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The category of affine connection control systems*

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Abstract

The category of affine connection control systems is one whose objects are control systems whose drift vector field is the geodesic spray of an affine connection, and whose control vector fields are vertical lifts to the tangent bundle of vector fields on configuration space. We investigate morphisms (feedback transformations) in this category.

Keywords. control of mechanical systems, affine connections

AMS Subject Classifications (2020). 53B05, 70Q05, 93B29

1. Introduction

It is apparent that the study of what we will in this paper call "affine connection control systems" has a significant rôle to play in the field of mechanical control systems. In a series of papers, [e.g., Bullo, Leonard, and Lewis 2000, Lewis 1998, Lewis 1999, Lewis and Murray 1997a, Lewis and Murray 1997b], the author and various coauthors have shown how the affine connection framework is useful in looking at mechanical systems whose Lagrangian is the kinetic energy with respect to a Riemannian metric, possibly in the presence of constraints linear in velocity [e.g., Lewis 1997, Lewis 2000]. In such an investigation, there appears to be no particular advantage to work with affine connections which come from physics, i.e., from the Riemannian metric and the constraints. Therefore, in this paper we deal with general affine connections.

The emphasis here is to lay a groundwork for the investigation of ways in which one can simplify or alter affine connection control systems using feedback. That simplification of these systems is important can be seen in the work of the author [Lewis 2000] where even simple physical systems yield rather complicated expressions for the system's affine connection. An example of where feedback has been used to simplify equations for constrained systems can be found in the work of Krishnaprasad and Tsakiris [2001] on the roller racer. Apart from the matter of simplification, one might also wish to use feedback to change the system into one whose characteristics are more desirable. The idea of restricting the types of feedback so that one remains in a certain class of systems is not new. In [Bloch, Krishnaprasad, Marsden, and Sánchez de Alvarez 1992] the authors retain the Hamiltonian structure of their system through feedback, and in work initiated in [Bloch,

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Leonard, and Marsden 1997] (see also [Hamberg 1999]), the emphasis is on retaining the Riemannian structure through "kinetic shaping." Our focus in this work is on equivalence which maintains the affine connection structure. For general control affine systems, the issues we address here are reviewed in the paper of Elkin [1998].

Besides introducing the basic notion of equivalence, we also look at how one may investigate subsystems and factor systems. In the former case, one wishes to determine when the dynamics of a given affine connection control systems are "contained in" the controlled dynamics of another. For factor systems, one wishes to project the dynamics of an affine connection control system onto another affine connection control system. Scenarios such as this arise, for example, when one is dealing with systems with symmetry and can perform a reduction of some sort.

The matters we address in this paper are technically challenging ones in practice. For example, the matter of equivalence typically produces a set of overdetermined nonlinear partial differential equations which one must solve. However, we hope that by illuminating some of the special structure in the class of affine connection control systems, we can point the way for certain profitable lines of investigation.

2. Relevant affine differential geometry

If M is a smooth manifold we denote by $C^\infty(M)$ the C^∞ functions on M, by $\Gamma^\infty(TM)$ the C^∞ vector fields on M, and by $\Gamma^\infty(T^*M)$ the set of one-forms on M. For a map $\phi\colon M \to N$ of manifolds M and N, we denote by $T\phi\colon TM \to TN$ the derivative of ϕ, and by $T_x\phi\colon T_xM \to T_{\phi(x)}N$ the restriction of $T\phi$. The zero section of TM we denote by $Z(TM)$, and note that this is diffeomorphic to M.

We refer the reader to [Kobayashi and Nomizu 1963] for details of our following brief discussion of affine differential geometry. Let Q be a finite-dimensional manifold. An ***affine connection*** on Q assigns to every pair of vector fields X and Y on Q a vector field $\nabla_X Y$ with the assignment satisfying

AC1. the map $(X, Y) \mapsto \nabla_X Y$ is $\mathbb{R}$-bilinear,

AC2. $\nabla_{fX} Y = f\nabla_X Y$ for $f \in C^\infty(Q)$ and $X, Y \in \Gamma^\infty(TQ)$, and

AC3. $\nabla_X(fY) = f\nabla_X Y + (\mathscr{L}_X f)Y$ for $f \in C^\infty(Q)$ and $X, Y \in \Gamma^\infty(TM)$.

Let us give a simple example of an affine connection.

2.1 Example: We let Q be an open subset of $\mathbb{R}^n$ and denote a vector field on Q by its principal part. Thus a vector field X is defined by $X(q) = (q, \boldsymbol{X}(q))$ for a map $\boldsymbol{X}\colon Q \to \mathbb{R}^n$. On Q define an affine connection by

$$(\nabla_X Y)_q = (q, \boldsymbol{DY}(q) \cdot \boldsymbol{X}(q))$$

where $\boldsymbol{DY}(q)$ denotes the Jacobian of $\boldsymbol{Y}\colon Q \to \mathbb{R}^n$. We call this the ***standard affine connection*** on an open subset of $\mathbb{R}^n$. •

Given an affine connection ∇ and an interval $I \subset \mathbb{R}$, it turns out to be possible to differentiate a vector field $Y\colon I \to TQ$ along a differentiable curve $c\colon I \to Q$. The result is a vector field along c which is denoted $t \mapsto \nabla_{c'(t)}Y(t)$. The vector field $Y\colon I \to TQ$ along

$c\colon I \to Q$ is ***parallel*** if $\nabla_{c'(t)}Y(t) = 0$ for each $t \in I$. It will be convenient to call a pair (c, Y) with c a curve and Y a vector field along c a ***parallel pair*** when Y is parallel along c. We may compute $\nabla_{c'(t)}c'(t)$ provided c is of class C^2. Given a curve $c\colon I \to Q$ and a vector $V \in T_{c(t_0)}Q$ for some $t_0 \in I$, there exists a unique vector field Y_V along c with the properties that $Y_V(t_0) = V$ and (c, Y_V) is a parallel pair. The curves for which $\nabla_{c'(t)}c'(t) = 0$ are called ***geodesics***. For a given affine connection ∇ on Q, there is a unique second-order vector field Z on TQ, called the ***geodesic spray***, with the property that integral curves of Z projected to Q are geodesics for ∇. An affine connection ∇ is ***complete*** if its corresponding geodesic spray is a complete vector field.

Given an affine connection ∇, its ***torsion tensor*** is the $(1, 2)$ tensor field T defined by

$$T(X, Y) = \nabla_X Y - \nabla_Y X - [X, Y]$$

where $[\cdot, \cdot]$ denote the Lie bracket of vector fields.

The following result will be useful to us, and it is an easy consequence of the defining properties of affine connections.

2.2 Lemma: *If ∇ and $\tilde{\nabla}$ are two affine connections on Q then the mapping sending the vector fields $X, Y \in \Gamma^\infty(TQ)$ to the vector field $\nabla_X Y - \tilde{\nabla}_X Y$ defines a $(1, 2)$ tensor field on Q.*

Note that this implies that if Q is an open subset of $\mathbb{R}^n$ then any affine connection on Q is given by

$$(\nabla_X Y)_q = (q, \boldsymbol{DY}(q) \cdot \boldsymbol{X}(q) + S(\boldsymbol{X}(q), \boldsymbol{Y}(q))) \tag{2.1}$$

for some $(1, 2)$ tensor field S on Q.

In talking about morphisms in our category of affine connection control systems, we will need to have a clear notion of what is meant by an affine mapping. We let ∇ be an affine connection on Q and $\tilde{\nabla}$ be an affine connection on $\tilde{Q}$. Given a curve $c\colon I \to Q$, a vector field $Y\colon I \to TQ$, and a smooth map $\phi\colon Q \to \tilde{Q}$, we define a curve $c_\phi\colon I \to \tilde{Q}$ by $c_\phi(t) = \phi \circ c(t)$, and a vector field $Y_\phi\colon I \to TQ$ along c_ϕ by $Y_\phi(t) = T_{c(t)}\phi(Y(t))$. The map $\phi\colon Q \to \tilde{Q}$ is an ***affine mapping*** provided that (c_ϕ, Y_ϕ) is a parallel pair on $\tilde{Q}$ for every parallel pair (c, Y) on Q. One may readily verify the following characterisation of affine mappings.

2.3 Lemma: *A map $\phi\colon Q \to \tilde{Q}$ is an affine mapping between affine connections ∇ and $\tilde{\nabla}$ if and only if*

$$T_q\phi(\nabla_X Y)_q = (\tilde{\nabla}_{\tilde{X}}\tilde{Y})_{\phi(q)}$$

where $\tilde{X}$ and $\tilde{Y}$ are vector fields on $\tilde{Q}$ which are ϕ-related to vector fields X and Y on Q.

We shall require a stronger type of mapping than an affine mapping. Thus we say a mapping $\phi\colon Q \to \tilde{Q}$ is ***totally geodesic mapping*** if

$$T_q\phi(\nabla_X X)_q = (\tilde{\nabla}_{\tilde{X}}\tilde{X})_{\phi(q)}$$

where $\tilde{X}$ is a vector field on $\tilde{Q}$ which is ϕ-related to X. Clearly a totally geodesic mapping has the property that it maps geodesics of ∇ to geodesics of $\tilde{\nabla}$. The converse is also true, and it is furthermore the case that ϕ is an affine mapping if and only if (1) it is a totally geodesic mapping and (2) ϕ commutes with the torsion tensors for ∇ and $\tilde{\nabla}$ [Vilms 1970].

The interaction of submanifolds and distributions with affine connections will arise when we talk about restricting affine connection control systems. Let us introduce here the necessary terminology. We let Q be a manifold with an affine connection ∇. A submanifold $N \subset Q$ is ***totally geodesic*** if for a geodesic $c\colon I \to Q$, $c'(t_0) \in T_{c(t_0)}N$ for some $t_0 \in I$ implies that $c'(t) \in T_{c(t)}N$ for every $t \in I$. Thus a submanifold N is totally geodesic if geodesics which start tangent to N remain tangent to N. In like manner, an integrable distribution D is ***totally geodesic*** if for a geodesic $c\colon I \to Q$, $c'(t_0) \in D_{c(t_0)}$ for some $t_0 \in I$ implies that $c'(t) \in D_{c(t)}$ for every $t \in I$. The notion of totally geodesic is a classical one in affine differential geometry. However, Lewis [1998] proposes the related, but weaker notion of a ***geodesically invariant*** distribution, whose definition reads just like that for a totally geodesic distribution, but the condition on integrability is not present. Lewis proves the following result.

2.4 Proposition: *A distribution D is geodesically invariant under an affine connection ∇ if and only if it is closed under the* ***symmetric product*** *which is defined by $\langle X : Y\rangle = \nabla_X Y + \nabla_Y X$.*

Our final matter to deal with concerns what we will have to do to factor affine connection control systems. Here we consider affine connections on a manifold Q in the presence of a totally geodesic surjective submersion $\phi\colon Q \to \tilde{Q}$. Although this subject is one of considerable research energy when ∇ is the Levi-Civita connection associated with a Riemannian metric [e.g., Vilms 1970], the situation for an arbitrary connection is not well studied. Nevertheless, we have the following result which gives a property of affine connections in the current context.

2.5 Proposition: *Let ∇ and $\tilde{\nabla}$ be affine connections on manifolds Q and $\tilde{Q}$. If $\phi\colon Q \to \tilde{Q}$ is a totally geodesic surjective submersion, then each of the submanifolds $\phi^{-1}(\tilde{q})$, $\tilde{q} \in \tilde{Q}$, is totally geodesic.*

Proof: If ϕ is totally geodesic, the geodesic of ∇ with initial condition $v_q \in TQ$ is mapped to the geodesic of $\tilde{\nabla}$ with initial condition $T_q\phi(v_q)$. In particular, if $T_q\phi(v_q) = 0$, the geodesic with initial condition v_q is mapped to the trivial geodesic fixing the point $\tilde{q} = \phi(q) \in \tilde{Q}$. But this is precisely the statement that geodesics of ∇ with initial velocities tangent to the submanifold $\phi^{-1}(\tilde{q})$ will evolve in that same submanifold. ■

Finally, we say that an affine connection ∇ on Q is ***geodesically ϕ-projectable*** if for geodesics $c_1, c_2\colon I \to Q$ with initial conditions $v_1 = c_1'(0)$ and $v_2 = c_2'(0)$, the condition $T_{c_1(0)}\phi(v_1) = T_{c_2(0)}\phi(v_2)$ implies that $c_{1,\phi} = c_{2,\phi}$. One may verify that the projected geodesics for ∇ are then the geodesics of an affine connection $\tilde{\nabla}$ on $\tilde{Q}$, and $\tilde{\nabla}$, if specified to have zero torsion, is uniquely defined. Furthermore, with $\tilde{\nabla}$ so defined, the mapping ϕ is a totally geodesic mapping from ∇ to $\tilde{\nabla}$.

