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Towards $F = ma$ in a general setting for Lagrangian mechanics

Andrew D. Lewis*

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Abstract

By using a suitably general definition of a force, one may geometrically cast the Euler-Lagrange equations in a "force balance" form. The key ingredient in such a construction is the Euler-Lagrange 2-force which is a bundle map from the bundle of two-jets into the first contact system. This 2-force can be used as the basis for a geometric presentation of Lagrangian mechanics with external forces and constraints. Also described is the precise correspondence between this 2-force and the Poincaré-Cartan two-form.

Keywords. Lagrangian mechanics, nonholonomic constraints, jet bundle geometry.

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1. Introduction

In making a differential geometric presentation of Lagrangian mechanics, one of the cumbersome aspects has always been that the Euler-Lagrange equations are not themselves the components of a tensor field, and so one cannot easily assign to the equations a geometric object. One way of circumventing this has been to use the symplectic-like formalism of the Poincaré-Cartan two-form. This is done in the time-independent case, e.g., by Abraham and Marsden [1978] and Libermann and Marle [1987], by pulling-back the symplectic form on the cotangent bundle via the Legendre transformation. In the time-dependent case one has to modify this construction to use the natural almost tangent-like structure on the bundle of one-jets [de León, Marrero, and Martín de Diego 1997b]. In either case, the Euler-Lagrange equations themselves are somewhat obscured "inside" the two-form.

We offer here an alternative to these two-form based geometric formulations of Lagrangian mechanics by making a general notion of a force, and then, with this general notion in hand, assigning to a Lagrangian function a force which may be thought of as a generalisation of the "inertial force" ma (i.e., mass×acceleration) in Newtonian mechanics. We call this force the ***Euler-Lagrange 2-force***. Using this generalised inertial force, we can provide easy, intuitive characterisations of the Euler-Lagrange equations. For example, if the Lagrangian is regular then we can define the Euler-Lagrange vector field in a

*Professor, Department of Mathematics and Statistics, Queen's University, Kingston, ON K7L 3N6, Canada
Email: `andrew.lewis@queensu.ca`, URL: `http://www.mast.queensu.ca/~andrew/`
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straightforward manner. Our presentation includes external forces, and a general class of constraints. The geometric object we define here is a fairly natural one to consider. However, it is interesting to see exactly how it is related to the standard constructions in Lagrangian mechanics. In particular, the exact relationship between the Euler-Lagrange 2-force and the Poincaré-Cartan two-form requires some work to make precise (see Theorem 6.1).

Another aspect of our presentation is that we use infinite-dimensional manifolds throughout. Besides increasing the scope of the work, this avoids the preponderance of indices that results from a strictly finite-dimensional presentation.

The standard formalism of jet bundles, which is an appropriate general setting for Lagrangian mechanics, is readily adapted to infinite-dimensional manifolds, and the basic ideas and local representations are given in Section 2. Using jet bundles as the starting point, we provide some basic background to Lagrangian mechanics in Section 3. We investigate the geometry which can be associated with a Lagrangian function, as well as provide suitably general definitions of a force and a constraint. It is when defining forces that we provide the framework which allows for the definition of the "Euler-Lagrange 2-force" in Section 4. In finite-dimensions, this is essentially a force whose coefficients are the components of the Euler-Lagrange equations. It is in this way that we give geometric meaning to the Euler-Lagrange equations themselves, and not through other devices such as the two-form formalism. When one mimics the Hamiltonian formulation using the Poincaré-Cartan two-form, the way one defines the Euler-Lagrange vector field, in the case when the Lagrangian is regular, follows just as it does in the Hamiltonian case (unsurprisingly). In Section 5 we indicate how to use the Euler-Lagrange 2-force to define the Euler-Lagrange vector field in cases when L is regular. We also indicate how to handle the forced and constrained cases within our "force balance" framework. It should not be surprising that the Poincaré-Cartan two-form and the Euler-Lagrange 2-force which we define are, when suitably interpreted, equivalent. The exact form of this equivalence is given in Section 6. It is somewhat non-trivial to derive the Poincaré-Cartan two-form from the Euler-Lagrange 2-force.

2. Jet bundle geometry

In this section we provide a review of the geometric tools we will use in the paper. We shall for the most part adopt the notations and conventions of Abraham, Marsden, and Ratiu [1988]. In particular, we shall work within the category of C^∞ reflexive Banach manifolds. However, where clarity is assisted, the finite-dimensional coordinate formulas are provided.

Here is a list of notation we use, in roughly alphabetical order. We shall define many objects upon their first usage, but all terminology should be found in this list in any case.

$]a, b[$	: the open interval in $\mathbb{R}$ with endpoints a and b with $a < b$
$\alpha \cdot e$	: the natural pairing of $\alpha \in E^*$ with $e \in E$
$X \triangleq Y$	: X is defined to be equal to Y
$V\|M$	: the restriction to $M \subset B$ of a vector bundle $\pi\colon V \to B$
$\operatorname{ann}(F)$	: the annihilator in E^* of $F \subset E$
$B_{r,e}$	: the open ball of radius $r > 0$ in E centred at e
$c'(t)$	: $Tc(t) \cdot 1$, where $c\colon [t_1, t_2] \to M$ is a curve
$\operatorname{coann}(\Sigma)$	: the elements in E annihilated by $\Sigma \subset E^*$

$\boldsymbol{D}^k f(u)$	: the kth derivative at u of a map $f\colon U \to F$ with U an open subset of a Banach space E and F a Banach space
$\boldsymbol{D}_k f(u_1, \dots, u_m)$	: the kth partial derivative at $(u_1, \dots, u_m)$ of a map $f\colon U_1 \times \cdots \times U_m \to F$ where U_i is an open subset of a Banach space E_i, $i = 1, \dots, m$, and F is a Banach space
$\boldsymbol{d}_k f$	: the k-jet derivative of $f\colon J^k\mathcal{Q} \to \mathbb{R}$
E, F	: typical Banach spaces
$\Gamma^\infty(V)$	: the sections of a vector bundle with total space V
$\mathrm{T}\bigwedge^k(V)$	: the bundle of exterior k-forms for a vector bundle V
$\mathrm{L}(E; F)$	: the set of continuous linear maps between Banach spaces E and F
$Tf\colon TM \to TN$	: the derivative of a mapping $f\colon M \to N$ between manifolds M and N
$\pi_{TM}\colon TM \to M$	: the tangent bundle projection
$\tau_{k,l}\colon J^k\mathcal{Q} \to J^l\mathcal{Q}$	: the natural projection for $k > l$ (we denote $\tau_k = \tau_{k,0}$)
$\boldsymbol{U}$	: the image of U under ϕ for a manifold chart (U, ϕ)
$\nu_{\mathcal{Q}}\colon V\mathcal{Q} \to \mathcal{Q}$	: the restriction of $\pi_{T\mathcal{Q}}$ to $V\mathcal{Q}$
η	: the pull-back of $\boldsymbol{d}t$ to $J^1\mathcal{Q}$

Our notation $\mathrm{ann}(F)$ and $\mathrm{coann}(\Sigma)$ is non-standard—normally one sees the notation F^0 and $\Sigma^\perp$. Also, we shall sometimes write $A \cdot e$ for the evaluation of $A \in \mathrm{L}(E; F)$ on $e \in E$, especially if A itself depends on other arguments. This avoids awkward double parentheses like $A(x)(e)$.

2.1. Jet bundles. We shall work in the strictly time-dependent formulation. Thus we consider a locally trivial fibre bundle $\pi\colon \mathcal{Q} \to \mathbb{R}$ which is generally not provided with a global trivialisation. A special case, of course, is that when $\mathcal{Q} = \mathbb{R} \times Q$ where Q is the ***configuration manifold*** and π is projection onto the first factor. In the general case, we call $\pi\colon \mathcal{Q} \to \mathbb{R}$ the ***configuration bundle***. This point of view of using a non-trivial bundle is taken, for example, by Giachetta [1992] and de León, Marrero, and Martín de Diego [1997b]. Various authors [Hermann 1982, Massa and Pagani 1994] use the jet bundle formalism, but consider trivial bundles. Employing non-trivial bundles offers no practical advantages, but often simplifies the exposition by disallowing certain confusing identifications which can be made in the trivial case. Also, the employment of fibre bundles perhaps makes easier any future generalisations to Lagrangian field theory.

We shall frequently work in an ***adapted chart*** for $\pi\colon \mathcal{Q} \to \mathbb{R}$. Since the base space $\mathbb{R}$ is equipped with a natural chart, we shall always assume our adapted charts, which we denote (U, ϕ), are chosen so that $\boldsymbol{U} \triangleq \mathrm{image}(\phi) =]a, b[\, \times U'$, and so that the induced chart on the base space is the identity chart $(\,]a, b[\,, \mathrm{id}_{]a,b[}\,)$. Here U' is an open subset of a Banach space we will usually denote by E. We shall write coordinates in an adapted chart as $(t, u) \in]a, b[\, \times U'$. When we write finite-dimensional coordinate formulas, we denote coordinates by (t, q^i). In the finite-dimensional case, we shall suppose that $\dim(\mathcal{Q}) = n + 1$ where $n \geq 1$.

The vertical subbundle of the fibration $\pi\colon \mathcal{Q} \to \mathbb{R}$ we denote by $V\mathcal{Q}$ and recall that it is the kernel of the projection $T\pi\colon T\mathcal{Q} \to T\mathbb{R}$. We denote the projection from $V\mathcal{Q}$ to $\mathcal{Q}$ by $\nu_{\mathcal{Q}}$.

Associated with the configuration bundle $\pi\colon \mathscr{Q} \to \mathbb{R}$ are its jet bundles [Saunders 1989]. By convention define $J^0\mathscr{Q} = \mathscr{Q}$. Fix $t \in \mathbb{R}$ and $q \in \pi^{-1}(t)$. Two local sections c_1 and c_2 are ***equivalent to order one*** at q if $c_1(t) = c_2(t) = q$ and $c_1'(t) = c_2'(t)$. We write the equivalence class to order one at q containing c by $[c(t)]_1$. The set of all such equivalence classes we denote by $J^1\mathscr{Q}$ which is the ***bundle of one-jets***. We denote by $\tau_1\colon J^1\mathscr{Q} \to \mathscr{Q}$ the natural projection which forgets first-order equivalence. If $c\colon [t_1, t_2] \to \mathscr{Q}$ is a local section, we define a local section j^1c of $\pi \circ \tau_1\colon J^1\mathscr{Q} \to \mathbb{R}$ by assigning to $t \in [t_1, t_2]$ the equivalence class $[c(t)]_1$. Now let $t \in \mathbb{R}$ and $v \in (\pi \circ \tau_1)^{-1}(t)$. Two local sections c_1 and c_2 are ***equivalent to order two*** at $v \in J^1\mathscr{Q}$ if they are equivalent to order one with $[c_1(t)]_1 = [c_2(t)]_1 = v$, and if $(j^1c_1)'(t) = (j^1c_2)'(t)$. We denote the equivalence class to order two at q containing c by $[c(t)]_2$. The set of these equivalence classes we denote by $J^2\mathscr{Q}$ and call the ***bundle of two-jets***. The map $\tau_{2,1}\colon J^2\mathscr{Q} \to J^1\mathscr{Q}$ is the natural projection which forgets equivalence to order two, but remembers equivalence to order one. Given a local section $c\colon [t_1, t_2] \to \mathscr{Q}$ we define a local section j^2c of $J^2\mathscr{Q}$ over $\mathbb{R}$ by assigning to $t \in [t_1, t_2]$ the equivalence class $[c'(t)]_2$. One may inductively proceed in this way, defining the higher order jet bundles $\tau_{k,k-1}\colon J^k\mathscr{Q} \to J^{k-1}\mathscr{Q}$. We shall denote by $\tau_{k,l}\colon J^k\mathscr{Q} \to J^l\mathscr{Q}$ the natural projection for $l < k$, and we shall adopt the convention that $\tau_k = \tau_{k,0}$. It may be shown that the fibre bundle $\tau_{k,k-1}\colon J^k\mathscr{Q} \to J^{k-1}\mathscr{Q}$ is an affine bundle modelled on the pull-back vector bundle $\tau_{k-1}^*\nu_{\mathscr{Q}}\colon \tau_{k-1}^*V\mathscr{Q} \to J^{k-1}\mathscr{Q}$. It is also true that the pull-back bundle $\tau_k^*\nu_{\mathscr{Q}}\colon \tau_k^*V\mathscr{Q} \to J^k\mathscr{Q}$ is naturally isomorphic to $\ker(T\tau_{k,k-1})$, and so is a subbundle of $T(J^k\mathscr{Q})$.

If (U, ϕ) is an adapted chart for $\pi\colon \mathscr{Q} \to \mathbb{R}$, we have induced natural charts for $T\mathscr{Q}$ and the jet bundles $J^k\mathscr{Q}$ which we denote by $(TU, T\phi)$ and $(J^kU, j^k\phi)$. If ϕ is a bijection from $U \subset \mathscr{Q}$ to $\boldsymbol{U} =]a, b[\times U' \subset \mathbb{R} \times E$, then $T\phi$ takes its values in $\boldsymbol{U} \times \mathbb{R} \times E$, and $j^k\phi$ takes its values in $\boldsymbol{U} \times E \times \cdots \times E$ (where there are k of the factors E). We shall write coordinates for $T\mathscr{Q}$ as $((t, u), (\tau, v))$, and coordinates for $J^k\mathscr{Q}$ as $(t, u, u_1, \dots, u_k)$. If we wish to express finite-dimensional coordinate formulas we write coordinates for $T\mathscr{Q}$ as $((t, q^i), (\tau, v^j))$ and coordinates for $J^2\mathscr{Q}$ as (t, q^i, v^j, a^k) (we shall not be employing anything higher than two-jets). The adapted chart (U, ϕ) also induces a natural chart $(T^*U, T^*\phi)$ for $T^*\mathscr{Q}$, and we denote coordinates here by $((t, u), (\lambda, \alpha))$ in infinite-dimensions, and by $((t, q^i), (\lambda, p_i))$ in finite-dimensions.

There are natural inclusions of $J^k\mathscr{Q}$ in $T(J^{k-1}\mathscr{Q})$ for $k \geq 1$. In natural coordinates, for $k = 1, 2$ the inclusions are given by

$$\begin{aligned}(t, u, u_1) &\mapsto ((t, u), (1, u_1))\\ (t, u, u_1, u_2) &\mapsto ((t, u, u_1), (1, u_1, u_2)).\end{aligned}$$

We also note that the local form of a vector in the vertical subbundle $V\mathscr{Q}$ is $((t, u), (0, e))$.

2.2. The first contact system. Associated with the jet bundles are the contact systems. The kth contact system is a subbundle of $T^*(J^k\mathscr{Q})$. We shall only use the first contact system which we now describe using the characterisation of Gardner and Shadwick [1987]. We define the fibre of $\mathsf{C}^1(\mathscr{Q})$, the first contact system, at $j^1c(t) \in J^1\mathscr{Q}$ by

$$\mathsf{C}^1_{j^1c(t)}(\mathscr{Q}) = \left\{(T_{j^1c(t)}\tau_1)^*\beta - (T_tc \circ T_{j^1c(t)}(\pi \circ \tau_1))^*\beta \;\middle|\; \beta \in T^*_{c(t)}\mathscr{Q}\right\}. \tag{2.1}$$

If one works through this definition, it may be seen that $\mathsf{C}^1(\mathcal{Q})$ consists in a natural chart of those elements of $T^*(J^1\mathcal{Q})$ of the form

$$((t, u, u_1), (-\alpha \cdot u_1, \alpha, 0)) \tag{2.2}$$

for some $\alpha \in E^*$ (here $\alpha \cdot u_1$ means the natural pairing of $\alpha \in E^*$ with $u_1 \in E$). In turn, one readily sees that this corresponds to the usual local basis $\{\mathrm{d}q^1 - v^1\mathrm{d}t, \ldots, \mathrm{d}q^n - v^n\mathrm{d}t\}$ in finite-dimensions. Gardner and Shadwick [1987] show that the definition (2.1) gives $\mathsf{C}^1(\mathcal{Q})$ the property that a local section $\sigma\colon [t_1, t_2] \to J^1\mathcal{Q}$ has the property that $\sigma'(t)$ is annihilated by $\mathsf{C}^1(\mathcal{Q})$ if and only if $\sigma = j^1c$ for a local section $c\colon [t_1, t_2] \to \mathcal{Q}$.

