Linearization of affine connection control system*

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2002/09/22

Abstract

A simple mechanical system is a triple (Q, g, V) where Q is a configuration space, g is a Riemannian metric on Q, and V is the potential energy. The Lagrangian associated with a simple mechanical system is defined by the kinetic energy minus the potential energy. The equations of motion given by the Euler-Lagrange equations for a simple mechanical system without potential energy can be formulated as an affine connection control system. If these systems are underactuated then they do not provide a controllable linearization about their equilibrium points. Without a controllable linearization it is not entirely clear how one should deriving a set of controls for such systems.

There are recent results that define the notion of kinematic controllability and its required set of conditions for underactuated systems. If the underactuated system in question satisfies these conditions, then a set of open-loop controls can be obtained for specific trajectories. These open-loop controls are susceptible to unmodeled environmental and dynamic effects. Without a controllable linearization a feedback control is not readily available to compensate for these effects.

This report considers linearizing affine connection control systems with zero potential energy along a reference trajectory. This linearization yields a linear second-order differential equation from the properties of its integral curves. The solution of this differential equation measures the variations of the system from the desired reference trajectory. This second-order differential equation is then written as a control system. If it is controllable then it provides a method for adding a feedback law. An example is provided where a feedback control is implemented.

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*Report for project in fulfilment of requirements for MSc

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Work performed while a graduate student at Queen's University.
Research supported in part by a grant from the Natural Sciences and Engineering Research Council of Canada.

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1. Introduction

In this report we investigate linearizing affine connection control systems with zero potential energy along a reference trajectory. We begin with two chapters of background material in attempt to make this report as self-contained as possible. A reader with a background in differential geometry and knowledge of linear systems may wish to skip to section 4.2, where the main results of the report are stated.

The objective of this report is to develop a linear equation which measures the variations of an underactuated mechanical system along a specified trajectory. The strategy taken consists of writing the affine connection control system as a first-order system on TQ. Then, employing the tangent lift, we form the linearization of the system on TTQ. We continue to use our geometry toolbox by employing an Ehersmann connection presented in [Bullo and Lewis 2005b, Chapter S4] to provide a splitting of each tangent space $T_{X_{v_x}}TTQ$.

This allows us to develop a second-order linear differential equation that is satisfied by the horizontal part of the integral curves corresponding to the system's linearization.

This second-order linear equation is used to form a linear control system along trajectories of an underactuated kinematically controllable planar rigid body. We will refer to this system as the hovercraft. The second-order linear differential equation for the hovercraft yields a time-varying system which is controllable. For this time-varying system, an optimal control problem can be stated in a linear quadratic regulator formulation. The derived optimal control will then provide a feedback law. The feedback control is incorporated with the origin open-loop reference controls. Simulations of the system with deviations in initial conditions are run. The simulations show the feedback control rejecting the initial conditions disturbances.

The hovercraft is an ongoing project at Queen's University and is a driving force behind the results presented in this report. The current implementation uses a set of open-loop controls. The open-loop controls as mentioned above are very susceptible to environmental and dynamic modeling errors. This is quite evident when observing the behavior of the hovercraft during a commanded run. We regards the results presented here as a step towards adding a possible feedback to the physical model and thus improving the current controller's performance.

2. Linear systems

Linear systems have been widely studied and are well understood compared to nonlinear systems. Since the main objective of this report is to develop a linear control equation for the deviations along system trajectories we offer a brief review of the necessary linear theory background.Note that within this section bold characters will denote vectors and matrices.

2.1. Linearization of a general system about a trajectory. In this report we will be talking about the linearization of mechanical systems. This section is quick reminder of the details of the linearization process for a general nonlinear system.

A general nonlinear system can be written as a vector differential equation

$$\dot{\boldsymbol{x}} = \boldsymbol{f}(\boldsymbol{x}, \boldsymbol{u}) \tag{2.1}$$

where $\boldsymbol{f}(\boldsymbol{x}, \boldsymbol{u}) = (f_1(\boldsymbol{x}, \boldsymbol{u}), \ldots, f_n(\boldsymbol{x}, \boldsymbol{u}))$ is a vector of smooth functions. To linearize equation (2.1) along a trajectory, $(\boldsymbol{x}_{ref}(t), \boldsymbol{u}_{ref})$, we take the Jacobian of $\boldsymbol{f}$ with respect to the state ,$\boldsymbol{x}(t)$, and the control, $\boldsymbol{u}(t)$. Then, both are evaluated along $(\boldsymbol{x}_{ref}(t), \boldsymbol{u}_{ref})$. Thus, the coefficient matrices of the linear system are

$$\boldsymbol{A}(t) = \begin{pmatrix} \frac{\partial f_1}{\partial x_1}(\boldsymbol{x}_{ref}(t), \boldsymbol{u}_{ref}) & \frac{\partial f_1}{\partial x_2}(\boldsymbol{x}_{ref}(t), \boldsymbol{u}_{ref}) & \ldots & \frac{\partial f_1}{\partial x_n}(\boldsymbol{x}_{ref}(t), \boldsymbol{u}_{ref}) \\ \frac{\partial f_2}{\partial x_1}(\boldsymbol{x}_{ref}(t), \boldsymbol{u}_{ref}) & \frac{\partial f_2}{\partial x_2}(\boldsymbol{x}_{ref}(t), \boldsymbol{u}_{ref}) & \ldots & \frac{\partial f_2}{\partial x_n}(\boldsymbol{x}_{ref}(t), \boldsymbol{u}_{ref}) \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial f_n}{\partial x_1}(\boldsymbol{x}_{ref}(t), \boldsymbol{u}_{ref}) & \frac{\partial f_n}{\partial x_2}(\boldsymbol{x}_{ref}(t), \boldsymbol{u}_{ref}) & \ldots & \frac{\partial f_n}{\partial x_n}(\boldsymbol{x}_{ref}(t), \boldsymbol{u}_{ref}) \end{pmatrix}$$

