the dynamics evolve on $\mathbb{R}^2 \times SO(3)$ we introduce a constraint given by

$$\dot{R} = \hat{\boldsymbol{\xi}} R \tag{3.7}$$

where $\hat{\boldsymbol{\xi}}$ is the skew-symmetric matrix

$$\hat{\boldsymbol{\xi}} = \begin{bmatrix} 0 & -\xi^3 & \xi^2 \\ \xi^3 & 0 & -\xi^1 \\ -\xi^2 & \xi^1 & 0 \end{bmatrix}.$$

We also still have the constraint that the ball roll without slipping.

In Appendix D we show that the equations of motion for the vakonomic ball are Lagrange's equation for the Lagrangian

$$\mathscr{L} = \frac{1}{2} I((\xi^1)^2 + (\xi^2)^2 + (\xi^3)^2) + \frac{1}{2} m(\dot{x}^2 + \dot{y}^2) - \\ \lambda_1(\dot{x} - r\xi^2 + \Omega y) - \lambda_2(\dot{y} + r\xi^1 - \Omega x) - \operatorname{tr}(\Lambda(\dot{R} - \hat{\boldsymbol{\xi}} R)). \tag{3.8}$$

on $\mathbb{R}^2 \times \mathbb{R}^3 \times \mathbb{R}^{3\times 3} \times \mathbb{R}^2 \times \mathbb{R}^{3\times 3}$ where $(x, y) \in \mathbb{R}^2$, $\boldsymbol{\xi} \in \mathbb{R}^3$, $R \in \mathbb{R}^{3\times 3}$, $(\lambda_1, \lambda_2) \in \mathbb{R}^2$, and $\Lambda \in \mathbb{R}^{3\times 3}$. Here Λ is a 3×3 matrix of Lagrange multipliers. We compute Lagrange's equations to be equivalent to

$$m\ddot{x} - \dot{\lambda}_1 - \Omega\lambda_2 = 0 \tag{3.9a}$$

$$m\ddot{y} - \dot{\lambda}_2 + \Omega\lambda_1 = 0 \tag{3.9b}$$

$$\dot{\Lambda}^T - \hat{\boldsymbol{\xi}}\Lambda^T = 0 \tag{3.9c}$$

$$\boldsymbol{v}^T(I\boldsymbol{\xi} + \lambda_1 r\boldsymbol{e}_2 - \lambda_2 r\boldsymbol{e}_1) + \operatorname{tr}(\Lambda\hat{\boldsymbol{v}}R) = 0, \qquad \forall\ \boldsymbol{v} \in \mathbb{R}^3, \tag{3.9d}$$

plus the constraint equations

$$\dot{x} - r\xi^2 + \Omega y = 0 \tag{3.10a}$$

$$\dot{y} + r\xi^1 - \Omega x = 0 \tag{3.10b}$$

$$\dot{R} - \hat{\boldsymbol{\xi}}R = 0 \tag{3.10c}$$

Here $\{\boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3\}$ are the standard basis vectors for $\mathbb{R}^3$. Now we differentiate (3.9d) with $\boldsymbol{v} \in \mathbb{R}^3$ arbitrary to get

$$\begin{aligned}
&\boldsymbol{v}^T(I\dot{\boldsymbol{\xi}} + \dot{\lambda}_1 r\boldsymbol{e}_2 - \dot{\lambda}_2 r\boldsymbol{e}_1) + \operatorname{tr}(\dot{\Lambda}\hat{\boldsymbol{v}}R - \Lambda\hat{\boldsymbol{v}}\dot{R}) = 0 \\
\Longrightarrow\ &\boldsymbol{v}^T I\dot{\boldsymbol{\xi}} - \operatorname{tr}(\Lambda(\hat{\boldsymbol{\xi}}\hat{\boldsymbol{v}} - \hat{\boldsymbol{v}}\hat{\boldsymbol{\xi}})R) + \boldsymbol{v}^T(\dot{\lambda}_1 r\boldsymbol{e}_2 - \dot{\lambda}_2 r\boldsymbol{e}_1) = 0 && \text{by (3.9c) and (3.7)} \\
\Longrightarrow\ &\boldsymbol{v}^T I\dot{\boldsymbol{\xi}} - \operatorname{tr}(\Lambda(\widehat{\hat{\boldsymbol{\xi}}\boldsymbol{v}})R) + \boldsymbol{v}^T(\dot{\lambda}_1 r\boldsymbol{e}_2 - \dot{\lambda}_2 r\boldsymbol{e}_1) = 0 \\
\Longrightarrow\ &\boldsymbol{v}^T I\dot{\boldsymbol{\xi}} - \boldsymbol{v}^T\hat{\boldsymbol{\xi}}(I\boldsymbol{\xi} + \lambda_1 r\boldsymbol{e}_2 - \lambda_2 r\boldsymbol{e}_1) + \boldsymbol{v}^T(\dot{\lambda}_1 r\boldsymbol{e}_2 - \dot{\lambda}_2 r\boldsymbol{e}_1) = 0 && \text{by (3.9d)} \\
\Longrightarrow\ &\boldsymbol{v}^T(I\dot{\boldsymbol{\xi}} + \hat{\boldsymbol{\xi}}(\lambda_2 r\boldsymbol{e}_1 - \lambda_1 r\boldsymbol{e}_2) + \dot{\lambda}_1 r\boldsymbol{e}_2 - \dot{\lambda}_2 r\boldsymbol{e}_1) = 0.
\end{aligned}$$

Thus

$$I\dot{\boldsymbol{\xi}} + \hat{\boldsymbol{\xi}}(\lambda_2 r\boldsymbol{e}_1 - \lambda_1 r\boldsymbol{e}_2) + \dot{\lambda}_1 r\boldsymbol{e}_2 - \dot{\lambda}_2 r\boldsymbol{e}_1 = 0. \tag{3.11}$$

The final thing to be done to get equations that are in a form for simulation is solving for $\dot{\lambda}_1, \dot{\lambda}_2$. From (3.11) we get

$$\begin{pmatrix}\dot{\lambda}_1 \\ \dot{\lambda}_2\end{pmatrix} = \begin{pmatrix}-\frac{I}{r}\dot{\xi}^2 - \lambda_2\xi^3 \\ \frac{I}{r}\dot{\xi}^1 + \lambda_1\xi^3\end{pmatrix}.$$

From the constraints and (3.9a), and (3.9b) we have

$$\begin{aligned}
r\dot{\xi}^1 &= \Omega\dot{x} - \ddot{y} = \frac{1}{m}(\lambda_1\Omega - \dot{\lambda}_2) + \Omega\dot{x} \\
r\dot{\xi}^2 &= \Omega\dot{y} + \ddot{x} = \frac{1}{m}(\lambda_2\Omega + \dot{\lambda}_1) + \Omega\dot{y}.
\end{aligned}$$