The map which in local coordinates is given by $((t, u, u_1), (-\alpha \cdot u_1, \alpha, 0)) \mapsto ((t, u, u_1), \alpha)$ is a vector bundle isomorphism of $\mathsf{C}^1(\mathcal{Q})$ with $\tau_1^* V^*\mathcal{Q}$ where $V^*\mathcal{Q}$ is the dual bundle to $V\mathcal{Q}$. Note that $V^*\mathcal{Q}$ is not naturally a subbundle of $T^*\mathcal{Q}$. We shall make frequent use of this natural identification of $\tau_1^* V^*\mathcal{Q}$ with $\mathsf{C}^1(\mathcal{Q})$. Roughly speaking, it is often more convenient to represent objects in a chart by using $\tau_1^* V^*\mathcal{Q}$, but more natural intrinsically to use $\mathsf{C}^1(\mathcal{Q})$. In finite-dimensions, when we regard $\mathsf{C}^1(\mathcal{Q})$ as the dual bundle to $\tau_1^* V\mathcal{Q} \simeq \ker(T\tau_1)$, the local basis $\{\mathrm{d}q^1 - v^1\mathrm{d}t, \ldots, \mathrm{d}q^n - v^n\mathrm{d}t\}$ for $\mathsf{C}^1(\mathcal{Q})$ is dual to the local basis $\{\frac{\partial}{\partial v^1}, \ldots, \frac{\partial}{\partial v^n}\}$ for $\tau_1^* V\mathcal{Q}$.

2.3. The jet derivatives. Lewis [1996] introduced the idea of the "acceleration derivative" of a function on $J^2\mathcal{Q}$ as a generalisation of the fibre derivative of a function on a vector bundle (also see [Gràcia 1998]). Here we extend this to the ***k-jet derivative*** of a function f defined on $J^k\mathcal{Q}$. The definition goes as follows. Fix $\xi \in J^{k-1}\mathcal{Q}$ and let f_ξ denote the restriction of f to the fibre of $J^k\mathcal{Q}$ over ξ. Since this fibre is an affine space modelled on the vector space $V_{\tau_{k-1}(\xi)}\mathcal{Q}$, its derivative $\boldsymbol{d}f_\xi$ may be regarded as taking its values in $V^*_{\tau_{k-1}(\xi)}\mathcal{Q}$. But as we just saw, $V^*_{\tau_{k-1}(\xi)}\mathcal{Q}$ is naturally isomorphic to $\mathsf{C}^1_{\tau_{k-1,1}(\xi)}(\mathcal{Q})$, and so in this way we construct a map $\boldsymbol{d}_k f\colon J^k\mathcal{Q} \to \mathsf{C}^1(\mathcal{Q})$ so that the diagram

$$\begin{tikzcd}
J^k\mathcal{Q} \arrow[rr, "\boldsymbol{d}_k f"] \arrow[dr, "\tau_{k,1}"'] & & \mathsf{C}^1(\mathcal{Q}) \arrow[dl] \\
& J^1\mathcal{Q} &
\end{tikzcd}$$

commutes. In a natural chart we have

$$\boldsymbol{d}_k f(t, u, u_1, \ldots, u_k) = ((t, u, u_1), \boldsymbol{D}_{k+2} f(t, u, u_1, \ldots, u_k))$$

where $\boldsymbol{D}_{k+2}$ denotes the $(k+2)$nd partial derivative.

2.4. The almost tangent-like structure on $J^1\mathcal{Q}$. Recall that on the tangent bundle of the manifold $\mathcal{Q}$ there is a natural almost tangent structure which we denote by $S_\mathcal{Q}$. Explicitly, $S_\mathcal{Q}$ is the $(1,1)$ tensor field on $T\mathcal{Q}$ given by

$$S_\mathcal{Q}(v_q)(X) = \mathrm{vlft}_{v_q}(T_{v_q}\pi_{T\mathcal{Q}}(X))$$

where $v_q \in T_q\mathcal{Q}$, $X \in T_{v_q}T\mathcal{Q}$, $\pi_{T\mathcal{Q}}\colon T\mathcal{Q} \to \mathcal{Q}$ is the tangent bundle projection, and vlft_{v_q} is the vertical lift defined by

$$\mathrm{vlft}_{v_q}(u_q) = \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} (v_q + tu_q)$$

for $u_q \in T_q\mathscr{Q}$.[1] It turns out that $S_{\mathscr{Q}}|J^1\mathscr{Q}$ leaves $T(J^1\mathscr{Q}) \subset TT\mathscr{Q}$ invariant and so defines a $(1,1)$ tensor field on $J^1\mathscr{Q}$ which we denote by $\tilde{S}_{\mathscr{Q}}$. In natural coordinates one may determine that

$$\tilde{S}_{\mathscr{Q}}(t,u,u_1)\cdot(\tau,e_1,e_2) = (0,0,e_1-\tau u_1).$$

The dual endomorphism $\tilde{S}^*_{\mathscr{Q}}$ of $T^*(J^1\mathscr{Q})$ has local representative

$$\tilde{S}^*_{\mathscr{Q}}(t,u,u_1)\cdot(\lambda,\alpha^1,\alpha^2) = (-\alpha^2\cdot u_1,\alpha^2,0). \tag{2.3}$$

Given our local representation (2.2) of $\mathsf{C}^1(\mathscr{Q})$, this shows that $\tilde{S}^*_{\mathscr{Q}}$ is a surjective map onto $\mathsf{C}^1(\mathscr{Q})$. In finite-dimensions we have

$$\tilde{S}_{\mathscr{Q}} = \frac{\partial}{\partial v^i}\otimes(\mathrm{d}q^i - v^i\mathrm{d}t).$$

2.5. Second-order vector fields. Since $J^2\mathscr{Q}$ is naturally a subset of $T(J^1\mathscr{Q})$, it makes sense to define a ***second-order vector field*** to be a vector field on $J^1\mathscr{Q}$ which takes its values in $J^2\mathscr{Q}$. Thus, in a natural chart $(J^1U, j^1\phi)$, a second-order vector field has representative

$$(t,u,u_1)\mapsto((t,u,u_1),(1,u_1,X(t,u,u_1)))$$

for some $X\colon \boldsymbol{U}\times E\to E$. Note that every second-order vector field is annihilated by $\mathsf{C}^1(\mathscr{Q})$, and that if we add a section of $\tau_1^*V\mathscr{Q}\simeq\ker(T\tau_1)$ to a second-order vector field, we get another second-order vector field. If M is a submanifold of $J^1\mathscr{Q}$, a ***second-order vector field on $\boldsymbol{M}$*** is a vector field on M taking values in $(J^2\mathscr{Q}|M)\cap TM$. Of course, for a general submanifold M, it is possible that there will be no second-order vector fields on M.

3. The components of Lagrangian mechanics

In this section we review some common terminology from Lagrangian mechanics. We wish to give some properties of Lagrangian functions, as well as present general definitions of forces and constraints.

3.1. Lagrangians. A ***Lagrangian*** is a $\mathbb{R}$-valued function L on $J^1\mathscr{Q}$. For $X\in\Gamma^\infty(V\mathscr{Q})$ define a function L_X on $J^1\mathscr{Q}$ by $L_X(v_q) = \langle \boldsymbol{d}_1L(v_q);X(q)\rangle$. Since X depends only on $\mathscr{Q}$ and since the derivative $\boldsymbol{d}_1$ is taken only with respect to the fibre in $J^1\mathscr{Q}$, $\boldsymbol{d}_1L_X(v_q)$ depends only on the value of X at q, and not on the derivative of X. As a consequence we may define a symmetric $(0,2)$ tensor g_L on the pull-back bundle $\tau_1^*V\mathscr{Q}$ as follows:

$$g_L(v_q)(X(q),Y(q)) = \langle \boldsymbol{d}_1L_X(v_q);Y(q)\rangle$$

where Y is a another vertical vector field on $\mathscr{Q}$. In a natural chart we have

$$g_L(t,u,u_1) = \boldsymbol{D}_3^2L(t,u,u_1).$$

[1] Note that this definition of $S_{\mathscr{Q}}$ has nothing to do with $\mathscr{Q}$ being the total space of a locally trivial fibre bundle over $\mathbb{R}$.

In finite-dimensions this yields the familiar formula

$$g_L = \frac{\partial^2 L}{\partial v^i \partial v^j}(\mathrm{d}q^i - v^i \mathrm{d}t) \otimes (\mathrm{d}q^j - v^j \mathrm{d}t)$$

where we identify $\tau_1^* V^* \mathscr{Q}$ with $\mathsf{C}^1(\mathscr{Q})$. Associated to g_L is a bundle map $g_L^\flat \colon \tau_1^* V\mathscr{Q} \to \mathsf{C}^1(\mathscr{Q})$ over the identity on $J^1\mathscr{Q}$. L is ***weakly regular*** if $g_L(v)$ is a weakly nondegenerate form (i.e., $g_L(v)(X,Y) = 0$ for all Y implies $X = 0$) for each $v \in J^1\mathscr{Q}$, and ***regular*** if $g_L^\flat$ is a vector bundle isomorphism from $\tau_1^* V\mathscr{Q}$ to $\tau_1^* V^*\mathscr{Q} \simeq \mathsf{C}^1(\mathscr{Q})$. L is ***positive-definite*** (resp. ***negative-definite***) if $g_L(v)$ is a positive-definite (resp. negative-definite) form for each $v \in J^1\mathscr{Q}$. If L is regular, we denote the inverse of $g_L^\flat$ by $g_L^\sharp \colon \mathsf{C}^1(\mathscr{Q}) \to \tau_1^* V\mathscr{Q}$.

3.2. Forces. We provide a definition of a force which allows dependence on time, configuration, and any finite number of derivatives of configuration with respect to time. One should think of a force as a mechanism for inhibiting motion in certain directions—thus it is intuitive to regard a force as taking values in $\tau_1^* V^*\mathscr{Q} \simeq \mathsf{C}^1(\mathscr{Q})$. Precisely, a ***k-force*** is a smooth map $\Phi \colon J^k\mathscr{Q} \to \mathsf{C}^1(\mathscr{Q})$ so that the diagram

$$\begin{tikzcd} J^k\mathscr{Q} \arrow[rr, "\Phi"] \arrow[dr, "\tau_{k,1}"'] & & \mathsf{C}^1(\mathscr{Q}) \arrow[dl] \\ & J^1\mathscr{Q} & \end{tikzcd}$$

commutes. In a natural chart a k-force is represented by

$$\Phi(t, u, u_1, \dots, u_k) = ((t, u, u_1), \mathbf{\Phi}(t, u, u_1, \dots, u_k))$$

for some map $\mathbf{\Phi} \colon \boldsymbol{U} \times E \times \cdots \times E \to E^*$. Here, for the sake of making a shorter formula, we have identified $\mathsf{C}^1(\mathscr{Q})$ with $\tau_1^* V^*\mathscr{Q}$. Most forces one encounters are 1-forces (i.e., they depend on time, configuration, and velocity), but the main new idea of this paper, presented in Section 4, is that of a 2-force which we associate with a Lagrangian. Note that a 1-force is simply a $\mathsf{C}^1(\mathscr{Q})$-valued one-form on $J^1\mathscr{Q}$.

If $c \colon [t_1, t_2] \to \mathscr{Q}$ is a local section of $\pi \colon \mathscr{Q} \to \mathbb{R}$, we may define a ***force along c*** to be a map $\Phi \colon [t_1, t_2] \to \mathsf{C}^1(\mathscr{Q})$ so that the diagram

$$\begin{tikzcd} & {[t_1, t_2]} \arrow[dl, "\Phi"'] \arrow[dr, "c"] & \\ \mathsf{C}^1(\mathscr{Q}) \arrow[rr] & & \mathscr{Q} \end{tikzcd}$$

commutes.

3.3. Constraints. We also wish to consider constraints in our formulation of mechanics. We shall consider a fairly general notion of a constraint, but one which is nonetheless convenient for proving an existence result for solutions of the corresponding constrained Euler-Lagrange equations. It is necessary to devote some effort to providing local descriptions for the constraints we consider, so a significant portion of this section will be devoted to this task.

The dividends will be reaped in the proof of Theorem 5.5 where the notation we introduce shortly will prove useful. The reader will find our notation applied to a simple example in Section 7.

A ***constraint*** is a pair (M, Λ) where M is a submanifold of $J^1\mathscr{Q}$ and Λ is a subbundle of $T^*(J^1\mathscr{Q})$. We shall see that all that is actually required is the restriction, $\Lambda|M$, of Λ to M. Any extension of Λ off M will suffice since only algebraic constructions are performed with the sections of Λ. A local section $c\colon [t_1, t_2] \to \mathscr{Q}$ ***satisfies*** the constraint (M, Λ) if $j^1c(t) \in M$ and if $(j^1c)'(t) \lrcorner \Lambda_{j^1c(t)} = 0$ for $t \in [t_1, t_2]$. Given a constraint (M, Λ), we associate to it a subset $\mathscr{F}_{(M,\Lambda)}$ of $\mathsf{C}^1(\mathscr{Q})|M$ defined by $\mathscr{F}_{(M,\Lambda)} = \tilde{S}^*_{\mathscr{Q}}(\Lambda|M)$. This makes sense since $\tilde{S}^*_{\mathscr{Q}}$ is $\mathsf{C}^1(\mathscr{Q})$-valued by (2.2) and (2.3). A ***constraint force*** is defined to be a force taking values in $\mathscr{F}_{(M,\Lambda)}$.[2]

The above general notion of a constraint is too unstructured. With this definition it is possible, for example, that constrained problems have no solutions. To enable us to state a general result concerning existence of solutions for constrained systems, we need to consider a class of constraints which is rather more restrictive. Even though it is restrictive, it contains the types of constraints most often considered—for example, as we indicate below, constraints which are affine in velocity are of the form we now introduce. Let us define a subset Λ_M of T^*M by defining its fibre at $v \in M$ to be

$$\Lambda_{M,v} = T^*_v i_M(\Lambda_{i_M(v)})$$

where $i_M\colon M \to J^1\mathscr{Q}$ is the inclusion. For a general submanifold M, Λ_M cannot be expected to be a subbundle. The constraint (M, Λ) is called ***ideal*** if the following conditions hold:

IC1. $\mathscr{F}_{(M,\Lambda)}$ is a subbundle of $\mathsf{C}^1(\mathscr{Q})|M$;

IC2. $\ker(\tilde{S}^*_{\mathscr{Q}}|M) \cap \Lambda|M$ is a subbundle of $\Lambda|M$;

IC3. $J^2\mathscr{Q}|M \cap \operatorname{coann}(\Lambda_M)$ is a non-trivial affine subbundle of $J^2\mathscr{Q}|M$ modelled on the vector subbundle $\operatorname{coann}(\mathscr{F}_{(M,\Lambda)}) \cap (\tau_1|M)^*V\mathscr{Q}$ of $(\tau_1|M)^*V\mathscr{Q}$.

The condition IC1 is a natural one as it asks that the set in which the constraint forces take their values be a subbundle. Condition IC2 will allow us to make a local decomposition of $\Lambda|M$ which renders the map $\tilde{S}^*_{\mathscr{Q}}|M$ in a particularly simple form. This in turn is helpful in the existence proof for solutions to the constrained problem. The final condition, IC3, provides a reasonable target set—an affine subbundle—for a second-order vector field which describes the constrained motion. As we shall see in Theorem 5.5, this general notion of an ideal constraint is sufficient to establish the existence of a second-order vector field on M whose integral curves are solutions of the constrained equations of motion. These conditions are not, however, necessary in order for the constrained system to have solutions. Note that in the event that $\mathscr{Q}$ is modelled on a Hilbert space rather than a Banach space, the assumptions that various subsets be subbundles bear only on their having constant rank (or the infinite-dimensional equivalent). However, for Banach manifolds there is the additional hypothesis that the subspaces involved be split.

[2] Just why one should define a constraint force in this way is not a simple matter to justify—it is really the key ingredient to the nature of the Euler-Lagrange equations for systems with nonlinear constraints. This matter is discussed by Chetaev [1989].

It will be important for us to have explicit local representations for a constraint (M, Λ), so let us introduce our notation for this. Let (U, ϕ) be an adapted chart for $\pi\colon \mathscr{Q} \to \mathbb{R}$. We suppose that $\boldsymbol{U} = \phi(U) \subset \mathbb{R} \times E$ for some Banach space E. This induces natural charts $(J^kU, j^k\phi)$ for $J^k\mathscr{Q}$, $k = 1, 2$, and a natural chart $(T^*(J^1U), T^*(j^1\phi))$ for $T^*(J^1\mathscr{Q})$. Since Λ is a subbundle of $T^*(J^1\mathscr{Q})$, Λ_v is split in $T^*_v(J^1\mathscr{Q})$ for each $v \in J^1\mathscr{Q}$. Therefore, by choosing U sufficiently small, there exists a vector bundle chart $(T^*(J^1U), \psi)$ for $T^*(J^1\mathscr{Q})$ which is adapted to this subbundle. This means that the following hold:

1. ψ is a bijection from $T^*(J^1U)$ onto $\boldsymbol{U} \times E \times F_1^* \times F_2^*$ for Banach spaces F_1 and F_2;
2. $\psi(\Lambda)_{(t,u,u_1)} = \{(t, u, u_1)\} \times F_1^* \times \{0\}$;
3. the overlap map from $T^*(j^1\phi)(T^*(J^1U)) = \boldsymbol{U} \times E \times \mathbb{R}^* \times E^* \times E^*$ to $\psi(T^*(J^1U)) = \boldsymbol{U} \times E \times F_1^* \times F_2^*$ has the form

$$\begin{aligned} h\colon ((t,u,u_1),(\lambda,\alpha^1,\alpha^2)) \mapsto ((t,u,u_1),\\ (A_{10}(t,u,u_1)\cdot\lambda + A_{11}(t,u,u_1)\cdot\alpha^1 + A_{12}(t,u,u_1)\cdot\alpha^2,\\ A_{20}(t,u,u_1)\cdot\lambda + A_{21}(t,u,u_1)\cdot\alpha^1 + A_{22}(t,u,u_1)\cdot\alpha^2)) \end{aligned}$$

for maps $A_{j0}\colon \boldsymbol{U} \times E \to \mathrm{L}(\mathbb{R}^*; F_j^*)$ and $A_{ij}\colon \boldsymbol{U} \times E \to \mathrm{L}(E^*; F_j^*)$, $i, j = 1, 2$.