3. The category of control affine systems

What we shall call "affine connection control systems" are examples of a commonly studied class of control systems: those which are affine in their controls. A clear discussion of this class of systems from a category theory perspective may be found in [Elkin 1998]. It is an unfortunate clash of common notation that we will use the word "affine" in two rather different contexts here; in one case we mean the general class of control systems affine in the controls, and in the other we means those specific systems whose drift vector field is the geodesic spray of an affine connection.

An ***object*** in the category CAS is a pair $\Sigma = (M, \mathscr{F})$ where M is a finite-dimensional smooth differentiable manifold, and $\mathscr{F}$ is a collection of vector fields $\mathscr{F} = \{f_0, f_1, \dots, f_m\}$. We say the object $\Sigma = (M, \mathscr{F})$ is ***regular*** if the vector fields $\{f_1, \dots, f_m\}$ generate a distribution of constant rank (with this rank necessarily being at most m) and ***irreducible*** if the vector fields $\{f_1, \dots, f_m\}$ are linearly independent. Associated with an object $\Sigma = (M, \mathscr{F})$ in CAS is a control affine system

$$\dot{x}(t) = f_0(x(t)) + u^a(t) f_a(x(t)). \tag{3.1}$$

(In this equation we employ a summation convention where there is an implied summation over repeated indices.) For such a system one typically considers a collection $\mathscr{U}$ of ***admissible controls*** $u\colon I \to \mathbb{R}^m$ (here $I \subset \mathbb{R}$ is an interval). One often wishes to consider $\mathscr{U}$ to be the collection of measurable, essentially bounded inputs defined on arbitrary intervals. We shall suppose that $\mathscr{U}$ contains the set of piecewise constant controls. A ***controlled trajectory*** for $\Sigma = (M, \mathscr{F})$ is a pair (c, u) with $c\colon I \to M$ absolutely continuous and $u\colon I \to \mathbb{R}^m$ in $\mathscr{U}$ satisfying (3.1) for each t in the interval $I \subset \mathbb{R}$.

If $\Sigma = (M, \mathscr{F})$ is an object in CAS and if U is an open submanifold of M, then we have the ***restricted object*** $\Sigma|U \triangleq (U, \mathscr{F}|U)$ where $\mathscr{F}|U = \{f_0|U, f_1|U, \dots, f_m|U\}$.

As is always the case with a category, we need to specify its morphisms. We suppose that we have two objects $\Sigma = (M, \mathscr{F})$ and $\tilde{\Sigma} = (\tilde{M}, \tilde{\mathscr{F}})$ where $\tilde{\mathscr{F}} = \{\tilde{f}_0, \tilde{f}_1, \dots, \tilde{f}_{\tilde{m}}\}$. We let $\mathrm{L}(\mathbb{R}^m; \mathbb{R}^{\tilde{m}})$ denote the set of linear maps from $\mathbb{R}^m$ to $\mathbb{R}^{\tilde{m}}$. A **CAS *morphism*** sending Σ to $\tilde{\Sigma}$ is a triple $(\psi, \lambda_0, \Lambda)$ with the following properties:

CAS1. $\psi\colon M \to N$ is a smooth mapping;

CAS2. $\lambda_0\colon M \to \mathbb{R}^{\tilde{m}}$ and $\Lambda\colon M \to \mathrm{L}(\mathbb{R}^m; \mathbb{R}^{\tilde{m}})$ are smooth mappings satisfying

(a) $T_x\psi(f_a(x)) = \Lambda^\alpha_a(x)\tilde{f}_\alpha(\psi(x))$, $a = 1, \dots, m$ and

(b) $T_x\psi(f_0(x)) = \tilde{f}_0(\psi(x)) + \lambda^\alpha_0(x)\tilde{f}_\alpha(\psi(x))$.

An essential feature of this class of morphisms is the given by the following result. Recall our notation that if c is a curve on M and $\phi\colon M \to \tilde{M}$ is a smooth map, the curve $\psi \circ c$ on $\tilde{M}$ is written as c_ψ.

3.1 Proposition: *If $(\psi, \lambda_0, \Lambda)$ is a morphism in CAS which maps $\Sigma = (M, \mathscr{F})$ to $\tilde{\Sigma} = (\tilde{M}, \tilde{\mathscr{F}})$ and if (c, u) is a controlled trajectory for Σ, then $(c_\psi, \tilde{u})$ is a controlled trajectory for $\tilde{\Sigma}$ where $\tilde{u}(t) = \lambda_0(c(t)) + \Lambda(c(t))u(t)$.*

Conversely, suppose that $\psi\colon M \to \tilde{M}$ is a smooth mapping which has the property that for every controlled trajectory (c, u) of Σ there exists an admissible input $\tilde{u}$ so that $(c_\psi, \tilde{u})$ is a controlled trajectory for $\tilde{\Sigma}$. Then there exists smooth mappings $\lambda_0\colon M \to \mathbb{R}^{\tilde{m}}$ and $\Lambda\colon M \to \mathrm{L}(\mathbb{R}^m; \mathbb{R}^{\tilde{m}})$ so that $(\psi, \lambda_0, \Lambda)$ is a CAS morphism sending Σ to $\tilde{\Sigma}$.

Proof: We have

$$\begin{aligned}(c_\psi)'(t) &= T_{c(t)}\psi(c'(t)) \\ &= T_{c(t)}\psi(f_0(c(t)) + u^a(t)f_a(c(t))) \\ &= \tilde{f}_0(\psi(c(t))) + \lambda_0^\alpha(c(t))\tilde{f}_\alpha(\psi(c(t))) + \Lambda_a^\alpha(c(t))u^a(t)\tilde{f}_\alpha(\psi(c(t))) \\ &= \tilde{f}_0(c_\psi(t)) + \tilde{u}^\alpha\tilde{f}_\alpha(c_\psi(t))\end{aligned}$$

where $\tilde{u}(t) = \lambda_0(c(t)) + \Lambda(c(t))u(t)$.

For the converse, begin by considering the system Σ with the zero control. In this case, the hypotheses guarantee that for each $x \in M$ there exists an admissible control $\tilde{u}_0(x)$ with the property that

$$T_x\psi(f_0(x)) = \tilde{f}_0(\psi(x)) + \tilde{u}_0^\alpha(x)\tilde{f}_\alpha(\psi(x)).$$

Moreover, since the dependence on x of all other terms in this relation is smooth, we may choose $x \mapsto \tilde{u}_0(x)$ to be a smooth mapping. We then define λ_0 by declaring that $\lambda_0(x) = \tilde{u}_0(x)$. Next we consider the situation when the control for Σ is the constant control $\tilde{u}_a\colon t \mapsto \boldsymbol{e}_a$ where $a \in \mathbb{R}^m$ is the ath standard basis vector. In this case, our hypotheses provide that for each $x \in M$ there exists an admissible control $\tilde{u}_a(x)$ so that

$$T_x\psi(f_0(x) + f_a(x)) = \tilde{f}_0(\psi(x)) + \tilde{u}_a^\alpha(x)\tilde{f}_\alpha(\psi(x)).$$

Since we have already declared that

$$T_x\psi(f_0(x)) = \tilde{f}_0(\psi(x)) + \lambda_0^\alpha(x)\tilde{f}_\alpha(\psi(x)),$$

we then have

$$\begin{aligned} & \tilde{f}_0(\psi(x)) + \lambda_0^\alpha(x)\tilde{f}_\alpha(\psi(x)) + T_x\psi(f_a(x)) = \tilde{f}_0(\psi(x)) + \tilde{u}_a^\alpha(x)\tilde{f}_\alpha(\psi(x)) \\ \implies\quad & T_x\psi(f_a(x)) = \big(\tilde{u}_a^\alpha(x) - \lambda_0^\alpha(x)\big)\tilde{f}_\alpha(\psi(x)).\end{aligned}$$

Again, all other terms in this expression have a smooth dependence on x, so we may suppose that the dependence of $\tilde{u}_a^\alpha$ on x may also be assumed to be smooth. We then declare that Λ is defined by $\Lambda_a^\alpha(x) = \tilde{u}_a^\alpha(x)$. It readily follows that with λ_0 and Λ defined as we have defined them, the triple $(\psi, \lambda_0, \Lambda)$ is a CAS morphism sending Σ to $\tilde{\Sigma}$. ■

4. The category of affine connection control systems

Now we can properly discuss the actual subject of the paper. What we consider in this section is a special class of control affine systems. We begin with a discussion of the objects in this category, and note that it is precisely the systems described here which form the basis for the work of the author and coauthors on "simple mechanical control systems."

4.1. Objects in ACCS. Now we turn to investigating that special subset of CAS which is of particular interest to us. We shall denote by ACCS the ***category of affine connection control systems***. An ***object*** in this category is a triple $\Sigma_{\mathrm{aff}} = (Q, \nabla, \mathscr{Y})$ where Q is a finite-dimensional manifold, ∇ is an affine connection on Q, and $\mathscr{Y} = \{Y_1, \dots, Y_m\}$ is a collection of vector fields on Q. We call such an object ***regular*** if the vector fields

$\{Y_1, \ldots, Y_m\}$ generate a distribution of constant rank, and ***irreducible*** is the vector fields $\{Y_1, \ldots, Y_m\}$ are linearly independent.

To an affine connection control system $\Sigma_{\text{aff}} = (Q, \nabla, \mathscr{Y})$ we associate a control system given by

$$\nabla_{c'(t)} c'(t) = u^a(t) Y_a(c(t)). \tag{4.1}$$

A ***controlled trajectory*** for $\Sigma_{\text{aff}} = (Q, \nabla, \mathscr{Y})$ is a pair (c, u) with $c \colon I \to Q$ having the property that its derivative $t \mapsto c'(t)$ is an absolutely continuous curve on TQ, and $u \colon I \to \mathbb{R}^m$ is an admissible control such that together c and u satisfy (4.1).

As with control affine systems, if $U \subset Q$ is an open submanifold, we may define the ***restricted object*** $\Sigma_{\text{aff}}|U = (U, \nabla|U, \mathscr{Y}|U)$.

An affine connection control system is ***fully actuated*** if $\operatorname{span}_{\mathbb{R}}(Y_1(q), \ldots, Y_m(q)) = T_qQ$ for each $q \in Q$. The following result is clear.

4.1 Lemma: *An affine connection control system $\Sigma_{\text{aff}} = (Q, \nabla, \mathscr{Y})$ is fully actuated if and only if for every curve $c \colon I \to Q$ with the property that c' is absolutely continuous, there exists an admissible input u with the property that (c, u) is a controlled trajectory for Σ_{aff}.*

A ***trivial*** affine connection control system is one where $Q = \mathbb{R}^n$, ∇ is the standard affine connection on $\mathbb{R}^n$, and $\mathscr{Y} = \{\frac{\partial}{\partial q^1}, \ldots, \frac{\partial}{\partial q^n})$ is the collection of standard coordinate vector fields on $\mathbb{R}^n$. We denote the trivial affine connection control system by $\Sigma_{\text{aff}}^{\text{can}}$. In standard coordinates $(q^1, \ldots, q^n)$ for $\mathbb{R}^n$ the trivial affine connection control system looks like

$$\begin{aligned} \ddot{q}^1(t) &= u^1(t) \\ &\vdots \\ \ddot{q}^n(t) &= u^n(t). \end{aligned}$$

The above manner of representing a control system associated with a triple $(Q, \nabla, \mathscr{Y})$ emphasises that the system essentially evolves on the configuration manifold Q. However, since the equations (4.1) are second-order, one may also think of this as a first-order system on TQ, and so as a control affine system. Let us see what this control affine system looks like. First of all, let us state that the object in CAS will have the form $(TQ, \mathscr{F})$. That is, its state space is the tangent bundle TQ. The vector field f_0 is defined to be the geodesic spray Z corresponding to the affine connection ∇. We also need to regard the vector fields $Y_1, \ldots, Y_m$ as vector fields on TQ in the appropriate manner. To do this, given a vector field X on Q, define the ***vectical lift*** of X to be the vector field $\operatorname{vlft}(X)$ on TQ defined by

$$\operatorname{vlft}(X)(v_q) = \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} (v_q + X(q))$$

where $v_q \in T_qM$. In coordinates, if $X = X^i \frac{\partial}{\partial q^i}$, we have $\operatorname{vlft}(X) = X^i \frac{\partial}{\partial v^i}$. It will also be convenient for us to have the notation of a map $\operatorname{vlft}_{v_q} \colon T_qQ \to T_{v_q}TQ$ which sends a vector to its vertical lift. We then define $f_a = \operatorname{vlft}(Y_a)$, $a = 1, \ldots, m$. Thus, to an affine connection control system $\Sigma_{\text{aff}} = (Q, \nabla, \{Y_1, \ldots, Y_m\})$ we associate the control affine system $\Sigma = (TQ, \{Z, \operatorname{vlft}(Y_1), \ldots, \operatorname{vlft}(Y_m)\})$. The associated first-order control-affine system on TQ is then

$$\dot{v}(t) = Z(v(t)) + u^a(t) \operatorname{vlft}(Y_a(v(t))).$$

4.2. Morphisms in ACCS. Now let us look at morphisms in the category ACCS. Thus we need to specify a way to send an affine connection control system to another affine connection control system. We consider morphisms which are special forms of morphisms in CAS. This is sensible since, as we noted in the previous section, we may think of ACCS as a subset of the category CAS. We let $\mathrm{TS}^2(TQ)$ denote the bundle of symmetric $(0,2)$ tensors on Q, and we denote by $\mathbb{R}^m_Q$ the trivial vector bundle $Q\times\mathbb{R}^m$ over Q. If $S\in\mathbb{R}^m_Q\otimes\mathrm{TS}^2(TQ)$ then for $a=1,\dots,m$ we define $S^a\in\mathrm{TS}^2(TQ)$ by

$$S^a(X,Y)=S(\boldsymbol{e}_a\otimes(X,Y))$$

where $\boldsymbol{e}_a$ is the ath standard basis vector for $\mathbb{R}^m$.