and

$$\boldsymbol{B}(t) = \begin{pmatrix} \frac{\partial f_1}{\partial u_1}(\boldsymbol{x}_{ref}(t), \boldsymbol{u}_{ref}) & \frac{\partial f_1}{\partial u_2}(\boldsymbol{x}_{ref}(t), \boldsymbol{u}_{ref}) & \ldots & \frac{\partial f_1}{\partial u_m}(\boldsymbol{x}_{ref}(t), \boldsymbol{u}_{ref}) \\ \frac{\partial f_2}{\partial u_1}(\boldsymbol{x}_{ref}(t), \boldsymbol{u}_{ref}) & \frac{\partial f_2}{\partial u_2}(\boldsymbol{x}_{ref}(t), \boldsymbol{u}_{ref}) & \ldots & \frac{\partial f_2}{\partial u_m}(\boldsymbol{x}_{ref}(t), \boldsymbol{u}_{ref}) \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial f_n}{\partial u_1}(\boldsymbol{x}_{ref}(t), \boldsymbol{u}_{ref}) & \frac{\partial f_n}{\partial u_2}(\boldsymbol{x}_{ref}(t), \boldsymbol{u}_{ref}) & \ldots & \frac{\partial f_n}{\partial u_m}(\boldsymbol{x}_{ref}(t), \boldsymbol{u}_{ref}) \end{pmatrix}.$$

The linearation is then a linear vector differential equation

$$\dot{\boldsymbol{\xi}}(t) = \boldsymbol{A}(t)\boldsymbol{\xi}(t) + \boldsymbol{B}(t)\boldsymbol{u}(t). \tag{2.2}$$

2.2. Time varying systems. The general equations for a continuous time linear time-varying system is a set of linear differential equations. We will consider time-varying systems of the form

$$\frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{x}(t) = \boldsymbol{A}(t)\boldsymbol{x}(t) + \boldsymbol{B}(t)\boldsymbol{u}(t)$$
$$\boldsymbol{y}(t) = \boldsymbol{C}(t)\boldsymbol{x}(t).$$

The *state transition matrix* $\boldsymbol{\Phi}(t, t_0)$ is defined by it having the property that it gives the solution to the homogeneous equation,

$$\frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{x}(t) = \boldsymbol{A}(t)\boldsymbol{x}(t), \quad \boldsymbol{x}(t_0) = \boldsymbol{x}_0, \ \textit{as } \boldsymbol{x}(t) = \boldsymbol{\Phi}(t, t_0)\boldsymbol{x_0}.$$

It is sometimes useful to know the following properties the transition matrix:

1. $\frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{\Phi}(t, t_0) = \boldsymbol{A}(t)\boldsymbol{\Phi}(t, t_0)$ with initial condition $\boldsymbol{\Phi}(t_0, t_0)$=$\boldsymbol{I}_{n\times n}$, where n=dimension of the state space.
2. $\boldsymbol{\Phi}(t, \tau)\boldsymbol{\Phi}(\tau, t_0)$= $\boldsymbol{\Phi}(t, t_0)$
3. $(\boldsymbol{\Phi}(t, \tau))^{-1}$=$\boldsymbol{\Phi}(\tau, t)$

The transition matrix is calculated from the following expression:

$$\begin{aligned}\boldsymbol{\Phi}(t, t_0) \quad &= \quad \boldsymbol{I}_{n\times n} + \int_{t_0}^{t} \boldsymbol{A}(\sigma_1)d\sigma_1 + \int_{t_0}^{t} \boldsymbol{A}(\sigma_1)\int_{t_0}^{\sigma_1} \boldsymbol{A}(\sigma_2)d\sigma_2 d\sigma_1 \\ &\quad + \int_{t_0}^{t} \boldsymbol{A}(\sigma_1)\int_{t_0}^{\sigma_1} \boldsymbol{A}(\sigma_2)\int_{t_0}^{\sigma_2} \boldsymbol{A}(\sigma_3)d\sigma_3 d\sigma_2 d\sigma_1 + \dots\end{aligned} \tag{2.3}$$

The reader may notice that if $\boldsymbol{A}(t)$ is replaced with the time invariant coefficient matrix $\boldsymbol{A}$ the above expression amounts to the power series expansion for an exponential.

$$\boldsymbol{\Phi}(t, t_0) = e^{\boldsymbol{A}(t-t_0)}$$

It should be noted that in the the time varying case, it may not be possible to find an exact expression for $\boldsymbol{\Phi}(t, t_0)$.

The solution of the inhomogeneous system is know as the *variation of constants* and has the form,

$$\boldsymbol{x}(t) = \boldsymbol{\Phi}(t, t_0)\boldsymbol{x}_0 + \int_{t_0}^{t} \boldsymbol{\Phi}(t, \sigma)\boldsymbol{B}(\sigma)\boldsymbol{u}(\sigma)d\sigma.$$

Controllability. Later, once we have a linear control system in hand, it will be important to check the linearization's controllability. For a full derivation of the following statements on controllability the reader is referred to [Brockett 1970].