Let us denote the inverse of the overlap map by

$$\begin{aligned} h^{-1}\colon ((t,u,u_1),(\nu^1,\nu^2)) \mapsto ((t,u,u_1),\\ (B_{01}(t,u,u_1)\cdot\nu^1 + B_{02}(t,u,u_1)\cdot\nu^2, B_{11}(t,u,u_1)\cdot\nu^1 + B_{12}(t,u,u_1)\cdot\nu^2,\\ B_{21}(t,u,u_1)\cdot\nu^1, B_{22}(t,u,u_1)\cdot\nu^2)) \end{aligned}$$

for maps $B_{0j}\colon \boldsymbol{U} \times E \to \mathrm{L}(F_j^*; \mathbb{R}^*)$ and $B_{ij}\colon \boldsymbol{U} \times E \to \mathrm{L}(F_j^*; E^*)$, $i, j = 1, 2$. Thus the constraint codistribution Λ is locally given in the natural chart $(T^*(J^1U), T^*(j^1\phi))$ by

$$\Lambda_{(t,u,u_1)} = \{((t,u,u_1),(B_{01}(t,u,u_1)\cdot\nu^1, B_{11}(t,u,u_1)\cdot\nu^1, B_{21}(t,u,u_1)\cdot\nu^1)) \mid \nu^1 \in F_1^*\}.$$

Using the local form (2.3) of $\tilde{S}^*_{\mathscr{Q}}$, we see that the local form of $\tilde{S}^*_{\mathscr{Q}}|\Lambda$ is given by

$$\begin{aligned} ((t,u,u_1),(B_{01}(t,u,u_1)\cdot\nu^1, B_{11}(t,u,u_1)\cdot\nu^1, B_{21}(t,u,u_1)\cdot\nu^1)) \mapsto\\ ((t,u,u_1), B_{21}(t,u,u_1)\cdot\nu^1) \end{aligned} \tag{3.1}$$

where we think of $\mathrm{C}^1(\mathscr{Q}) \simeq \tau_1^* V^*\mathscr{Q}$.

The local decomposition we have just presented is not really sufficient for our purposes. The problem arises because $B_{21}(t, u, u_1)$ may not be injective, corresponding to the fact that $\tilde{S}^*_{\mathscr{Q}}|\Lambda$ may not be injective. To overcome this difficulty, we make a further refinement of Λ which is appropriately adapted to the mapping $\tilde{S}^*_{\mathscr{Q}}$. Let us suppose that $\mathscr{F}_{(M,\Lambda)}$ is a subbundle of $\mathrm{C}^1(\mathscr{Q})|M$ and that $\ker(\tilde{S}^*_{\mathscr{Q}}|M) \cap \Lambda|M$ is a subbundle of $\Lambda|M$ (as, for example, when (M, Λ) is ideal). In this case the map (3.1) is a local vector bundle mapping whose image and kernel are subbundles. Thus (see [Abraham, Marsden, and Ratiu 1988, Proposition 3.4.18]) we may further refine our local representation of $\Lambda|M$. Indeed, supposing that $M \cap J^1U \neq \varnothing$, we may choose a vector bundle chart $(T^*(J^1U), \psi')$ for $T^*(J^1\mathscr{Q})$ with the following properties:

1. ψ' is a bijection from $T^*(J^1\mathscr{Q})$ onto $\boldsymbol{U} \times E \times F_{11}^* \times F_{12}^* \times F_2^*$ for Banach spaces F_{11}, F_{12}, and F_2;
2. $\psi'(\Lambda)_{(t,u,u_1)} = \{(t,u,u_1)\} \times F_{11}^* \times F_{12}^* \times \{0\}$;
3. $\psi'(\ker(\tilde{S}_{\mathscr{Q}}^*|M) \cap \Lambda|M) = \{(t,u,u_1)\} \times \{0\} \times F_{12}^* \times \{0\}$ if $\phi^{-1}(t,u,u_1) \in M$;
4. the overlap map from $T^*j^1\phi(T^*(J^1U)) = \boldsymbol{U} \times E \times \mathbb{R}^* \times E^* \times E^*$ to $\psi'(T^*(J^1U)) = \boldsymbol{U} \times E \times F_{11}^* \times F_{12}^* \times F_2^*$ has the form

$$\begin{aligned} h' \colon ((t,u,u_1),(\lambda,\alpha^1,\alpha^2)) \mapsto ((t,u,u_1),& \\ (A_{110}(t,u,u_1)\cdot\lambda + A_{111}(t,u,u_1)\cdot\alpha^1 + A_{112}(t,u,u_1)\cdot\alpha^2,& \\ (A_{120}(t,u,u_1)\cdot\lambda + A_{121}(t,u,u_1)\cdot\alpha^1 + A_{122}(t,u,u_1)\cdot\alpha^2,& \\ A_{20}(t,u,u_1)\cdot\lambda + A_{21}(t,u,u_1)\cdot\alpha^1 + A_{22}(t,u,u_1)\cdot\alpha^2))& \end{aligned}$$

for maps $A_{1j0} \colon \boldsymbol{U} \times E \to \mathrm{L}(\mathbb{R}^*; F_{1j})$, $A_{1ij} \colon \boldsymbol{U} \times E \to \mathrm{L}(E^*; F_{1i}^*)$, $A_{20} \colon \boldsymbol{U} \times E \to \mathrm{L}(\mathbb{R}^*; F_2^*)$ and $A_{2j} \colon \boldsymbol{U} \times E \to \mathrm{L}(E^*; F_j^*)$, $i,j = 1,2$;
5. if the inverse of h' is

$$\begin{aligned} h'^{-1} \colon ((t,u,u_1),(\nu^{11},\nu^{12},\nu^2)) \mapsto ((t,u,u_1),& \\ (B_{011}(t,u,u_1)\cdot\nu^{11} + B_{012}(t,u,u_1)\cdot\nu^{12} + B_{02}(t,u,u_1)\cdot\nu^2,& \\ B_{111}(t,u,u_1)\cdot\nu^{11} + B_{112}(t,u,u_1)\cdot\nu^{12} + B_{12}(t,u,u_1)\cdot\nu^2,& \\ B_{211}(t,u,u_1)\cdot\nu^{11} + B_{212}(t,u,u_1)\cdot\nu^{12} + B_{22}(t,u,u_1)\cdot\nu^2))& \end{aligned}$$

for maps $B_{01j} \colon \boldsymbol{U} \times E \to \mathrm{L}(F_{1j}^*; \mathbb{R}^*)$, $B_{i1j} \colon \boldsymbol{U} \times E \to \mathrm{L}(F_{1j}^*; E^*)$, $B_{02} \colon \boldsymbol{U} \times E \to \mathrm{L}(F_2^*; \mathbb{R}^*)$, and $B_{j2} \colon \boldsymbol{U} \times E \to \mathrm{L}(F_2^*; E^*)$, $i,j = 1,2$, then, when $\phi^{-1}(t,u,u_1) \in M$, the following hold:
   (a) $B_{211}(t,u,u_1)$ is injective with split image;
   (b) $B_{212}(t,u,u_1) = 0$;
   (c) the map $\nu^{12} \mapsto B_{012}(t,u,u_1)\cdot\nu^{12} + B_{112}(t,u,u_1)\cdot\nu^{12}$ is injective with split image.

With this refined splitting we locally have

$$\mathscr{F}_{(M,\Lambda)} = \{((t,u,u_1), B_{211}(t,u,u_1)\cdot\nu^{11}) \mid \nu^{11} \in F_{11}^*, \ (t,u,u_1) \in j^1\phi(M \cap J^1U)\}$$

and

$$\begin{aligned} \ker(\tilde{S}_{\mathscr{Q}}^*|M) \cap \Lambda|M = \big\{((t,u,u_1),(B_{012}(t,u,u_1)\cdot\nu^{12},& \\ B_{112}(t,u,u_1)\cdot\nu^{12},0)) \mid \ \nu^{12} \in F_{12}^*, \ (t,u,u_1) \in j^1\phi(M \cap J^1U)\big\}.& \end{aligned}$$

To emphasise how this refined splitting is adapted to $\tilde{S}_{\mathscr{Q}}^*|\Lambda$, let us explicitly state that the local form of this vector bundle map is

$$\begin{aligned} ((t,u,u_1),(B_{011}(t,u,u_1)\cdot\nu^{11} + B_{012}(t,u,u_1)\cdot\nu^{12}, B_{111}(t,u,u_1)\cdot\nu^{11} + B_{112}(t,u,u_1)\cdot\nu^{12},& \\ B_{211}(t,u,u_1)\cdot\nu^{11} + B_{212}(t,u,u_1)\cdot\nu^{12})) \mapsto ((t,u,u_1), B_{211}(t,u,u_1)\cdot\nu^{11}).& \end{aligned}$$

This describes one half of a constraint (M, Λ). To see how the submanifold M may be locally added to the mix, suppose that the chart (U, ϕ) is chosen so that $V \triangleq M \cap J^1U$ is open in M and forms the domain of a chart $(\tilde{U}, \chi)$ for M. We shall suppose that χ is $\tilde{E}$-valued for some Banach space $\tilde{E}$. Let $\boldsymbol{i_M} : \tilde{\boldsymbol{U}} \to \boldsymbol{U} \times E$ be the local representative of i_M which we write as

$$\boldsymbol{i_M}(\tilde{u}) = (C_0(\tilde{u}), C_1(\tilde{u}), C_2(\tilde{u}))$$

for maps $C_0 : \tilde{\boldsymbol{U}} \to \boldsymbol{U} \cap \mathbb{R}$, $C_1 : \tilde{\boldsymbol{U}} \to \boldsymbol{U} \cap E$, and $C_2 : \tilde{\boldsymbol{U}} \to E$.

In finite-dimensions, the above constructions may be described in terms of local bases for the various subbundles. We choose a local basis $\{\beta^1, \dots, \beta^{2n+1}\}$ for $T^*(J^1\mathscr{Q})$ with the property that $\beta^1, \dots, \beta^m$ generate Λ. We shall write the forms β^a, $a = 1, \dots, m$, as

$$\beta^a = \beta^a_0 \mathrm{d}t + \beta^a_i \mathrm{d}q^i + \hat{\beta}^a_i \mathrm{d}v^i. \tag{3.2}$$

One readily sees that the one-forms

$$\hat{\beta}^a_i(\mathrm{d}q^i - v^i \mathrm{d}t), \quad a = 1, \dots, m$$

when restricted to M, locally generate $\mathscr{F}_{(M,\Lambda)}$. However, they will not in general be linearly independent. If $\mathscr{F}_{(M,\Lambda)}$ is a subbundle, then we may choose the one-forms β^a, $a = 1, \dots, m$, in such a way that the one-forms

$$\hat{\beta}^a_i(\mathrm{d}q^i - v^i \mathrm{d}t), \quad a = 1, \dots, \tilde{m}, \quad \tilde{m} \leq m,$$

form a basis for $\mathscr{F}_{(M,\Lambda)}$ when restricted to M, and the one-forms β^a, $a = \tilde{m}+1, \dots, m$ form a basis for $\ker(\tilde{S}^*_{\mathscr{Q}}|M) \cap \Lambda|M$ when restricted to M. If $(x^1, \dots, x^r)$ are coordinates for M, we may write the inclusion i_M locally as

$$(x^1, \dots, x^r) \mapsto (C_0(x), C^i_1(x), C^j_2(x)), \quad i, j = 1, \dots, n.$$

In this case, a local section given in coordinates by $t \mapsto (t, q^i(t))$ satisfies the constraint if and only if

$$C_0(x(t)) = t, \quad C^i_1(x(t)) = q^i(t), \quad C^j_2(x(t)) = \dot{q}^j(t), \quad i, j = 1, \dots, n$$

for some curve $t \mapsto (x^\alpha(t))$ in M, and

$$\hat{\beta}^a_i \ddot{q}^i + \beta^a_i \dot{q}^i + \beta^0 = 0, \quad a = 1, \dots, m.$$

Let us now employ our local notation to characterise ideal constraints. The local description of $\Lambda_M \subset T^*M$ is

$$\begin{aligned}\Lambda_M = \big\{ &(\tilde{u}, \boldsymbol{D}C_0(\tilde{u})^* \cdot (B_{011}(t, u, u_1) \cdot \nu^{11} + B_{012}(t, u, u_1) \cdot \nu^{12}) + \\ &\boldsymbol{D}C_1(\tilde{u})^* \cdot (B_{111}(t, u, u_1) \cdot \nu^{11} + B_{112}(t, u, u_1) \cdot \nu^{12}) + \boldsymbol{D}C_2(\tilde{u})^* \cdot B_{211}(t, u, u_1) \cdot \nu^{11}) \mid \\ &\qquad \nu^{11} \in F^*_{11},\ \nu^{12} \in F^*_{12},\ (t, u, u_1) = \boldsymbol{i_M}(\tilde{u}) \big\}.\end{aligned} \tag{3.3}$$

Therefore,

$$\begin{aligned}J^2\mathscr{Q}|M \cap \mathrm{coann}(\Lambda_M) = \big\{ &(\tilde{u}, \tilde{e}) \mid \ \boldsymbol{D}C_0(\tilde{u}) \cdot \tilde{e} = 1,\ \boldsymbol{D}C_1(\tilde{u}) \cdot \tilde{e} = u_1, \\ &B^*_{011}(t, u, u_1) \cdot 1 + B^*_{111}(t, u, u_1) \cdot u_1 + B^*_{211}(t, u, u_1) \circ \boldsymbol{D}C_2(\tilde{u}) \cdot \tilde{e} = 0, \\ &B^*_{012}(t, u, u_1) \cdot 1 + B^*_{112}(t, u, u_1) \cdot u_1 = 0,\ (t, u, u_1) = \boldsymbol{i_M}(\tilde{u}) \big\}.\end{aligned} \tag{3.4}$$

We also have

$$\operatorname{coann}(\mathscr{F}_{(M,\Lambda)}) \cap (\tau_1|M)^*V\mathscr{Q} = \{(t,u,u_1),(0,0,e) \mid \ B^*_{112}(t,u,u_1)\cdot e = 0\}. \tag{3.5}$$

If $J^2\mathscr{Q} \cap \operatorname{coann}(\Lambda_M)$ is an affine subbundle modelled on $\operatorname{coann}(\mathscr{F}_{(M,\Lambda)}) \cap (\tau_1|M)^*V\mathscr{Q}$ we must have

$$\boldsymbol{D}C_2(\tilde{u})\cdot\tilde{e} + e \in \operatorname{image}(\boldsymbol{D}C_2(\tilde{u})) \tag{3.6}$$

for each e which satisfies the relation in (3.5). This shows that for ideal constraints we have

$$\operatorname{coann}(\mathscr{F}_{(M,\Lambda)}) \cap (\tau_1|M)^*V\mathscr{Q} \subset TM.$$

3.1 Remarks: 1. The case of constraints for which $M = J^1\mathscr{Q}$ is taken up by Giachetta [1992]. In this case Giachetta calls a constraint "ideal" when $\tilde{S}^*_{\mathscr{Q}}|\Lambda$ is a vector bundle monomorphism. One then readily checks that $(M = J^1\mathscr{Q}, \Lambda)$ is ideal in our sense.