This then gives

$$\begin{bmatrix}1 + \frac{I}{mr^2} & 0 \\ 0 & 1 + \frac{I}{mr^2}\end{bmatrix}\begin{pmatrix}\dot{\lambda}_1 \\ \dot{\lambda}_2\end{pmatrix} = \begin{pmatrix}-\frac{I\Omega}{mr^2}\lambda_2 - \frac{I\Omega}{r^2}\dot{y} - \lambda_2\xi^3 \\ \frac{I\Omega}{mr^2}\lambda_1 + \frac{I\Omega}{r^2}\dot{x} + \lambda_1\xi^3\end{pmatrix}.$$

Therefore, the equations which we simulate are

$$m\ddot{x} - \dot{\lambda}_1 - \Omega\lambda_2 = 0 \tag{3.12}$$
$$m\ddot{y} - \dot{\lambda}_2 + \Omega\lambda_1 = 0 \tag{3.13}$$
$$\boldsymbol{I}\dot{\boldsymbol{\xi}} + \hat{\boldsymbol{\xi}}(\lambda_2 r\boldsymbol{e}_1 - \lambda_1 r\boldsymbol{e}_2) + \dot{\lambda}_1 r\boldsymbol{e}_2 - \dot{\lambda}_2 r\boldsymbol{e}_1 = 0 \tag{3.14}$$
$$\begin{bmatrix} 1 + \frac{I}{mr^2} & 0 \\ 0 & 1 + \frac{I}{mr^2} \end{bmatrix} \begin{pmatrix} \dot{\lambda}_1 \\ \dot{\lambda}_2 \end{pmatrix} = \begin{pmatrix} -\frac{I\Omega}{mr^2}\lambda_2 - \frac{I\Omega}{r^2}\dot{y} - \lambda_2\xi^3 \\ \frac{I\Omega}{mr^2}\lambda_1 + \frac{I\Omega}{r^2}\dot{x} + \lambda_1\xi^3 \end{pmatrix}. \tag{3.15}$$

These may be further simplified using the relation $I = \frac{2}{5}mr^2$ for a solid spherical ball.

3.5. Analysis and simulation of the vakonomic ball. Not much can be easily done in the way of concrete analysis of the vakonomic equations for the ball on the rotating table. However, there is one important observation that we can make which illustrates that, for this system, the nonholonomic and vakonomic methods are fundamentally different. We state this as a lemma.

3.1 Lemma: *Let* $q_0 = (x_0, y_0, \xi_0^1, \xi_0^2, \xi_0^3) \in \mathbb{R}^2 \times \mathbb{R}^3$ *and let*

$$c_{q_0} \colon t \mapsto (x(t), y(t), \xi^1(t), \xi^2(t), \xi^3(t))$$

be an integral curve for the nonholonomic equations of motion through q_0 *at* $t = 0$*. Then we may choose* q_0 *so that* c_{q_0} *is not a solution of the vakonomic equations of motion for any choice of initial conditions for the Lagrange multipliers.*

Proof: Substituting (3.15) into (3.12) and (3.13) we get

$$m\ddot{x} + \frac{mI\Omega}{I + mr^2}\dot{y} + \Omega\left(\frac{I}{I+mr^2} - 1\right)\lambda_2 + \frac{mr^2}{I+mr^2}\lambda_2\xi^3 = 0$$
$$m\ddot{y} - \frac{mI\Omega}{I + mr^2}\dot{x} + \Omega\left(1 - \frac{I}{I+mr^2}\right)\lambda_1 - \frac{mr^2}{I+mr^2}\lambda_1\xi^3 = 0.$$

The nonholonomic equations for x, y may be written as

$$m\ddot{x} + \frac{mI\Omega}{I+mr^2}\dot{y} = 0$$
$$m\ddot{y} - \frac{mI\Omega}{I+mr^2}\dot{x} = 0.$$

We may easily see that these equations will give the same motions in x and y only if

$$\lambda_2(\xi^3 - \Omega) = 0$$
$$\lambda_1(\xi^3 - \Omega) = 0.$$

Let us choose q_0 so that $\xi_0^3 \neq \Omega$. This means that we must have $\xi^3(t) \neq \Omega$ for all t since $\dot{\xi}^3 = 0$ in the nonholonomic equations. Therefore we must have $\lambda_1(t) = \lambda_2(t) = 0$ for all t. From equations (3.15) this means that we must have $\dot{x}(t) = \dot{y}(t) = 0$ for all t if a vakonomic solution is to agree with the nonholonomic solution. To prove the lemma we then choose initial conditions so that $\dot{x}(0)^2 + \dot{y}(0)^2 \neq 0$. ∎

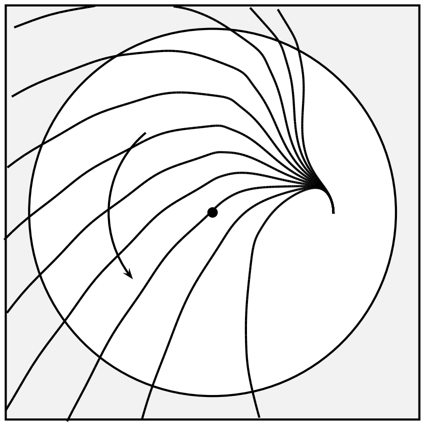

Figure 11. Vakonomic trajectories with $\lambda_2(0) = \omega_3(0) = 0$ and $\lambda_1(0)$ varying.

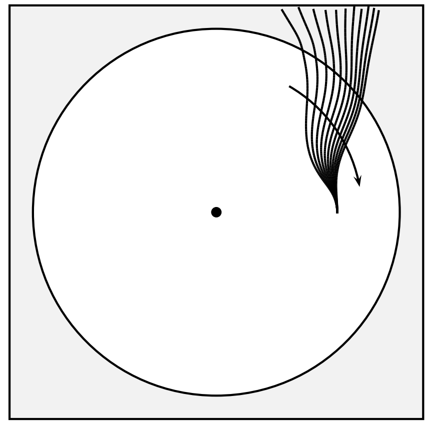

Figure 12. Vakonomic trajectories with $\lambda_1(0) = \omega_3(0) = 0$ and $\lambda_2(0)$ varying.

Note that the lemma is true for an open set of initial conditions q_0.