2.1 Definition: A system is **controllable** if for each $\boldsymbol{x}_0$,$\boldsymbol{x}_1 \in \mathbb{R}^n$ there exists a suitable control $\boldsymbol{u}(t)$ which moves $\frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{x}(t) = \boldsymbol{A}(t)\boldsymbol{x}(t) + \boldsymbol{B}(t)\boldsymbol{u}(t)$ from state $\boldsymbol{x}_0$ at time $t = t_0$ to $\boldsymbol{x}_1$ at $t = t_1 > t_0$.

2.2 Theorem: *A linear time-varying system is controllable if and only if* $\boldsymbol{x}_0 - \boldsymbol{\Phi}(t_0, t_1)\boldsymbol{x}_1$ *belongs to the range space of the controllability Gramian.*

$$\boldsymbol{W}(t_0, t_1) = \int_{t_0}^{t_1} \boldsymbol{\Phi}(t_0, \sigma)\boldsymbol{B}(\sigma)\boldsymbol{B}^*(\sigma)\boldsymbol{\Phi}^*(t_0, \sigma)d\sigma.$$

In particular, the system is controllable if $\boldsymbol{W}(t_0, t_1)$ *is full rank.*

2.3. The linear quadratic regulator. The linear quadratic regulator (LQR) arises from solving an optimization problem. Let $\boldsymbol{F}$,$\boldsymbol{Q}$, and $\boldsymbol{R}$ be the cost matrices for the terminal state, the system output, and the control effort respectivitly. Then, minimizing η,

$$\eta = \boldsymbol{x}^*(T)\boldsymbol{F}\boldsymbol{x}(T) + \int_0^T (\boldsymbol{C}\boldsymbol{x}(t))^*\boldsymbol{Q}(\boldsymbol{C}\boldsymbol{x}(t)) + \boldsymbol{u}(t)^*\boldsymbol{R}\boldsymbol{u}(t)dt$$

subject to the following constraints,

$$\frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{x}(t) = \boldsymbol{A}(t)\boldsymbol{x}(t) + \boldsymbol{B}(t)\boldsymbol{u}(t)$$

$$\boldsymbol{y}(t) = \boldsymbol{C}(t)\boldsymbol{x}(t)$$

an optimal control, $\boldsymbol{u}(t)$, that minimizes the output of the system is obtained. The form of the optimal control can be derived based on a simple but algebraically messy completing the square exercise [Davis 2002].

2.3 Theorem: *Given that the solution of the differential Riccati equation,*

$$-\frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{K}(t) = \boldsymbol{A}^*(t)\boldsymbol{K}(t) + \boldsymbol{K}(t)\boldsymbol{A}(t) - \boldsymbol{K}(t)\boldsymbol{B}(t)\boldsymbol{R}^{-1}\boldsymbol{B}^*(t)\boldsymbol{K}(t) + \boldsymbol{C}^*(t)\boldsymbol{Q}\boldsymbol{C}(t) \tag{2.4}$$

with final conditions $\boldsymbol{K}(T) = \boldsymbol{C}^*(T)\boldsymbol{F}\boldsymbol{C}(T)$, *exists on the interval* $[0, T]$, *then there exists a control minimizing,*

$$\eta = \boldsymbol{x}^*(T)\boldsymbol{F}\boldsymbol{x}(T) + \int_0^T (\boldsymbol{C}\boldsymbol{x}(t))^*\boldsymbol{Q}(\boldsymbol{C}\boldsymbol{x}(t)) + \boldsymbol{u}(t)^*\boldsymbol{R}\boldsymbol{u}(t)dt.$$

The optimal control satisfies:

$$\boldsymbol{u}(t) + \boldsymbol{R}^{-1}\boldsymbol{B}^*(t)\boldsymbol{K}(t)\boldsymbol{x}(t) = 0. \tag{2.5}$$

The existence of the optimal control rests solely on whether the solution to the Riccati equation exists on the desired time interval. The solutions of the Riccati equation can become unbound in finite time. To rule this out, the cost matrix $\boldsymbol{Q}$ must be chosen such that it is positive-definite.

2.4. Numerical integration of the Riccati equation. To implement an optimal control for a linear time-varying system, it requires the differential Riccati equation to be numerically integrated. This is not so straightforward because the Riccati equation solution is specified by final conditions. To rewrite this as an initial value problem we make a change of variables $\tau = T - t,\ t \in [0, T]$ to obtain

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{K}(T-t) \quad = \quad & \boldsymbol{A}^*(T-t)\boldsymbol{K}(T-t) + \boldsymbol{K}(T-t)\boldsymbol{A}(T-t) \\ & -\boldsymbol{K}(T-t)\boldsymbol{B}(T-t)\boldsymbol{R}^{-1}\boldsymbol{B}^*(T-t)\boldsymbol{K}(T-t) \\ & +\boldsymbol{C}^*(T-t)\boldsymbol{Q}\boldsymbol{C}(T-t)\end{aligned} \tag{2.6}$$

with initial conditions $\boldsymbol{K}(0) = \boldsymbol{C}^*(0)\boldsymbol{F}\boldsymbol{C}(0)$.