2. de León, Marrero, and Martín de Diego [1997a] provide a notion of an "admissible" constraint as a pair (M, D) (here D is a distribution on M) for which $\tilde{S}^*_{\mathscr{Q}}|\operatorname{ann}(D)$ is a vector bundle monomorphism. Taking Λ so that $\Lambda|M = \operatorname{coann}(D)$, this implies, but is not equivalent to, the conditions IC2 and IC1. This notion of "admissible" is not adequate to ensure solutions for the constrained dynamics; these are provided as separate conditions by de León, Marrero, and Martín de Diego. •

An important class of constraints is ***affine constraints*** which are defined by a codistribution Λ_0 on $\mathscr{Q}$. A local section $c\colon [t_1,t_2] \to \mathscr{Q}$ ***satisfies*** the affine constraint Λ_0 if $c'(t) \lrcorner \Lambda_{0,c(t)} = 0$ for $t \in [t_1,t_2]$. Let us see how we can construct a general constraint from an affine constraint. First of all, we define the submanifold M to be $\operatorname{coann}(\Lambda_0) \cap J^1\mathscr{Q}$. This set will not always be non-empty. Lewis [1998] shows that if $\boldsymbol{d}t \wedge \Lambda_0 \neq 0$ then M defined in this way is non-empty, and is further an affine subbundle of $J^1\mathscr{Q}$. Let us say Λ_0 is ***compatible with*** $\boldsymbol{\pi}$ if $\boldsymbol{d}t\wedge\Lambda_0 \neq 0$. Also, one can lift Λ_0 to a subbundle $j^1\Lambda_0$ of $T^*(J^1\mathscr{Q})$ as follows. For a section β of Λ_0 define a function f_β on $T\mathscr{Q}$ by $f_\beta(v_q) = \boldsymbol{d}\beta(q)\cdot v_q$. We then define a subset Λ_0^T of $T^*T\mathscr{Q}$ whose fibre over $v_q \in J^1\mathscr{Q}$ is

$$\Lambda^T_{0,v_q} = \{\boldsymbol{d}f_\beta(v_q) \mid \ \beta \in \Gamma^\infty(\Lambda_0)\}.$$

Lewis [1998] shows that Λ_0^T is a subbundle of $T^*T\mathscr{Q}$. If $\iota_{\mathscr{Q}}\colon J^1\mathscr{Q} \to T\mathscr{Q}$ is the inclusion, Lewis further shows that if Λ_0 is compatible with π then the map $T^*_v\iota_{\mathscr{Q}}\colon T^*_vT\mathscr{Q} \to T^*_v(J^1\mathscr{Q})$ restricted to $\Lambda^T_{0,v}$ is an injection. Thus, if Λ_0 is compatible with π then Λ_0^T restricts to a well-defined subbundle of $T^*(J^1\mathscr{Q})$ which we denote by $j^1\Lambda_0$. In this way, given an affine constraint Λ_0 which is compatible with π we can define a constraint of general type given by $(\operatorname{coann}(\Lambda_0) \cap J^1\mathscr{Q}, j^1\Lambda_0)$. In finite-dimensions we have $\operatorname{rank}(j^1\Lambda_0) = 2\operatorname{rank}(\Lambda_0)$. If we have a local basis for Λ_0 given by one-forms

$$\beta^a = \beta^a_0\mathrm{d}t + \beta^a_i\mathrm{d}q^i, \quad a = 1,\dots,m,$$

then Λ_0 is compatible with π if and only if the matrix β^a_i, $a = 1,\dots,m$, $i = 1,\dots,n$, has maximal rank (i.e., rank m). In this case the one-forms

$$\left(\frac{\partial\beta^a_0}{\partial t} + \frac{\partial\beta^a_i}{\partial t}v^i\right)\mathrm{d}t + \left(\frac{\partial\beta^a_0}{\partial q^j} + \frac{\partial\beta^a_i}{\partial q^j}v^i\right)\mathrm{d}q^j + \beta^a_i\mathrm{d}v^i, \quad a = 1,\dots,m, \tag{3.7}$$

together with the local basis for Λ_0, form a local basis for $j^1\Lambda_0$. An infinite-dimensional version of this is given by Lewis [1998], where it is also shown that $(\mathrm{coann}(\Lambda_0) \cap J^1\mathscr{Q}, j^1\Lambda_0)$ is ideal. A coordinate definition of $j^1\Lambda_0$ is given by de León, Marrero, and Martín de Diego [1997b]. We also give an example of a system, treated in our framework, with affine constraints, and we refer to Section 7 for some further remarks on these systems.

4. The Euler-Lagrange 2-force

Recall the unforced, unconstrained Euler-Lagrange equations in their classical form:

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{q}^i}\right) - \frac{\partial L}{\partial q^i} = 0, \quad i = 1, \dots, n.$$

To motivate our definition of the Euler-Lagrange 2-force, it is convenient to expand the Euler-Lagrange equations to

$$\frac{\partial^2 L}{\partial \dot{q}^j \partial \dot{q}^i}\ddot{q}^j + \frac{\partial^2 L}{\partial q^j \partial \dot{q}^i}\dot{q}^j + \frac{\partial^2 L}{\partial t \partial \dot{q}^i} - \frac{\partial L}{\partial q^i} = 0, \quad i = 1, \dots, n.$$

With this form of the Euler-Lagrange equations in front of us, it is natural to define the ***Euler-Lagrange 2-force*** to be the 2-force Φ_L on $\mathscr{Q}$ which is given in a natural chart for $J^2\mathscr{Q}$ by

$$\begin{aligned}\Phi_L(t, u, u_1, u_2) = ((t, u, u_1), \boldsymbol{D}_3^2 L(t, u, u_1) \cdot u_2 + \boldsymbol{D}_2\boldsymbol{D}_3 L(t, u, u_1) \cdot (u_1, \cdot) +\\ \boldsymbol{D}_1\boldsymbol{D}_3 L(t, u, u_1) \cdot (1, \cdot) - \boldsymbol{D}_1 L(t, u, u_1)),\end{aligned}$$

regarding Φ_L as $\tau_1^* V^*\mathscr{Q}$-valued. We shall find the "place-holder" notation convenient. Thus, for example, $\boldsymbol{D}_2\boldsymbol{D}_3 L(t, u, u_1) \cdot (u_1, \cdot)$ is the element of E^* defined by

$$e \mapsto \boldsymbol{D}_2\boldsymbol{D}_3 L(t, u, u_1) \cdot (u_1, e).$$

Also note that we match the arguments with the partial derivatives in the same order—that is, the leftmost partial derivative takes the first argument, and so on. A partial derivative with respect to "t" will be supposed to be evaluated on the vector "1" unless otherwise indicated.

This definition for Φ_L needs to be shown to be independent of natural chart. It is well-known that the Euler-Lagrange equations are independent of coordinates in the sense that a solution in one set of coordinates will still be a solution when we make a change of coordinates. But we can do better than this with Φ_L.

4.1 Proposition: Φ_L *obeys the transformation property of a 2-force.*

Proof: This is a straightforward but tedious exercise in differential calculus, and we shall only outline the main points. Let (U, ϕ) and $(U, \tilde{\phi})$ be adapted charts for $\mathscr{Q}$ with the same domain U. We let $\phi(U) =]a, b[\times U'$ and $\phi(\tilde{U}) =]a, b[\times \tilde{U}'$. These charts induce natural charts $(J^2U, j^2\phi)$ and $(J^2U, j^2\tilde{\phi})$ for $J^2\mathscr{Q}$. Given our assumption on the form of adapted charts for $\pi \colon \mathscr{Q} \to \mathbb{R}$, the overlap map $\tilde{\phi} \circ \phi^{-1}$ is given by $(t, u) \mapsto (t, \psi(t, u))$ for some map $\psi \colon\ ]a, b[\times U' \to \tilde{U}'$. Let us denote coordinates in the chart $(J^2U, j^2\phi)$ by (t, u, u_1, u_2) and coordinates in the chart $(J^2U, j^2\tilde{\phi})$ by $(\tilde{t}, \tilde{u}, \tilde{u}_1, \tilde{u}_2)$. The transformation property for the overlap map $j^2\tilde{\phi} \circ (j^2\phi)^{-1}$ (i.e., the transformation law for 2-jets) is given by

$$\begin{aligned} \tilde{u}_1 &= \boldsymbol{D}_1\psi(t,u) + \boldsymbol{D}_2\psi(t,u)\cdot u_1, \\ \tilde{u}_2 &= \boldsymbol{D}_1^2\psi(t,u) + 2\boldsymbol{D}_2\boldsymbol{D}_1\psi(t,u)\cdot u_1 + \boldsymbol{D}_2^2\psi(t,u)\cdot(u_1,u_1) + \boldsymbol{D}_2\psi(t,u)\cdot u_2. \end{aligned}$$

For brevity let us denote $\chi = j^1\tilde{\phi} \circ (j^1\phi)^{-1}$. We compute

$$\begin{aligned} \boldsymbol{D}_1(L\circ\chi)(t,u,u_1) = {} & \boldsymbol{D}_1L(\chi(t,u,u_1)) + \boldsymbol{D}_2L(\chi(t,u,u_1))\circ\boldsymbol{D}_1\psi(t,u) + \\ & \boldsymbol{D}_3L(\chi(t,u,u_1))\circ\boldsymbol{D}_1\boldsymbol{D}_2\psi(t,u)\cdot u_1 + \boldsymbol{D}_3L(\chi(t,u,u_1))\circ\boldsymbol{D}_1^2\psi(t,u), \end{aligned}$$

and

$$\begin{aligned} \boldsymbol{D}_2(L\circ\chi)(t,u,u_1) = {} & \boldsymbol{D}_2L(\chi(t,u,u_1))\circ\boldsymbol{D}_2\psi(t,u) + \boldsymbol{D}_3L(\chi(t,u,u_1))\circ\boldsymbol{D}_2^2\psi(t,u)\cdot u_1 + \\ & \boldsymbol{D}_3L(\chi(t,u,u_1))\circ\boldsymbol{D}_1\boldsymbol{D}_2\chi(t,u), \end{aligned}$$

and

$$\boldsymbol{D}_3(L\circ\chi)(t,u,u_1) = \boldsymbol{D}_3L(\chi(t,u,u_1))\circ\boldsymbol{D}_2\psi(t,u).$$

From these we readily compute

$$\begin{aligned} \boldsymbol{D}_1\boldsymbol{D}_3(L\circ\chi)(t,u,u_1) = {} & \boldsymbol{D}_1\boldsymbol{D}_3L(\chi(t,u,u_1))\circ\boldsymbol{D}_2\psi(t,u) + \\ & \boldsymbol{D}_2\boldsymbol{D}_3L(\chi(t,u,u_1))\cdot(\boldsymbol{D}_1\psi(t,u), \boldsymbol{D}_2\psi(t,u)\cdot(\cdot)) + \\ & \boldsymbol{D}_3L(\chi(t,u,u_1))\circ\boldsymbol{D}_1\boldsymbol{D}_2\psi(t,u) + \\ & \boldsymbol{D}_3^2L(\chi(t,u,u_1))\cdot(\boldsymbol{D}_1\boldsymbol{D}_2\psi(t,u)\cdot u_1, \boldsymbol{D}_2\psi(t,u)\cdot(\cdot)) + \\ & \boldsymbol{D}_3^2L(\chi(t,u,u_1))\cdot(\boldsymbol{D}_1^2\psi(t,u), \boldsymbol{D}_2\psi(t,u)\cdot(\cdot)), \end{aligned}$$

and

$$\begin{aligned} \boldsymbol{D}_2\boldsymbol{D}_3(L\circ\chi)(t,u,u_1) = {} & \boldsymbol{D}_2\boldsymbol{D}_3L(\chi(t,u,u_1))\circ(\boldsymbol{D}_2\psi(t,u)\times\boldsymbol{D}_2\psi(t,u)) + \\ & \boldsymbol{D}_3^2L(\chi(t,u,u_1))\circ(\boldsymbol{D}_2^2\psi(t,u)\cdot u_1\times\boldsymbol{D}_2\psi(t,u)) + \\ & \boldsymbol{D}_3L(\chi(t,u,u_1))\circ\boldsymbol{D}_2^2\psi(t,u) \\ & \boldsymbol{D}_3^2L(\chi(t,u,u_1))\circ(\boldsymbol{D}_1\boldsymbol{D}_2\psi(t,u)\times\boldsymbol{D}_2\psi(t,u)), \end{aligned}$$

and

$$\boldsymbol{D}_3^2(L\circ\chi)(t,u,u_1) = \boldsymbol{D}_3^2L(\chi(t,u,u_1))\circ(\boldsymbol{D}_2\psi(t,u)\times\boldsymbol{D}_2\psi(t,u)),$$

where, for example,

$$\begin{aligned} & \boldsymbol{D}_3^2L(\chi(t,u,u_1))\cdot(\boldsymbol{D}_1^2\psi(t,u), \boldsymbol{D}_2\psi(t,u)\cdot(\cdot))\cdot e = \\ & \qquad \boldsymbol{D}_3^2L(\chi(t,u,u_1))\cdot(\boldsymbol{D}_1^2\psi(t,u), \boldsymbol{D}_2\psi(t,u)\cdot e), \end{aligned}$$

and

$$\begin{aligned} & \boldsymbol{D}_3^2L(\chi(t,u,u_1))\circ(\boldsymbol{D}_2\psi(t,u)\times\boldsymbol{D}_2\psi(t,u))\cdot(e_1,e_2) = \\ & \qquad \boldsymbol{D}_3^2L(\chi(t,u,u_1))\cdot(\boldsymbol{D}_2\psi(t,u)\cdot e_1, \boldsymbol{D}_2\psi(t,u)\cdot e_2). \end{aligned}$$

Collecting all this together, and using the transformation rule for $J^2\mathscr{Q}$ gives

$$\begin{aligned}
&\boldsymbol{D}_3^2(L\circ\chi)(t,u,u_1)\cdot u_2 + \boldsymbol{D}_2\boldsymbol{D}_3(L\circ\chi)(t,u,u_1)\cdot u_1 + \\
&\qquad \boldsymbol{D}_1\boldsymbol{D}_3(L\circ\chi)(t,u,u_1) - \boldsymbol{D}_2(L\circ\chi)(t,u,u_1) = \\
&\quad (\boldsymbol{D}_2\psi(t,u))^*\boldsymbol{D}_3^2L(\chi(t,u,u_1))\cdot\tilde{u}_2 + (\boldsymbol{D}_2\psi(t,u))^*\boldsymbol{D}_2\boldsymbol{D}_3L(\chi(t,u,u_1))\cdot\tilde{u}_1 + \\
&\qquad (\boldsymbol{D}_2\psi(t,u))^*\boldsymbol{D}_1\boldsymbol{D}_3L(\chi(t,u,u_1)) - (\boldsymbol{D}_2\psi(t,u))^*\boldsymbol{D}_2L(\chi(t,u,u_1)),
\end{aligned}$$

That is to say

$$\boldsymbol{\Phi_L}(\tilde{t},\tilde{u},\tilde{u}_1,\tilde{u}_2) = (\boldsymbol{D}_2\psi(t,u))^*(\boldsymbol{\Phi_L}(t,u,u_1,u_2)).$$

It remains to show that this is how a 2-force should transform. Since a 2-force is $\mathsf{C}^1(\mathscr{Q})$-valued, we need to see how sections of $\mathsf{C}^1(\mathscr{Q})$ transform under the change of coordinates χ. Recall from (2.2) that elements of $\mathsf{C}^1(\mathscr{Q})$ are locally of the form $((t,u,u_1),(-\alpha\cdot u_1,\alpha,0))$ for some $\alpha\in E^*$. For $(\tau,e_1,e_2)\in\mathbb{R}\times E\times E$ we compute

$$\begin{aligned}
&(-\alpha\cdot\tilde{u}_1,\alpha,0)\cdot(\boldsymbol{D}\chi(t,u,u_1)\cdot(\tau,e_1,e_2)) = \\
&\qquad (-((\boldsymbol{D}_2\psi(t,u))^*\cdot\alpha)\cdot u_1,(\boldsymbol{D}_2\psi(t,u))^*\cdot\alpha,0)\cdot(\tau,e_1,e_2)
\end{aligned}$$

This shows that Φ_L transforms as do sections of $\mathsf{C}^1(\mathscr{Q})$ which completes the proof. ■

4.2 Remarks: 1. In finite-dimensions we have

$$\Phi_L = \left(\frac{\partial^2 L}{\partial v^i\partial v^j}a^j + \frac{\partial^2 L}{\partial v^i\partial q^j}v^j + \frac{\partial^2 L}{\partial v^i\partial t} - \frac{\partial L}{\partial q^i}\right)(\mathrm{d}q^i - v^i\mathrm{d}t).$$

2. In some sense, all the work of the above result is unnecessary as we shall see an intrinsic way of defining the Euler-Lagrange 2-force in Section 6. However, if we are to enable the Euler-Lagrange 2-force to stand on its own two feet, so to speak, then it is necessary to ensure that its definition is coordinate independent.

3. Of course, the Euler-Lagrange 2-force has been conceived of in various guises in the past. That is to say, it is possible to think intrinsically of the Euler-Lagrange equations in a manner different from what we do here. For example, Tulczyjew [1976] provides a discussion of the "Lagrange differential." •

With the Euler-Lagrange 2-force, it is a simple matter to provide an intuitive characterisation of solutions to the Euler-Lagrange equations. Since we are considering external forces and constraints, let us be precise about this. A ***Lagrangian system*** on $\mathscr{Q}$ is a triple $(L,\Phi,(M,\Lambda))$ where L is a Lagrangian, Φ is a 1-force, and (M,Λ) is a constraint. A local section $c\colon[t_1,t_2]\to\mathscr{Q}$ is a ***solution*** for $(L,\Phi,(M,\Lambda))$ if c satisfies the constraint (M,Λ) and if there exists a constraint force λ along c so that

$$\Phi_L(j^2c(t)) = \Phi(j^1c(t)) + \lambda(t)$$

for $t\in[t_1,t_2]$. In finite-dimensions, let

$$\beta^a = \beta_0^a\mathrm{d}t + \beta_i^a\mathrm{d}q^i + \hat{\beta}_i^a\mathrm{d}v^i,\quad a=1,\dots,m,$$

be a local basis for Λ and suppose the inclusion of M into $J^1\mathscr{Q}$ has the form

$$(x^1,\dots,x^r)\mapsto(C_0(x),C_1^i(x),C_2^j(x)).$$

If a section c has the local form $t\mapsto(t,q^i(t))$, then c is a solution for $(L,\Phi,(M,\Lambda))$ if and only if

1. there exists a curve $t\mapsto(x^\alpha(t))$ in M so that
$$C_0(x(t))=t,\quad C_1^i(x(t))=q^i(t),\quad C_2^j(x(t))=\dot q^i(t),$$
for $i,j=1,\dots,n$,
2. the relations $\hat\beta_i^a\ddot q^i+\beta_i^a\dot q^i+\beta_0^a=0$ hold for $a=1,\dots,m$, and
3. the equality
$$\left(\frac{\partial^2L}{\partial v^i\partial v^j}\ddot q^j+\frac{\partial^2L}{\partial v^i\partial q^j}\dot q^j+\frac{\partial^2L}{\partial v^i\partial t}-\frac{\partial L}{\partial q^i}\right)(\mathrm{d}q^i-\dot q^i\mathrm{d}t)=(\Phi_i+\lambda_i)(\mathrm{d}q^i-\dot q^i\mathrm{d}t)$$
holds for some 1-force λ along c.

