Affine connection control systems

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Abstract

The affine connection formalism provides a useful framework for the investigation of a large class of mechanical systems. Mechanical systems with kinetic energy Lagrangians and possibly with nonholonomic constraints are fit naturally into the formalism, and some results are stated in the areas of controllability and optimal control for affine connection control systems.

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1. Introduction

Control theory for mechanical systems is a topic which has received a certain degree of attention in the past decade. Apart from the fact that many control applications are mechanical in nature, the differential geometric flavour of aspects of both mechanics and nonlinear control theory provides compelling theoretical motivation for this interest.

When one begins to think about studying control theory for mechanical systems, one must in some sense choose something from each of the two subjects—control theory and mechanics—in order to initiate the investigation. In the author's own work, the choice from control theory was nonlinear controllability, and the choice from mechanics was so-called "simple mechanical systems," those whose Lagrangians are kinetic minus potential energy. When this choice is made, the equipment made available by the choice often dictates the nature of the results one obtains. For example, in the author's initial work in the area, the investigation of a certain type of controllability for simple mechanical systems led to the "symmetric product." A readable overview of this work with Richard Murray may be found in a recent SIAM Review paper [Lewis and Murray 1999]. Interestingly, the symmetric product also appears in the somewhat unrelated work of Crouch [1981]. The symmetric product is an object which one might consider in terms of affine differential geometry, quite apart from any mechanical or control theoretic context. This is done, along with other related work, in the paper [Lewis 1998]. This differential geometric interpretation of the symmetric product may then be brought *back* to control theory, and provides an interesting interpretation of reachable sets for simple mechanical control systems [Lewis and Murray 1997b].

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Recent work of the author has centred on optimal control theory for mechanical systems, again utilising the affine connection framework. Here one can produce a geometric version of the Maximum Principle where the essential ingredient is the so-called "adjoint Jacobi equation" which forms that part of the equation describing the evolution of the adjoint vector which is independent of the cost function, i.e., that part which depends only on the control system. This equation is, as the name suggests, related to the Jacobi equation of geodesic variation. The full development is somewhat lengthy, and here we present an abbreviated form of these results, noting that their full statement has not yet appeared in the literature.

The impression might then be gotten that there is a connection, possibly a deep one, between affine differential geometry and control theory for simple mechanical systems. This impression has been reinforced by other work in this area, for example [Baillieul 1999, Bloch and Crouch 1995b, Bullo 1999, Bullo 2002, Bullo, Leonard, and Lewis 2000, Crouch and Silva Leite 1991, Lewis 1999, Lewis 2000a, Noakes, Heinzinger, and Paden 1989]. We will touch on the content of some of these and other papers as they comes up in the sequel.

2. Mechanical systems as affine connection control systems

We begin by motivating a discussion of what we shall in Section 3 refer to as "affine connection control systems." We do this by showing how affine connections naturally arise when discussing mechanical systems with kinetic energy Lagrangians. Thus we have a configuration manifold Q which possesses a Riemannian metric g giving rise to the Lagrangian $L(v_q) = \frac{1}{2}g(v_q, v_q)$. Often, of course, one is interested in including potential forces in the Lagrangian, and it is indeed possible to do this. For example, the issue of potential shaping is touched upon in some recent work [Bloch, Leonard, and Marsden 1999, Bullo 2002, Weibel and Baillieul 1998]. The initial work on controllability of Lewis and Murray [1997a] also includes potential forces.

Our aim is to show, in as concise a manner as possible, how one makes the step from mechanics to affine differential geometry. To do this we use local coordinates $(q^1, \ldots, q^n)$ for Q and remark that a simple calculation shows that if we take $L = \frac{1}{2}g_{ij}\dot{q}^i\dot{q}^j$ (here we use the summation convention where repeated indices are summed) we obtain the equivalence

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{q}^i}\right) - \frac{\partial L}{\partial q^i} = 0, \ \ i = 1, \ldots, n \quad \Longleftrightarrow \quad \ddot{q}^i + \overset{g}{\Gamma}^i_{jk}\dot{q}^j\dot{q}^j = 0, \ \ i = 1, \ldots, n, \qquad (2.1)$$

where

$$\overset{g}{\Gamma}^i_{jk} = \frac{1}{2}g^{i\ell}\left(\frac{\partial g_{\ell j}}{\partial q^k} + \frac{\partial g_{\ell k}}{\partial q^j} - \frac{\partial g_{jk}}{\partial q^\ell}\right).$$