Simulations of the vakonomic equations were performed. Since a choice of initial conditions for the Lagrange multipliers was required, we chose to display the effects of these initial conditions on the trajectories keeping the other initial conditions fixed. In fact, the initial conditions for x, y, ξ^1, ξ^2 are the same as for the nonholonomic simulations. The effects of the choice of initial conditions for ω_3 were also explored.

In Figure 11 we fix $\lambda_2(0) = 0$ and $\omega_3(0) = 0$. The initial condition for λ_1 was varied from 0 to 100 in increments of 10. The arrow on the plot show the direction of increasing $\lambda_1(0)$. The same thing was done for $\lambda_2(0)$ with $\lambda_1(0)$ and $\omega_3(0)$ fixed at 0. The results are shown in Figure 12. In Figure 13 we show $\lambda_1(0)$ and $\lambda_2(0)$ fixed at 0 and $\omega_3(0)$ varying. The value of $\omega_3(0)$ was varied from 0 to 10 in increments of 1. In Figure 12 and Figure 13 the arrows indicate the direction of increasing $\lambda_2(0)$ and $\omega_3(0)$, respectively.

3.6. The experimental setup. In an effort to get some conclusive answers regarding the validity of either of our two models of the rolling ball on a spinning table, a simple experimental apparatus was put together. The spinning table was provided by a turntable with available rotational velocities of $33\frac{1}{3}$ rpm and 45 rpm. A plexiglass plate was fabricated as the surface on which the ball rolls.

The following objects were used as balls:

1. Steel balls of diameters 0.635 cm, 1.905 cm, and 2.54 cm,
2. a 2.54 cm diameter aluminium ball,
3. nylon balls of diameters 1.905 cm, and 2.54 cm,
4. a ping pong ball, and
5. a "super" ball.

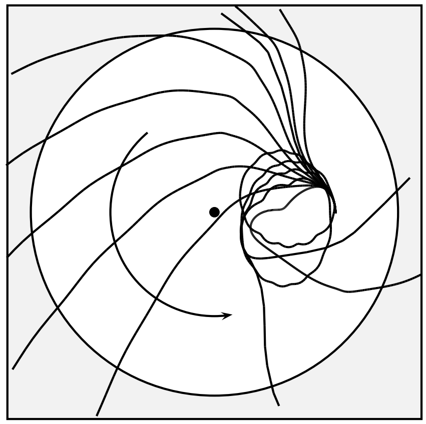

Figure 13. Vakonomic trajectories with $\lambda_1(0) = \lambda_2(0) = 0$ and $\omega_3(0)$ varying.

3.7. Comparison of experimental data and simulations. The most consistent results were obtained with the most massive balls. We shall present here data obtained from experiments using the 2.54 cm diameter steel ball. This was the data used in both the nonholonomic and vakonomic simulations. Since we were not able to produce any vakonomic simulations which resembled the experimental observations, we will limit ourselves to comparisons of the experiment with the nonholonomic simulations.

The comparisons we make are:

1. *Period:* Here we extend a ray from the centre of rotation of the table through the initial condition and measure the time it takes for the trajectory to cross this line again. When this period varies (for example, when the amplitude grows with the addition of dissipation in the simulations), the average period from the initial time until the ball leaves the table is measured. If the ball did not leave the table, the average period of the first five crossings was measured.

2. *Amplitude:* We use the same ray as described above and now we measure the distance between crossings of that ray. We shall enumerate the first five crossings of the ray and give the corresponding distances as $d\,\mathrm{i}$, $i = 1, 2, 3, 4, 5$.

In Table 1 we display the data.

We also make some general observations about how the experiment compares with simulations.

1. The experimental apparatus was noticed to be quite sensitive to deviations of the rolling surface from level. If the surface was not made level before an experiment was started, the ball typically left in short order (two or three revolutions).

2. If good level was obtained, however, the experimental results were quite good (in that they agreed with simulations) and repeatable. On some occasions, when the level was particularly good, the ball would exhibit equilibrium behaviour if given zero

	Period(sec)	$d1$(cm)	$d2$(cm)	$d3$(cm)	$d4$(cm)	$d5$(cm)
Experiment	4.38	-1.84	-0.64	0.0	0.64	1.84
$\delta = \mu = \nu = 0$	4.67	0.0	0.0	0.0	0.0	0.0
$\delta = \mu = 0$, $\nu = 5.0$	4.65	-0.8	-0.63	-0.49	-0.39	-0.3
$\delta = \nu = 0$, $\mu = 6.0$	4.67	-0.47	-0.29	-0.11	0.05	0.21
$\mu = \nu = 0$, $\delta = 0.005$	4.67	1.48	1.61	1.67	1.72	1.75
$\nu = 0$, $\delta = 0.005$, $\mu = 6.0$	4.66	0.79	0.73	0.67	0.61	0.57

Table 1. Display of experimental and numerical data.

initial velocity. This would indicate that rolling friction and rotational viscous friction are negligible in these instances since these dissipational effects will not preserve this equilibrium in general.

3. Also, when good level was attained, the general behaviour was in good agreement with the nonholonomic simulations with rotational friction.

There are several sources of error that may contribute to deviations between experimental observations and simulated data. We feel that the most significant of these is deviation of the rotating surface from level. This error has the effect of supplying an unmodelled potential field to the experimental dynamics. Another source of a similar type of error is unevenness in the surface of the rotating plate. Errors are also incurred in imparting initial velocity to the ball. This was not done precisely in the experiment. This difference in initial condition should not affect the comparison of periods between the experiment and the simulations, but will affect other quantitative comparisons.

These sources of error notwithstanding, we may make the following conclusion.

Conclusion: The nonholonomic equations of motion provide a good model for the ball rolling on a rotating table. The agreement between the model and observations may be further improved with addition of appropriate dissipative effects to the model. •

4. Future work

In this paper we have discussed some of the differences in the nonholonomic and vakonomic methods for deriving equations of motion for mechanical systems with constraints. By performing a simple experiment and comparing observed data with the two sets of equations of motion, we have provided evidence that may lead one to conclude that, at least for the system studied, the nonholonomic equations of motion do a reasonable job of predicting physical reality. This is especially true if we include some well-motivated friction effects in the nonholonomic model. However, there are still some things that should be resolved before the book can be closed on this issue.

Certainly a more careful end exhaustive experimental effort on systems other than the ball on the rotating table would be valuable in providing data which would allow for a fair comparison of the nonholonomic and vakonomic methods. Such experimentation may include a more careful determination of the friction forces which are inevitably present and which affect, in no small way, the behaviour of nonholonomic systems.