We consider affine connection control systems denoted $\Sigma_{\mathrm{aff}}=(Q,\nabla,\mathscr{Y})$ and $\tilde{\Sigma}_{\mathrm{aff}}=(\tilde{Q},\tilde{\nabla},\tilde{\mathscr{Y}})$ with $\mathscr{Y}=\{Y_1,\dots,Y_m\}$ and $\tilde{\mathscr{Y}}=\{\tilde{Y}_1,\dots,\tilde{Y}_{\tilde{m}}\}$. Recall our notation that if c is a curve on Q, Y is a vector field along c, and $\phi\colon Q\to\tilde{Q}$ is a smooth map, we define a curve c_ϕ on $\tilde{Q}$ by $c_\phi=\phi\circ c$ and a vector field Y_ϕ along c by $Y_\phi=T\phi\circ Y$. An **ACCS *morphism*** sending Σ_{aff} to $\tilde{\Sigma}_{\mathrm{aff}}$ is a triple (ϕ,S,Λ) with the following properties:

ACCS1. $\phi\colon Q\to\tilde{Q}$ is a smooth mapping;

ACCS2. S is a smooth section of $\mathbb{R}^{\tilde{m}}_Q\otimes\mathrm{TS}^2(TQ)$ and $\Lambda\colon Q\to\mathrm{L}(\mathbb{R}^m;\mathbb{R}^{\tilde{m}})$ is a smooth map which together satisfy the following conditions:

(a) $T_q\phi(Y_a(q))=\Lambda^\alpha_a(q)\tilde{Y}_\alpha(\phi(q))$;

(b) $T_q\phi(\nabla_XX)_q=(\tilde{\nabla}_{\tilde{X}}\tilde{X})_{\phi(q)}+S^\alpha(X(q),X(q))\tilde{Y}_\alpha(\phi(q))$ where $\tilde{X}$ is a vector field on $\tilde{Q}$ which is ϕ-related to a vector field X on Q.

The ***identity morphism*** which sends $\Sigma_{\mathrm{aff}}=(Q,\nabla,\mathscr{Y})$ to itself is defined by $(\mathrm{id}_Q,0,q\mapsto\mathrm{id}_{\mathbb{R}^m})$. If $\Lambda(q)$ is an isomorphism for each $q\in Q$ then the ACCS morphism (ϕ,S,Λ) is called ***control nondegenerate***. If

$$T_q\phi(\operatorname{span}(Y_1(q),\dots,Y_m(q)))=\operatorname{span}(\tilde{Y}_1(\phi(q)),\dots,\tilde{Y}_{\tilde{m}}(\phi(q)))$$

for all $q\in Q$, the ACCS morphism (ϕ,S,Λ) is called ***complete***.

Let us look at what are isomorphisms in this category. Let $\Sigma_{\mathrm{aff}}=(Q,\nabla,\mathscr{Y})$ and $\tilde{\Sigma}_{\mathrm{aff}}=(\tilde{Q},\tilde{\nabla},\tilde{\mathscr{Y}})$ be two affine connection control systems, and suppose that (ϕ,S,Λ) sends Σ_{aff} to $\tilde{\Sigma}_{\mathrm{aff}}$. We say that (ϕ,S,Λ) is an ***isomorphism*** of Σ_{aff} and $\tilde{\Sigma}_{\mathrm{aff}}$ if $\phi\colon Q\to\tilde{Q}$ is a diffeomorphism and if ϕ^{-1} has the property that for every controlled trajectory $(\tilde{c},\tilde{u})$ of $\tilde{\Sigma}_{\mathrm{aff}}$, there exists an admissible input u for Σ_{aff} so that $(\tilde{c}_{\phi^{-1}},u)$ is a controlled trajectory of Σ_{aff}. The following result indicates how one may define the inverse isomorphism, and follows from Proposition 4.5 below. The symbol δ^a_b denotes the Kronecker delta.

4.2 Proposition: *If (ϕ,S,Λ) is an ACCS isomorphism which maps $\Sigma_{\mathrm{aff}}=(Q,\nabla,\mathscr{Y})$ to $\tilde{\Sigma}_{\mathrm{aff}}=(\tilde{Q},\tilde{\nabla},\tilde{\mathscr{Y}})$, then there exists an ACCS morphism $(\tilde{\phi},\tilde{S},\tilde{\Lambda})$ which satisfies the conditions*

(i) $\tilde{\phi}=\phi^{-1}$,

(ii) $\tilde{S}^a=\phi_*(S^\alpha\tilde{\Lambda}^a_\alpha)$, *and*

(iii) $\tilde{\Lambda}^a_\alpha(\tilde{q})\Lambda^\alpha_b(\phi^{-1}(\tilde{q}))=\delta^a_b$, $a,b=1,\dots,m$,

and which is an ACCS morphism which maps $\tilde{\Sigma}_{\mathrm{aff}}$ to Σ_{aff}.

4.3 Remark: Given two affine connection control systems $\Sigma_{\text{aff}} = (U, \nabla, \mathscr{Y})$ and $\tilde{\Sigma}_{\text{aff}} = (\tilde{U}, \tilde{\nabla}, \tilde{\mathscr{Y}})$, we say that they are ***locally equivalent by*** $(\boldsymbol{\phi}, \boldsymbol{S}, \boldsymbol{\Lambda})$ if for each $q \in Q$ there exists a neighbourhood U of q and a neighbourhood $\tilde{U}$ of $\phi(q)$ so that $(\phi|U, S|U, \Lambda|U)$ is an isomorphism from $\Sigma_{\text{aff}}|U$ to $\tilde{\Sigma}_{\text{aff}}|\tilde{U}$. Given an affine connection control system Σ_{aff}, one is often interested in what types of affine connection control systems are locally equivalent to Σ_{aff}. The issue of deciding what properties locally equivalent affine connection control systems ought to possess is apt to be one which is difficult to resolve. The intent here is to lay the foundation for this investigation. We note that the idea of "kinetic shaping" in [Bloch, Leonard, and Marsden 1998] can be thought of as a problem in local equivalence of affine connection control systems, with the further proviso that one wishes to stay in the subcategory of affine connection control systems whose affine connection is the Levi-Civita connection associated to a Riemannian metric. •

As a very simple example of local equivalence we have the following result.

4.4 Proposition: *If $\Sigma_{\text{aff}} = (Q, \nabla, \mathscr{Y})$ is a fully actuated affine connection control system with $\dim(Q) = n$, then for each $q \in Q$ there exists a neighbourhood U of Q, a neighbourhood $\tilde{U}$ of $0 \in \mathbb{R}^n$, and an ACCS isomorphism from $\Sigma_{\text{aff}}|U$ to $\Sigma_{\text{aff}}^{\text{can}}|\tilde{U}$.*

Proof: For $q_0 \in Q$ let (U, ϕ) be a chart around q_0 so that $\phi(q_0) = 0 \in \mathbb{R}^n$. Denote $\tilde{U} = \phi(U)$, and on $\tilde{U}$ define an affine connection $\tilde{\nabla}$ by

$$(\tilde{\nabla}_{\tilde{X}} \tilde{Y})_{\tilde{q}} = T_{\phi^{-1}(\tilde{q})}\phi(\nabla_{\phi^* \tilde{X}} \phi^* \tilde{Y})_{\phi^{-1}(\tilde{q})}.$$

Thus $\tilde{\nabla}$ is defined so as to make ϕ an affine diffeomorphism. By (2.1) there exists a smooth section $\hat{S}$ of $\mathbb{R}^n_U \otimes \mathrm{TS}^2(TU)$ so that

$$\tilde{\nabla}_{\tilde{X}} \tilde{Y} = \Big(\frac{\partial \tilde{Y}^i}{\partial \tilde{q}^j} \tilde{X}^j + \hat{S}^i(\phi^* \tilde{X}(\phi^{-1}(\tilde{q})), \phi^* \tilde{Y}(\phi^{-1}(\tilde{q}))\Big) \frac{\partial}{\partial \tilde{q}^i}.$$

We also define control vector field $\tilde{Y}_1, \dots, \tilde{Y}_m$ on $\tilde{U}$ by $\tilde{Y}_a = \phi_* Y_a$, $a = 1, \dots, m$. Since Σ_{aff} is fully actuated, the $n \times m$ matrix with components $\tilde{Y}^i_a(\tilde{q})$, $i = 1, \dots, n$, $a = 1, \dots, m$, is surjective for each $\tilde{q} \in \tilde{U}$. Therefore, for each $\tilde{q} \in \tilde{U}$ there exists an $m \times n$ functions $X^a_i(\tilde{q})$ so that $X^a_i(\tilde{q}) Y^j_b(\tilde{q}) = \delta^j_i$ for $\tilde{q} \in \tilde{U}$. It is then a straightforward matter to verify that (ϕ, S, Λ) is an ACCS isomorphism from $\Sigma_{\text{aff}}|U$ to $\Sigma_{\text{aff}}^{\text{can}}|\tilde{U}$ when $S = \hat{S}$ and $\Lambda^i_a(q) = Y^i_a(\phi(q))$. ■

4.3. Properties of ACCS morphisms.

We begin by giving an essential property for ACCS morphisms, mirroring the result Proposition 3.1 which we have for morphisms in CAS.

4.5 Proposition: *If (ϕ, S, Λ) is an ACCS morphism sending $\Sigma_{\text{aff}} = (Q, \nabla, \mathscr{Y})$ to $\tilde{\Sigma}_{\text{aff}} = (\tilde{Q}, \tilde{\nabla}, \tilde{\mathscr{Y}})$ and if (c, u) is a controlled trajectory for Σ_{aff} then $(c_\phi, \tilde{u})$ is a controlled trajectory for $\tilde{\Sigma}_{\text{aff}}$ where $\tilde{u}(t) = \Lambda(c(t))u(t) - S^\alpha(c'(t), c'(t))\tilde{Y}_\alpha(c_\phi(t))$.*

Conversely, suppose that $\phi\colon Q \to \tilde{Q}$ is a smooth mapping with the property that if (c, u) is a controlled trajectory for Σ_{aff}, then there exists an admissible input $\tilde{u}$ for $\tilde{\Sigma}_{\text{aff}}$ so that $(c_\phi, \tilde{u})$ is a controlled trajectory for $\tilde{\Sigma}_{\text{aff}}$. Then there exists a smooth section S of $\mathbb{R}^{\tilde{m}}_Q \otimes \mathrm{TS}^2(TQ)$ and a smooth mapping $\Lambda\colon Q \to \mathrm{L}(\mathbb{R}^m; \mathbb{R}^{\tilde{m}})$ so that (ϕ, S, Λ) is an ACCS morphism sending Σ_{aff} to $\tilde{\Sigma}_{\text{aff}}$.