The n^3 functions $\overset{g}{\Gamma}^i_{jk}, i, j, k = 1, \ldots, n$ are the **Christoffel symbols** for an affine connection $\overset{g}{\nabla}$ called the **Levi-Civita connection**. The equation (2.1) asserts that the solutions of the Euler-Lagrange equations are exactly **geodesics** for the affine connection ∇. A thorough discussion of affine connections may be found in Kobayashi and Nomizu [1963a, 1963b], but we shall say a few cursory words on the subject in the next section.

Interestingly, it is also true that one may use the affine connection formalism to describe the motion of a system with a kinetic energy Lagrangian, and with constraints linear in velocity. This idea seems to originate with Synge [1928], and the author was first made

aware of it via the paper of Bloch and Crouch [1995a] at the CDC in 1995. Briefly the setup is this. One has a distribution D on Q and the system's velocities are constrained to lie in D. To write the equations of motion for such systems, one uses the Lagrange-d'Alembert principle, and doing so yields equations of the form

$$\ddot{q}^i + \Gamma^i_{jk}\dot{q}^j\dot{q}^k = 0, \ \ i = 1,\ldots,n$$

where the Christoffel symbols Γ^i_{jk} are those for the affine connection ∇ defined by

$$\nabla_X Y = \overset{g}{\nabla}_X Y + (\overset{g}{\nabla}_X P^\perp)(Y) \tag{2.2}$$

where $P^\perp$ is the orthogonal projection onto the orthogonal complement of D. Some sense can perhaps be made of the expression (2.2) after reading the next section in which affine connections are characterised. Nonetheless, details aside, the bottom line is that for mechanical systems, constrained or unconstrained, with kinetic energy Lagrangians, the unforced equations are geodesic equations, and as such are typically written $\nabla_{c'(t)}c'(t) = 0$ for a curve $c\colon I \to Q$ with $I \subset \mathbb{R}$ an interval.

The preceding discussion is of a purely mechanical nature, and has naught to do with control theory. To make the mechanical systems into *control* systems, we add forces to the picture. The idea is to select one input force associated with each direction in which one may apply a force, and take as the control force a linear combination of these forces. We make the assumption that the directions in which one may apply forces vary only with the configuration of the system, and not with, for example, velocity or time. Doing so means that control forces may be modelled as vector fields $\{Y_1,\ldots,Y_m\}$ on Q, and the control system we consider is this one:

$$\nabla_{c'(t)}c'(t) = u^a(t)Y_a(c(t)). \tag{2.3}$$

This control system forms the basis of discussion for the remainder of the paper.

3. Affine connection control systems

The motivation of Section 2 serves to provide a mechanical backdrop for this section, where we look formally, but briefly, at affine connections and control systems formed by them. We refer to the bibliography, principally Kobayashi and Nomizu [1963a, 1963b], for details on the plethora of under-justified assertions in this section.

3.1. Affine differential geometry. An *affine connection* on Q assigns to each pair of vector fields X and Y on Q a vector field $\nabla_X Y$ with the assignment satisfying

1. the map $(X,Y) \mapsto \nabla_X Y$ is $\mathbb{R}$-bilinear,

2. $\nabla_{fX}Y = f\nabla_X Y$, and

3. $\nabla_X(fY) = f\nabla_X Y + (\mathscr{L}_X f)Y$

for all vector fields X and Y on Q, all functions f on Q, and where $\mathscr{L}_X$ denotes the Lie derivative with respect to X. The association of this abstract object with the Christoffel

symbols of the previous section occurs when we choose local coordinates $(q^1, \ldots, q^n)$. Then we may apply the affine connection to a pair of coordinate vector fields $\frac{\partial}{\partial q^i}$:

$$\nabla_{\frac{\partial}{\partial q^j}} \frac{\partial}{\partial q^k} = \Gamma^i_{jk} \frac{\partial}{\partial q^i},$$