Equation (2.6) can now be solved using a standard numerical integrator. Notice that (2.6) will be solved backwards in time, whereas the desired optimal control will be evolving in the forward direction. This direction conflict can be rectified by another integration. Once the Riccati equation is integrated backwards with the change of variables $\tau = T - t,\ t \in [0, T]$, the solution at the final time $t = T$ is the initial conditions for the forwards integration of equation (2.4). With these initial condition (2.4) will integrate forwards in time with the control as needed.

3. Differential geometry

This chapter assumes the reader has some understanding of differential geometry, as it is brief and will cover only the necessary material for the subsequent sections. The geometric objects presented here will be used later to define the linearized control equations. For a more in depth coverage the reader is referred to [Kobayashi and Nomizu 1963].

We will assume all manifolds are C^∞ and finite-dimensional. Also, this report uses a repeated index summation convention. The summation sign Σ is replaced by a repeated index, one being a superscript and the other being a subscript. It should be noted that a superscript in a denominator is a subscript.

3.1. Riemannian metrics. A Riemannian metric, g, on a manifold Q is a smooth assignment of an inner product to each tangent space T_qQ. We use g to define the kinetic energy, a function on TQ given by

$$K(v_q) = \frac{1}{2} g(v_q, v_q).$$

If we pick a chart (U, ψ) where $U \subset Q$ is an open subset and $\psi\colon U \to \psi(U) \subset \mathbb{R}^n$ is a bijection, we may represent a Riemannian metric in coordinates. Given a set of coordinates $(q^1, ..., q^n)$ for the chart (U, ψ) we define n^2 numbers, $g_{ij}(q)$, by giving the metric two basis vectors from the set of basis vectors for T_qQ.

$$g_{ij}(q) = g(q)\left(\frac{\partial}{\partial q^i}\Big|_q, \frac{\partial}{\partial q^j}\Big|_q\right)$$

The Riemannian metric is a map, $g(q)\colon T_qQ \times T_qQ \to \mathbb{R}$ and in coordinates using the repeated index summation convention we have:

$$g(q) = g_{ij}(q)(dq^i\big|_q \otimes dq^j\big|_q).$$

Later, to define the input vector fields of our control system we will need two maps: the *sharp* and the *flat* maps that are associated with the Riemannian metric.

$$g^\flat : TQ \to T^*Q$$
$$g^\sharp : T^*Q \to TQ.$$

In coordinates,

$$\begin{aligned} g^\flat\left(\frac{\partial}{\partial q^i}\right) &= g_{ij}dq^j; \\ g^\sharp(dq^i) &= g^{ij}\frac{\partial}{\partial q^j}. \end{aligned}$$

3.2. Affine connections. Let Q be a manifold, and X and Y be a pair of vector fields on Q. An *affine connection* is an assignment of each pair of vector fields to a new vector field $\nabla_X Y$. This vector field, $\nabla_X Y$, is called the *covariant derivative* of Y with respect to X.

An affine connection has the following properties:

1. The map $(X, Y) \mapsto \nabla_X Y$ is $\mathbb{R}$-bilinear;
2. $\nabla_{fX} Y = f\nabla_X Y$;
3. $\nabla_X(fY) = f\nabla_X Y + (\mathcal{L}_X f)Y$;

where f is a C^∞ function on Q and where $\mathcal{L}_X f$ is the Lie derivative with respect to X: $\mathcal{L}_X f = X^i\frac{\partial f}{\partial q^i}$. For computational purposes it is sometimes useful to work with coordinate expressions. To obtain the covariant derivative of two vector fields on Q, let (U, ψ) be a chart for Q with coordinates $(q^1, ..., q^n)$. Choosing a pair of coordinate vector fields, $\frac{\partial}{\partial q^i}$ and $\frac{\partial}{\partial q^j}$, we may write their covariant derivative as a linear combination of the basis vector fields on Q. This defines n^3 functions, the *Christoffel symbols* by

$$\nabla_{\frac{\partial}{\partial q^i}}\frac{\partial}{\partial q^j} = \Gamma^k_{ij}\frac{\partial}{\partial q^k}. \tag{3.1}$$

It is then easily verified using the properties of the affine connection and equation (3.1) that for any two vector fields X and Y on Q we have

$$\nabla_X Y = \left(\frac{\partial Y^k}{\partial q^j}X^j + \Gamma^k_{ij}Y^iX^j\right)\frac{\partial}{\partial q^k}.$$

Later on it will be necessary to differentiate vector fields along curves. This can be done as follows. Let $c\colon I \to Q$ be a curve on Q, and let S be a vector field along c. Let X and Y be vector fields such that the curve c is a integral curve of X, and $Y(c(t)) = S(t)$ for $t \in I$. Then we define the covariant derivative of S along c to be the vector field along c as

$$\nabla_{c'(t)}S(t) = \nabla_X Y(c(t)).$$

Levi-Civita connection. Given a Reimannian metric g on Q, there is a unique affine connection $\overset{g}{\nabla}$ called the *Levi-Civita connection*, having the following properties:

1. $\overset{g}{\nabla}_X Y + \overset{g}{\nabla}_Y X = [X, Y]$ for all vector fields X and Y on Q;
2. $\mathcal{L}_Z(g(X,Y)) = g(\overset{g}{\nabla}_Z X, Y) + g(X, \overset{g}{\nabla}_Z Y)$.