Proof: We are given $\nabla_{c'(t)}c'(t) = u^a(t)Y_a(c(t))$. Using the properties of ACCS morphisms we compute

$$\begin{aligned}
T_{c(t)}\phi\big(\nabla_{c'(t)}c'(t) - u^a(t)Y_a(c(t))\big) &= \\
&\tilde{\nabla}_{c'_\phi(t)}c'_\phi(t) + S^\alpha_{c(t)}(c'(t), c'(t))\tilde{Y}_\alpha(c_\phi(t)) - u^a(t)T_{c(t)}\phi\big(Y_a(c(t))\big) \\
&= \tilde{\nabla}_{c'_\phi(t)}c'_\phi(t) + S^\alpha_{c(t)}(c'(t), c'(t))\tilde{Y}_\alpha(c_\phi(t)) - u^a(t)\Lambda^\alpha_a(c(t))\tilde{Y}_\alpha(c_\phi(t)) \\
&= \tilde{\nabla}_{c'_\phi(t)}c'_\phi(t) - \tilde{u}^\alpha\tilde{Y}_\alpha(c_\phi(t))
\end{aligned}$$

where $\tilde{u}(t) = \Lambda(c(t))u(t) - S^\alpha(c'(t), c'(t))\tilde{Y}_\alpha(c_\phi(t))$, as desired.

For the converse, we first look at the case of the zero input for Σ_{aff}. In this case, given $v_q \in TQ$, we let c be the geodesic which passes through v_q at $t = 0$. Our hypotheses guarantee the existence of an admissible input $\tilde{u}_0$ for $\tilde{\Sigma}_{\text{aff}}$ so that

$$\tilde{\nabla}_{c'_\phi(t)}c'_\phi(t) = \tilde{u}^\alpha_0(t)\tilde{Y}_\alpha(c_\phi(t)).$$

Since c is a geodesic, we may write

$$\tilde{\nabla}_{c'_\phi(t)}c'_\phi(t) - T_{c(t)}\phi(\nabla_{c'(t)}c'(t)) = \tilde{u}^\alpha_0(t)\tilde{Y}_\alpha(c_\phi(t)). \tag{4.2}$$

As a consequence of Lemma 2.2, the expression on the left-hand side is bilinear in $(c'(t), c'(t))$. Therefore, since the right-hand side is a vector field along c_ϕ taking values in $\operatorname{span}_{C^\infty(\tilde{Q})}\big(\tilde{Y}_1, \dots, \tilde{Y}_{\tilde{m}}\big)$, there exists a tensor $S_{c(t)} \in \mathbb{R}^{\tilde{m}} \otimes \mathrm{TS}^2(T_{c(t)}Q)$ so that

$$S^\alpha_{c(t)}(c'(t), c'(t))\tilde{Y}_\alpha(c_\phi(t)) = \tilde{u}^\alpha_0(t)\tilde{Y}_\alpha(c_\phi(t)).$$

In particular, at $t = 0$ we have

$$S^\alpha_q(v_q, v_q)\tilde{Y}_\alpha(\phi(q)) = \tilde{u}^\alpha_0(0)\tilde{Y}_\alpha(\phi(q)).$$

Since the terms in (4.2) vary smoothly as we vary v_q, the mapping $q \mapsto S_q$ defines a smooth section of $\mathbb{R}^{\tilde{m}}_Q \otimes \mathrm{TS}^2(TQ)$. Next we look at the situation when the input for Σ_{aff} is the constant input $u(t) = \boldsymbol{e}_a$. In this case we have a curve c through $v_q \in TQ$ satisfying $\nabla_{c'(t)}c'(t) = Y_a(c(t))$. Our hypotheses assert the existence of an admissible input $\tilde{u}_a$ for $\tilde{\Sigma}_{\text{aff}}$ so that

$$\begin{aligned}
&\tilde{\nabla}_{c'_\phi(t)}c'_\phi(t) = \tilde{u}^\alpha_a(t)\tilde{Y}_\alpha(c_\phi(t)) \\
\implies\quad &\tilde{\nabla}_{c'_\phi(t)}c'_\phi(t) - T_{c(t)}\phi(\nabla_{c'(t)}c'(t)) + T_{c(t)}\phi(Y_a(c(t))) = \tilde{u}^\alpha_a(t)\tilde{Y}_\alpha(c_\phi(t)) \\
\implies\quad &S^\alpha(c'(t), c'(t))\tilde{Y}_\alpha(c_\phi(t)) + T_{c(t)}\phi(Y_a(c(t))) = \tilde{u}^\alpha_a(t)\tilde{Y}_\alpha(c_\phi(t)).
\end{aligned}$$

Evaluating this at $t = 0$ gives

$$S^\alpha(v_q, v_q)\tilde{Y}_\alpha(\phi(q)) + T_q\phi(Y_a(q)) = \tilde{u}^\alpha_a(0)\tilde{Y}_\alpha(\phi(q))$$

Since v_q can be selected as an arbitrary vector in T_qQ, this implies that $\tilde{u}_a(0)$ is a sum of two components, one which is bilinear in (v_q, v_q) (let us denote this by $\tilde{v}_a(v_q)$) and another which is independent of the velocity v_q, and only depends on the configuration q (let us denote this by $\tilde{w}_a(q)$). The bilinear component must then be $\tilde{v}_a(v_q) = S^\alpha(v_q, v_q)\tilde{Y}_\alpha(\phi(q))$, leaving the term independent of velocity to satisfy

$$T_q\phi(Y_a(q)) = \tilde{w}_a^\alpha(q)\tilde{Y}_\alpha(\phi(q)).$$

Let us define $\Lambda(q) \in \mathrm{L}(\mathbb{R}^m; \mathbb{R}^{\tilde{m}})$ by $\Lambda_a^\alpha(q) = \tilde{w}_a^\alpha(q)$. As usual, we may choose Λ so that $q \mapsto \Lambda(q)$ is smooth. With the S and Λ we have defined, one then easily verifies that (ϕ, S, Λ) is an ACCS morphism sending Σ_{aff} to $\tilde{\Sigma}_{\mathrm{aff}}$. ■

Since we can think of ACCS as a subcategory of CAS, it follows that ACCS morphisms can be realised as CAS morphisms. This is easy to do, and the following result states the resulting correspondence.

4.6 Proposition: *Let $\Sigma_{\mathrm{aff}} = (Q, \nabla, \mathscr{Y})$ and $\tilde{\Sigma}_{\mathrm{aff}} = (\tilde{Q}, \tilde{\nabla}, \tilde{\mathscr{Y}})$ be affine connection control systems with $\Sigma = (TQ, \mathscr{F})$ and $\tilde{\Sigma} = (T\tilde{Q}, \tilde{\mathscr{F}})$ the corresponding control affine systems. If (ϕ, S, Λ) is an ACCS morphism sending Σ_{aff} to $\tilde{\Sigma}_{\mathrm{aff}}$, then $(\psi, \lambda_0, \Lambda')$ is a CAS morphism sending Σ to $\tilde{\Sigma}$ where*

(i) $\psi = T\phi$,

(ii) $\lambda_0^\alpha(v_q) = S^\alpha(v_q, v_q)$, *and*

(iii) $\Lambda'(v_q) = \Lambda(q)$.

The converse question here is not so clear. That is, if one has a CAS morphism $(\psi, \lambda_0, \Lambda')$ sending an object in ACCS $\subset$ CAS to another object in ACCS $\subset$ CAS, is it necessarily the case that $(\psi, \lambda_0, \Lambda')$ is derived from an ACCS morphism as described in Proposition 4.6? The following example answers the question in the negative.

4.7 Example: We take $Q = \tilde{Q} = \mathbb{R}^2$. The standard coordinates for Q will be denoted (q^1, q^2) and for $\tilde{Q}$ will be denoted $(\tilde{q}^1, \tilde{q}^2)$. Canonical tangent bundle coordinates for TQ and $T\tilde{Q}$ are denoted (q^1, q^2, v^1, v^2) and $(\tilde{q}^1, \tilde{q}^2, \tilde{v}^1, \tilde{v}^2)$, respectively. As in Example 2.1 we write vector fields in terms of their principal parts, and so on Q we define the affine connection ∇ by

$$(\nabla_X Y)_q = (q, \boldsymbol{D}\boldsymbol{Y}(q) \cdot \boldsymbol{X}(q)).$$

We let $\tilde{\nabla}$ be the same affine connection on $T\tilde{Q}$:

$$\tilde{\nabla}_{\tilde{X}}\tilde{Y} = (\tilde{q}, \boldsymbol{D}\tilde{\boldsymbol{Y}}(\tilde{q}) \cdot \tilde{\boldsymbol{X}}(\tilde{q})).$$

We consider a single input for these systems defined again by the same vector field; we take as input vector fields

$$Y_1(q) = (q, (1, 0)), \quad \tilde{Y}_1 = (\tilde{q}, (1, 0)),$$

on Q and $\tilde{Q}$, respectively. Thus we have defined two identical single-input affine connection control systems, $\Sigma_{\mathrm{aff}} = (Q, \nabla, \{Y_1\})$ and $\tilde{\Sigma}_{\mathrm{aff}} = (\tilde{Q}, \tilde{\nabla}, \{\tilde{Y}_1\})$.

One readily determines that the corresponding control affine systems are $\Sigma = (TQ, \{f_0, f_1\})$ and $\tilde{\Sigma} = (T\tilde{Q}, \{\tilde{f}_0, \tilde{f}_1\})$ where

$$f_0(q, v) = ((q, v), (v, 0)), \quad f_1(q, v) = ((q, v), (0, 0, 1, 0)),$$
$$\tilde{f}_0(\tilde{q}, \tilde{v}) = ((\tilde{q}, \tilde{v}), (\tilde{v}, 0)), \quad \tilde{f}_1(\tilde{q}, \tilde{v} = ((\tilde{q}, \tilde{v}), (0, 0, 1, 0)).$$

We define claim that the triple $(\psi, \lambda_0, \Lambda')$ is a CAS morphism sending Σ to $\tilde{\Sigma}$ when ψ is defined by

$$\psi(q^1, q^2, v^1, v^2) = (q^1, q^2 + v^2, v^1, v^2),$$

λ_0 is defined by $\lambda_0(q, v) = 0$, and Λ' is defined by $\Lambda'(q, v) = 1$. To check this we note that

$$T\psi((q, v), (u, w)) = ((q^1, q^2 + v^2, v^1, v^2), (u^1, u^2 + w^2, w^1, w^2)),$$

and so we readily compute

$$\begin{aligned} T\psi(f_0(q, v)) &= ((q^1, q^2 + v^2, v^1, v^2), (v^1, v^2, 0, 0)), \\ T\psi(f_1(q, v)) &= ((q^1, q^2 + v^2, v^1, v^2), (0, 0, 1, 0)). \end{aligned}$$

We also readily compute

$$\begin{aligned} \tilde{f}_0(\psi(q, v)) &= ((q^1, q^2 + v^2, v^1, v^2), (v^1, v^2, 0, 0)), \\ \tilde{f}_1(\psi(q, v)) &= ((q^1, q^2 + v^2, v^1, v^2), (0, 0, 1, 0)). \end{aligned}$$

Thus we see that

$$\begin{aligned} T\psi(f_0(q, v)) &= \tilde{f}_0(\psi(q, v)) + \lambda_0 \tilde{f}_1(\psi(q, v)), \\ T\psi(f_1(q, v)) &= \Lambda' \tilde{f}_1(\psi(q, v)), \end{aligned}$$

and so $(\psi, \lambda_0, \Lambda')$ is a CAS morphism sending Σ to $\tilde{\Sigma}$ as claimed.

However, we note that ψ is not a bundle mapping, and so in particular cannot be of the form $\psi = T\phi$ for some mapping $\phi\colon Q \to \tilde{Q}$. Therefore, there is no ACCS morphism which gives rise to the CAS morphism $(\psi, \lambda_0, \Lambda')$ in the manner described in Proposition 4.6. •

Thus ACCS morphisms are indeed a smaller class than are CAS morphisms. What's more, since the previous example exhibits a CAS morphism $(\psi, \lambda_0, \Lambda')$ for which ψ is not even a bundle mapping, there appears to be little hope of obtaining a nice description of CAS morphisms which map affine connection control systems to other affine connection control systems. On these grounds, we propose that ACCS morphisms are useful entities to study when looking at how one transforms affine connection control systems.

The following result gives a property possessed by an ACCS morphism which is not necessarily possessed by a CAS morphism.

4.8 Proposition: *Let $\Sigma_{\mathrm{aff}} = (Q, \nabla, \mathscr{Y})$ and $\tilde{\Sigma}_{\mathrm{aff}} = (\tilde{Q}, \tilde{\nabla}, \tilde{\mathscr{Y}})$ be affine connection control systems. Suppose that (ϕ, S, Λ) is an ACCS morphism which maps Σ to $\tilde{\Sigma}_{\mathrm{aff}}$, and that $\Lambda(q) \in \mathrm{L}(\mathbb{R}^m; \mathbb{R}^{\tilde{m}})$ is an epimorphism for each $q \in Q$ with right inverse denoted by $\Theta(q)$. On Q define an affine connection $\overline{\nabla}$ by*

$$(\overline{\nabla}_X Y)_q = (\nabla_X Y)_q - S^\alpha(X(q), Y(q))\Theta^a_\alpha(q) Y_a(\phi(q)).$$

Then $\phi\colon Q \to \tilde{Q}$ is a totally geodesic mapping between $\overline{\nabla}$ and $\tilde{\nabla}$. Furthermore, there exists an ACCS isomorphism from Σ_{aff} to $\overline{\Sigma}_{\mathrm{aff}} = (Q, \overline{\nabla}, \mathscr{Y})$.