Matching the coefficients of the basis for $\mathsf{C}^1(\mathscr{Q})$ in this last expression gives the usual Lagrange multiplier form of the Euler-Lagrange equations. Of course, our discussion here of solutions for a Lagrangian system does nothing to assert the existence of such. Let us now deal with exactly this question.

5. The Euler-Lagrange vector field

In Section 4 we wrote down the forced, constrained Euler-Lagrange equations associated with a Lagrangian system $(L,\Phi,(M,\Lambda))$ on the total space of a locally trivial fibre bundle $\pi\colon\mathscr{Q}\to\mathbb{R}$. Now we wish to assert, under restrictions on the Lagrangian system, that the Euler-Lagrange equations have unique solutions. One way to do this is to construct a second-order vector field on $M\subset J^1\mathscr{Q}$ whose integral curves are solutions of the Euler-Lagrange equations. This is indeed the route we, along with many others, choose when characterising Lagrangian systems which possess unique solutions. However, our characterisation explicitly uses the Euler-Lagrange 2-force rather than a two-form formalism.

5.1. The unforced, unconstrained case. We investigate the unforced, unconstrained case first. That is, we consider a Lagrangian system of the form $(L,0,(J^1\mathscr{Q},\{0\}))$. The following result is, of course, well-known in that it asserts the existence of the Euler-Lagrange vector field when the Lagrangian is regular.

5.1 Theorem: *If L is a regular Lagrangian then there exists a unique second-order vector field X_L on $J^1\mathscr{Q}$ with the property that $\Phi_L\circ X_L=0$. We call X_L the **Euler-Lagrange vector field** for the regular Lagrangian L.*

Proof: We work locally with an adapted chart (U,ϕ) for $\pi\colon\mathscr{Q}\to\mathbb{R}$. We take ϕ as being $\mathbb{R}\times E$-valued for a Banach space E. Throughout we regard $J^2\mathscr{Q}$ as a subset of $T(J^1\mathscr{Q})$, and we regard forces as being $\tau_1^*V^*\mathscr{Q}$-valued. It will be convenient to write

$$\Phi_L((t,u,u_1),(1,u_1,u_2))=((t,u,u_1),A_L^{-1}(t,u,u_1)\cdot u_2+\xi_L(t,u,u_1))$$

where $A_L(t,u,u_1)\colon E^* \to E$ is the inverse of the map

$$e \mapsto \boldsymbol{D}_3^2 L(t,u,u_1)\cdot e$$

(this inverse exists since we are assuming L regular) and

$$\xi_L(t,u,u_1) = \boldsymbol{D}_2\boldsymbol{D}_3 L(t,u,u_1)\cdot(u_1,\cdot) + \boldsymbol{D}_1\boldsymbol{D}_3 L(t,u,u_1)\cdot(1,\cdot) - \boldsymbol{D}_2 L(t,u,u_1).$$

Suppose a second-order vector field X on $J^1\mathscr{Q}$ has local representative

$$(t,u,u_1) \mapsto ((t,u,u_1),(1,u_1,\boldsymbol{X}(t,u,u_1))).$$

Then $\Phi_L \circ X = 0$ if and only if

$$\boldsymbol{X}(t,u,u_1) = -A_L(t,u,u_1)\cdot\xi_L(t,u,u_1)$$

Thus we choose X_L to have local representative

$$(t,u,u_1) \mapsto ((t,u,u_1),(1,u_1,-A_L(t,u,u_1)\cdot\xi_L(t,u,u_1)))$$

which shows that X_L exists. That X_L is the *unique* second-order vector field with the stated property is a consequence of all of the above statements being "if and only if." ■

The proof immediately yields the following local form of X_L.

5.2 Corollary: *Let L be a regular Lagrangian with X_L the Euler-Lagrange vector field, and let (U,ϕ) be an adapted chart for $\pi\colon \mathscr{Q} \to \mathbb{R}$. The local representative of X_L is*

$$\begin{aligned}(t,u,u_1) \mapsto ((t,u,u_1),(1,u_1,-A_L(t,u,u_1)\cdot \boldsymbol{D}_2\boldsymbol{D}_3 L(t,u(t),u_1)\cdot u_1 -{}\\ A_L(t,u,u_1)\cdot\boldsymbol{D}_1\boldsymbol{D}_3 L(t,u(t),u_1) + A_L(t,u,u_1)\cdot\boldsymbol{D}_2 L(t,u(t),u_1)))\end{aligned}$$

where $A_L(t,u,u_1)$ is the inverse of the map $e \mapsto \boldsymbol{D}_3^2 L(t,u,u_1)\cdot e$.

In finite-dimensions we have

$$X_L = \frac{\partial}{\partial t} + v^i\frac{\partial}{\partial q^i} + g_L^{ij}\left(-\frac{\partial^2 L}{\partial q^k\partial v^j}v^k - \frac{\partial^2 L}{\partial t\partial v^j} + \frac{\partial L}{\partial q^j}\right)\frac{\partial}{\partial v^i}$$

where the matrix with components g_L^{ij}, $i,j = 1,\dots,n$, is the inverse of the matrix with components

$$\frac{\partial^2 L}{\partial v^i\partial v^j},\quad i,j = 1,\dots,n. \tag{5.1}$$

5.2. The forced, unconstrained case. Let us now consider the addition of a 1-force Φ to the problem data. That is, we consider Lagrangian systems of the form $(L,\Phi,(J^1\mathscr{Q},\{0\}))$. As in the unforced case, we can formulate a purely geometric result which defines for us the required second-order vector field. The proof of the following result is a simple adaptation of that of Theorem 5.1.

5.3 Proposition: *Let L be a regular Lagrangian on $\mathscr{Q}$ and let Φ be a 1-force on $\mathscr{Q}$. There exists a unique second-order vector field $X_{L,\Phi}$ on $J^1\mathscr{Q}$ with the property that $\Phi_L \circ X_{L,\Phi} = \Phi$. We call $X_{L,\Phi}$ the **forced Euler-Lagrange vector field** for the regular Lagrangian L and the 1-force Φ.*

It is now natural to ask whether $X_{L,\Phi}$ is related to X_L, and if so in what way. Answering this is the following result.

5.4 Proposition: *If L is a regular Lagrangian and Φ is a 1-force then $X_{L,\Phi} = X_L + g_L^\sharp \circ \Phi$.*

Proof: This is a simple matter of examining the local representatives of X_L, $g_L^\sharp \circ \Phi$, and $X_{L,\Phi}$ which are

$$(t, u, u_1) \mapsto ((t, u, u_1), (1, u_1, -A_L(t, u, u_1) \cdot \xi_L(t, u, u_1)))$$
$$(t, u, u_1) \mapsto ((t, u, u_1), (0, 0, A_L(t, u, u_1) \cdot \mathbf{\Phi}(t, u, u_1)))$$
$$(t, u, u_1) \mapsto ((t, u, u_1), (1, u_1, -A_L(t, u, u_1) \cdot \xi_L(t, u, u_1) + A_L(t, u, u_1) \cdot \mathbf{\Phi}(t, u, u_1))),$$

respectively. The result follows directly. ■

Note that the vector field $g_L^\sharp \circ \Phi$ takes its values in $\tau_1^* V\mathscr{Q}$ and so when we add it to a second-order vector field, the result will be another second-order vector field.

5.3. The forced, constrained case.

We now consider the general situation when we have a full Lagrangian system $(L, \Phi, (M, \Lambda))$. The construction of a vector field describing nonlinearly constrained dynamics is dealt with, for example, by Marle [1997] in the Hamiltonian context and by de León, Marrero, and Martín de Diego [1997a] in the Lagrangian. Neither of these papers allow external forces, although it would not be difficult in either case to work out how this could be done. In our case, we expect that we will require at least a regular Lagrangian. The following result asserts that if, in addition, we ask that L be definite and that the constraints be ideal, then we may establish the existence of a vector field having certain properties. In Proposition 5.6 below, we show that integral curves of this vector field are in 1–1 correspondence with solutions of $(L, \Phi, (M, \Lambda))$. Recall from Section 3 the definition of the subbundle Λ_M of T^*M.

5.5 Theorem: *Let $(L, \Phi, (M, \Lambda))$ be a Lagrangian system with L a regular Lagrangian and (M, Λ) a constraint which is ideal. Further assume that $g_L|(\operatorname{coann}(\mathscr{F}_{(M,\Lambda)}) \cap \tau_1^* V\mathscr{Q})$ is strongly nondegenerate. Then there exists a unique second-order vector field $X_{L,\Phi,(M,\Lambda)}$ on M having the following two properties:*

(i) $X_{L,\Phi,(M,\Lambda)}(M) \subset (\operatorname{coann}(\Lambda_M) \cap J^2\mathscr{Q}|M)$;

(ii) $(\Phi_L \circ X_L - \Phi)(J^1\mathscr{Q}) \subset \mathscr{F}_{(M,\Lambda)}$.

In particular, if L is definite (i.e., positive or negative-definite) then $X_{L,\Phi,(M,\Lambda)}$ exists and is uniquely determined by (i) and (ii).

Proof: We work locally, and borrow the notation of Section 3.3 and Theorem 5.1. Thus suppose (U, ϕ) to be an adapted chart for $\pi\colon \mathscr{Q} \to \mathbb{R}$ taking values in $\mathbb{R} \times E$ for a Banach space E. We assume that $\tilde{U} = M \cap J^1U$ is the domain for a chart $(\tilde{U}, \chi)$ for M with χ taking values in $\tilde{E}$. Throughout the proof, we write a typical point in $\tilde{\boldsymbol{U}}$ as $\tilde{u}$ and a typical point in $\boldsymbol{i_M}(\tilde{\boldsymbol{U}})$ as (t, u, u_1). We shall always be considering points (t, u, u_1) lying in the image of $\boldsymbol{i_M}$. If we write $\tilde{u}$ and (t,u,u_1) in the same equation, it will always be the case that $(t,u,u_1) = \boldsymbol{i_M}(\tilde{u})$.

The local representative of Φ_L will be taken to have the form

$$\Phi_L(t,u,u_1,u_2) = ((t,u,u_1), A_L^{-1}(t,u,u_1)\cdot u_2 + \xi_L(t,u,u_1))$$

and Φ has local representative

$$(t,u,u_1) \mapsto ((t,u,u_1), \boldsymbol{\Phi}(t,u,u_1)).$$

As in (3.4), the local model for $\operatorname{coann}(\Lambda_M) \cap J^2\mathscr{Q}$ is

$$\begin{aligned}\big\{((t,u,u_1), (\boldsymbol{D}C_0(\tilde{u})\cdot\tilde{e}, \boldsymbol{D}C_1(\tilde{u})\cdot\tilde{e}, \boldsymbol{D}C_2(\tilde{u})\cdot\tilde{e}))\ |\ \ &\boldsymbol{D}C_0(\tilde{u})\cdot\tilde{e} = 1,\ \ \boldsymbol{D}C_1(\tilde{u})\cdot\tilde{e} = u_1,\\ (B_{011}^*(t,u,u_1)\circ\boldsymbol{D}C_0(\tilde{u}) + B_{111}^*(t,u,u_1)\circ\boldsymbol{D}C_1(\tilde{u}) &+ B_{211}^*(t,u,u_1)\circ\boldsymbol{D}C_2(\tilde{u}))\cdot\tilde{e} = 0,\\ (B_{012}^*(t,u,u_1)\circ\boldsymbol{D}C_0(\tilde{u}) &+ B_{112}^*(t,u,u_1)\circ\boldsymbol{D}C_1(\tilde{u}))\cdot\tilde{e} = 0\big\}\end{aligned}$$

for Banach spaces F_{11} and F_{12}, and maps $B_{01j}\colon \boldsymbol{U}\times E \to \mathrm{L}(F_{1j}^*;\mathbb{R}^*)$, $B_{i1j}\colon \boldsymbol{U}\times E \to \mathrm{L}(F_{1j}^*;E^*)$, $B_{02}\colon \boldsymbol{U}\times E \to \mathrm{L}(F_2^*;\mathbb{R}^*)$, and $B_{j2}\colon \boldsymbol{U}\times E \to \mathrm{L}(F_2^*;E^*)$, $i,j = 1,2$, and $\boldsymbol{D}C_0\colon \tilde{\boldsymbol{U}} \to \mathrm{L}(\tilde{E};\mathbb{R})$ $\boldsymbol{D}C_i\colon \tilde{\boldsymbol{U}} \to \mathrm{L}(\tilde{E};E)$, $i = 1,2$. Also, the local model for $\mathscr{F}_{(M,\Lambda)}$ is

$$\{((t,u,u_1), B_{211}(t,u,u_1)\cdot\nu^{11})\ |\ \nu^{11}\in F_{11}^*\}.$$

Recall that, by definition, $B_{211}(t,u,u_1)$ is injective with split image for each $\tilde{u}\in\tilde{\boldsymbol{U}}$. Let X be a second-order vector field on M with local representative

$$\tilde{u} \mapsto (\tilde{u}, \boldsymbol{X}(\tilde{u})).$$

X satisfies (i) if and only if for each $\tilde{u}\in\tilde{\boldsymbol{U}}$

$$\begin{gathered}\boldsymbol{D}C_0(\tilde{u})\cdot\boldsymbol{X}(\tilde{u}) = 1,\\ \boldsymbol{D}C_1(\tilde{u})\cdot\boldsymbol{X}(\tilde{u}) = u_1,\\ B_{011}^*(t,u,u_1)\cdot 1 + B_{111}^*(t,u,u_1)\cdot u_1 + B_{211}^*(t,u,u_1)\circ\boldsymbol{D}C_2(\tilde{u})\cdot\boldsymbol{X}(\tilde{u}) = 0,\\ B_{011}^*(t,u,u_1)\cdot 1 + B_{111}^*(t,u,u_1)\cdot u_1 = 0.\end{gathered}\tag{5.2}$$

Since (M,Λ) is ideal, we know that such an X exists. X satisfies (ii) if and only if, for each $(t,u,u_1)\in\operatorname{image}(\boldsymbol{i_M})$, there exists some $\tilde{\lambda}(t,u,u_1)\in F_{11}^*$ with the property that

$$\begin{aligned}A_L^{-1}(t,u,u_1)\circ\boldsymbol{D}C_2(\tilde{u})\cdot\boldsymbol{X}(t,u,u_1) + \xi_L(t,u,u_1) - \boldsymbol{\Phi}(t,u,u_1) = \\ B_{211}(t,u,u_1)\cdot\tilde{\lambda}(t,u,u_1).\end{aligned}\tag{5.3}$$

Note that this implies that *if* X satisfies (ii) then it is uniquely determined by (5.3) as $A_L^{-1}(t,u,u_1)\circ\boldsymbol{D}C_2(\tilde{u})$ is injective. Furthermore, the same equation uniquely specifies $\tilde{\lambda}(t,u,u_1)$ since $B_{211}(t,u,u_1)$ is injective.

To establish existence of $X_{L,\Phi,(M,\Lambda)}$ we will explicitly determine $\tilde{\lambda}$. Fix $(t,u,u_1) = \boldsymbol{i_M}(\tilde{u})$. By our assumption that $g_L|(\operatorname{coann}(\mathscr{F}_{(M,\Lambda)})\cap\tau_1^*V\mathscr{Q})$ is strongly nondegenerate, the map

$$B_{211}^*(t,u,u_1)\circ A_L(t,u,u_1)\circ B_{211}(t,u,u_1)\in\mathrm{L}(F_{11}^*;F_{11})\tag{5.4}$$

is a Banach isomorphism. It is then a straightforward to check that if we choose $\tilde{\lambda}(t,u,u_1) \in F_{11}^*$ to be the unique solution of

$$\begin{aligned}B_{211}^*(t,u,u_1) \circ A_L(t,u,u_1) \circ B_{211}(t,u,u_1) \cdot \tilde{\lambda}(t,u,u_1) = \\ B_{211}^*(t,u,u_1) \circ A_L(t,u,u_1) \cdot \xi_L(t,u,u_1) - B_{211}^*(t,u,u_1) \circ A_L(t,u,u_1) \cdot \mathbf{\Phi}(t,u,u_1) - \\ B_{111}^*(t,u,u_1) \cdot u_1 - B_{011}^*(t,u,u_1) \cdot 1\end{aligned}$$

then X chosen as in (5.3) satisfies (5.2). Therefore, choosing $X_{L,\Phi,(M,\Lambda)} = X$ in this way, we see that $X_{L,\Phi,(M,\Lambda)}$ satisfies (i). Moreover, $X_{L,\Phi,(M,\Lambda)}$ was constructed by requiring it to satisfy (ii). This establishes existence of $X_{L,\Phi,(M,\Lambda)}$ and so completes the proof of the first assertion as uniqueness has already been proved.