which provides the definition of the Christoffel symbols for an arbitrary affine connection. The vector field $\nabla_X Y$ is called the **covariant derivative** of Y with respect to X, and if we define ∇_X on smooth functions by $\nabla_X f = \mathscr{L}_X f$, then we may extend ∇_X to a derivation on the tensor algebra over Q in the usual manner. That is, it is possible to define the covariant derivative $\nabla_X A$ of an (r, s) tensor field A with respect to X. The **torsion tensor** and the **curvature tensor** for an affine connection are the $(1, 2)$ tensor field T on Q and the $(1, 3)$ tensor field R on Q defined by

$$T(X, Y) = \nabla_X Y - \nabla_Y X - [X, Y]$$
$$R(X, Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z,$$

respectively. The affine connection $\overset{g}{\nabla}$ associated with the Riemannian metric g is then the unique torsion-free affine connection with the property that $\overset{g}{\nabla}_X g = 0$ for every vector field X on Q. Thus far the objects we have discussed are classical to affine differential geometry. However the control considerations of [Lewis and Murray 1997a] led to the symmetric product of two vector fields which we define by $\langle X : Y \rangle = \nabla_X Y + \nabla_Y X$. The geometric meaning of the symmetric product has been provided by the author [Lewis 1998], and we refer to [Crouch 1981] for an appearance of the symmetric product in another setting.

A **geodesic** for an affine connection ∇ is a curve $c : I \to Q$ from an interval $I \subset \mathbb{R}$ which has the property that $\nabla_{c'(t)} c'(t) = 0$ for $\in I$. If $c_s : I \to Q$ is a smooth family of geodesics defined for $s \in]-\epsilon, \epsilon[$, and which has the property that $c_0 = c$, we define a **Jacobi field** along c to be any vector field along c of the form

$$\xi(t) = \left. \frac{\mathrm{d}}{\mathrm{d}s} \right|_{s=0} c_s(t).$$

Jacobi fields may be shown to satisfy the **Jacobi equation**:

$$\nabla^2_{c'(t)} \xi(t) + R(\xi(t), c'(t)) c'(t) + \nabla_{c'(t)} (T(\xi(t), c'(t))) = 0.$$

Thus the Jacobi equation may be thought of as the equation of "geodesic variation."

Yet another piece of equipment is the **geodesic spray** associated with an affine connection ∇. This is the second-order vector field with the property that the projection of its integral curves to Q are geodesics for ∇. In coordinates we have

$$Z = v^i \frac{\partial}{\partial q^i} - \Gamma^i_{jk} v^j v^k \frac{\partial}{\partial v^i}.$$

If one wishes to treat the system (2.3) as a first-order control affine nonlinear control system on TQ, the vector field Z is the drift vector field. In a treatment such as this, the control vector fields are the m vector fields on TQ denoted Y_a^{lift}, $a = 1, \ldots, m$ where

$$Y_a^{\text{lift}}(v_q) = \left. \frac{\mathrm{d}}{\mathrm{d}t} \right|_{t=0} \left(v_q + t Y_a(q) \right)$$

With all this notation the control system (2.3) is given by

$$\dot{v}(t) = Z(v(t)) + u^a(t)Y_a^{\mathrm{lift}}(v(t))$$

in first-order form on TQ.

3.2. Control system definitions. An ***affine connection control system*** is comprised of a triple $(Q, \nabla, \mathscr{Y})$ where Q is a finite-dimensional manifold, ∇ is an affine connection on Q, and $\mathscr{Y} = \{Y_1, \ldots, Y_m\}$ is a collection of vector fields on Q. To an affine connection control system we obviously associate a control system of the form (2.3). The inputs $u \colon I \to \mathbb{R}^m$ we consider are measurable functions from an interval $I \subset \mathbb{R}$, and we denote this set of inputs by $\mathscr{U}$. A ***controlled trajectory*** is then a pair (c, u) where $u \colon I \to \mathbb{R}^m$ is a map from the set of inputs $\mathscr{U}$ and where $c \colon I \to Q$ satisfies (2.3). A ***controlled arc*** is a controlled trajectory defined on a compact interval. Let $q_0, q_1 \in Q$ and let $v_{q_0} \in T_{q_0}Q$ and $v_{q_1} \in T_{q_1}Q$. We denote by $\mathrm{Carc}(\Sigma, q_0, q_1)$ (resp. $\mathrm{Carc}(\Sigma, v_{q_0}, v_{q_1})$) the set of controlled arcs (u, c) for which $c(a) = q_0$ (resp. $c'(a) = v_{q_0}$) and $c(b) = q_1$ (resp. $c'(b) = v_{q_1}$) where (u, c) is defined on some interval $[a, b]$. If we wish only to consider controlled arcs from $\mathrm{Carc}(\Sigma, q_0, q_1)$ (resp. $\mathrm{Carc}(\Sigma, vq_0, v_{q_1})$) defined on a fixed $[a, b]$ we write $\mathrm{Carc}(\Sigma, q_0, q_1, [a, b])$ (resp. $\mathrm{Carc}(\Sigma, vq_0, v_{q_1}, [a, b])$). For $q \in Q$, U a neighbourhood of q, and $T > 0$ define