One of the major drawbacks of the vakonomic method was determined to be its requiring initial conditions for the Lagrange multipliers. It was pointed out that for the penny rolling upright on a stationary table, the nonholonomic equations of motion may be regarded as a subset of the vakonomic solutions. It would be interesting, and perhaps useful, to know exactly when this can be done. It certainly cannot be done for all systems, given Lemma 3.1. A discussion along these lines takes place in [Cardin and Favretti 1996].

Finally we mention that the nonholonomic method is implicit in many aspects nonholonomic control theory. In particular, in [M'Closkey and Murray 1994] a mobile robot towing a cart is modelled as a nonholonomic system, and there is good agreement between the theory and the experiments.

References

Abraham, R. and Marsden, J. E. [1978] *Foundations of Mechanics*, 2nd edition, Addison Wesley: Reading, MA, ISBN: 978-0-8218-4438-0.

Abraham, R., Marsden, J. E., and Ratiu, T. S. [1988] *Manifolds, Tensor Analysis, and Applications*, number 75 in Applied Mathematical Sciences, Springer-Verlag: New York/-Heidelberg/Berlin, ISBN: 978-0-387-96790-5.

Arnol′d, V. I. [1988] *Dynamical Systems*, volume 3, Encyclopedia of Mathematical Sciences, Springer-Verlag: New York/Heidelberg/Berlin, ISBN: 978-3-662-02535-2.

Bidwell, J. B., editor [1962], Symposium on Rolling Contact Phenomena, (Warren, MI, Oct. 1960), Elsevier Publishing Company: Amsterdam/London/New York, ISBN: 978-0-4444-0051-2.

Bloch, A. M. and Crouch, P. E. [1995] *Nonholonomic and vakonomic control systems on Riemannian manifolds*, SIAM Journal on Control and Optimization, **33**(1), pages 126–148, ISSN: 0363-0129, DOI: 10.1137/S036301299223533X.

Bloch, A. M., Krishnaprasad, P. S., Marsden, J. E., and Murray, R. M. [1996] *Nonholonomic mechanical systems with symmetry*, Archive for Rational Mechanics and Analysis, **136**(1), pages 21–99, ISSN: 0003-9527, DOI: 10.1007/BF02199365.

Cardin, F. and Favretti, M. [1996] *On nonholonomic and vakonomic dynamics of mechanical systems with nonintegrable constraints*, Journal of Geometry and Physics, **18**(4), pages 295–325, ISSN: 0393-0440, DOI: 10.1016/0393-0440(95)00016-X.

Flom, D. G. [1962] *Dynamic mechanical losses in rolling contacts*, in *Rolling Contact Phenomena*, Symposium on Rolling Contact Phenomena, (Warren, MI, Oct. 1960), edited by J. B. Bidwell, pages 97–112, Elsevier Publishing Company: Amsterdam/London/New York, ISBN: 978-0-4444-0051-2.

Kharlomov, P. V. [1992] *A critique of some mathematical models of mechanical systems with differential constraints*, Journal of Applied Mathematics and Mechanics, Translation of

Rossiĭskaya Akademiya Nauk. Prikladnaya Matematika i Mekhanika, **56**(4), pages 584–594, issn: 0021-8928, doi: 10.1016/0021-8928(92)90016-2.

Klingenberg, W. [1995] *Riemannian Geometry*, 2nd edition, number 1 in Studies in Mathematics, Walter de Gruyter: Berlin/New York, isbn: 978-3-11-014593-9.

Koizumi, T., Shibazaki, H., Nishio, T., and Nishiwaki, N. [1983] *Study of rolling friction-behaviour of the small displacement of starting rolling friction*, Wear, **88**(3), pages 285–296, issn: 0043-1648, doi: 10.1016/0043-1648(83)90299-5.

Kozlov, V. V. [1983] *Realization of nonintegrable constraints in classical mechanics*, Soviet Physics. Doklady. Translation of the physics sections of *Doklady Akademii Nauk SSSR*, **28**, pages 594–600.

— [1992] *The problem of realizing constraints in dynamics*, Journal of Applied Mathematics and Mechanics, Translation of *Rossiĭskaya Akademiya Nauk. Prikladnaya Matematika i Mekhanika*, **56**(4), pages 594–600, issn: 0021-8928, doi: 10.1016/0021-8928(92)90017-3.

M'Closkey, R. T. and Murray, R. M. [1994] *Experiments in exponential stabilization of a mobile robot towing a trailer*, in *Proceedings of the 1994 American Control Conference*, American Control Conference, (Baltimore, MD, June 1994), Institute of Electrical and Electronics Engineers, pages 988–993, doi: 10.1109/ACC.1994.751893.

Pars, L. A. [1965] *A Treatise on Analytical Dynamics*, John Wiley and Sons: NewYork, NY, isbn: 978-0-918024-07-7.

Appendices

A. Variations and Hamilton's method

In this appendix we introduce the basic tools for studying variational principles. The main purpose of the discussion is to get the reader acquainted with the techniques we shall be using to pose and solve the variational problems considered. In particular, we introduce the notion of a ***variation*** of a curve c and an ***infinitesimal variation***. The classical functional, J, is defined here as well.

A.1. Unconstrained variations. We will typically be considering curves, $c\colon [a,b] \to Q$, which connect two points, q_1 and q_2, in the configuration manifold Q. These curves may be subject to some constraints, but let us initially deal with the unconstrained case for the sake of concreteness. The set of all such curves which are C^2 will be denoted by $C^2(q_1,q_2,[a,b])$. It may be demonstrated that this set is a smooth infinite-dimensional manifold (see [Klingenberg 1995]). The tangent space at a point $c \in C^2(q_1,q_2,[a,b])$ may be shown to be given by

$$T_cC^2(q_1,q_2,[a,b]) = \{u\colon [a,b] \to TQ \mid \ u \text{ is } C^2, \\ \tau_Q \circ u = c,\ u(a) = 0 \text{ and } u(b) = 0\}.$$

We may think of a tangent vector, u, at c as being a vector field along c which vanishes at the endpoints (see Figure 14). Since u is a tangent vector we may write it as the tangent vector to a curve which passes through c. A curve in $C^2(q_1,q_2,[a,b])$ will be written as

$$\mathbb{R} \ni s \mapsto c_s \in C^2(q_1,q_2,[a,b]).$$

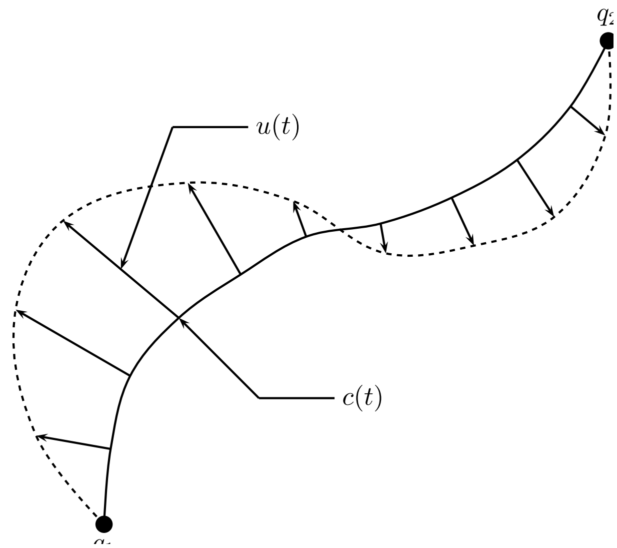

Figure 14. An infinitesimal variation.