The Christoffel symbols for $\overset{g}{\nabla}$ in a set of coordinates $(q^1, ..., q^n)$ are given by

$$\overset{g}{\Gamma}{}^i_{jk} = \frac{1}{2} g^{il} \left(\frac{\partial g_{kl}}{\partial q^k} + \frac{\partial g_{jl}}{\partial q^j} - \frac{\partial g_{jk}}{\partial q^l} \right).$$

It is not so obvious from the above how the Christoffel symbols come about. To derive the Christoffel symbols, let the vector fields in property 2 be coordinate vector fields.

$$X = \frac{\partial}{\partial q^i}, \quad Y = \frac{\partial}{\partial q^j}, \quad Z = \frac{\partial}{\partial q^k}$$

Then they are cyclicly permuted to obtain,

1. $\mathcal{L}_Z(g(X,Y)) = g(\overset{g}{\nabla}_Z X, Y) + g(X, \overset{g}{\nabla}_Z Y)$;
2. $\mathcal{L}_X(g(Y,Z)) = g(\overset{g}{\nabla}_X Y, Z) + g(Y, \overset{g}{\nabla}_X Z)$;
3. $\mathcal{L}_Y(g(Z,X)) = g(\overset{g}{\nabla}_Y Z, X) + g(Z, \overset{g}{\nabla}_Y X)$.

Now by adding 2 and 3, then subtracting the result from 1 and noting that the Riemannian metric is symmetric we get,

$$\mathcal{L}_X(g(Y,Z)) + \mathcal{L}_Y(g(Z,X)) - \mathcal{L}_Z(g(X,Y)) = 2g(\overset{g}{\nabla}_X Y, Z).$$

By substituting for X,Y, and Z we obtain the desired result

$$\begin{array}{rcl} \left(\frac{\partial g_{jk}}{\partial q^i} + \frac{\partial g_{ik}}{\partial q^j} - \frac{\partial g_{ij}}{\partial q^k}\right) & = & 2g(\Gamma^l_{ij} \frac{\partial}{\partial q^l}, \frac{\partial}{\partial q^k}) \\ & = & 2g_{lk}\Gamma^l_{ij}. \end{array}$$

The Levi-Civita connection is important in that it provides a link between Lagrangian mechanics and affine connection control systems.

3.1 Proposition: *Let (Q,g,V) be a simple mechanical system with associated Lagrangian L. Let $c\colon I \to Q$ be a curve which is represent by $t \mapsto (q^1(t), ..., q^n(t))$ in a coordinate chart (U, ψ). The following are equivalent:*

(i) $t \mapsto (q^1(t), ..., q^n(t))$ *satisfies the Euler-Lagrange equations for the Lagrangian L;*

(ii) $t \mapsto (q^1(t), ..., q^n(t))$ *satisfies the second-order differential equation* $\ddot{q}^i + \Gamma^i_{jk}\dot{q}^j\dot{q}^k = -(\mathrm{grad}V)^i$,

where Γ^i_{jk}, i,j,k=1,...,n, are functions of q defined by

$$\overset{g}{\Gamma}{}^i_{jk} = \frac{1}{2} g^{il} \left(\frac{\partial g_{kl}}{\partial q^k} + \frac{\partial g_{jl}}{\partial q^j} - \frac{\partial g_{jk}}{\partial q^l} \right).$$

Proof: In coordinates the Lagrangian is,

$$L = \frac{1}{2} g_{jk} v^k v^j - V(q).$$

Then, the *Euler-Lagrange* equations, $\frac{\mathrm{d}}{\mathrm{d}t}\frac{\mathrm{d}L}{\mathrm{d}v^i} - \frac{\mathrm{d}L}{\mathrm{d}q^i} = 0$, are

$$g_{lj}\ddot{q}^j + \left(\frac{\partial g_{lj}}{\partial q^k} - \frac{1}{2}\frac{\partial g_{jk}}{\partial q^l}\right)\dot{q}^j\dot{q}^k + \frac{\partial V}{\partial q^l} = 0.$$

Then noticing that $\dot{q}^j\dot{q}^k$ is symmetric with respect to transposing the j and k indices, we see that only the symmetric part of $A_{ljk} = \left(\frac{\partial g_{lj}}{\partial q^k} - \frac{1}{2}\frac{\partial g_{jk}}{\partial q^l}\right)$ will contribute to the expression

$$\left(\frac{\partial g_{lj}}{\partial q^k} - \frac{1}{2}\frac{\partial g_{jk}}{\partial q^l}\right)\dot{q}^j\dot{q}^k.$$

The symmetric part is

$$\frac{1}{2}(A_{ljk} + A_{lkj}) = \frac{1}{2}\left(\frac{\partial g_{kl}}{\partial q^k} + \frac{\partial g_{jl}}{\partial q^j} - \frac{\partial g_{jk}}{\partial q^l}\right).$$

And now multiplying through by g^{il} we obtain the desired result,

$$\ddot{q}^j + \frac{1}{2} g^{il}\left(\frac{\partial g_{kl}}{\partial q^k} + \frac{\partial g_{jl}}{\partial q^j} - \frac{\partial g_{jk}}{\partial q^l}\right)\dot{q}^j\dot{q}^k = -g^{il}\frac{\partial V}{\partial q^l}.$$

■

Geodesics. Geodesics of an affine connection ∇ are curves $c\colon I \to Q$ which satisfy $\nabla_{c'(t)}c'(t) = 0$. Thus, in coordinates a geodesic satisfies the second-order differential equation

$$\ddot{q}^i + \Gamma^i_{jk}\,\dot{q}^j\dot{q}^k = 0.$$

This second-order differential equation defines a second-order vector field Zon TQ called the *geodesic spray* of ∇. It will be seen later that the geodesic spray is an important vector field in this report. In coordinates we have

$$Z = v^i\frac{\partial}{\partial q^i} - \Gamma^i_{jk} v^j v^k \frac{\partial}{\partial v^i}.$$

3.3. Torsion and Curvature. We now define two tensor fields, the torsion and curvature, both of which are important in obtaining our linear control system.