$$\mathcal{R}_Q^U(q, T) = \{c(T) \mid \, \exists \, u \in \mathscr{U} \text{ so that } (u, c) \text{ is a controlled trajectory}$$
$$\text{defined on } [0, T] \text{ with } c'(0) = 0_q \text{ and } c(t) \in U\}.$$

Thus $R_Q^U(q, T)$ are those configurations reachable in exactly time T from q starting with zero initial velocity (0_q is the zero vector in T_qQ). Note that we do not restrict the final velocity. We also define

$$\mathcal{R}_Q^U(q, \leq T) = \bigcup_{0 \leq t \leq T} \mathcal{R}_Q^U(q, t).$$

4. Controllability for affine connection control systems

The initial impetus for the investigation of the class of systems we are describing was controllability theory. Here one wishes to exploit the special structure of the system, in conjunction with well-known techniques in nonlinear controllability, to derive useful controllability tests. In this section we suppose that we have an analytic affine connection control system $(Q, \nabla, \mathscr{Y} = \{Y_1, \ldots, Y_m\})$. The controllability tests from nonlinear control theory which we adapt are those from standard accessibility theory [e.g., Sussmann and Jurdjevic 1972] and the small-time local controllability results of Sussmann [1987]. Because affine connection control systems, although they have a state space of TQ, are defined in terms of objects on the configuration manifold Q, one would like to obtain results which are expressed in terms of conditions on Q. Furthermore, it makes a great deal of sense to formulate controllability definitions on the configuration manifold.

We make the following controllability definitions.

4.1 Definition: Let $\Sigma = (Q, \nabla, \mathscr{Y})$ be an affine connection control system.

   (i) Σ is ***locally configuration accessible*** at q if for each neighbourhood U of q there exists $T > 0$ so that $\mathcal{R}_Q^U(q, t)$ has nonempty interior for each $0 < t \leq T$.

  (ii) Σ is ***locally configuration controllable*** at q if it is locally configuration accessible at q and if for each neighbourhood U of q there exists $T > 0$ so that $q \in \text{int}(\mathcal{R}_Q^U(q, t))$ for each $0 < t \leq T$.

 (iii) Σ is ***equilibrium controllable*** if for each $q_1, q_2 \in Q$ there exists a controlled trajectory (u, c) defined on $[0, T]$ so that $q_1 = c(0)$, $q_2 = c(T)$, $c'(0) = 0_{q_1}$ and 0_{q_2}.   •

One wishes to study this so-called "configuration controllability" for a couple of reasons. One of the most compelling is that it is possible for a system to be locally configuration accessible (resp. controllable) and *not* be locally accessible (resp. controllable) in state space. If one is interested only in what is happening to configurations anyway, it makes sense to have controllability definitions and tests which reflect this. Also, as we shall see, it is possible to provide simple tests for the configuration controllability definitions we provide.

Let us first look at the accessibility conditions which were first presented in complete form in [Lewis and Murray 1997a]. We let $\text{Sym}(\mathscr{Y})$ be the smallest subspace of vector fields on Q which contains $\mathscr{Y}$ and which is closed under symmetric product, and we let $\text{Lie}(\text{Sym}(\mathscr{Y}))$ be the smallest subspace of vector fields on Q which contains $\text{Sym}(\mathscr{Y})$ and which is closed under Lie bracket. We then define

$$\text{Lie}(\text{Sym}(\mathscr{Y}))_q = \{X(q) \mid X \in \text{Lie}(\text{Sym}(\mathscr{Y}))\}.$$

For analytic systems we have the following sharp result for configuration accessibility.

4.2 Theorem: *An analytic affine connection control system* $\Sigma = (Q, \nabla, \mathscr{Y})$ *is locally configuration accessible at* $q \in Q$ *if and only if* $\text{Lie}(\text{Sym}(\mathscr{Y}))_q = T_q Q$.