Proof: Let X be a vector fields on Q with $\tilde{X}$ a vector field on $\tilde{Q}$ which is ϕ-related to X. We compute

$$\begin{aligned}
T_q\phi((\overline{\nabla}_X X)_q) &= T_q\phi((\nabla_X X)_q - S^\alpha(X(q),X(q))\Theta^a_\alpha(q)Y_a(\phi(q))) \\
&= T_q\phi(\nabla_X X)_q - S^\alpha(X(q),X(q))\Theta^a_\alpha(q)T_q\phi(Y_a(q)) \\
&= (\tilde{\nabla}_{\tilde{X}}\tilde{X})_{\phi(q)} + S^\alpha(X(q),X(q))\tilde{Y}_\alpha(\phi(q)) - S^\alpha(X(q),X(q))\tilde{Y}_\alpha(\phi(q)) \\
&= (\tilde{\nabla}_{\tilde{X}}\tilde{X})_{\phi(q)}.
\end{aligned}$$

By the definition of a totally geodesic mapping, the first part of the proposition follows.

For the second assertion, consider the morphism the section $\bar{S}$ of $\mathbb{R}^m \otimes \mathrm{TS}^2(TQ)$ defined by

$$\bar{S}^a(q) = S^\alpha(q)\Theta^a_\alpha(q).$$

A straightforward computation verifies that this makes $(\mathrm{id}_{\mathbb{R}}, \bar{S}, q \mapsto \mathrm{id}_{\mathbb{R}^m})$ an ACCS morphism which sends Σ_{aff} to $\overline{\Sigma}_{\mathrm{aff}}$. ■

4.9 Remarks: 1. The second assertion of the above proposition is really very simple, of course. The key point is that the $(1,2)$ tensor $S^\alpha(q)\Theta^a_\alpha(q)Y_a(q)$ takes its values in the distribution spanned by the controls, and so when we subtract this from ∇ to get the affine connection $\overline{\nabla}$, the geodesics of the resulting affine connection can be following by controlled trajectories of Σ_{aff}.

2. In the next section we shall call the morphism of the second part of the result a CACCS morphism.

3. The surjectivity of Λ in the above result is essential. One may weaken somewhat this condition by instead requiring that the morphism (ϕ, S, Λ) simply be complete, at least in the case when $\tilde{\Sigma}_{\mathrm{aff}}$ is regular. In this case we can always find a local basis $\mathscr{Y}' = \{Y'_1, \dots, Y'_{m'}\}$ of vector fields for the distribution spanned by the vector fields $\tilde{\mathscr{Y}}$. One can then locally replace $\tilde{\mathscr{Y}}$ with $\mathscr{Y}'$, and in so doing ensure that Λ is surjective. •

4.4. Compositions and decompositions of ACCS morphisms. We now wish to determine conditions under which a morphism in ACCS can be written as a product of two simpler ACCS morphisms. Obviously, to do this we need to say how one forms the product of ACCS morphisms.

4.10 Proposition: *Let $\Sigma_{\mathrm{aff},1} = (Q^1, \nabla^1, \mathscr{Y}^1)$, $\Sigma_{\mathrm{aff},2} = (Q^2, \nabla^2, \mathscr{Y}^2)$, and $\Sigma_{\mathrm{aff},3} = (Q^3, \nabla^3, \mathscr{Y}^3)$ be affine connection control systems, and let (ϕ_1, S_1, Λ_1) and (ϕ_2, S_2, Λ_2) be ACCS morphisms which send $\Sigma_{\mathrm{aff},1}$ to $\Sigma_{\mathrm{aff},2}$ and $\Sigma_{\mathrm{aff},2}$ to $\Sigma_{\mathrm{aff},3}$, respectively. Then $(\phi_{21}, S_{21}, \Lambda_{21})$ is an ACCS morphism sending $\Sigma_{\mathrm{aff},1}$ to $\Sigma_{\mathrm{aff},3}$ where*

(i) $\phi_{21} = \phi_2 \circ \phi_1$,

(ii) $S^\sigma_{21}(X(q), Y(q)) = S^\sigma_2(T_q\phi_1(X(q)), T_q\phi_1(Y(q))) + S^\alpha_1(X(q), Y(q))(\Lambda_2)^\sigma_\alpha(\phi_1(q))$, *and*

(iii) $(\Lambda_{21})^\sigma_a(q) = (\Lambda_1)^\sigma_\alpha(q)(\Lambda_2)^\sigma_\alpha(\phi_1(q))$.

The ACCS morphism $(\phi_{21}, S_{21}, \Lambda_{21})$ is called the ***composition*** *of (ϕ_1, S_1, Λ_1) with (ϕ_2, S_2, Λ_2).*

Proof: Let X_1 be a vector field on Q^1 with X_2 a vector field on Q^2 which is ϕ_1-related to X_1, and with X_2 a vector field on Q^3 which is ϕ_2-related to X_2. One then checks that with $(\phi_{21}, S_{21}, \Lambda_{21})$ defined as in (i)-(iii) we have

$$T_q\phi_{21}(\nabla^1_{X_1}X_1)_q = (\nabla^3_{X_3}X_3)_{\phi_{21}(q)} + S^\sigma_{21}(X_1(q), X_1(q))Y^3_\sigma(\phi_{21}(q)),$$

and

$$T_q\phi_{21}(Y^1_a(q)) = (\Lambda_{21})^\sigma_a(q)Y^3_\sigma(\phi_{12}(q)),$$

where one uses the definitions of ACCS morphisms. ■

One can verify that this notion of composition has the property that if one composes an ACCS morphism with its inverse as given in Proposition 4.2, the resulting ACCS morphism is the identity morphism.

Now let us define the special classes of ACCS morphisms one may consider. An ACCS morphism (ϕ, S, Λ) which maps $\Sigma_{\mathrm{aff}} = (Q, \nabla, \mathscr{Y})$ to $\tilde{\Sigma}_{\mathrm{aff}} = (\tilde{Q}, \tilde{\nabla}, \tilde{\mathscr{Y}})$ is a ***morphism over controls*** if $Q \subset \tilde{Q}$ and if $\phi\colon Q \to \tilde{Q}$ is the inclusion map. The category whose objects are affine connection control systems and whose morphisms are ACCS morphisms which are morphisms over controls we denote by CACCS. The idea is that a morphism over controls does essentially nothing to the system's states, and alters the only controls. Moreover, a morphism over controls is an algebraic operation since one only alters the controls by a map which is affine in control.

An ACCS morphism (ϕ, S, Λ) is a ***morphism over configurations*** if $S_q = 0$ and $\Lambda(q) = \mathrm{id}_{\mathbb{R}^m}$ for each $q \in Q$. We denote by QACCS the category whose objects are affine connection control systems and whose morphisms are ACCS morphisms which are morphisms over configurations. The idea here is that one leaves the controls alone, and alters only the configuration spaces. The following result is clear.

4.11 Proposition: *A triple (ϕ, S, Λ) is a QACCS morphism mapping $\Sigma_{\mathrm{aff}} = (Q, \nabla, \mathscr{Y})$ to $\tilde{\Sigma}_{\mathrm{aff}} = (\tilde{Q}, \tilde{\nabla}, \tilde{\mathscr{Y}})$ if and only if the following two conditions hold:*

(i) $\phi\colon Q \to \tilde{Q}$ is a totally geodesic mapping between ∇ and $\tilde{\nabla}$;

(ii) each control vector field $\tilde{Y}_a$ on $\tilde{Q}$ is ϕ-related to the control vector field Y_a on Q.

4.12 Remark: When one is working with Levi-Civita connections, the discussion surrounding QACCS morphisms acquires more structure. For example, one can add the requirement that the mappings ϕ comprising a QACCS morphism are isometries of the underlying Riemannian metrics. In this case, Vilms [1970] shows that the mapping ϕ can be decomposed into the product of a totally geodesic immersion with a submersion which is an isometry. •

Let us give a few simple results concerning decompositions of ACCS morphisms. The first result deals with the case when the transformation of inputs is invertible.

4.13 Proposition: *A control nondegenerate ACCS morphism (ϕ, S, Λ) is a composition of a CACCS isomorphism with a QACCS morphism.*

Proof: Let $\Theta(q)$ denote the inverse of $\Lambda(q)$ for $q \in Q$. If (ϕ, S, Λ) maps $\Sigma_{\mathrm{aff}} = (Q, \nabla, \mathscr{Y})$ to $\tilde{\Sigma}_{\mathrm{aff}} = (\tilde{Q}, \tilde{\nabla}, \tilde{\mathscr{Y}})$, then define a CACCS morphism (ϕ_1, S_1, Λ_1) by $\phi_1 = \mathrm{id}_Q$, $S^a_1(q) = S^b(q)\Theta^a_b(q)$, and $\Lambda_1 = \Lambda$. Note that by Proposition 4.8, (ϕ_1, S_1, Λ_1) is a morphism which sends Σ_{aff} to the affine connection control system $\Sigma'_{\mathrm{aff}} = (Q, \nabla', \mathscr{Y}')$ where

$$(\nabla'_X Y)_q = (\nabla_X Y)_q - S^a(X(q), Y(q))\Theta^b_a(q)Y_b(q),$$

and where $Y_b'(q) = \Theta_b^a(q)Y_b(q)$. Next we define a QACCS morphism (ϕ_2, S_2, Λ_2) by asking that $\phi_2 = \phi$, $S_2 = 0$, and $\Lambda_2(q) = \mathrm{id}_{\mathbb{R}^m}$, $q \in Q$. It is straightforward to see that (ϕ_2, S_2, Λ_2) maps Σ'_{aff} to $\tilde{\Sigma}_{\mathrm{aff}}$. Therefore, the composition of (ϕ_1, S_1, Λ_1) with (ϕ_2, S_2, Λ_2) is an ACCS morphism which maps Σ_{aff} to $\tilde{\Sigma}_{\mathrm{aff}}$. ■

In the previous result, the inputs were assumed to be in 1-1 correspondence for Σ_{aff} and $\tilde{\Sigma}_{\mathrm{aff}}$. Now we look at the situation when the configurations are in 1-1 correspondence.

4.14 Proposition: *An ACCS isomorphism (ϕ, S, Λ) is a composition of a QACCS isomorphism with a CACCS isomorphism.*

Proof: We suppose that (ϕ, S, Λ) maps $\Sigma_{\mathrm{aff}} = (Q, \nabla, \mathscr{Y})$ to $\tilde{\Sigma}_{\mathrm{aff}} = (\tilde{Q}, \tilde{\nabla}, \tilde{\mathscr{Y}})$. We first define a QACCS isomorphism by (ϕ_1, S_1, Λ_1) where $\phi_1 = \phi$, $S_1 = 0$, and $\Lambda_1(q) = \mathrm{id}_{\mathbb{R}^m}$, $q \in Q$. One verifies that (ϕ_1, S_1, Λ_1) maps Σ_{aff} to $\Sigma'_{\mathrm{aff}} = (\tilde{Q}, \nabla', \mathscr{Y}')$ where

$$(\nabla'_{\tilde{X}}\tilde{Y})_{\tilde{q}} = (\tilde{\nabla}_{\tilde{X}}\tilde{Y})_{\tilde{q}} + S^\alpha(T_{\tilde{q}}\phi^{-1}(\tilde{X}(\tilde{q}))), T_{\tilde{q}}\phi^{-1}(\tilde{Y}(\tilde{q}))\tilde{Y}_\alpha(\tilde{q}),$$

and $Y_a'(\tilde{q}) = \Lambda_a^\alpha(\phi^{-1}(\tilde{q}))\tilde{Y}_\alpha(\tilde{q})$. Now one defines a CACCS morphism (ϕ_2, S_2, Λ_2) by letting $\phi_2 = \mathrm{id}_{\tilde{Q}}$, $S_2^\alpha = \phi_* S^\alpha$, and $\Lambda_2(\tilde{q}) = \Lambda(\phi^{-1}(\tilde{q}))$. With these definitions, it is a simple matter to verify that (ϕ_2, S_2, Λ_2) maps Σ'_{aff} to $\tilde{\Sigma}_{\mathrm{aff}}$, and so the composition of (ϕ_1, S_1, Λ_1) with (ϕ_2, S_2, Λ_2) maps Σ_{aff} to $\tilde{\Sigma}_{\mathrm{aff}}$ as desired. ■

The final result we give in this section deals with a somewhat more general situation, but in consequence we sacrifice a global decomposition of the morphism.