The torsion, T, of an affine connection ∇ is the (1,2)-tensor field given by

$$T(X,Y) = \nabla_X Y - \nabla_Y X - [X,Y],$$

In coordinates the components of T are

$$T^i_{jk} = \Gamma^i_{jk} - \Gamma^i_{kj}.$$

The curvature R of an affine connection ∇ is the (1,3)-tensor field given by

$$R(X,Y)\xi = \nabla_X\nabla_Y\xi - \nabla_Y\nabla_X\xi - \nabla_{[X,Y]}\xi,$$

In coordinates the components of R are

$$R^i_{jkl} = \frac{\partial\Gamma^i_{lj}}{\partial q^k} - \frac{\partial\Gamma^i_{kj}}{\partial q^l} + \Gamma^i_{km}\Gamma^m_{lj} - \Gamma^i_{lm}\Gamma^m_{kj}.$$

The definitions do not make it apparent that these objects are tensors at all. To shed some light on this we will consider the torsion, but the same arguments also work for the curvature. To show that the torsion is a tensor there are at least two approaches. The first is to show the torsion changes coordinates in the right way. The derivation is simple but a little messy; the only tools required are the chain rule and the definition of an affine connection.

To avoid the messiness we take a different approach. We show that given two C^∞ functions f and g, the mapping $(X,Y)\mapsto T(X,Y)$ is C^∞-linear. That is,

$$T(fX,gY) = fgT(X,Y).$$

To start we use the above definition of the torsion,

$$T(fX,gY) = \nabla_{fX}gY - \nabla_{gY}fX - [fX,gY].$$

This can be expanded using the properties of an affine connection and the Lie bracket:

$$T(fX,gY) = f(g\nabla_XY + (\mathscr{L}_Xg)Y) - g(f\nabla_YX + (\mathscr{L}_Yf)X) - [fX,gY].$$

Since

$$[fX,gY] = fg[X,Y] + f(\mathscr{L}_Xg)Y - g(\mathscr{L}_Yf)X$$

we have our result,

$$T(fX,gY) = fg\nabla_XY - fg\nabla_YX - fg[X,Y].$$

3.4. Tangent bundles.

The canonical involution *TTQ*. Let B be a neighbourhood of $(0,0)\in\mathbb{R}^2$. Then let $\rho_1\colon B\to Q$ and $\rho_2\colon B\to Q$ be maps that are at least C^2. Choosing a set of coordinates (x_1,x_2) for $\mathbb{R}^2$ we say two maps are *equivalent* if:

1. $\rho_1(0,0) = \rho_2(0,0)$;
2. $\frac{\partial\rho_1}{\partial x_1}(0,0) = \frac{\partial\rho_2}{\partial x_1}(0,0)$;
3. $\frac{\partial\rho_1}{\partial x_2}(0,0) = \frac{\partial\rho_2}{\partial x_2}(0,0)$;
4. $\frac{\partial^2\rho_1}{\partial x_1\partial x_2}(0,0) = \frac{\partial^2\rho_2}{\partial x_1\partial x_2}(0,0)$.

We then may define an equivalence class $[\rho]$ and associate to it points in local TTQ coordinates,

$$\left(\rho(0,0), \frac{\partial\rho}{\partial x_1}(0,0), \frac{\partial\rho}{\partial x_2}(0,0), \frac{\partial^2\rho}{\partial x_1\partial x_2}(0,0)\right).$$

From this assoication we define the *canonical involution*, $I_Q\colon TTQ \to TTQ$, which is given by

$$I_Q([\rho]) = [\bar{\rho}],$$

where $\bar{\rho}\colon B \to Q$ is defined by $\rho(x_1, x_2) = \bar{\rho}(x_2, x_1)$. In coordinates,

$$I_Q((q,v),(u,w)) = ((q,u),(v,w)).$$

Tangent lift. Let $X\colon M \to TM$ be a vector field on M. We define a vector field on TM, $X^T\colon TM \to TTM$ as the tangent lift of X. In coordinates,

$$X^T = X^i\frac{\partial}{\partial x^i} + \frac{\partial X^i}{\partial x^j}v^j\frac{\partial}{\partial v^i}.$$

The tangent lift of a vector field X is the linearization of X in sense that $X^T(v_x)$ measures the deviations in the integral curves of X in the direction of v_x. This is shown in the following way.

Let $c\colon I \to M$ be a integral curve of X through $x \in M$ at time $t = a$ and let $c^T\colon I \to TM$ be an integral curve of X^T with initial conditions $v_x \in T_xM$ at time $t = a$. Choose a smooth one-parameter family of deformations $\sigma\colon I \times [-\epsilon, \epsilon] \to M$ of c with the following properties:

1. $s \mapsto \sigma(t,s)$ is differentiable for $t \in I$;
2. for $s \in [-\epsilon, \epsilon]$, $t \mapsto \sigma(t,s)$ is the integral curve of X through $\sigma(a,s)$ at time $t = a$;
3. $\sigma(t,0) = c(t)$ for $t \in I$;
4. $v_x = \frac{\mathrm{d}}{\mathrm{d}s}\big|_{s=0}\sigma(0,s)$.