To prove the final assertion, we need to show that if g_L is definite then (5.4) defines a Banach isomorphism. But if g_L is positive-definite then the pairing

$$(\nu^{11}, \tilde{\nu}^{11}) \mapsto \langle A_L(t,u,u_1) \circ B_{211}(t,u,u_1)(\nu^{11}); B_{211}(t,u,u_1)(\tilde{\nu}^{11})\rangle$$

defines an inner product on F_{11}^* (it is symmetric since $A_L^*(t,u,u_1) = A_L(t,u,u_1)$). If g_L is negative definite we may obtain an inner product by multiplying by -1. Therefore, by the Riesz Representation Theorem, the map which sends ν^{11} to the element

$$\tilde{\nu}^{11} \mapsto \langle A_L(t,u,u_1) \circ B_{211}(t,u,u_1)(\tilde{\nu}^{11}); B_{211}(t,u,u_1)(\nu^{11})\rangle$$

of F_{11} is a Banach isomorphism. In other words, the map (5.4) is a Banach isomorphism. ■

For linear constraints in finite-dimensions, the computations we perform in the proof of the theorem are standard [Murray, Li, and Sastry 1994, Section 6.1.2].

Note that the requirement that $g_L|(\mathrm{coann}(\mathscr{F}_{(M,\Lambda)}) \cap \tau_1^* V\mathscr{Q})$ be strongly nondegenerate can fail in finite-dimensions, even when L is regular, as can be seen with a simple example. We let $\mathscr{Q} = \mathbb{R} \times Q$ with $Q = \mathbb{R}^2$. Denote natural coordinates for $J^1\mathscr{Q}$ with respect to Cartesian coordinates (x,y) on $\mathbb{R}^2$ by (t,x,y,v_x,v_y). The Lagrangian

$$L = \frac{1}{2}(v_x^2 - v_y^2)$$

is regular, and the constraint $(M = J^1\mathscr{Q}, \Lambda)$ where Λ is generated by the one-form

$$\beta = \mathrm{d}v_x - \mathrm{d}v_y$$

is ideal. In this case $\mathscr{F}_{(M,\Lambda)}$ is generated by the one-form $\mathrm{d}x - v_x\mathrm{d}t - (\mathrm{d}y - v_y\mathrm{d}t)$ and so $\mathrm{coann}(\mathscr{F}_{(M,\Lambda)}) \cap \tau_1^* V\mathscr{Q}$ is spanned by $\frac{\partial}{\partial v_x} + \frac{\partial}{\partial v_y}$. Therefore $g_L|(\mathrm{coann}(\mathscr{F}_{(M,\Lambda)}) \cap \tau_1^* V\mathscr{Q})$ is zero.

The following characterisation of $X_{L,\Phi,(M,\Lambda)}$ establishes that the integral curves of $X_{L,\Phi,(M,\Lambda)}$ are indeed solutions for $(L,\Phi,(M,\Lambda))$.

5.6 Proposition: *If $(L,\Phi,(M,\Lambda))$ is a Lagrangian system satisfying the hypotheses of Theorem 5.5, then $X_{L,\Phi,(M,\Lambda)}$ is the unique second-order vector field on M with the property that a local section $t \mapsto j^1c(t) \in M$ is an integral curve of $X_{L,\Phi,(M,\Lambda)}$ if and only if $t \mapsto c(t)$ is a solution of $(L,\Phi,(M,\Lambda))$.*

Proof: We work locally with the notation of Theorem 5.5. We take a curve on M which is locally given by $t \mapsto \tilde{u}(t)$. This is a solution of $(L, \Phi, (M, \Lambda))$ if and only if

1. $C_0(\tilde{u}(t)) = t$, $C_1(\tilde{u}(t)) = u(t)$, and $C_2(\tilde{u}(t)) = \dot{u}(t)$ (which defines a curve $t \mapsto (t, u(t))$ in $\boldsymbol{U}$),
2. $B^*_{211}(t, u(t), \dot{u}(t)) \cdot \ddot{u}(t) + B^*_{111}(t, u(t), \dot{u}(t)) \cdot \dot{u}(t) + B^*_{011}(t, u(t), \dot{u}(t)) = 0$ and $B^*_{112}(t, u(t), \dot{u}(t)) \cdot \dot{u}(t) + B^*_{012}(t, u(t), \dot{u}(t)) = 0$, and
3. $A_L^{-1}(t, u(t), \dot{u}(t)) \cdot \ddot{u}(t) + \xi_L(t, u(t), \dot{u}(t)) - \boldsymbol{\Phi}(t, u(t), \dot{u}(t)) = B_{211}(t, u(t), \dot{u}(t)) \cdot \tilde{\lambda}(t)$ for some $t \mapsto \tilde{\lambda}(t) \in F^*_{11}$.

We now proceed exactly as in the proof of Theorem 5.5 and ascertain that $\tilde{\lambda}(t)$ is uniquely defined by its being the unique solution of

$$\begin{aligned} B^*_{211}(t, u(t), \dot{u}(t)) \circ A_L(t, u(t), \dot{u}(t)) \circ B_{211}(t, u(t), \dot{u}(t)) \cdot \tilde{\lambda}(t) = \\ B^*_{211}(t, u(t), \dot{u}(t)) \circ A_L(t, u(t), \dot{u}(t)) \cdot \xi_L(t, u(t), \dot{u}(t)) - \\ B^*_{211}(t, u(t), \dot{u}(t)) \circ A_L(t, u(t), \dot{u}(t)) \cdot \boldsymbol{\Phi}(t, u(t), \dot{u}(t)) - \\ B^*_{111}(t, u(t), \dot{u}(t)) \cdot \dot{u} - B^*_{011}(t, u(t), \dot{u}(t)) \cdot 1 \end{aligned}$$

This completes the proof. ■

Note that under the hypotheses of Theorem 5.5, the constraint forces have more structure than that of forces along a curve—they are determined by a well-defined 1-force on M. In finite-dimensions, we may be explicit about writing this constraint 1-force. Take a local basis

$$\beta^a_0 \mathrm{d}t + \beta^a_i \mathrm{d}q^i + \hat{\beta}^a_i \mathrm{d}v^i, \quad a = 1, \dots, m,$$

for Λ with the property that, when restricted to M, the forms

$$\hat{\beta}^a_i (\mathrm{d}q^i - v^i \mathrm{d}t), \quad a = 1, \dots, \tilde{m},$$

form a basis for $\mathscr{F}_{(M,\Lambda)}$. Then a 1-force which is a constraint force for Λ will have the form $\hat{\beta}^a_i \tilde{\lambda}_a (\mathrm{d}q^i - v^i \mathrm{d}t)$ for some functions $\tilde{\lambda}_a$, $a = 1, \dots, \tilde{m}$. For the constraint 1-force these functions are given by

$$\tilde{\lambda}_a = C_{ab} \left(\hat{\beta}^b_i (t, u, v) g^{ij}_L \left(\frac{\partial^2 L}{\partial q^k \partial v^j} v^k + \frac{\partial^2 L}{\partial t \partial v^j} - \frac{\partial L}{\partial q^j} - \Phi_j \right) - \beta^b_i v^i - \beta^b_0 \right), \tag{5.5}$$

$a = 1, \dots, \tilde{m}$, where C_{ab}, $a, b = 1, \dots, \tilde{m}$ is the matrix whose components are formed by the inverse of the matrix with components $\hat{\beta}^a_i \hat{\beta}^b_j g^{ij}_L$, $a, b = 1, \dots, \tilde{m}$, and, as usual, g^{ij}_L, $i, j = 1, \dots, n$, are the components of the inverse of the matrix whose components are given by (5.1).

6. The relationship of Φ_L to the Poincaré-Cartan two-form

In Section 4 and Section 5 we formulated the Euler-Lagrange equations and derived the Euler-Lagrange vector field (when possible) using as our primary tool the Euler-Lagrange 2-force Φ_L. This is not necessarily the standard way to accomplish these tasks. One more common way to do this [de León, Marrero, and Martín de Diego 1997b, Hermann 1982] is to construct a two-form where solutions are then defined by their making the two-form vanish upon taking interior products (we will be precise about this shortly). This approach may be seen as borrowing from the Hamiltonian formulation. In this section we make precise the relationship between this standard approach and our approach as presented in Section 5.

6.1. The unforced, unconstrained case. To avoid confusion as to where the one-form $\boldsymbol{d}t$ lives, let us define $\eta = (\pi \circ \tau_1)^* \boldsymbol{d}t$ as its pull-back to $J^1\mathscr{Q}$. For a Lagrangian L on $\mathscr{Q}$ define the ***Poincaré-Cartan one-form*** on $J^1\mathscr{Q}$ by

$$\Theta_L = L\eta + \tilde{S}^*_{\mathscr{Q}}(\boldsymbol{d}L)$$

and define the ***Poincaré-Cartan two-form*** on $J^1\mathscr{Q}$ by $\Omega_L = -\boldsymbol{d}\Theta_L$. The one-form Θ_L was first introduced by Cartan [1971]. In infinite-dimensions, one uses the definition of the exterior derivative in terms of the Lie derivative by Palais [1954]. This gives the local representations for the Poincaré-Cartan forms as

$$\Theta_L(t,u,u_1) = ((t,u,u_1),(L - \boldsymbol{D}_3 L(t,u,u_1)\cdot u_1, \boldsymbol{D}_3 L(t,u,u_1), 0)), \tag{6.1}$$

and

$$\begin{aligned}\Omega_L(t,u,u_1)\cdot((\tau_1,e_1,f_1),(\tau_2,e_2,f_2)) = \tau_1 \boldsymbol{D}_2 L\cdot e_2 - \tau_2 \boldsymbol{D}_2 L \cdot e_1 - \\ \tau_1 \boldsymbol{D}_2\boldsymbol{D}_3 L\cdot(e_2,u_1) + \tau_2 \boldsymbol{D}_2\boldsymbol{D}_3 L\cdot(e_1,u_1) - \tau_1 \boldsymbol{D}_3^2 L\cdot(u_1,f_2) + \tau_2 \boldsymbol{D}_3^2 L\cdot(u_1,f_1) + \\ \boldsymbol{D}_1\boldsymbol{D}_3 L\cdot(\tau_2,e_1) - \boldsymbol{D}_1\boldsymbol{D}_3 L\cdot(\tau_1,e_2) + \boldsymbol{D}_2\boldsymbol{D}_3 L\cdot(e_2,e_1) - \boldsymbol{D}_2\boldsymbol{D}_3 L\cdot(e_1,e_2) + \\ \boldsymbol{D}_3^2 L\cdot(e_1,f_2) - \boldsymbol{D}_3^2 L\cdot(e_2,f_1)\end{aligned} \tag{6.2}$$

where all derivatives in (6.2) are evaluated at (t,u,u_1). In finite-dimensions these read

$$\Theta_L = L\mathrm{d}t + \frac{\partial L}{\partial v^i}(\mathrm{d}q^i - v^i \mathrm{d}t)$$

and

$$\Omega_L = -\left(\boldsymbol{d}\left(\frac{\partial L}{\partial v^i}\right) - \frac{\partial L}{\partial q^i}\mathrm{d}t\right) \wedge (\mathrm{d}q^i - v^i\mathrm{d}t).$$

It is our goal to relate Ω_L with Φ_L. To do this requires the following exterior algebraic construction. We denote by $\mathrm{T}\bigwedge^k(T(J^1\mathscr{Q}))$ the bundle of exterior k-forms on $J^1\mathscr{Q}$. Associated with the subbundle $\mathrm{C}^1(\mathscr{Q})$ of $T^*(J^1\mathscr{Q})$ is the subbundle $(\mathrm{C}^1(\mathscr{Q}))^k$ of $\mathrm{T}\bigwedge^k(T(J^1\mathscr{Q}))$ whose fibre at $v \in J^1\mathscr{Q}$ is

$$(\mathrm{C}^1(\mathscr{Q}))^k_v = \big\{\alpha \in \mathrm{T}\textstyle\bigwedge^k(T_v(J^1\mathscr{Q})) \mid \\ \alpha(X_1,\dots,X_k) = 0 \text{ for all } X_1,\dots,X_k \in \mathrm{coann}(\mathrm{C}^1_v(\mathscr{Q}))\big\}.$$

Thus the sections of $\oplus_{k\geq 1}(\mathrm{C}^1(\mathscr{Q}))^k$ form the algebraic ideal in $\Gamma^\infty(\mathrm{T}\bigwedge(J^1\mathscr{Q}))$ corresponding to the subbundle $\mathrm{C}^1(\mathscr{Q})$ of $T^*(J^1\mathscr{Q})$. We can now establish an *exact* correspondence between Φ_L and Ω_L.

6.1 Theorem: *Let L be a Lagrangian on $\mathscr{Q}$. The following statements hold.*

(i) There exists a unique two-form Ω on $J^1\mathscr{Q}$ with the following properties:

(a) Ω is closed;

(b) $\Omega \in \Gamma^\infty((\mathsf{C}^1(\mathscr{Q}))^2)$;

(c) for every second-order vector field X on $J^1\mathscr{Q}$, $\Phi_L \circ X = -X \lrcorner \Omega$.

Furthermore, this unique two-form is exactly Ω_L.

(ii) Conversely, there exists a unique 2-force Φ on $\mathscr{Q}$ for which $\Phi \circ X = -X \lrcorner \Omega_L$, and this 2-force is exactly Φ_L.

Proof: (i) Let (U, ϕ) be an adapted chart for $\pi \colon \mathscr{Q} \to \mathbb{R}$ with $(J^1U, j^1\phi)$ the corresponding natural chart for $J^1\mathscr{Q}$. Take ϕ to be $\mathbb{R} \times E$-valued, and suppose that $\boldsymbol{U} =]a, b[\times B_{r,0}$ for some $a, b \in \mathbb{R}$ and $r > 0$ (here $B_{r,0}$ is the open ball of radius r centred at 0). In this case, $\boldsymbol{U}$ and $\boldsymbol{U} \times E$ are contractible. Suppose Ω is a closed two-form on $\boldsymbol{U} \times E$. By the Poincaré lemma, Ω is exact, so suppose $\Omega = \boldsymbol{d}\Theta$ and that Θ is given by

$$\Theta(t, u, u_1) = ((t, u, u_1), (A_0(t, u, u_1), A_1(t, u, u_1), A_2(t, u, u_1)))$$

for $A_0 \colon \boldsymbol{U} \times E \to \mathbb{R}^*$ and $A_i \colon \boldsymbol{U} \times E \to E^*$, $i = 1, 2$. A computation gives

$$\begin{aligned}(\tau, e_1, e_2) \lrcorner \boldsymbol{d}\Theta(t, u, u_1) = ((t, u, u_1), (&\boldsymbol{D}_2 A_0 \cdot (e_1, \cdot) - \boldsymbol{D}_1 A_1 \cdot (\cdot, e_1) + \\ &\boldsymbol{D}_3 A_0 \cdot (e_2, \cdot) - \boldsymbol{D}_1 A_2 \cdot (\cdot, e_2), -\boldsymbol{D}_2 A_0 \cdot (\cdot, \tau) + \boldsymbol{D}_1 A_1 \cdot (\tau, \cdot) + \\ &\boldsymbol{D}_2 A_1 \cdot (e_1, \cdot) - \boldsymbol{D}_2 A_1 \cdot (\cdot, e_1) + \boldsymbol{D}_3 A_1 \cdot (e_2, \cdot) - \boldsymbol{D}_2 A_2(\cdot, e_2), \\ &- \boldsymbol{D}_3 A_0 \cdot (\cdot, \tau) + \boldsymbol{D}_1 A_2 \cdot (\tau, \cdot) - \boldsymbol{D}_3 A_1 \cdot (\cdot, e_1) + \boldsymbol{D}_2 A_2 \cdot (e_1, \cdot) + \\ &\boldsymbol{D}_3 A_2 \cdot (e_2, \cdot) - \boldsymbol{D}_3 A_2(\cdot, e_2))).\end{aligned}$$

Now let us require (b) to hold. Elements of $\operatorname{coann}(\mathsf{C}^1(\mathscr{Q}))$ at (t, u, u_1) have the form $(\tau, \tau u_1, e) \in \mathbb{R} \times E \times E$. An element of $\mathsf{C}^1(\mathscr{Q})$ at (t, u, u_1) has the form $(-\alpha \cdot u_1, \alpha, 0)$ for some $\alpha \in E^*$. Thus $\Omega(t, u, u_1)$ satisfies (a) and (b) if and only if

$$\begin{aligned}\tau \boldsymbol{D}_2 A_0 \cdot (u_1, \cdot) - \tau \boldsymbol{D}_1 A_1 \cdot (\cdot, u_1) + \boldsymbol{D}_3 A_0 \cdot (e, \cdot) - \boldsymbol{D}_1 A_2 \cdot (\cdot, e) &= -\alpha \cdot u_1 \\ -\boldsymbol{D}_2 A_0 \cdot (\cdot, \tau) + \boldsymbol{D}_1 A_1 \cdot (\tau, \cdot) + \tau \boldsymbol{D}_2 A_1 \cdot (u_1, \cdot) - \tau \boldsymbol{D}_2 A_1 \cdot (\cdot, u_1) + & \\ \boldsymbol{D}_3 A_1 \cdot (e, \cdot) - \boldsymbol{D}_2 A_2(\cdot, e) &= \alpha \\ -\boldsymbol{D}_3 A_0 \cdot (\cdot, \tau) + \boldsymbol{D}_1 A_2 \cdot (\tau, \cdot) - \tau \boldsymbol{D}_3 A_1 \cdot (\cdot, u_1) + \tau \boldsymbol{D}_2 A_2 \cdot (u_1, \cdot) + & \\ \boldsymbol{D}_3 A_2 \cdot (e, \cdot) - \boldsymbol{D}_3 A_2(\cdot, e) &= 0\end{aligned} \tag{6.3}$$

for some $\alpha \in E^*$ and for every $(\tau, e) \in \mathbb{R} \times E$. Here all derivatives have been evaluated at some $(t, u, u_1) \in \boldsymbol{U} \times E$. Note that α is allowed to vary as a function of τ and e, as well as of (t, u, u_1). In what follows, we shall write $\alpha(\tau, e)$ and take the (t, u, u_1) dependence for granted.