4.15 Proposition: *Let $\Sigma_{\mathrm{aff}} = (Q, \nabla, \mathscr{Y})$ and $\tilde{\Sigma}_{\mathrm{aff}} = (\tilde{Q}, \tilde{\nabla}, \tilde{\mathscr{Y}})$ be affine connection control systems, and suppose that (ϕ, S, Λ) is a complete ACCS morphism and that $m = \tilde{m}$. Then for each $q \in Q$ there is a neighbourhood U of q and a neighbourhood $\tilde{U}$ of $\phi(q)$ so that the morphism $(\phi|U, S|U, \Lambda|U)$ from $\Sigma_{\mathrm{aff}}|U$ to $\tilde{\Sigma}_{\mathrm{aff}}|\tilde{U}$ is the composition of a CACCS isomorphism and a QACCS morphism.*

Proof: Since (ϕ, S, Λ) is complete and since $m = \tilde{m}$, around any point $q \in Q$ we may define vector fields $\mathscr{Y}' = \{Y_1', \dots, Y_m'\}$ defined on a neighbourhood U with the property that $T_q\phi(Y_a'(q)) = \tilde{Y}_a(\phi(q))$, $a = 1, \dots, m$, $q \in U$. We also define an affine connection ∇' on U by

$$(\nabla'_X Y)_q = (\nabla_X Y)_q - S^a(X(q), Y(q))Y_a'(q).$$

With these objects, we define an affine connection control system $\Sigma'_{\mathrm{aff}} = (U, \nabla', \mathscr{Y}')$. Defining (ϕ_1, S_1, Λ_1) so that $\phi_1 = \mathrm{id}_U$, $S_1 = S|U$, and $\Lambda_1 = \Lambda|U$, we may easily check that (ϕ_1, S_1, Λ_1) is a CACCS isomorphism which maps $\Sigma_{\mathrm{aff}}|U$ to Σ'_{aff}. Also, if we define (ϕ_2, S_2, Λ_2) by $\phi_2 = \phi$, $S_2 = 0$, and $\Lambda_2(q) = \mathrm{id}_{\mathbb{R}^m}$ for $q \in Q$, we readily see that (ϕ_2, S_2, Λ_2) is a QACCS morphism which maps Σ'_{aff} to $\tilde{\Sigma}_{\mathrm{aff}}$. Our result now holds for any neighbourhood $\tilde{U}$ of $\phi(q)$ for which $\phi(U) \subset \tilde{U}$. ■

4.16 Remark: In each of the above results concerning decomposition of ACCS morphisms, we were concerned with decomposing a morphism into a product of a CACCS morphism with a QACCS morphism, or vice versa. The idea is that one breaks the study of morphisms into a part which concerns only algebraic operations on controls (CACCS morphisms) and a part which concerns mappings (QACCS morphisms). •

5. Restricted systems

Let us now turn to the question of describing ACCS morphisms (ϕ, S, Λ) for which the map ϕ has certain properties. Our program here mirrors that of [Elkin 1998], but we have to take into account the special structure of affine connection control systems. We begin with a description of the situation when the dynamics of one affine connection control system are “contained in” the dynamics of another.

5.1. Restrictions of affine connection control systems and invariance. An affine connection control system $\Sigma_{\mathrm{aff}} = (Q, \nabla, \mathscr{Y})$ is a ***subsystem*** in the category ACCS of another affine connection control system $\tilde{\Sigma}_{\mathrm{aff}} = (\tilde{Q}, \tilde{\nabla}, \tilde{\mathscr{Y}})$ if there is an ACCS morphism (ϕ, S, Λ) for which $\phi\colon Q \to \tilde{Q}$ is an embedding. We say the affine connection control system $\tilde{\Sigma}_{\mathrm{aff}} = (\tilde{Q}, \tilde{\nabla}, \tilde{\mathscr{Y}})$ ***restricts*** in the category ACCS to a submanifold $N \subset \tilde{Q}$ if there exists an affine connection control system $\Sigma_{\mathrm{aff}} = (Q, \nabla, \mathscr{Y})$ which is an ACCS subsystem of $\tilde{\Sigma}_{\mathrm{aff}}$ and where $N = \mathrm{image}(\phi)$. The idea is that the controlled dynamics of a subsystem can be contained in those of the full system. We also have the notion of subsystems in the category CACCS and QACCS by considering morphisms which are further restricted to be morphisms in the respective categories.

In the category QACCS subsystems have a very particular structure.

5.1 Proposition: *An affine connection control system $\tilde{\Sigma}_{\mathrm{aff}} = (\tilde{Q}, \tilde{\nabla}, \tilde{\mathscr{Y}})$ restricts in the category QACCS to a submanifold $N \subset \tilde{Q}$ if and only if the following two conditions are satisfied:*

(i) N is a totally geodesic submanifold of $\tilde{Q}$;

(ii) the vector fields in $\tilde{\mathscr{Y}}$ are all tangent to N.

Proof: First suppose that $\tilde{\Sigma}_{\mathrm{aff}}$ restricts in QACCS to a subsystem Σ_{aff} via the QACCS morphism $(\phi, 0, q \mapsto \mathrm{id}_{\mathbb{R}^m})$. Then all geodesics of ∇ must be mapped to geodesics of $\tilde{\nabla}$ by ϕ. Since ϕ is a diffeomorphism onto its image, this implies that all geodesics of $\tilde{\nabla}$ which are somewhere tangent to N are everywhere tangent to N. Thus (i) holds. By the definition of a QACCS morphism we must also have $\tilde{Y}_a(\tilde{q}) = T_{\phi^{-1}(\tilde{q})}\phi(Y_a(\phi^{-1}(\tilde{q})))$, from which follows (ii).

For the converse, suppose that $\tilde{\Sigma}_{\mathrm{aff}}$ satisfies (i) and (ii). In this case we can define $Q = N$ and we note that (i) and (ii) imply that $\nabla = \tilde{\nabla}|Q$ and $\mathscr{Y} = \tilde{\mathscr{Y}}|N$ are well-defined. Then it is easy to see that if one takes $\phi\colon N \to \tilde{Q}$ to be inclusion, the QACCS morphism $(\phi, 0, q \mapsto \mathrm{id}_{\mathbb{R}^m})$ renders $\Sigma_{\mathrm{aff}} = (Q, \nabla, \mathscr{Y})$ a subsystem of $\tilde{\Sigma}_{\mathrm{aff}}$. ■

Note that if an affine connection control system $\tilde{\Sigma}_{\mathrm{aff}} = (\tilde{Q}, \tilde{\nabla}, \tilde{\mathscr{Y}})$ admits a restriction to $N \subset \tilde{Q}$ in either of the categories ACCS or CACCS, this does not imply that the control system $\tilde{\Sigma}_{\mathrm{aff}}$ leaves invariant the submanifold $\mathrm{image}(\phi) \subset \tilde{Q}$. Indeed, it is easy to construct examples of systems $\tilde{\Sigma}_{\mathrm{aff}} = (\tilde{Q}, \tilde{\nabla}, \tilde{\mathscr{Y}})$ which possess ACCS or QACCS subsystems on a submanifold of $N \subset \tilde{Q}$, but where the dynamics of the affine connection control system $\tilde{\Sigma}_{\mathrm{aff}}$ do not leave N invariant. Thus we introduce the notion of invariance. For an affine connection control system $\tilde{\Sigma}_{\mathrm{aff}} = (\tilde{Q}, \tilde{\nabla}, \tilde{\mathscr{Y}})$, a submanifold $N \subset \tilde{Q}$ is ***invariant*** if the properties (i) and (ii) of Proposition 5.1 are satisfied.

The following result indicates that the term “invariant” is justified as we have used it.

5.2 Proposition: *A manifold N is an invariant manifold for $\tilde{\Sigma}_{\text{aff}} = (\tilde{Q}, \tilde{\nabla}, \tilde{\mathscr{Y}})$ if and only if any controlled trajectory (c, u) for $\tilde{\Sigma}_{\text{aff}}$ which has the property that $c'(t_0) \in T_{c(t_0)}N$ for some t_0 on the domain of definition of c also has the property that $c'(t) \in T_{c(t)}N$ for every t in the domain of definition of c.*

Proof: Let $\tilde{\Sigma} = (T\tilde{Q}, \tilde{\mathscr{F}})$ be the control affine system corresponding to $\tilde{\Sigma}_{\text{aff}}$, and suppose that conditions (i) and (ii) hold. If (i) holds, then this implies that $TN \subset TQ$ is an invariant manifold for the geodesic spray for $\tilde{\nabla}$. Condition (ii) on the other hand implies that $\operatorname{vlft}(\tilde{Y}_\alpha)$ is tangent to TN. Thus TN is an invariant manifold for the control affine system $\tilde{\Sigma}$. This means that N is invariant under all curves which are projections from TN to N of controlled trajectories for $\tilde{\Sigma}|TN$. However, these projected curves are precisely the controlled trajectories for $\tilde{\Sigma}_{\text{aff}}$ whose initial conditions are tangent to N. Thus we have shown that any controlled trajectory of $\tilde{\Sigma}_{\text{aff}}$ which starts tangent to N remains tangent to N.

Now suppose that every controlled trajectory which starts tangent to N remains tangent to N. In particular, every geodesic of $\tilde{\nabla}$ which starts tangent to N remains tangent to N. Thus N is totally geodesic. By Proposition 2.4 this implies that $\nabla_{c'(t)}c'(t) \in T_{c(t)}N$ for every curve c which is tangent to N. This also means that for $a = 1, \dots, m$, $\tilde{\nabla}_{c'(t)}c'(t) - \tilde{Y}_\alpha(c(t)) \in T_{c(t)}N$ for any curve c which is tangent to N, which implies that $Y_a(c(t)) \in T_{c(t)}N$ for any curve c which is tangent to N. This means that for $a = 1, \dots, m$ the vector field $\tilde{Y}_a$ is tangent to N, and so N is then an invariant manifold for $\tilde{\Sigma}_{\text{aff}}$. ■

In the category QACCS it is clear that restriction and invariance are indistinguishable notions.

Let us determine the manner in which we can factor morphisms which give rise to restrictions in ACCS.

5.3 Proposition: *Let $\Sigma_{\text{aff}} = (Q, \nabla, \mathscr{Y})$ and $\tilde{\Sigma}_{\text{aff}} = (\tilde{Q}, \tilde{\nabla}, \tilde{\mathscr{Y}})$ be affine connection control systems with Σ_{aff} a subsystem of $\tilde{\Sigma}_{\text{aff}}$ via the ACCS morphism (ϕ, S, Λ). Then there exists a neighbourhood $\tilde{U}$ of $N = \operatorname{image}(\phi)$ in $\tilde{Q}$ and a CACCS subsystem $\Sigma'_{\text{aff}} = (Q', \nabla', \mathscr{Y}')$ of $\tilde{\Sigma}_{\text{aff}}|\tilde{U}$ with the property that there is a QACCS morphism $(\phi', 0, q \mapsto \mathrm{id}_{\mathbb{R}^m})$ which renders Σ_{aff} a QACCS subsystem of Σ'_{aff}.*

Proof: If we let $\tilde{U}$ be a tubular neighbourhood of N, we may regard $\tilde{U}$ as an open subset of the zero section of a vector bundle $\pi \colon E \to N$. We make this identification, and write points in $\tilde{U}$ as $e_{\tilde{q}}$ for some $\tilde{q} \in N$. We also let $HE \subset TE$ be a linear connection on $\pi \colon E \to N$ which allows us to define a complement to $\ker(T_{e_{\tilde{q}}}\pi)$ for each $e_{\tilde{q}} \in E$. Recall that such connections always exist, and that they have the property that $HE|N = TN$ [Kolář, Michor, and Slovák 1993, §11.10]. We call vectors ***vertical*** which are in $\ker(T_{e_{\tilde{q}}}\pi)$ and ***horizontal*** which are in $H_{e_{\tilde{q}}}E$.