For any $u \in T_c C^2(q_1, q_2, [a, b])$ we may write

$$u = \left.\frac{\mathrm{d}c_s}{\mathrm{d}s}\right|_{s=0}.$$

We shall refer to the curve c_s in $C^2(q_1, q_2, [a, b])$ as a ***variation*** of $c = c_0$ and we shall refer to u as an ***infinitesimal variation*** of c.

A.2. Constrained variations. Now we place an affine constraint, (D, γ), on Q. For $q_1, q_2 \in Q$ we define

$$C^2(q_1, q_2, [a, b], D, \gamma) = \{c\colon [a, b] \to Q \mid \; c \text{ is a } C^2 \text{ curve}, \\ c(a) = q_1,\ c(b) = q_2, \text{ and } \dot{c}(t) - \gamma(c(t)) \in D(c(t)) \text{ for } t \in [a, b]\}.$$

It is possible that this subset of $C^2(q_1, q_2, [a, b])$ is empty, but let us suppose that it is not.

We will now define, in the presence of affine constraints, a special class of infinitesimal variations. In the classical literature these are commonly referred to as ***virtual displacements***. Let $c \in C^2(q_1, q_2, [a, b], D, \gamma)$. Define

$$X_c(q_1, q_2, [a, b], D) = \{u \in T_c C^2(q_1, q_2, [a, b]) \mid \\ \dot{c}(t) + u(t) - \gamma(c(t)) \in D(c(t))\}.$$

In words, $X_c(q_1, q_2, [a, b], D)$ is the set of infinitesimal variations which, when added to $\dot{c}$, still satisfy the affine constraints. Clearly, since $c \in C^2(q_1, q_2, [a, b], D, \gamma)$, $u \in X_c(q_1, q_2, [a, b], D)$ if and only if $u(t) \in D(c(t))$, i.e., if u satisfies the non-affine constraints. This is why no reference to γ appears in the name of $X_c(q_1, q_2, [a, b], D)$.

A.3. The functional J. Since we are on a manifold, we may speak of smooth functions which may be differentiated. We therefore know what it means for a function to have a critical point. We will only define the functional for unconstrained systems. It is given by

$$\begin{aligned} J\colon C^2(q_1, q_2, [a,b]) &\to \mathbb{R} \\ c &\mapsto \int_a^b L(\dot{c}(t), t)\,\mathrm{d}t \end{aligned} \tag{A.1}$$

where L is a Lagrangian on Q. Note that $\boldsymbol{d}J(c) = 0$ if and only if $\boldsymbol{d}J(c) \cdot u = 0$ for every $u \in T_c C^2(q_1, q_2, [a,b])$. It is convenient to write

$$\boldsymbol{d}J(c) \cdot u = \left.\frac{\mathrm{d}}{\mathrm{d}s}\right|_{s=0} J(c_s).$$

With J as given by (A.1) we have

$$\boldsymbol{d}J(c) \cdot u = \frac{\mathrm{d}}{\mathrm{d}s} \int_a^b L(\dot{c}_s(t), t)\,\mathrm{d}t \bigg|_{s=0} = \int_a^b \frac{\mathrm{d}}{\mathrm{d}s} L(\dot{c}_s(t), t)\bigg|_{s=0} \mathrm{d}t.$$

We wish to evaluate this expression in local coordinates for Q. By the chain rule we have

$$\boldsymbol{d}J(c) \cdot u = \int_a^b \left(\frac{\partial L}{\partial q^i} \frac{\partial q^i}{\partial s} + \frac{\partial L}{\partial \dot{q}^i} \frac{\partial \dot{q}^i}{\partial s} \right) \bigg|_{s=0} \mathrm{d}t.$$

A.4. Hamilton's Principle. As an example of how to apply the above concepts we present ***Hamilton's Principle***. This establishes a correspondence between solutions of Lagrange's equations and the solution of a variational problem. We present this as a proposition whose proof goes much like the one by Abraham and Marsden [1978]. Recall that the ***Legendre transformation*** for a Lagrangian, L, is the map defined in coordinates by

$$\begin{aligned} \boldsymbol{F}L\colon TQ &\to T^*Q \\ (q^1, \dots, q^n, v^1, \dots, v^n) &\mapsto \left(q^1, \dots, q^n, p_1 = \frac{\partial L}{\partial v^1}, \dots, p_n = \frac{\partial L}{\partial v^n} \right). \end{aligned}$$

We say that L is a ***regular*** Lagrangian if $\boldsymbol{F}L$ is a local diffeomorphism.

A.1 Proposition: (Hamilton's Principle) *Let L be a regular Lagrangian on Q. A curve, $c\colon [a,b] \to Q$, joining q_1 with q_2 in Q is a solution to Lagrange's equations,*

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{q}^i}\right) - \frac{\partial L}{\partial q^i} = 0, \qquad i = 1, \dots, n,$$

if and only if $\boldsymbol{d}J(c) = 0$.