We then have $c^T(t) = \frac{\mathrm{d}}{\mathrm{d}s}\big|_{s=0}\sigma(t,s)$.

Also, later we will need the vertical lift of X, denoted by $\mathrm{vlft}\,(X)$, on TM. This vector field is given by,

$$\mathrm{vlft}\,X(t,v_x) = \frac{\mathrm{d}}{\mathrm{d}s}\bigg|_{s=0}(v_x + sX(t,x)).$$

By picking a set of local coordinates for M and a vector field $X = X^i\frac{\partial}{\partial x^i}$ on M, then $\mathrm{vlft}\,(X) = X^i\frac{\partial}{\partial v^i}$ on TM.

The almost tangent structure. To define an almost tangent structure we first must define a $(1,2)$ tensor, the Nijenhuis tensor.

3.2 Definition: Let A be a $(1,1)$ tensor field on a manifold, M. Then the *Nijenhuis tensor* associated with A is a $(1,2)$ tensor field N_A such that:

$$N_A(X,Y) = [AX,AY] + A^2[X,Y] - A[AX,Y] - A[X,AY],$$

where X and Y are vector fields on M. In local coordinates,

$$(N_A)^k_{ij} = A^l_i \frac{\partial A^k_j}{\partial x^l} - A^l_j \frac{\partial A^k_i}{\partial x^l} + A^k_l \frac{\partial A^l_i}{\partial x^j} - A^k_l \frac{\partial A^l_j}{\partial x^i}.$$

To verify this is a actually tensor, its linearity with respect to multiplication by C^∞ functions can be checked as was done for the torsion.

3.3 Definition: An *almost tangent structure* S is a $(1,1)$ tensor field on a manifold M having the following properties:

1. $\ker(S)$ is a subbundle of TM;
2. image(S)=ker(S);
3. $N_A = 0$.

The canonical almost tangent structure. Let J_M be a (1,1)-tensor field on TM defined by,

$$J_M(X_{v_x}) = \text{vlft}_{v_x}(T_{v_x}\tau_M(X_{v_x})).$$

In coordinates (x,v) for TM,

$$J_M = \frac{\partial}{\partial v_i} \otimes dx^i.$$

J_M is called the *canonical almost tangent structure*. It can be checked that J_M satisfies the properties of an almost tangent structure.

3.4 Proposition: *Given the (1,1)-tensor field J_M on TM, J_M provides an almost tangent structure on TM.*

Proof: To prove this we shall first pick a set of coordinates on TM. Let $(TU, T\psi)$ be chart for TM such that $v_x \in TM$. We choose a vector field $X_{v_x} \in T_{v_x}TM$, with

$$X = f^i_1 \frac{\partial}{\partial x^i} + f^j_2 \frac{\partial}{\partial v^j}.$$

Feeding this to our (1,1) tensor field yields

$$\begin{array}{rcl} J_M(X) & = & \frac{\partial}{\partial v_i} \otimes dx^i (f^i_1 \frac{\partial}{\partial x^i} + f^j_2 \frac{\partial}{\partial v^j}) \\ & = & f^i_1 \frac{\partial}{\partial v^i} \end{array}$$

We see immediately that $J_M(J_M(X)) = 0$, thus image(J_M)=ker(J_M) and 2 is satisfied.

Now we notice that ker(J_M)=span$\{\frac{\partial}{\partial v^1}, \dots, \frac{\partial}{\partial v^n}\}$. The set $\{\frac{\partial}{\partial v^1}, \dots, \frac{\partial}{\partial v^n}\}$ is coordinate basis of the vertical subbundle VTM. This satisfies 1.

To finish the proof we write J_M in matrix form using the basis $(\frac{\partial}{\partial x^1}, \dots, \frac{\partial}{\partial x^n}, \frac{\partial}{\partial v^1}, \dots, \frac{\partial}{\partial v^n})$. In doing so we obtain the constant matrix

$$J_M = \left(\begin{array}{cc} 0 & 0 \\ Id_{TM} & 0 \end{array} \right).$$

Using the coordinate version of the Nijenhuis tensor of definition 3.2, N_{J_M} vanishes and 3 is satisfied. ■

3.5. Ehresmann connections.

The Ehresmann connection associated with a second-order vector field. An additional structure can be given to each tangent space $T_{v_x}TM$ by a second-order vector field on TM. Let S be a second-order vector field on TM and (x,v) be the natural coordinates for TM. We write S as,

$$S = v^i \frac{\partial}{\partial x^i} + S^i(x,v)\frac{\partial}{\partial v^i},$$

which has the property that $T\tau_M \circ S = \mathrm{Id}_{TM}$. Now recall the canonical almost tangent structure J_M. By taking its Lie derivative with respect to a second-order vector field S we obtain a new $(1,1)$ tensor $\mathcal{L}_S J_M$. We then define a distribution,

$$D_{v_x} = \{w \in T_{v_x}TM | \mathcal{L}_S J_M(w) = -w\}.$$

This distribution is a subbundle complementary to

$$V_{v_x}TM = \{w \in T_{v_x}TM | \mathcal{L}_S J_M(w) = w\}$$

and is denoted as $H_{v_x}TM$. This provides a splitting of $T_{v_x}TM$ and thus is an *Ehresmann connection.*

3.5 Proposition: *Given a second-order vector field S on TM we are provided with an Ehresmann connection which splits $T_{v_x}TM$.*