For C^∞ systems, the condition $\text{Lie}(\text{Sym}(\mathscr{Y}))_q = T_q Q$ is sufficient but not necessary for local configuration accessibility. This is explored in detail in the original paper of Lewis and Murray—the proof requires delving into detail the bracket computations for an affine connection control system when thought of as a nonlinear control system. In that paper, a good deal of effort is also devoted to deriving the conditions for local configuration accessibility for systems with potential energy; this significantly complicates the statement of the result, so we shall not go into this here. We also mention that the geometric interpretation of the symmetric product [see Lewis 1998] may be applied to give an interpretation of Theorem 4.2 [Lewis and Murray 1997b].

The configuration controllability result requires that we look at the behaviour of certain types of symmetric product. A symmetric product from the set $\mathscr{Y} = \{Y_1, \ldots, Y_m\}$ is ***bad*** if it contains an even number of each of the vector fields Y_a, $a = 1, \ldots, m$, and is ***good*** if it is not bad. The ***degree*** of a symmetric product is the total number of vector fields of which it comprised.[1] For example, the symmetric product $\langle Y_a : \langle Y_b : Y_a \rangle \rangle$ is good and of degree 3 and the symmetric product $\langle \langle Y_a : Y_b \rangle : \langle Y_a : Y_b \rangle \rangle$ is bad and of degree 4.

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[1]Of course, we are speaking imprecisely here—to do this rigorously requires that one work with free algebraic quantities, and this is explained in detail by Lewis and Murray [1997a].

4.3 Theorem: *Let $\Sigma = (Q, \nabla, \mathscr{Y})$ be an affine connection control system.*

(i) Σ is locally configuration controllable at $q \in Q$ if every bad symmetric product P can be written at q as

$$P(q) = \sum_{i=1}^{k} \xi^i P_i(q)$$

for good symmetric products $P_1, \ldots, P_k$ of degree lower than that of P.

(ii) Σ is equilibrium controllable if the conditions of (i) are satisfied at each $q \in Q$.

In practice one need only check the condition (i) on a set of symmetric products which form a basis for

$$\mathrm{Sym}(\mathscr{Y})_q = \{X(q) \mid X \in \mathrm{Sym}(\mathscr{Y})\}.$$

These conditions are a watered down version of the general controllability conditions proved by Sussmann [1987]. It would be interesting indeed if one were able to provide sharp conditions for analytic systems in terms only of symmetric products (and potentially Lie brackets) of the input vector fields $\mathscr{Y}$. That the control Lie algebra for affine connection control systems is simpler than that for generic nonlinear control systems may be observed by noting that sharp conditions exist for single-input systems [Lewis 1997]. In this case one can show that the system is locally configuration controllable if and only if $\dim(Q) = 1$, i.e., we have controllability only in the uninteresting case when the system is fully actuated.

Unfortunately, space does not permit the presentation of examples which illustrate the range of behaviours relating to configuration accessibility and controllability, and we refer the reader to the references for details concerning some simple physical examples.

5. Optimal control for affine connection control systems

The structure of mechanical systems in some sense makes the problem of optimal control a natural one. In particular, if one possesses a kinetic energy Riemannian metric, this encourages the definition of some natural cost functions. It turns out to be possible to formulate for affine connection control systems a powerful version of the Maximum Principle. The general buildup is rather substantial, and we refer to [Bullo and Lewis 2005, Chapter S4] for details. The referenced work relies on a wonderful paper by Sussmann [1998] which provides a general and geometric formulation for the Maximum Principle.

To eliminate a significant part of the generality of [Bullo and Lewis 2005, Chapter S4] we choose a specific cost function, and the one with which it is the simplest to deal. We let $\Sigma = (Q, \nabla, \mathscr{Y} = \{Y_1, \ldots, Y_m\})$ be a C^∞ affine connection control system and we suppose that we have a Riemannian metric g on Q. We define a cost function $F \colon \mathbb{R}^m \times Q \to \mathbb{R}$ by $F(u, q) = g(u^a Y_a(q), u^b Y_b(q))$. The objective of the optimal control problem is to minimise

$$J(u, c) = \int_I F(u(t), c(t)) \, \mathrm{d}t$$

over a class of controlled trajectories (u, c) defined on an interval I.