Now let X be a second order vector field on $\boldsymbol{U} \times E$ defined by $(t, u, u_1) \mapsto ((t, u, u_1), (1, u_1, e))$ for some $e \in E$ which varies with (t, u, u_1). Then

$$\begin{aligned}\Phi_L \circ X(t, u, u_1) = ((t, u, u_1), (&-\boldsymbol{D}_3^2 L(t, u, u_1) \cdot (e, u_1) - \xi_L(t, u, u_1) \cdot u_1, \\ &\boldsymbol{D}_3^2 L(t, u, u_1) \cdot e + \xi_L(t, u, u_1), 0))\end{aligned}$$

where

$$\xi_L(t,u,u_1) = \boldsymbol{D}_1\boldsymbol{D}_3L(t,u,u_1)\cdot(1,\cdot) + \boldsymbol{D}_2\boldsymbol{D}_3L(t,u,u_1)\cdot(u_1,\cdot) - \boldsymbol{D}_2L.$$

Thus Ω satisfies (a) and (c) if and only if

$$\begin{gathered}
\boldsymbol{D}_2A_0\cdot(u_1,\cdot) - \boldsymbol{D}_1A_1\cdot(\cdot,u_1) + \boldsymbol{D}_3A_0\cdot(e,\cdot) - \boldsymbol{D}_1A_2\cdot(\cdot,e) = \\
\boldsymbol{D}_3^2L(t,u,u_1)\cdot(u_1,e) + \xi_L(t,u,u_1)\cdot u_1 \\
-\boldsymbol{D}_2A_0\cdot(\cdot,1) + \boldsymbol{D}_1A_1\cdot(1,\cdot) + \boldsymbol{D}_2A_1\cdot(u_1,\cdot) - \boldsymbol{D}_2A_1\cdot(\cdot,u_1) + \\
\boldsymbol{D}_3A_1\cdot(e,\cdot) - \boldsymbol{D}_2A_2(\cdot,e) = -\boldsymbol{D}_3^2L(t,u,u_1)\cdot e - \xi_L(t,u,u_1) \\
-\boldsymbol{D}_3A_0\cdot(\cdot,1) + \boldsymbol{D}_1A_2\cdot(1,\cdot) - \boldsymbol{D}_3A_1\cdot(\cdot,u_1) + \boldsymbol{D}_2A_2\cdot(u_1,\cdot) + \\
\boldsymbol{D}_3A_2\cdot(e,\cdot) - \boldsymbol{D}_3A_2(\cdot,e) = 0
\end{gathered} \tag{6.4}$$

for every $e\in E$.

Let us take the third equation from (6.3) with $\tau=0$:

$$\boldsymbol{D}_3A_2\cdot(e,\cdot) - \boldsymbol{D}_3A_2\cdot(\cdot,e) = 0$$

for every $e\in E$. This means that for each fixed $(t,u)\in\boldsymbol{U}$ the one-form on E defined by $u_1\mapsto A_2(t,u,u_1)$ is closed, and so by the Poincaré lemma exact. Therefore there exists a function $F\colon\boldsymbol{U}\times E\to\mathbb{R}$ so that

$$A_2(t,u,u_1) = \boldsymbol{D}_3F(t,u,u_1). \tag{6.5}$$

If we take the second of equations (6.3) with $\tau=1$ and subtract from it the second of equations (6.4), we get

$$\alpha(1,e) = -\boldsymbol{D}_3^2L(t,u,u_1)\cdot e - \xi_L(t,u,u_1).$$

From the second of equations (6.3) we see that $\alpha(\tau,e) + \boldsymbol{D}_2A_2\cdot(e,\cdot) - \boldsymbol{D}_3A_1\cdot(\cdot,e)$ is a linear function of τ and is independent of e; let us write

$$\alpha(\tau,e) = \boldsymbol{D}_3A_1\cdot(\cdot,e) - \boldsymbol{D}_2A_2\cdot(e,\cdot) + \tau\beta$$

where β depends only on (t,u,u_1). We then have

$$-\boldsymbol{D}_3^2L(t,u,u_1)\cdot e - \xi_L(t,u,u_1) = \boldsymbol{D}_3A_1\cdot(\cdot,e) - \boldsymbol{D}_2A_2\cdot(e,\cdot) + \beta$$

for every $e\in E$. Therefore

$$\boldsymbol{D}_3A_1\cdot(\cdot,e) - \boldsymbol{D}_2A_2\cdot(e,\cdot) = -\boldsymbol{D}_3^2L(t,u,u_1)\cdot e \tag{6.6}$$

and $\beta = -\xi_L(t,u,u_1)$ and so

$$\alpha(\tau,e) = -\boldsymbol{D}_3^2L(t,u,u_1)\cdot e - \tau\xi_L(t,u,u_1).$$

Now we substitute (6.5) into (6.6) to get

$$\boldsymbol{D}_3A_1\cdot(\cdot,e) = \boldsymbol{D}_2\boldsymbol{D}_3F\cdot(e,\cdot) - \boldsymbol{D}_3^2L(t,u,u_1)\cdot e.$$

Therefore we must have

$$A_1(t,u,u_1) = \boldsymbol{D}_2F(t,u,u_1) - \boldsymbol{D}_3L(t,u,u_1) + G(t,u) \tag{6.7}$$

where F is as above and for some $G\colon \boldsymbol{U} \to E^*$.

Now we substitute (6.5) and (6.7) into the third of equations (6.4) to get

$$-\boldsymbol{D}_3A_0 \cdot (\cdot,1) + \boldsymbol{D}_1\boldsymbol{D}_3F \cdot (1,\cdot) + \boldsymbol{D}_3^2L(t,u,u_1) \cdot u_1 = 0$$

Note that

$$\boldsymbol{D}_3^2L(t,u,u_1) \cdot u_1 = \boldsymbol{D}_3(\boldsymbol{D}_3L(t,u,u_1) \cdot u_1) - \boldsymbol{D}_3L(t,u,u_1).$$

This implies that

$$A_0(t,u,u_1) = \boldsymbol{D}_1F(t,u,u_1) \cdot 1 + \boldsymbol{D}_3L(t,u,u_1) \cdot u_1 - L(t,u,u_1) + H(t,u) \tag{6.8}$$

for some $H\colon \boldsymbol{U} \to \mathbb{R}$.

Now substitute (6.5), (6.7), and (6.8) into the first of equations (6.4) with $e = 0$ to get

$$(\boldsymbol{D}_2H - \boldsymbol{D}_1G \cdot 1) \cdot u_1 = 0.$$

Since G and H are independent of u_1 and $u_1 \in E$ is arbitrary we get

$$\boldsymbol{D}_1G \cdot 1 = \boldsymbol{D}_2H. \tag{6.9}$$

Next we substitute (6.5), (6.7), (6.8), and (6.9) into the second of equations (6.4) with $e = 0$ to get

$$\boldsymbol{D}_2G \cdot (u_1,\cdot) - \boldsymbol{D}_2G \cdot (\cdot,u_1) = 0.$$

Since G is independent of u_1 and $u_1 \in E$ is arbitrary, this implies that for each fixed $t \in]a,b[$ the one-form $u \mapsto G(t,u)$ is closed. By the Poincaré lemma we may then write $G(t,u) = \boldsymbol{D}_2R$ for some function R on $\boldsymbol{U}$. In the expression (6.7) for A_1, we can simply absorb R into F and write $A_1(t,u,u_1) = \boldsymbol{D}_2F - \boldsymbol{D}_3L$ for some function F on $\boldsymbol{U}$. By (6.9) we have $\boldsymbol{D}_2H(t,u) = \boldsymbol{D}_2\boldsymbol{D}_1R(t,u)$ which means that $H(t,u) = \boldsymbol{D}_1R(t,u) + S(t)$ for some function $S\colon\ ]a,b[\to \mathbb{R}$.

As a result of the above computations we have $\Theta = \Theta_1 + \Theta_2$ where

$$\begin{aligned}
\Theta_1(t,u,u_1) &= ((t,u,u_1),(\boldsymbol{D}_3L(t,u,u_1) \cdot u_1 - L, -\boldsymbol{D}_3L(t,u,u_1),0)),\\
\Theta_2(t,u,u_1) &= ((t,u,u_1),(\boldsymbol{D}_1F(t,u) \cdot 1 + \boldsymbol{D}_1R(t,u) + S(t), \boldsymbol{D}_2F(t,u), \boldsymbol{D}_3F)).
\end{aligned}$$

By (6.1), Θ_1 is exactly the local representative of $-\Theta_L$, and it is a straightforward computation to check that Θ_2 is closed. Therefore, we have shown that the conditions (a), (b), and (c) imply that $\Omega = -\boldsymbol{d}\Theta_L$. One may further check that Ω so defined does indeed satisfy the conditions (a), (b), and (c).

(ii) As above, let X be the second order vector field on $\boldsymbol{U} \times E$ taking the value $(1,u_1,e)$ at (t,u,u_1) for $e \in E$. Now consider a local 2-force Φ whose value at $((t,u,u_1),(1,u_1,u_2))$ is $(-\alpha(t,u,u_1,u_2) \cdot u_1, \alpha(t,u,u_1,u_2), 0)$ for $\alpha(t,u,u_1,u_2) \in E^*$. Given the coordinate form for Ω_L in (6.2) we compute

$$\begin{aligned}
-(1,u_1,e) \lrcorner\, \Omega_L(t,u,u_1) = (&\boldsymbol{D}_2L \cdot u_1 - \boldsymbol{D}_2\boldsymbol{D}_3L \cdot (u_1,u_1) - \boldsymbol{D}_3^2L \cdot (u_1,e) -\\
&\boldsymbol{D}_1\boldsymbol{D}_3L \cdot (1,u_1), \boldsymbol{D}_1\boldsymbol{D}_3L \cdot (1,\cdot) + \boldsymbol{D}_2\boldsymbol{D}_3L(u_1,\cdot) + \boldsymbol{D}_3^2L \cdot e - \boldsymbol{D}_2L, 0).
\end{aligned} \tag{6.10}$$

We also compute

$$\Phi \circ X(t,u,u_1) = (-\alpha(t,u,u_1,e)\cdot u_1, \alpha(t,u,u_1,e), 0). \tag{6.11}$$

Thus $\Phi\circ X = -X \lrcorner\, \Omega$ if and only if all components of (6.10) and (6.11) match. In particular, matching the second component gives

$$\alpha(t,u,u_1,e) = \boldsymbol{D}_1\boldsymbol{D}_3L\cdot(1,\cdot) + \boldsymbol{D}_2\boldsymbol{D}_3L(u_1,\cdot) + \boldsymbol{D}_3^2L\cdot e - \boldsymbol{D}_2L$$

for every $e\in E$. But this α makes Φ exactly the local representative of Φ_L. One may easily see that this α also makes the first components of (6.10) and (6.11) match. ■

6.2 Remarks: 1. Thus we have an *exact* correspondence between Φ_L and Ω_L in that given Φ_L we can compute Ω_L using the geometry of $J^1\mathscr{Q}$ and the condition that Ω_L be closed, and given Ω_L we can directly determine Φ_L. As the proof of Theorem 6.1 suggests, determining Ω_L from Φ_L is not altogether trivial, even though the statement of the correspondence is quite benign.

2. One can use Theorem 6.1(ii) as an intrinsic definition of Φ_L.

3. None of the three conditions in Theorem 6.1(i) can be omitted if the result is to be true. Hermann [1982] gives a characterisation of Ω_L, but his conditions are not enough to determine it uniquely.

4. An obvious consequence of Theorem 6.1 is that a local section $c\colon [t_1,t_2]\to\mathscr{Q}$ is a solution for $(L,0,(J^1\mathscr{Q},\{0\}))$ if and only if $(j^1c)'(t)\lrcorner\,\Omega_L(j^1c(t)) = 0$ for every $t\in[t_1,t_2]$.

5. When L is regular, the characteristic distribution of Ω_L, whose fibre at $v\in J^1\mathscr{Q}$ is

$$D(\Omega_L)_v = \{X\in T_v(J^1\mathscr{Q}) \mid X\lrcorner\,\Omega_L(v) = 0\},$$

is a subbundle of rank 1, and is generated by X_L. In particular, if $\mathscr{Q}$ is finite-dimensional then Ω_L defines a contact structure on $J^1\mathscr{Q}$. •

6.2. The forced, unconstrained case. It is a fairly simple matter to add an external force to the formulation above. Recall that a 1-force Φ may be regarded as a $\mathsf{C}^1(\mathscr{Q})$-valued one-form on $J^1\mathscr{Q}$. The ***forced Poincaré-Cartan two-form*** is the two-form on $J^1\mathscr{Q}$ given by

$$\Omega_{L,\Phi} = \Omega_L - \Phi\wedge\eta.$$

In finite-dimensions one readily computes

$$\Omega_{L,\Phi}(t,q,v) = -\left(\boldsymbol{d}\left(\frac{\partial L}{\partial v^i}\right) - \frac{\partial L}{\partial q^i}\mathrm{d}t - \Phi_i\mathrm{d}t\right)\wedge(\mathrm{d}q^i - v^i\mathrm{d}t).$$

Given Theorem 6.1, the following result is natural. Its proof follows very much along the lines of that of the theorem.

6.3 Proposition: *Let L be a Lagrangian and let Φ be a 1-force on $\mathscr{Q}$. There exists a unique two-form Ω on $J^1\mathscr{Q}$ with the properties:*

(i) $\Omega + \Phi \wedge \eta$ *is closed;*

(ii) $\Omega \in \Gamma^\infty((\mathrm{C}^1(\mathscr{Q}))^2)$*;*

(iii) $\Phi_L \circ X - \Phi = -X \lrcorner\, \Omega$ *for every second-order vector field X on $J^1\mathscr{Q}$.*

Furthermore, this unique two-form is precisely $\Omega_{L,\Phi}$.

6.4 Remarks: 1. It certainly need not be the case that $\Omega_{L,\Phi}$ be closed as is Ω_L. In particular, $\Omega_{L,\Phi}$ will not generally be exact. These issues are discussed by Hermann [1982, §13].

2. By Proposition 5.3 and property (iii) of Proposition 6.3, a local section $c\colon [t_1, t_2] \to \mathscr{Q}$ is a solution for $(L, \Phi, (J^1\mathscr{Q}, \{0\}))$ if and only if $(j^1c)'(t) \lrcorner\, \Omega_{L,\Phi}(j^1c(t)) = 0$ for $t \in [t_1, t_2]$.

3. When L is regular, the characteristic distribution of $\Omega_{L,\Phi}$ is a subbundle of $T(J^1\mathscr{Q})$ of rank 1, and is generated by $X_{L,\Phi}$. Indeed, $X_{L,\Phi}$ is the unique second-order vector field for which $X_{L,\Phi} \lrcorner\, \Omega_{L,\Phi} = 0$. •

6.3. The forced, constrained case. Now let us use the Poincaré-Cartan two-form to develop conditions for solutions of $(L, \Phi, (M, \Lambda))$ when (M, Λ) is possibly non-trivial. What we do here is reminiscent of the development of de León, Marrero, and Martín de Diego [1997b].

As is to be expected, when one adds constraints, the way in which one characterises solutions changes somewhat. We do this in the following result.