Proof: We need to show that c is a solution to Lagrange's equations if and only if $\boldsymbol{d}J(c) \cdot u = 0$ for every $u \in T_c C^2(q_1, q_2, [a,b])$. Let c_s be a one-parameter family of curves in $C^2(q_1, q_2, [a,b])$ with $c_0 = c$. For any $u \in T_c C^2(q_1, q_2, [a,b])$ we may then write

$$u = \left.\frac{\mathrm{d}c_s}{\mathrm{d}s}\right|_{s=0}$$

for some such one-parameter family. Then we have

$$\begin{aligned} \boldsymbol{d}J(c)\cdot u &= \frac{\mathrm{d}}{\mathrm{d}s}J(c_s)\bigg|_{s=0} \\ &= \frac{\mathrm{d}}{\mathrm{d}s}\int_a^b L(c_s(t),)\,\mathrm{d}t\bigg|_{s=0}. \end{aligned}$$

The differentiation may be moved under the integral sign and in coordinates we have

$$\begin{aligned} \boldsymbol{d}J(c)\cdot u &= \int_a^b \frac{\mathrm{d}}{\mathrm{d}s}L(q(t,s),\dot{q}(t,s),t)\bigg|_{s=0}\,\mathrm{d}t \\ &= \int_a^b \left(\frac{\partial L}{\partial q^i}\frac{\partial q^i}{\partial s} + \frac{\partial L}{\partial \dot{q}^i}\frac{\partial \dot{q}^i}{\partial s}\right)\bigg|_{s=0}\,\mathrm{d}t. \end{aligned}$$

For the variation given we have

$$\frac{\partial q^i(t,s)}{\partial s}\bigg|_{s=0} = u^i(t), \quad \text{and} \quad \frac{\partial \dot{q}^i(t,s)}{\partial s}\bigg|_{s=0} = \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial q^i(t,s)}{\partial s}\bigg|_{s=0} = \dot{u}^i(t).$$

We thus have, using integration by parts,

$$\begin{aligned} \boldsymbol{d}J(c)\cdot u &= \int_a^b \left(\frac{\partial L}{\partial q^i}u^i + \frac{\partial L}{\partial \dot{q}^i}\dot{u}^i\right)\,\mathrm{d}t \\ &= \int_a^b \left(\frac{\partial L}{\partial q^i} - \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{q}^i}\right)u^i\,\mathrm{d}t + \frac{\partial L}{\partial \dot{q}^i}u^i\bigg|_a^b \\ &= \int_a^b \left(\frac{\partial L}{\partial q^i} - \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{q}^i}\right)u^i\,\mathrm{d}t. \end{aligned}$$

Clearly then $\boldsymbol{d}J(c)\cdot u = 0$ for every u if and only if Lagrange's equations are satisfied. This completes the proof. ■

Note that the condition that L be regular is present to ensure that Lagrange's equations have a solution.

B. The Principle of Virtual Work

This principle is classically presented as an axiom of mechanics which is not derivable from the other basic axioms. It is typically stated in terms as follows:

The Principle of Virtual Work: *The work done by the forces of constraint is zero on motions allowed by the constraints.*

When we say that a force does no work on motions allowed by the constraints we mean that, regarded as a differential one-form, the force annihilates tangent vectors in D. Thus the constraint force annihilates all vectors annihilated by the forms $\omega^1,\dots,\omega^{n-k}$. We shall say that the Principle of Virtual Work is satisfied by a curve, c, if there exists external forces F_i^c which do no work on the constraints and are such that

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{q}^i}\right) - \frac{\partial L}{\partial q^i} = F_i^c(t)$$

along c. In other words, regarded as a differential form, $F^c_i(t)\mathrm{d}q^i$ must lie in the span of $\omega^1(c(t)),\ldots,\omega^{n-k}(c(t))$. Thus, for each $t\in\mathbb{R}$ which is in the domain of definition of c, there must exist constants $\lambda_1(t),\ldots,\lambda_{n-k}(t)$ such that

$$F^c_i(t)\mathrm{d}q^i = \lambda_a(t)\omega^a(c(t)) = \lambda_a(t)\omega^a_i(c(t))\mathrm{d}q^i$$

which means that $F^c_i(t) = \lambda_a(t)\omega^a_i(c(t))$ for some constants $\lambda_1(t),\ldots,\lambda_{n-k}(t)$. Thus Lagrange's equations may be written as

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot q^i}\right) - \frac{\partial L}{\partial q^i} = \lambda_a\omega^a_i, \qquad i=1,\ldots,n$$

and we are to solve for the ***Lagrange multipliers*** $\lambda_1,\ldots,\lambda_{n-k}$ as part of the solution. To get the right number of equations for the number of unknowns we append the constraint equations

$$\omega^a_i\dot q^i = \omega^a_i\gamma^i, \qquad a=1,\ldots,n-k.$$

We have the following easy result which relates the Principle of Virtual Work to the nonholonomic constrained variational problem discussed above.

B.1 Proposition: *A curve, $c\in C^2(q_1,q_2,[a,b],D,\gamma)$, is a solution of the nonholonomic constrained variational problem if and only if the Principal of Virtual Work is satisfied by the curve c.*

Proof: We must show that

$$\left[\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot q^i}\right) - \frac{\partial L}{\partial q^i}\right]u^i(t) = 0$$

for every $u\in X_cC^2(q_1,q_2,[a,b],D)$ if and only if

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot q^i}\right) - \frac{\partial L}{\partial q^i} = F^c_i(t)$$

along c, where the forces F^c_i do no work on motion allowed by the constraints. By definition, the forces F^c_i do no work on motions allowed by the constraints if and only if

$$F^c_i(t)u^i(t) = 0$$

for every $u\in X_cC^2(q_1,q_2,[a,b],D)$ and $t\in[a,b]$. Thus the proposition is proved. ■

This gives a way of determining equations of motions for solutions to the nonholonomic constrained variational problem. Existence and uniqueness of solutions of these equations of motion is not something we shall take up here.

C. Derivation of the vakonomic equations of motion

Since the vakonomic method is simply a constrained minimisation problem, we need some results from that field. The main one we shall use is the Lagrange Multiplier Theorem, the version which we use being taken from [Abraham, Marsden, and Ratiu 1988].

C.1 Lemma: (The Lagrange Multiplier Theorem) *Let M be a smooth manifold and let E be a Banach space with $g\colon M \to E$ a smooth submersion so that $N = g^{-1}(0)$ is a submanifold of M. Let $f\colon M \to \mathbb{R}$ be a smooth function. Then $n \in N$ is a critical point of $f \mid N$ if and only if there exists $\lambda \in E^*$ such that n is a critical point of $f - \lambda \circ g$.*

To utilise this lemma we must further examine the structure of $C^2(q_1, q_2, [a,b], D, \gamma)$, which was defined when we introduced the nonholonomic problem. If E is a real Banach space we denote by $\mathscr{F}([a,b], E)$ be the Banach space of C^2, E-valued functions on the interval $[a,b]$. Suppose that the distribution D is annihilated by $n-k$ one-forms, $\omega^1, \dots, \omega^{n-k}$. We define a function $g\colon C^2(q_1, q_2, [a,b]) \to \mathscr{F}([a,b], \mathbb{R}^{n-k})$ by

$$g(c) = \Big\{ t \mapsto \Big(\omega^1(\dot{c}(t)) - \omega^1(\gamma(c(t))), \dots, \omega^{n-k}(\dot{c}(t)) - \omega^{n-k}(\gamma(c(t))) \Big) \Big\}. \tag{C.1}$$

We shall assume that g is a smooth submersion. Note that

$$C^2(q_1, q_2, [a,b], D, \gamma) = g^{-1}(0, \dots, 0)$$

is a smooth submanifold with this assumption.