Let us precisely state the type of control problems we shall look at.

5.1 Definition: Let $\Sigma = (Q, \nabla, \mathscr{Y})$ be an affine connection control system, let $q_0, q_1 \in Q$, and let $v_{q_0} \in T_{q_0}Q$ and $v_{q_1} \in T_{q_1}Q$.

(i) A controlled arc (u_*, c_*) is a **solution of** $\mathscr{F}(\Sigma, v_{q_0}, v_{q_1})$ if $J(u_*, c_*) \leq J(u, c)$ for every $(u, c) \in \mathrm{Carc}(\Sigma, v_{q_0}, v_{q_1})$.

(ii) A controlled arc $\gamma = (u_*, c_*)$ is a **solution of** $\mathscr{F}_{[a,b]}(\Sigma, v_{q_0}, v_{q_1})$ if $J(u_*, c_*) \leq J(u, c)$ for every $(u, c) \in \mathrm{Carc}(\Sigma, v_{q_0}, v_{q_1}, [a, b])$.

(iii) A controlled arc $\gamma = (u_*, c_*)$ is a **solution of** $\mathscr{F}(\Sigma, q_0, q_1)$ if $J(u_*, c_*) \leq J(u, c)$ for every $(u, c) \in \mathrm{Carc}(\Sigma, q_0, q_1)$.

(iv) A controlled arc $\gamma = (u_*, c_*)$ is a **solution of** $\mathscr{F}_{[a,b]}(\Sigma, q_0, q_1)$ if $J(u_*, c_*) \leq J(u, c)$ for every $(u, c) \in \mathrm{Carc}(\Sigma, q_0, q_1, [a, b])$.   ●

Bullo and Lewis [2005, Chapter S4] provides a general statement of the Maximum Principle for affine connection control systems which we will here distill to the cost function at hand. We let $P \colon TQ \to TQ$ be the orthogonal projection onto the distribution spanned by the input vector fields $\mathscr{Y}$, and define a $(2, 0)$ tensor field h on Q by

$$h(\alpha, \beta) = g^{-1}(P^*(\alpha), P^*(\beta))$$

for one-forms α and β, where g^{-1} is the $(2, 0)$ tensor field associated with g (i.e., that one whose components in coordinates are the inverse of the components of g) and where $P^* \colon T^*Q \to T^*Q$ is the dual endomorphism of P. We also define some notation for the curvature and torsion tensors. For vector fields X, Y, and Z and a one-form α define R^* and T^* by

$$\langle T^*(\alpha, X); Y \rangle = \langle \alpha; T(Y, X) \rangle,$$
$$\langle R^*(\alpha, X)Y; Z \rangle = \langle \alpha; R(Z, X)Y \rangle$$

where $\langle \cdot; \cdot \rangle$ denotes the natural pairing of a one-form and a vector field.

The usual statement of the Maximum Principle requires the Hamiltonian associated with the optimal control problem. The Hamiltonian will be function $H^{\Sigma, F}$ on $\mathbb{R}^m \times T^*TQ$ defined by

$$H^{\Sigma, F}(u, \alpha_{v_q}) = F(u, q) + \left\langle \alpha_{v_q}; Z(v_q) + u^a Y_a^{\mathrm{lift}}(v_q) \right\rangle.$$

To represent this function in the most useful manner one needs to use a splitting which is adapted to the affine connection. This part of the construction requires some effort to reproduce, so let us just state where one ends up after the effort is expended. One obtains a splitting of each fibre $T^*_{v_q}TQ$ as $T^*_qQ \oplus T^*_qQ$. With this splitting we denote a typical point in T^*TQ by $\alpha_{v_q} \oplus \beta_{v_q} \in T^*_{v_q}TQ \simeq T^*_q \oplus T^*_qQ$. With this splitting chosen in the appropriate manner it turns out that

$$H^{\Sigma, F}(u, \alpha_{v_q} \oplus \beta_{v_q}) = F(u, q) + \alpha_{v_q} \cdot v_q + u^a \beta_{v_q} \cdot Y_a(q).$$

In this splitting, the Hamiltonian "decouples" and the geodesic spray Z has disappeared, absorbed into the splitting. In the Maximum Principle one fixes $\alpha_{v_q} \oplus \beta_{v_q} \in T^*TQ$ and seeks $u \in \mathbb{R}^m$ to minimise the Hamiltonian. This value of u is then substituted back into the Hamiltonian to yield the **minimum Hamiltonian** which in this case is determined to be

$$H_{\min}^{\Sigma, F}(\alpha_{v_q} \oplus \beta_{v_q}) = \alpha_{v_q} \cdot v_q - \frac{1}{2}h(\beta_{v_q}, \beta_{v_q}).$$

The normal extremals are integral curves for this Hamiltonian. The following result describes these normal extremals.