6.5 Proposition: *A local section $c\colon [t_1, t_2] \to \mathscr{Q}$ is a solution for the Lagrangian system $(L, \Phi, (M, \Lambda))$ if and only if c satisfies the constraint (M, Λ) and $(j^1c)'(t) \lrcorner\, \Omega_{L,\Phi}|M(j^1c(t)) \in \mathscr{F}_{(M,\Lambda)}$ for $t \in [t_1, t_2]$.*

Proof: By definition, c is a solution if and only if it satisfies the constraint and $\Phi_L \circ j^2c(t) - \Phi \circ j^1c(t) \in \mathscr{F}_{(M,\Lambda)}$ for $t \in [t_1, t_2]$. Since $(j^1c)'$ is a second-order vector field along j^1c, by Proposition 6.3(iii) this is equivalent to c satisfying the constraint and $(j^1c)'(t) \lrcorner\, \Omega_{L,\Phi}|M(j^1c(t)) \in \mathscr{F}_{(M,\Lambda)}$. ■

In the case when we are assured of solutions to (M, Λ) we have the following assertion which follows from Theorem 5.5 and Proposition 6.3(iii).

6.6 Proposition: *If $(L, \Phi, (M, \Lambda))$ is a Lagrangian system with L a definite Lagrangian, and (M, Λ) an ideal constraint, then there exists a unique second-order vector field X on M with the properties*

(i) $X(M) \subset (\mathrm{coann}(\Lambda_M) \cap J^2\mathscr{Q}|M)$*;*

(ii) $X \lrcorner\, \Omega_{L,\Phi}|M \in \mathscr{F}_{(M,\Lambda)}$.

Furthermore, this vector field is precisely $X_{L,\Phi,(M,\Lambda)}$.

7. An example

In order to illustrate the methodology of the paper, let us look at an example. The system we look at is a simple one with constraints linear in velocity, and our intention is to illustrate the concepts of the paper, in particular our general constructions with constraints in Section 3.3.

We look at a system with a trivial configuration bundle $\mathscr{Q} = \mathbb{R} \times Q$ where $Q = SE(2) \times SO(2)$. Here $SE(2)$ denotes the group of proper isometries of the two-dimensional Euclidean plane E^2, and $SO(2)$ denotes the special orthogonal group in two-dimensions. To coordinatise $SE(2)$ we fix a point $O \in E^2$ and attach to O an orthonormal frame $\{\boldsymbol{e}_1, \boldsymbol{e}_2\}$. An element $\Psi \in SE(2)$ will map the orthonormal frame $\{\boldsymbol{e}_1, \boldsymbol{e}_2\}$ to an orthonormal frame $\{\boldsymbol{f}_1, \boldsymbol{f}_2\}$ which is attached to a point $P \in E^2$. To Ψ we associated the coordinates (x, y, θ) where $(x, y) = P - O \in \mathbb{R}^2$ and $\theta \in \mathbb{R}$ has the property that $\boldsymbol{f}_i = R(\theta)\boldsymbol{e}_i$, $i = 1, 2$, where $R(\theta)$ is rotation by $\theta \bmod 2\pi$. We use ϕ to coordinatise $SO(2)$ in the usual manner, and this gives coordinates (x, y, θ, ϕ) for Q. Of course, these coordinates are not globally defined, but that will not bother us here.

On $J^1\mathscr{Q} \simeq \mathbb{R} \times TQ$ we consider the Lagrangian

$$L(t, x, y, \theta, \phi, v_x, v_y, v_\theta, v_\phi) = \frac{1}{2}m(v_x^2 + v_y^2) + \frac{1}{2}I_1 v_\theta^2 + \frac{1}{2}I_2 v_\phi^2$$

where m, I_1, and I_2 are positive real constants.

Let us now specify a constraint (M, Λ) for the system. We take

$$M = \{(t, x, y, \theta, \phi, v_x, v_y, v_\theta, v_\phi) \mid\ v_x = r\cos\theta v_\phi,\ \ v_y = r\sin\theta v_\phi\}$$

for some constant $r > 0$. This defines the subset of $J^1\mathscr{Q}$ within which motion will be constrained, and we use as coordinates for M the coordinates $(t, x, y, \theta, \phi, v_\theta, v_\phi)$. This defines the inclusion of M into $J^1\mathscr{Q}$ by

$$i_M(t, x, y, \theta, \phi, v_\theta, v_\phi) = (t, v, y, \theta, \phi, r\cos\theta v_\phi, r\sin\theta v_\phi, v_\theta, v_\phi).$$

Therefore, using notation from Section 3.3, we have

$$\begin{aligned}
\boldsymbol{D}C_0 &= \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \\
\boldsymbol{D}C_1 &= \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \end{bmatrix}, \\
\boldsymbol{D}C_2 &= \begin{bmatrix} 0 & 0 & 0 & -r\sin\theta v_\phi & 0 & 0 & r\cos\theta \\ 0 & 0 & 0 & r\cos\theta v_\phi & 0 & 0 & r\sin\theta \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}.
\end{aligned}$$

The constraints here are linear in velocity and so are defined by a codistribution $\tilde{\Lambda}_0$ on Q which pulls back to a codistribution Λ_0 on $\mathscr{Q}$. We take $\tilde{\Lambda}_0$ to be the codistribution generated by the one-forms

$$\mathrm{d}x - r\cos\theta\mathrm{d}\phi, \quad \mathrm{d}y - r\sin\theta\mathrm{d}\phi.$$

Of course, these are also the generators of Λ_0, but thought of as forms on $\mathscr{Q}$ rather that Q. Using (3.7), one then computes the generators of $\Lambda = j^1\Lambda_0$ to be

$$\begin{gathered} \mathrm{d}x - r\cos\theta \mathrm{d}\phi, \quad \mathrm{d}y - r\sin\theta \mathrm{d}\phi \\ r\sin\theta v_\phi \mathrm{d}\theta + \mathrm{d}v_x - r\cos\theta \mathrm{d}v_\phi, \quad -r\cos\theta v_\phi \mathrm{d}\theta + \mathrm{d}v_y - r\sin\theta \mathrm{d}v_\phi. \end{gathered}$$

This then gives the constraint (M, Λ) which we will consider in this section.

7.1 Remark: With this linear constraint, the system models a disk rolling upright on the plane. The coordinates (x, y) locate the point of contact of the disk with the plane, θ is the "heading angle" of the disk, and ϕ is the angle of rotation of the disk. The parameter m is the mass of the disk, I_1 is the disk's moment of inertia about the axis normal to the surface, I_2 is the moment of inertia of the disk about its centre of rotation, and r is the radius of the disk. •

Now let us compute the various subbundles associated with the constraint. To do so, we select a basis of one-forms on $T^*(J^1\mathscr{Q})$ as outlined in Section 3.3. Let us choose a basis

$$\begin{aligned} \beta^1 &= r\sin\theta v_\phi \mathrm{d}\theta + \mathrm{d}v_x - r\cos\theta \mathrm{d}v_\phi \\ \beta^2 &= -r\cos\theta v_\phi \mathrm{d}\theta + \mathrm{d}v_y - r\sin\theta \mathrm{d}v_\phi \\ \beta^3 &= \mathrm{d}x - r\cos\theta \mathrm{d}\phi \\ \beta^4 &= \mathrm{d}y - r\sin\theta \mathrm{d}\phi \\ \tau &= \mathrm{d}t \\ \alpha^1 &= \mathrm{d}v_\theta \\ \alpha^2 &= \mathrm{d}v_\phi \\ \alpha^3 &= \mathrm{d}\theta \\ \alpha^4 &= \mathrm{d}\phi. \end{aligned}$$

This basis is chosen so that $\{\beta^3, \beta^4\}$ form a basis for $\ker(\tilde{S}^*_{\mathscr{Q}}|M) \cap \Lambda|M$ and so that, in the notation of (3.2), the one-forms

$$\{\hat{\beta}^1_i(\mathrm{d}x^i - v^i\mathrm{d}t), \hat{\beta}^2_i(\mathrm{d}x^i - v^i\mathrm{d}t)\}$$

form a basis for $\mathscr{F}_{(M,\Lambda)}$. From the expressions for our adapted basis one-forms, we may ascertain that, in the notation of Section 3.3, we have

$$B_{011} = \begin{bmatrix} 0 & 0 \end{bmatrix}, \quad B_{012} = \begin{bmatrix} 0 & 0 \end{bmatrix}, \quad B_{02} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \end{bmatrix},$$

$$B_{111} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ r\sin\theta v_\phi & -r\cos\theta v_\phi \\ 0 & 0 \end{bmatrix}, \quad B_{112} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ -r\cos\theta & -r\sin\theta \end{bmatrix}, \quad B_{12} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix},$$

$$B_{211} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ -r\cos\theta & -r\sin\theta \end{bmatrix}, \quad B_{212} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}, \quad B_{22} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{bmatrix}$$

From (3.3) we compute generators for Λ_M to be

$$\mathrm{d}x - r\cos\theta\mathrm{d}\phi, \quad \mathrm{d}y - r\sin\theta\mathrm{d}\phi.$$

From (3.5) we compute generators for $\mathrm{coann}(\mathscr{F}_{(M,\Lambda)}) \cap (\tau_1|M)^*V\mathscr{Q}$ to be

$$\frac{\partial}{\partial v_\theta}, \quad r\cos\theta\frac{\partial}{\partial v_x} + r\sin\theta\frac{\partial}{\partial v_y} + \frac{\partial}{\partial v_\phi}. \tag{7.1}$$

One readily ascertains that these vector fields are tangent to M. Restricted to M, and using the coordinates for M, these vectors are exactly

$$\frac{\partial}{\partial v_\theta}, \quad \frac{\partial}{\partial v_\phi}. \tag{7.2}$$

There is a notational confusion here because the vectors in equation (7.1) live on $J^1\mathscr{Q}$ whereas the vectors in equation (7.2) live on M. Confusion arises since we are naming the coordinates for M the same as a subset of the coordinates for $J^1\mathscr{Q}$. In any event, using (3.4) we also derive

$$J^2\mathscr{Q}|M \cap \mathrm{coann}(\Lambda_M) = \\ \big\{\tfrac{\partial}{\partial t} + r\cos\theta v_\phi\tfrac{\partial}{\partial x} + r\sin\theta v_\phi\tfrac{\partial}{\partial y} + v_\theta\tfrac{\partial}{\partial\theta} + v_\phi\tfrac{\partial}{\partial\phi} + a\tfrac{\partial}{\partial v_\theta} + b\tfrac{\partial}{\partial v_\phi} \;\big|\; a, b \in \mathbb{R}\big\}.$$

With all these calculations, one readily checks that the conditions IC1, IC2, and IC3 are satisfied by the constraint (M, Λ). This is to be expected given the general fact that linear constraints are ideal.

7.2 Remark: For this example, we observe the following facts:

1. $\mathscr{F}_{(M,\Lambda)} = \Lambda_M = \ker(\tilde{S}^*_{\mathscr{Q}}|M) \cap (\Lambda|M)$;
2. the generators for these codistributions on M are "the same" as those for Λ_0 (keeping in mind that the forms live on different spaces);
3. $\mathrm{coann}(\mathscr{F}_{(M,\Lambda)}) \cap (\tau_1|M)^*V\mathscr{Q} = TM \cap (\tau_1|M)^*V\mathscr{Q}$;
4. in local coordinates, $[J^2\mathscr{Q}|M \cap \mathrm{coann}(\Lambda_M)]_X = (1, v) + T_XM \cap (\tau_1|M)^*V\mathscr{Q}$ for $X = ((t, q), (1, v)) \in M$.

These observations will generally hold for systems with compatible affine constraints. •

The Euler-Lagrange 2-force is easily computed:

$$\Phi_L = ma_x(\mathrm{d}x - v_x\mathrm{d}t) + ma_y(\mathrm{d}y - v_y\mathrm{d}t) + I_1v_\theta(\mathrm{d}\theta - v_\theta\mathrm{d}t) + I_2v_\phi(\mathrm{d}\phi - v_\phi\mathrm{d}t)$$

where $(a_x, a_y, a_\theta, a_\phi)$ are the fibre coordinates for $J^2\mathscr{Q}$ over $J^1\mathscr{Q}$. As a general constraint force has the form

$$\lambda_1(\mathrm{d}x - v_x\mathrm{d}t - r\cos\theta(\mathrm{d}\phi - v_\phi\mathrm{d}t)) + \lambda_2(\mathrm{d}y - v_y\mathrm{d}t - r\sin\theta(\mathrm{d}\phi - v_\phi\mathrm{d}t))$$

for some functions λ_1 and λ_2, we ascertain that the equations of motion subject to the external 1-force

$$\Phi = \Phi_x(\mathrm{d}x - v_x\mathrm{d}t) + \Phi_y(\mathrm{d}y - v_y\mathrm{d}t) + \Phi_\theta(\mathrm{d}\theta - v_\theta\mathrm{d}t) + \Phi_\phi(\mathrm{d}\phi - v_\phi\mathrm{d}t)$$

are

$$\begin{aligned} m\ddot{x} &= \Phi_x + \lambda_1 \\ m\ddot{y} &= \Phi_y + \lambda_2 \\ I_1\ddot{\theta} &= \Phi_\theta \\ I_2\ddot{\phi} &= \Phi_\phi - r\cos\theta\lambda_1 - r\sin\theta\lambda_2 \end{aligned}$$

which are algebro-differential equations when combined with the constraint equations

$$\dot{x} = r\cos\theta\dot{\phi}, \quad \dot{y} = r\sin\theta\dot{\phi}.$$

We may also explicitly write the vector field $X_{L,\Phi,(M,\Lambda)}$. First we compute the Lagrange multipliers using (5.5). The matrix with components denoted C_{ab} in equation (5.5) is computed to be

$$C = \begin{bmatrix} \frac{m(I_2+mr^2\sin^2\theta)}{I_2+mr^2} & -\frac{m^2r^2\sin\theta\cos\theta}{I_2+mr^2} \\ -\frac{m^2r^2\sin\theta\cos\theta}{I_2+mr^2} & \frac{m(I_2+mr^2\cos^2\theta)}{I_2+mr^2} \end{bmatrix}.$$

We thus compute

$$\begin{aligned} \lambda_1 &= \frac{(mr^2(\cos 2\theta - 1) - 2I_2)\Phi_x + mr^2\sin 2\theta\Phi_y + 2mr\cos\theta\Phi_\phi - 2mr(I_2+mr^2)\sin\theta v_\theta v_\phi}{2(I_2+mr^2)} \\ \lambda_2 &= \frac{mr^2\sin 2\theta\Phi_x - (mr^2(\cos 2\theta + 1) + 2I_2)\Phi_y + 2mr\sin\theta\Phi_\phi + 2mr(I_2+mr^2)\cos\theta v_\theta v_\phi}{2(I_2+mr^2)}. \end{aligned}$$

Substituting these expressions for λ_1 and λ_2 into the vector field

$$\frac{\partial}{\partial t} + v_x\frac{\partial}{\partial x} + v_y\frac{\partial}{\partial y} + v_\theta\frac{\partial}{\partial \theta} + v_\phi\frac{\partial}{\partial \phi} + \frac{\Phi_x+\lambda_1}{m}\frac{\partial}{\partial v_x} + \frac{\Phi_y+\lambda_2}{m}\frac{\partial}{\partial v_y} + \\ \frac{\Phi_\theta}{I_1}\frac{\partial}{\partial v_\theta} + \frac{\Phi_\phi - r\cos\theta\lambda_1 - r\sin\theta\lambda_2}{I_2}\frac{\partial}{\partial v_\phi}$$

gives a vector defined on all of $J^1\mathscr{Q}$. We are, of course, only in the restriction of this vector field to M. First of all, one may readily check that the vector field *does* in fact restrict to M. In the coordinates $(t, x, y, \theta, \phi, v_\theta, v_\phi)$ for M this restricted vector field is given by

$$X_{L,\Phi,(M,\Lambda)} = \frac{\partial}{\partial t} + r\cos\theta v_\phi\frac{\partial}{\partial x} + r\sin\theta v_\phi\frac{\partial}{\partial y} + v_\theta\frac{\partial}{\partial \theta} + v_\phi\frac{\partial}{\partial \phi} + \\ \frac{\Phi_\theta}{I_1}\frac{\partial}{\partial v_\theta} + \frac{r\cos\theta\Phi_x + r\sin\theta\Phi_y + \Phi_\phi}{I_2+mr^2}\frac{\partial}{\partial v_\phi}.$$

Finally, for the sake of completeness, let us write the two-form $\Omega_{L,\Phi}$:

$$\Omega_{L,\Phi} = (m\mathrm{d}v_x - \Phi_x\mathrm{d}t)\wedge(\mathrm{d}x - v_x\mathrm{d}t) + (m\mathrm{d}v_y - \Phi_y\mathrm{d}t)\wedge(\mathrm{d}y - v_y\mathrm{d}t) + \\ (I_1\mathrm{d}v_\theta - \Phi_\theta\mathrm{d}t)\wedge(\mathrm{d}\theta - v_\theta\mathrm{d}t) + (I_2\mathrm{d}v_\phi - \Phi_\phi\mathrm{d}t)\wedge(\mathrm{d}\phi - v_\phi\mathrm{d}t).$$

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