To prove the result, it suffices to find the following objects:

1. an affine connection ∇' on $\tilde{U}$;
2. a family $\mathscr{Y}' = \{Y'_1, \dots, Y'_m\}$ of vector fields on $\tilde{U}$;
3. a smooth section $\tilde{S}$ of $\mathbb{R}^{\tilde{m}}_{\tilde{U}} \otimes \mathrm{TS}^2(T\tilde{U})$;
4. a smooth map $\tilde{\Lambda} \colon \tilde{U} \to \mathrm{L}(\mathbb{R}^m; \mathbb{R}^{\tilde{m}})$

with the properties

5. $\Sigma'_{\mathrm{aff}} = (\tilde{U}, \nabla', \mathscr{Y}')$ is a CACCS subsystem of $\tilde{\Sigma}_{\mathrm{aff}}|\tilde{U}$ via the CACCS morphism $(\tilde{\phi}, \tilde{S}, \tilde{\Lambda})$ where $\tilde{\phi}$ is the inclusion of $\tilde{U}$ in $\tilde{Q}$;

6. N is an invariant manifold for Σ'_{aff}.

To this end, for $e_q \in \tilde{U}$ we define $\tilde{S}^\alpha$ by

$$\tilde{S}^\alpha_{e_{\tilde{q}}}(V_1, V_2) = \begin{cases} 0, & V_1 \text{ and } V_2 \text{ are vertical} \\ S^\alpha_{\phi^{-1}(\tilde{q})}(T_{e_{\tilde{q}}}\pi(V_1), T_{e_{\tilde{q}}}\pi(V_2)), & V_1 \text{ and } V_2 \text{ are horizontal.} \end{cases}$$

Since $\tilde{S}^\alpha$ is symmetric, this suffices to define it for general vectors. We also define $\tilde{\Lambda}(e_{\tilde{q}}) = \Lambda(\phi^{-1}(\tilde{q})$ and $Y'_a(e_{\tilde{q}}) = Y_a(\phi^{-1}(\tilde{q}))$. The affine connection ∇' we define by

$$(\nabla'_{\tilde{X}}\tilde{Y})_{e_{\tilde{q}}} = (\tilde{\nabla}_{\tilde{X}}\tilde{Y})_{e_{\tilde{q}}} + \tilde{S}^\alpha(\tilde{X}(e_{\tilde{q}}), \tilde{Y}(e_{\tilde{q}}))\tilde{Y}_\alpha(e_{\tilde{q}}).$$

(In writing this equation we are identifying points in U with their image in $\tilde{Q}$ under $\tilde{\phi}$.)

With these definitions, let us check that condition 5 is satisfied. Since $\tilde{\phi}$ is the inclusion of $\tilde{U}$ in $\tilde{Q}$, it is obvious that

$$T_{e_{\tilde{q}}}\tilde{\phi}(\nabla'_{\tilde{X}}\tilde{Y})_{e_{\tilde{q}}} = (\tilde{\nabla}_{\tilde{X}}\tilde{Y})_{e_{\tilde{q}}} + \tilde{S}^\alpha(\tilde{X}(e_{\tilde{q}}), \tilde{Y}(e_{\tilde{q}}))\tilde{Y}_\alpha(e_{\tilde{q}}).$$

We also can verify that

$$T_{e_{\tilde{q}}}\tilde{\phi}(Y'_a(e_{\tilde{q}})) = \tilde{\Lambda}^\alpha_a(e_{\tilde{q}})\tilde{Y}_\alpha(e_{\tilde{q}})$$

using the definition of $\tilde{\Lambda}$ and the fact that (ϕ, S, Λ) maps Σ_{aff} to $\tilde{\Sigma}_{\mathrm{aff}}$. Thus 5 holds.

Now we verify that 6 holds with the definitions we have given. By Propositions 2.4 and 5.1 we first need to show that the vector field $\nabla'_{\tilde{X}}\tilde{X}$ is tangent to N for any vector field $\tilde{X}$ which is tangent to N. This follows from the definition of ∇', the definition of $\tilde{S}$, and the fact that (ϕ, S, Λ) renders Σ_{aff} a subsystem of $\tilde{\Sigma}_{\mathrm{aff}}|\tilde{U}$. To complete the proof we note that the vector fields Y'_a, $a = 1, \dots, m$, are tangent to N. ■

The idea here is that by a change of controls one arrives at the system Σ'_{aff} which possesses $\phi(Q)$ as an invariant manifold.

5.2. Integral manifolds for affine connection control systems. We now discuss a notion which is stronger than that of restriction. A submanifold $N \subset Q$ is an ***integral manifold*** for an affine connection control system $\Sigma_{\mathrm{aff}} = (Q, \nabla, \mathscr{Y})$ if (1) $T_qN \subset \mathrm{span}_{\mathbb{R}}(Y_1(q), \dots, Y_m(q))$ for each $q \in N$ and (2) N is totally geodesic. We have the following characterisation of integral manifolds.

5.4 Proposition: *N is an integral manifold for the affine connection control system $\tilde{\Sigma}_{\mathrm{aff}} = (\tilde{Q}, \tilde{\nabla}, \tilde{\mathscr{Y}})$ if and only if there exists a fully actuated affine connection control system $\Sigma_{\mathrm{aff}} = (Q, \nabla, \mathscr{Y})$ which is a subsystem of $\tilde{\Sigma}_{\mathrm{aff}}$ via an ACCS morphism (ϕ, S, Λ) which has the property that $N = \mathrm{image}(\phi)$.*

Proof: First suppose that N is an integral manifold for $\tilde{\Sigma}_{\text{aff}}$. To show that there is a fully actuated subsystem, by Lemma 4.1 it suffices to show that for any curve $c\colon I \to N$ whose tangent vector field c' is absolutely continuous, there exists an admissible control u so that (c, u) is a controlled trajectory of $\tilde{\Sigma}_{\text{aff}}$. This follows by Proposition 2.4 and the defining properties of integral manifolds.

For the converse, suppose that $\tilde{\Sigma}_{\text{aff}}$ possesses a fully actuated subsystem Σ_{aff} defined by the ACCS morphism (ϕ, S, Λ). By Lemma 4.1, since ϕ is a diffeomorphism onto its image, for any curve c on N with absolutely continuous tangent vector field c', there exists an admissible control u so that (c, u) is a controlled trajectory for $\tilde{\Sigma}_{\text{aff}}$. This means that N is an invariant manifold for the affine connection control system $\tilde{\Sigma}_{\text{aff}}$, and so by Proposition 5.1 it follows that N is a totally geodesic manifold for $\tilde{\nabla}$. To show that the condition $T_qN \subset \operatorname{span}_{\mathbb{R}}(Y_1(q), \dots, Y_m(q))$ holds, we note that the following lemma was proved by the author [Lewis 1999].

Lemma: *If N is a totally geodesic submanifold of $\tilde{Q}$ with respect to the affine connection $\tilde{\nabla}$, then for each $\tilde{q} \in N$ and each $X \in T_{\tilde{q}}N$, there exists $T > 0$ and a smooth curve $c\colon [0, T] \to \tilde{Q}$ with the properties*

(i) $c'(t) \in T_{c(t)}N$ *for* $t \in]0, T]$ *and*

(ii) $\tilde{\nabla}_{c'(0)}c'(0) = X$.

From the lemma and the fact that for every smooth curve is c, (c, u) is a controlled trajectory for some admissible input u, it follows that for any $\tilde{q} \in N$ we must have $T_qN \subset \operatorname{span}_{\mathbb{R}}(Y_1(q), \dots, Y_m(q))$. This shows that N is an invariant manifold, and so completes the proof. ■

5.5 Remark: We can formulate a weaker notion than that of an integral manifold without much trouble. Given an affine connection control system $\Sigma_{\text{aff}} = (Q, \nabla, \mathscr{Y})$, a submanifold N of Q, and a subbundle D of TN, one can call (N, D) an ***integral subbundle*** if (1) D is geodesically invariant, (2) if $D_q \subset \operatorname{span}_{\mathbb{R}}(Y_1(q), \dots, Y_m(q))$, and (3) D is maximally geodesic (i.e., the closure of the distribution D under Lie bracket is TN). One can then proceed exactly along the lines of Proposition 5.4 to show that (N, D) is an integral subbundle if and only if for any curve $c\colon I \to N$ with $c'(t) \in D_{c(t)}$, there exist an admissible input u so that (c, u) is a controlled trajectory for Σ_{aff}. •

6. Factor systems

In the previous section we looked at how the controlled dynamics of an affine connection control system can be embedded into the dynamics of another affine connection control system. Now we project the controlled dynamics of an affine connection onto those of another. Scenarios such as this arise, for example, when talking about reduction of affine connection control system. This is something for which a completely satisfactory theory does not yet exist, but we refer to the work of Ostrowski [1995] for a discussion of reduction for control systems with nonholonomic constraints.

6.1. Factorisation. If $\Sigma_{\text{aff}} = (Q, \nabla, \mathscr{Y})$ and $\tilde{\Sigma}_{\text{aff}} = (\tilde{Q}, \tilde{\nabla}, \tilde{\mathscr{Y}})$ are affine connection control systems, $\tilde{\Sigma}_{\text{aff}}$ is a ***factor system*** of Σ_{aff} if there exists a complete ACCS morphism (ϕ, S, Λ)

for which $\phi\colon Q \to \tilde{Q}$ is a surjective submersion. As usual, we may talk about ACCS, CACCS, or QACCS factor systems, depending on the character of the morphism (ϕ, S, Λ). We say that an affine connection control system $\Sigma_{\mathrm{aff}} = (Q, \nabla, \mathscr{Y})$ admits a ***factorisation to*** $\tilde{\boldsymbol{Q}}$ ***via*** $\boldsymbol{\phi}$ in the category ACCS (resp. CACCS or QACCS) if there exists an ACCS (resp. CACCS or QACCS) factor system $\tilde{\Sigma}_{\mathrm{aff}} = (\tilde{Q}, \tilde{\nabla}, \tilde{\mathscr{Y}})$ of Σ_{aff} with a morphism of the form (ϕ, S, Λ).

We have the usual terminology associated with a surjective submersion. That is, a vector $v_q \in T_qQ$ is ***vertical*** if $T_q\phi(v_q) = 0$. We denote the subbundle of vertical vectors by VQ. The set $\phi^{-1}(\tilde{q})$ is called the ***fibre*** over $\tilde{q} \in \tilde{Q}$. A vector field X on Q is $\boldsymbol{\phi}$***-projectable*** if $\phi(q_1) = \phi(q_2)$ implies that $T_{q_1}\phi(X(q_1)) = T_{q_2}\phi(X(q_2))$.

Let us begin our discussion of factor systems by indicating that ACCS morphisms which factor can indeed be thought of as epimorphisms in ACCS. The following result relies on a result of Blumenthal [1985] which states that a surjective submersion $\phi\colon Q \to \tilde{Q}$ has the path lifting property provided that Q and $\tilde{Q}$ are connected, and that Q possesses a complete affine connection.

6.1 Proposition: *Let $\Sigma_{\mathrm{aff}} = (Q, \nabla, \mathscr{Y})$ and $\tilde{\Sigma}_{\mathrm{aff}} = (\tilde{Q}, \tilde{\nabla}, \tilde{\mathscr{Y}})$ be affine connection control systems with ∇ and $\tilde{\nabla}$ affinely connected and ∇ complete. If (ϕ, S, Λ) is an ACCS morphism which makes $\tilde{\Sigma}_{\mathrm{aff}}$ a factor system of Σ_{aff}, then for every controlled trajectory $(\tilde{c}, \tilde{u})$ for $\tilde{\Sigma}_{\mathrm{aff}}$ there exists a controlled trajectory (c, u) for Σ_{aff} so that $\tilde{c} = c_\phi$.*

Proof: Let $(\tilde{c}, \tilde{u})$ be a controlled trajectory for $\tilde{\Sigma}_{\mathrm{aff}}$ defined on $I \subset \mathbb{R}$, and let $\bar{c}$ be a lift of $\tilde{c}$—thus c is a curve with the property that $\tilde{c} = \bar{c}_\phi$. Since (ϕ, S, Λ) is complete, we can define a bounded, essentially measurable map $u\colon I \to \mathbb{R}^m$ with the property

$$u^a(t)T_q\phi(Y_a(\bar{c}(t))) = \tilde{u}^\alpha\tilde{Y}_\alpha(\tilde{c}(t)),$$

for all $t \in \mathbb{R}$. To obtain a controlled trajectory (c, u) for Σ_{aff} with the property that $\tilde{c} = c_\phi$, we solve the time-dependent second-order differential equation

$$\nabla_{c'(t)}c'(t) = u^a(t)Y_a(c(t))$$

with initial condition $c'(0) = \bar{c}'(0)$. Since the time-dependence is through u *and* $\bar{c}$, it follows that the curve c so obtained will have the property that $\tilde{c} = c_\phi$. ■

Let us now provide a description of QACCS factor systems.

6.2 Proposition: *An affine connection control system $\Sigma_{\mathrm{aff}} = (Q, \nabla, \mathscr{Y})$ admits a QACCS factorisation if and only if there exists a manifold $\tilde{Q}$ and a surjective submersion $\phi\colon Q \to \tilde{Q}$ so that the two conditions*

(i) ∇ is geodesically ϕ-projectable and

(ii) the vector fields in $\mathscr{Y}$ are ϕ-projectable

are satisfied.