We shall need to have some idea of what elements of $\mathscr{F}([a,b], \mathbb{R}^{n-k})^*$ look like. We shall be purposefully formal here. Note that $\mathscr{F}([a,b], \mathbb{R}^{n-k})$ is naturally isomorphic to the $(n-k)$-fold direct product of $\mathscr{F}([a,b], \mathbb{R})$ with itself. Therefore $\mathscr{F}([a,b], \mathbb{R}^{n-k})^*$ will be naturally isomorphic to the $(n-k)$-fold direct product of $\mathscr{F}([a,b], \mathbb{R})^*$ with itself. Recall that elements of $\mathscr{F}([a,b], \mathbb{R})^*$ are ***distributions*** on $[a,b]$. We shall not depart from the tradition of denoting the pairing of elements of $\mathscr{F}([a,b], \mathbb{R})^*$ with elements of $\mathscr{F}([a,b], \mathbb{R})$ by

$$\langle \alpha; f \rangle = \int_a^b \alpha \cdot f(t)\, \mathrm{d}t.$$

We will at times regard elements of $\mathscr{F}([a,b], \mathbb{R})^*$ as elements of $\mathscr{F}([a,b], \mathbb{R})$ via the integral. The reader should be aware of what is taking place, and that it is not wholly precise. In any case, we may write the action of an element of $\mathscr{F}([a,b], \mathbb{R}^{n-k})^*$ on an element of $\mathscr{F}([a,b], \mathbb{R}^{n-k})$ as

$$\langle (\alpha^1, \dots, \alpha^{n-k}); (f_1, \dots, f_{n-k}) \rangle = \int_a^b \alpha^c \cdot f_c(t)\, \mathrm{d}t.$$

The following result gives the equations of motion for the vakonomic constrained variational problem.

C.2 Proposition: *Let L be a Lagrangian on Q, let (D, γ) be an affine constraint on Q, and let $\omega^1, \dots, \omega^{n-k}$ be $n-k$ linearly independent differential one-forms on Q which annihilate Q. Then $c \in C^2(q_1, q_2, [a,b], D)$ is a solution of the vakonomic constrained variational problem if and only if there exists $(\lambda_1, \dots, \lambda_{n-k}) \in \mathscr{F}([a,b], \mathbb{R}^{n-k})^*$ such that*

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial \mathscr{L}}{\partial \dot{q}^i} \right) - \frac{\partial \mathscr{L}}{\partial q^i} = 0, \qquad i = 1, \dots, n$$

where $\mathscr{L}\colon TQ \times \mathbb{R} \to \mathbb{R}$ is defined along c by

$$\mathscr{L}(\dot{c}(t), t) = L(\dot{c}(t), t) - \lambda_a(t)[\omega^a(\dot{c}(t)) - \omega^a(\gamma(c(t)))].$$

Proof: Let $(g^1(c), \ldots, g^{n-k}(c))$ denote the components of $g(c)$ under the identification of $\mathscr{F}([a,b], \mathbb{R}^{n-k})$ with $\mathscr{F}([a,b], \mathbb{R}) \times \cdots \times \mathscr{F}([a,b], \mathbb{R})$. By (C.1) we have

$$g^a(c) = \{t \mapsto \omega^a(\dot{c}(t)) - \omega^a(\gamma(c(t)))\}, \qquad a = 1, \ldots, n-k.$$

From the Lagrange Multiplier Theorem we know that c is a solution to the vakonomic constrained variational problem if and only if there exists $(\lambda_1, \ldots, \lambda_{n-k}) \in \mathscr{F}([a,b], \mathbb{R}^{n-k})^*$ such that c is a critical point of the function J_D on $C^2(q_1, q_2, [a,b], D)$ defined by

$$J_D(c) = \int_a^b L(\dot{c}(t), t)\,\mathrm{d}t - \lambda_a \cdot g^a(c).$$

Note that c is a critical point of J_D if and only if

$$\left.\frac{\mathrm{d}J_D(c_s)}{\mathrm{d}s}\right|_{s=0} = \frac{\mathrm{d}}{\mathrm{d}s}\int_a^b L(\dot{c}_s(t), t)\,\mathrm{d}t\bigg|_{s=0} - \frac{\mathrm{d}}{\mathrm{d}s}\lambda_a \cdot g^a(c_s)\bigg|_{s=0} = 0$$

for every variation c_s of c. Now we use the integral notation for the pairing of the distribution λ_a with the element $g^a(c_s)$ of $\mathscr{F}([a,b], \mathbb{R})$. This then gives

$$\left.\frac{J_D(c_s)}{ds}\right|_{s=0} = \int_a^b \frac{\mathrm{d}}{\mathrm{d}s}\left(L(\dot{c}_s(t)) - \lambda_a \cdot (\omega^a(\dot{c}_s(t)) - \omega^a(\gamma(c(t))))\right)\bigg|_{s=0} \mathrm{d}t.$$

The result now follows by the arguments used in the proof of Hamilton's Principle, Proposition A.1. ■

D. Derivation of the vakonomic equations for the rolling ball on the spinning table

Here we derive a simple set of equations whose solutions describe the motion of the vakonomic ball on the spinning table. To be somewhat precise about it we need to introduce some notation. Let $Q = \mathbb{R}^2 \times SO(3)$ and let $\tilde{Q} = \mathbb{R}^2 \times \mathbb{R}^3 \times \mathbb{R}^{3\times 3}$. Let $q_1 = (x_1, y_1, R_1), q_2 = (x_2, y_2, R_2) \in Q$ and define the set

$$\begin{aligned} X = \{c\colon [a,b] \to \tilde{Q} \mid \quad & c \text{ is } C^2,\ c(a) = (x_1, y_1, \xi^1, R_1), \text{ and} \\ & c(b) = (x_2, y_2, \xi^2, R_2) \text{ where } \xi^1, \xi^2 \in \mathbb{R}^3 \text{ are arbitrary}\}. \end{aligned}$$

If we write $c \in X$ as

$$t \mapsto (x(t), y(t), \boldsymbol{\xi}(t), R(t)),$$

we may define the following $\mathbb{R}^{3\times 3}$-valued function on X:

$$\begin{aligned} g\colon X &\to \mathbb{R}^{3\times 3} \\ c &\mapsto \dot{R}(t) - \hat{\boldsymbol{\xi}}(t)R(t). \end{aligned}$$