5.2 Theorem: *Let* $\Sigma = (Q, \nabla, \mathscr{Y})$ *be a* C^∞ *affine connection control system. Suppose that* (u, c) *is a solution of one of the four problems of Definition 5.1 with* u *and* c *defined on* $[a, b]$. *If* (u, c) *is a normal extremal then* c *is of class* C^∞ *and there exists a* C^∞ *one-form field* λ *along* c *so that* c *and* λ *together satisfy the differential equations*

$$\nabla_{c'(t)}c'(t) = -h^\sharp(\lambda(t))$$

$$\nabla^2_{c'(t)}\lambda(t) + R^*(\lambda(t), c'(t))c'(t) - T^*(\nabla_{c'(t)}\lambda(t), c'(t)) = \tfrac{1}{2}\nabla h(\lambda(t), \lambda(t)) - \qquad (5.1)$$
$$T^*(\lambda(t), h^\sharp(\lambda(t))).$$

If $\gamma = (u, c)$ *is a solution of* $\mathscr{F}_{[a,b]}(\Sigma, q_0, q_1)$ *or of* $\mathscr{F}(\Sigma, q_0, q_1)$ *then we additionally have* $\lambda(a) = 0$ *and* $\lambda(b) = 0$.

If we suppose $\mathscr{Y}$ contains vector fields which span T_qQ for each $q \in Q$, i.e., that the system is fully actuated, and that ∇ is the Levi-Civita connection associated with the Riemannian metric g used in the definition of the cost function, then, with some straightforward manipulations, we recover the results of Noakes, Heinzinger, and Paden [1989] and Crouch and Silva Leite [1991], namely that the necessary condition for minimisers is

$$\overset{g}{\nabla}{}^3_{c'(t)}c'(t) + R(\overset{g}{\nabla}_{c'(t)}c'(t))c'(t) = 0.$$

Of course one can define other natural cost functions, and explore other questions associated with optimal control for affine connection control systems. There is much to be done here, and doubtless some beautiful results await discovery.

6. Closing remarks

The idea of this paper is to give a flavour of the types of results which one may obtain using the affine connection formalism. That this formalism has an intimate relation to mechanics, as outlined in Section 2, makes the exploration of this affine connection setting a bit more enticing. The emphasis here was on affine connection control systems as a class of systems in and of itself, and dues to space constraints, not much attention has been paid to mechanics per se. However, the reader is invited to look into the references for examples of how the theory may be applied to physical examples. Even some simple examples exhibit surprisingly subtle behaviour.

Also, we have only touched on certain aspects of the author's own work. A potentially promising area which has not been discussed is that of whether affine connection control systems may simplify. In the paper [Lewis 1999] it is shown, for example, that an affine connection control system is in some sense reducible to a driftless control system provided the distribution spanned by the input vector fields $\mathscr{Y}$ is closed under symmetric product. In particular, this puts the lie to any possibility of generally reducing the study of a mechanical system to one which is driftless, even though this might be possible in specific cases. This line of thinking suggests the possibility of perhaps simplifying affine connection control systems using feedback transformations, and the setting for this is described by the author [Lewis 2000b].

Another possible avenue of exploration is that concerning the rôle of symmetry. In mechanics, symmetry plays an important rôle, but how this impinges on control theory, and in particular on the affine connection setting, has not been explored [but see Bloch

and Crouch 1995b, Bloch and Crouch 1998]. A case which *has* seen some attention is that when Q is a Lie group, and the problem data is left-invariant. In this case, Bullo, Leonard, and Lewis [2000] provide some explicit trajectory generation algorithms, including an exponential stabilisation algorithm. A different approach for systems with symmetry and nonholonomic constraints is taken by Ostrowski and Burdick [1997].

We hope we have explicated the value of the affine connection formalism in studying control theory for a class of mechanical systems.

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