Proof: Suppose that $\tilde{\Sigma}_{\mathrm{aff}}$ is a QACCS factor system of Σ_{aff} via $(\phi, 0, q \mapsto \mathrm{id}_{\mathbb{R}^m})$. Since ϕ maps geodesics of ∇ to geodesics of $\tilde{\nabla}$, it is true that $T_{c(t)}\phi(\nabla_{c'(t)}c'(t)) = \tilde{\nabla}_{c'_\phi(t)}c'_\phi(t)$ for any curve c on Q. Thus ∇ is geodesically ϕ-projectable. Also, if $\tilde{\Sigma}_{\mathrm{aff}}$ is a QACCS factor system of Σ_{aff}, this implies that $T_q\phi(Y_a(q)) = \tilde{Y}_a(\phi(q))$, $a = 1, \ldots, m$, for all $q \in Q$. This clearly implies that Y_a is ϕ-projectable for $a = 1, \ldots, m$.

For the converse, suppose that we have $\tilde{Q}$ and a surjective submersion $\phi\colon Q \to \tilde{Q}$ so that (i) and (ii) hold. Since ϕ-projectability of the vector fields Y_a implies that $T_q\phi(Y_a(q)) = \tilde{Y}_a(\phi(q))$, $a = 1, \dots, m$, for all $q \in Q$, we need only show that ϕ maps geodesics of ∇ to geodesics of some affine connection $\tilde{\nabla}$ on $\tilde{Q}$. However, this follows directly from the fact that ∇ is geodesically ϕ-projectable, and that the projected geodesics of ∇ are geodesics of some affine connection. ■

Let us investigate the manner in which we can decompose morphisms which give rise to factor objects in the category ACCS. As was the case with our factorisation result for subsystems, the result here is local.

6.3 Proposition: *Let $\phi\colon Q \to \tilde{Q}$ be a surjective submersion, and let $\Sigma_{\mathrm{aff}} = (Q, \nabla, \mathscr{Y})$ and $\tilde{\Sigma}_{\mathrm{aff}} = (\tilde{Q}, \tilde{\nabla}, \tilde{\mathscr{Y}})$ be affine connection control systems with $\tilde{\Sigma}_{\mathrm{aff}}$ a factor system of Σ_{aff} via an ACCS morphism (ϕ, S, Λ). Then for each $q \in Q$ there exists a neighbourhood U of q and a CACCS factor system $\Sigma'_{\mathrm{aff}} = (Q, \nabla', \mathscr{Y}')$ of $\Sigma_{\mathrm{aff}}|U$ with the property that there is a QACCS morphism $(\tilde{\phi}, 0, q \mapsto \mathrm{id}_{\mathbb{R}^{\tilde{m}}})$ which makes $\tilde{\Sigma}_{\mathrm{aff}}$ a QACCS factor system of Σ'_{aff}.*

Proof: Since the morphism (ϕ, S, Λ) is complete, in a neighbourhood U of each $q_0 \in Q$ it is possible to define a map $\Theta\colon U \to \mathrm{L}(\mathbb{R}^{\tilde{m}}; \mathbb{R}^m)$ with the property that

$$\Theta^a_\alpha(q)\Lambda^\beta_a(q) = \delta^\beta_\alpha, \quad q \in U.$$

So for $q_0 \in Q$, let U be such a neighbourhood. On U define an affine connection ∇' by

$$(\nabla'_X Y)_q = (\nabla_X Y)_q - S^\alpha(X(q), Y(q))\Theta^a_\alpha(q) Y_a(q),$$

and define a family of vector fields $\mathscr{Y}' = \{Y'_1, \dots, Y'_{\tilde{m}}\}$ on U by

$$Y'_\alpha(q) = \Theta^a_\alpha(q) Y_a(q).$$

Let $\tilde{\phi} = \phi|U$. One then verifies that

$$T_q\tilde{\phi}(\nabla'_X X)_q = (\tilde{\nabla}_{\tilde{X}}\tilde{X})_q, \quad q \in U,$$

where $\tilde{X}$ is $\tilde{\phi}$-related to the $\tilde{\phi}$-projectable vector field X. In particular, it follows from this that ∇' is geodesically $\tilde{\phi}$-projectable. We also note that

$$T_q\tilde{\phi}(Y'_\alpha(q)) = \tilde{Y}_\alpha(\tilde{\phi}(q)), \quad q \in U.$$

Therefore the vector fields from $\mathscr{Y}'$ are $\tilde{\phi}$-projectable which shows, by Proposition 6.2 that $\tilde{\Sigma}_{\mathrm{aff}}$ is a QACCS factor system of $\Sigma_{\mathrm{aff}}|U$. ■

The idea here is that a factoring morphism can, by algebraic transformations to the control, be converted into a system which projects to the factor system. The idea here is thus quite similar to the decomposition we saw for subsystems.

6.2. Special types of factor systems. Now we turn to the situation where the factor system has certain properties. We begin by looking at the case where there are no controls in the factor system.

6.4 Proposition: *Let $\phi\colon Q \to \tilde{Q}$ be a surjective submersion. If an affine connection control system Σ_{aff} possesses an ACCS factor system $\tilde{\Sigma}_{\mathrm{aff}} = (\tilde{Q}, \tilde{\nabla}, \tilde{\mathscr{Y}})$ with the property that $\mathrm{span}_{\mathbb{R}}(\tilde{Y}_1(\tilde{q}), \dots, \tilde{Y}_{\tilde{m}}(q)) = \{0\}$ for each $\tilde{q} \in \tilde{Q}$, then each of the fibres $\phi^{-1}(\tilde{q})$, $\tilde{q} \in \tilde{Q}$, is an invariant manifold for Σ_{aff}.*

Proof: Note that all controlled trajectories of $\tilde{\Sigma}_{\mathrm{aff}}$ are geodesics of $\tilde{\nabla}$. Therefore, if (c, u) is a controlled trajectory of Σ_{aff} with $c'(0) \in V_{c(0)}Q$, then c_ϕ is the geodesic satisfying $c'_\phi(0) = T_{c(0)}\phi(c'(0)) = 0$. Thus $c(t) \in \phi^{-1}(c_\phi(0))$ for all t in the domain of definition of c. In particular, every geodesic for ∇ with initial velocity tangent to a fibre remains in that fibre—that is, the fibres of ϕ are totally geodesic submanifolds. Similarly, for $a = 1, \dots, m$, a curve c satisfying $\nabla_{c'(t)}c'(t) = Y_a(c(t))$ having the property that $c'(0)$ is vertical will remain tangent to the fibre containing $c(0)$. Since this implies that $\nabla_{c'(t)}c'(t)$ is vertical by virtue of the fibres being totally geodesic, the vector field Y_a must also be vertical, which proves that each fibre of ϕ is an invariant manifold for Σ_{aff}. ■

The idea here is that if one can find an uncontrolled ACCS factor object, the implication is that the control system is essentially comprised of a family of invariant manifolds, and thus one can restrict ones attention to a particular one of these invariant manifolds.

Now we look at the opposite extreme—the case when the factor system is fully actuated. In this case it is convenient to work with factor morphisms (ϕ, S, Λ) for which $\phi\colon Q \to \tilde{Q}$ is a locally trivial fibre bundle. The reason for this being convenient is that in this case it is possible to ensure the existence of a bundle HQ which is complementary to the vertical bundle VQ. Such a subbundle is called a ***horizontal bundle*** or an ***Ehresmann connection***. Given a vector field X on Q we denote by $\mathrm{hor}(X)$ the projection of X to HQ and by $\mathrm{ver}(X)$ the projection to VQ.

Let us provide the consequences of factorisation to a fully actuated system. If $\mathscr{Y} = \{Y_1, \dots, Y_m\}$, we denote by $\mathscr{Y}_{\mathrm{hor}}$ (resp. $\mathscr{Y}_{\mathrm{ver}}$) the vector fields $\{\mathrm{hor}(Y_1), \dots, \mathrm{hor}(Y_m)\}$ (resp. $\{\mathrm{ver}(Y_1), \dots, \mathrm{ver}(Y_m)\}$).

6.5 Proposition: *Let $\phi\colon Q \to \tilde{Q}$ be a locally trivial fibre bundle with HQ an Ehresmann connection, and let $\Sigma_{\mathrm{aff}} = (Q, \nabla, \mathscr{Y})$ be an affine connection control system. Consider the following three statements:*

(i) the conditions

(a) there exists a symmetric $(1, 2)$ tensor field B on Q, taking its values in the distribution spanned by the vector fields from $\mathscr{Y}$, for which the affine connection

$$\overline{\nabla}_X Y = \nabla_X Y + B(X, Y)$$

is geodesically ϕ-projectable and

(b) $H_qQ = \mathrm{span}_{\mathbb{R}}(Y_1(q), \dots, Y_m(q))$ for each $q \in Q$

hold;

(ii) there exists a fully actuated affine connection control system $\tilde{\Sigma}_{\mathrm{aff}} = (\tilde{Q}, \tilde{\nabla}, \tilde{\mathscr{Y}})$ and an ACCS morphism (ϕ, S, Λ) which renders $\tilde{\Sigma}_{\mathrm{aff}}$ an ACCS factor system of Σ_{aff};

(iii) for any curve $\tilde{c}$ on $\tilde{Q}$ with the property that $\tilde{c}'$ is absolutely continuous, there exists a controlled trajectory (c, u) for Σ_{aff} so that $\tilde{c} = c_\phi$.

The following conclusions hold:

(iii) if $\tilde{Q}$ is trivialisable then (i) $\iff$ (ii).

(iv) (ii) $\implies$ (iii);

Proof: (iii) First we look at (i) $\implies$ (ii). Given a $(1,2)$ tensor field B as in the first condition of (i), we define a unique torsion-free affine connection $\tilde{\nabla}$ on $\tilde{Q}$ by

$$T_q\phi(\overline{\nabla}_X Y)_q = (\tilde{\nabla}_{\tilde{X}}\tilde{Y})_q,$$

where $\tilde{X}$ and $\tilde{Y}$ are vector fields on $\tilde{Q}$ which are ϕ-related to the vector fields X and Y on Q. That this definition makes sense follows since $\overline{\nabla}$ is geodesically ϕ-projectable. Since B is symmetric and takes its values in the distribution spanned by the vector fields $\mathscr{Y}$, there exists a smooth section $\tilde{S}$ of $\mathbb{R}^m_Q \otimes \mathrm{TS}^q(TQ)$ so that

$$B(X(q), Y(q)) = \tilde{S}^a(X(q), Y(q))Y_a(q).$$

Since $\tilde{Q}$ is trivialisable, we may choose a basis $\tilde{\mathscr{Y}} = \{\tilde{Y}_1, \dots, \tilde{Y}_{\tilde{n}}\}$ for the vector fields on $\tilde{Q}$. Because of the second condition of (i), for each $q \in Q$ there exists an epimorphism $\Lambda(q) \in \mathrm{L}(\mathbb{R}^m; \mathbb{R}^{\tilde{n}})$ with the property that

$$T_q\phi(Y_a(q)) = \Lambda^\alpha_a(q)\tilde{Y}_\alpha(\phi(q))$$

for each $a = 1, \dots, m$. One now verifies that if we take

$$S^\alpha(q) = -\tilde{S}^a(q)\Lambda^\alpha_a(q),$$

then (ϕ, S, Λ) is an ACCS factor morphism which sends Σ_{aff} to $\tilde{\Sigma}_{\mathrm{aff}} = (\tilde{Q}, \tilde{\nabla}, \tilde{\mathscr{Y}})$ with the definitions of $\tilde{\nabla}$ and $\tilde{\mathscr{Y}}$ we have provided.

Now we look at (ii) $\implies$ (i). By our Remark 4.9–3 we may assume that $\Lambda(q)$ is surjective for each $q \in Q$. We denote a right inverse of $\Lambda(q)$ by $\Theta(q)$. Now by Proposition 4.8, the affine connection $\overline{\nabla}$ defined by

$$(\overline{\nabla}_X Y)_q = (\nabla_X Y)_q - S^\alpha(X(q), Y(q))\Theta^a_\alpha(q)Y_a(q)$$

is geodesically ϕ-projectable. Thus taking

$$B(X(q), Y(q)) = -S^\alpha(X(q), Y(q))\Theta^a_\alpha(q)Y_a(q),$$

we have the first condition of (i). What's more, since (ϕ, S, Λ) is complete, the second condition of (i) is also true.

(iv) This follows from Proposition 6.1. ■

6.6 Remark: One cannot expect to be able to generally make implications from (iii) to either (i) or (ii). The reason for this is that the conditions (i) and (ii) give conditions on all of Q, whereas it may be possible for (iii) to imply such conditions on a set whose complement has positive measure. •

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