We claim that $g^{-1}(0)$ may be naturally identified with $C^2(q_1, q_2, [a,b])$. Indeed, it is easy to check that the identification is provided by

$$\begin{aligned} \rho\colon g^{-1}(0) &\to C^2(q_1, q_2, [a,b]) \\ (x(t), y(t), \boldsymbol{\xi}(t), R(t)) &\mapsto (x(t), y(t), R(t)). \end{aligned}$$

Now define Lagrangians on Q and $\tilde{Q}$ by

$$L(x, y, R, \dot{x}, \dot{y}, \dot{R}) = -\frac{1}{4} I \operatorname{tr}(\dot{R} R^T \dot{R} R^T) + \frac{1}{2} m(\dot{x}^2 + \dot{y}^2)$$

and

$$\tilde{L}(x, y, \boldsymbol{\xi}, R, \dot{x}\, \dot{y}, \dot{\boldsymbol{\xi}}, \dot{R}) = \frac{1}{2} I \|\boldsymbol{\xi}\| + \frac{1}{2} m(\dot{x}^2 + \dot{y}^2),$$

respectively. These determine functionals defined on $C^2(q_1, q_2, [a, b])$ and X defined by

$$J = \int_a^b L(x(t), y(t), R(t), \dot{x}(t), \dot{y}(t), \dot{R}(t))\, \mathrm{d}t$$

and

$$\tilde{J} = \int_a^b \tilde{L}(x(t), y(t), \boldsymbol{\xi}(t), R(t), \dot{x}(t), \dot{y}(t), \dot{\boldsymbol{\xi}}(t), \dot{R}(t))\, \mathrm{d}t,$$

respectively. Note that $J \circ \rho(c) = \tilde{J}(c)$ for all $c \in g^{-1}(0)$. Thus, determining a minimum of J is equivalent to determining the minimum of $\tilde{J}$ on the submanifold $g^{-1}(0)$. By the Lagrange Multiplier Theorem, c is a critical point of $\tilde{J} \mid g^{-1}(0)$ if and only if there exists $\Lambda \in \mathscr{F}([a, b], \mathbb{R}^{3\times 3})^*$ so that c is a critical point of

$$c \mapsto \tilde{J}(c) - \Lambda \cdot g(c).$$

If in the usual manner we write the action of $\mathscr{F}([a, b], \mathbb{R}^{3\times 3})^*$ on $\mathscr{F}([a, b], \mathbb{R}^{3\times 3})$ via the integral, we see that critical points of J are in correspondence with critical points of the functional

$$c \mapsto \int_a^b \Big(\tilde{L}(c(t)) - \operatorname{tr}(\Lambda \cdot (\dot{R}(t) - \hat{\boldsymbol{\xi}}(t) R(t))) \Big)\, \mathrm{d}t. \tag{D.1}$$

Hamilton's Principle does not directly apply here since our variations do not have fixed endpoints ($\boldsymbol{\xi}$ is arbitrary). Nevertheless critical points of (D.1) are solutions of Lagrange's equations on $\mathbb{R}^2 \times \mathbb{R}^3 \times \mathbb{R}^{3\times 3} \times \mathbb{R}^{3\times 3}$ with Lagrangian given by

$$\tilde{L}(x, y, \boldsymbol{\xi}, R, \dot{x}, \dot{y}, \dot{\boldsymbol{\xi}}, \dot{R}) - \operatorname{tr}(\Lambda(\dot{R} - \hat{\boldsymbol{\xi}} R)) \tag{D.2}$$

as the following lemma points out.

D.1 Lemma: *Let $\Lambda \in \mathscr{F}([a, b], \mathbb{R}^{3\times 3})^*$. A curve $c \in X$ is a critical point of the functional in (D.1) if and only if c is a solution to Lagrange's equations with Lagrangian given by (D.2).*

Proof: Let us denote by $\tilde{q}$ the coordinates $(x, y, \boldsymbol{\xi}, R)$ for $\mathbb{R}^2 \times \mathbb{R}^3 \times \mathbb{R}^{3\times 3}$. Let c_s be a variation of a curve $c \in X$ and let u be the corresponding infinitesimal variation. Then, as in the proof of Proposition A.1, we have

$$\boldsymbol{d}J(c) \cdot u = \int_1^b \left(\frac{\partial \tilde{L}}{\partial \tilde{q}^i} - \frac{\mathrm{d}}{\mathrm{d}t} \frac{\partial \tilde{L}}{\partial \dot{\tilde{q}}^i} \right) u^i\, \mathrm{d}t + \frac{\partial \tilde{L}}{\partial \dot{\tilde{q}}^i} u^i \bigg|_a^b. \tag{D.3}$$

Since $c \in X$, the infinitesimal variation must satisfy

$$u_x(t) = u_y(t) = u_{R_{ij}}(t) = 0, \qquad i, j = 1, 2, 3, \;\; t = a, b.$$

Also, since $\tilde{L}$ does not depend on $\dot{\boldsymbol{\xi}}$ we have

$$\frac{\partial \tilde{L}}{\partial \dot{\xi}^i} = 0, \qquad i = 1, 2, 3$$

for all t. Therefore the boundary term in (D.3) must vanish and the lemma follows from an arbitrary choice of u. ■

In the above discussion we have omitted any mention of the rolling constraints for the ball. One may easily see that they may simply be added on at each step so that they appear in the same manner in both the determination of J and $\tilde{J}$. Thus, we have sketched a proof of the following result.

D.2 Lemma: *A curve $t \mapsto (x(t), y(t), R(t))$ is a solution to the vakonomic equations for the ball rolling on the spinning table if and only if there exists a function $t \mapsto (\boldsymbol{\xi}(t), \boldsymbol{\lambda}(t), \Lambda(t))$ so that the curve $t \mapsto (x(t), y(t), \boldsymbol{\xi}(t), R(t), \boldsymbol{\lambda}(t), \Lambda(t))$ is a solution of Lagrange's equations on $\mathbb{R}^2 \times \mathbb{R}^3 \times \mathbb{R}^{3\times 3} \times \mathbb{R}^2 \times \mathbb{R}^{3\times 3}$ with Lagrangian*

$$\mathscr{L} = \frac{1}{2} I((\xi^1)^2 + (\xi^2)^2 + (\xi^3)^2) + \frac{1}{2} m(\dot{x}^2 + \dot{y}^2) - \\ \lambda_1(\dot{x} - r\xi^2 + \Omega y) + \lambda_2(\dot{y} + r\xi^1 - \Omega x) - \mathrm{tr}(\Lambda(\dot{R} - \hat{\boldsymbol{\xi}} R)).$$