Proof: Let $Y_{v_x} \in T_{v_x}TM$ and write

$$Y = Y_1^i \frac{\partial}{\partial x^i} + Y_2^j \frac{\partial}{\partial v^j}.$$

We now write $\mathcal{L}_S J_M(Y) = [S, J_M(Y)] - J_M([S,Y])$ using the properties of the Lie derivative. In coordinates we have,

$$\mathcal{L}_S J_M(Y) = -Y_1^i \frac{\partial}{\partial x^i} + \big(Y_2^j - \frac{\partial S^j}{\partial x^m} Y_1^m\big)\frac{\partial}{\partial v^j}.$$

Now consider the distribution,

$$D_{v_x} = \{Y_{v_x} \in T_{v_x}TM | \mathcal{L}_S J_M(Y_{v_x}) = -Y_{v_x}\}.$$

This can be shown to be

$$D_{v_x} = \{Y_{v_x} \in T_{v_x}TM | Y_2^j = \frac{1}{2}\frac{\partial S^j}{\partial x^m} Y_1^m\}.$$

We are now left to check it is complementary to VT_vM

$$\begin{aligned} V_{v_x}TM &= \{w \in T_{v_x}TM | \mathcal{L}_S J_M(w) = w\} \\ &= \{Y_{v_x} \in T_{v_x}TM | (Y_1^i, Y_2^j) = (-Y_1^i, Y_2^j - \tfrac{\partial S^j}{\partial x^m} Y_1^m)\} \\ &= \{Y_{v_x} \in T_{v_x}TM | Y_1^i = 0 \ \forall\ i \in \{1,\dots,n\}\} \end{aligned}$$

One sees that $D_{v_x} \cap V_{v_x}TM = \{0\}$. ∎

3.6. The Jacobi equation. For an affine connection ∇, let T and R be the torsion and curvature tensor fields, respectively. Let X be a vector field along a geodesic $c\colon I \to Q$. X is a *Jacobi field* if it satisfies

$$\nabla^2_{c'(t)} X(t) + \nabla_{c'(t)}(T(X, c'(t))) + R(X, c'(t))c'(t) = 0. \tag{3.2}$$

Equation (3.2) is a linear second-order differential equation call the *Jacobi equation*. A geometric interpretation of a Jacobi field is offered by [Kobayashi and Nomizu 1963]. However, before stating the theorem we make some definitions.

3.6 Definition: A *variation* along a geodesic $c(t)$ is a one-parameter family of geodesics $\sigma\colon I \times [-\epsilon, \epsilon] \to Q$ such that,

1. $s \mapsto \sigma(t, s)$ is differentiable for $t \in I$,
2. for $s \in [-\epsilon, \epsilon], t \mapsto \sigma(t, s)$ is a geodesic, and
3. $\sigma(t, 0){=}c(t)$ for $t \in I$.

3.7 Definition: An *infinitesimal variation* is a vector field along $c(t)$ defined by

$$X(t) = \left.\frac{\mathrm{d}}{\mathrm{d}s}\right|_{s=0} \sigma(t, s) \in T_{c(t)}Q.$$

3.8 Theorem: *A vector field X along a geodesic $c\colon I \to Q$ is a Jacobi field if and only if it is an infinitesimal variation of c.*

We shall see that the Jacobi equation emerges during the linearization of the affine connection control system.

3.9 Proposition: *The linearization of the system, $\nabla_{c'(t)} c'(t) = 0$ along a geodesic, measures the variations of the system along that geodesic.*

Proof: Let ξ be a Jacobi field along the geodesic $c\colon I \to Q$ and $(q^1, \ldots, q^n)$ be a set of local coordinates.

$$\begin{aligned}
\nabla_{\dot q}\nabla_{\dot q}\xi &= \tfrac{\partial}{\partial q^s}\big(\tfrac{\partial \xi^i}{\partial q^l}\,\dot q^l + \Gamma^i_{lm}\xi^m\,\dot q^l\,\big)\,\dot q^s + \Gamma^i_{sj}\big(\tfrac{\partial \xi^j}{\partial q^l}\,\dot q^l + \Gamma^j_{lm}\xi^m\,\dot q^l\,\big)\,\dot q^s \\
&= \tfrac{\partial^2 \xi}{\partial t^2} + \tfrac{\partial \Gamma^i_{lm}}{\partial q^s}\xi^m\,\dot q^l \dot q^s + \Gamma^i_{lm}\tfrac{\partial \xi^m}{\partial q^s}\,\dot q^l \dot q^s + \Gamma^i_{lm}\xi^m \tfrac{\partial \dot q^l}{\partial q^s}\,\dot q^s \\
&\quad + \Gamma^i_{sj}\tfrac{\partial \xi^j}{\partial q^l}\,\dot q^l \dot q^s + \Gamma^i_{sj}\Gamma^j_{lm}\xi^m\,\dot q^l \dot q^s
\end{aligned}$$

$$\begin{aligned}
\nabla_{\dot q} T(\xi, \dot q)^i &= \tfrac{\partial}{\partial q^l}(T^i_{jk}\xi^j\,\dot q^k)\,\dot q^l + \Gamma^i_{lm}(T^m_{jk}\xi^j\,\dot q^k)\,\dot q^l \\
&= \tfrac{\partial \Gamma^i_{jk}}{\partial q^l}\xi^j\,\dot q^k \dot q^l - \tfrac{\partial \Gamma^i_{kj}}{\partial q^l}\xi^j\,\dot q^k \dot q^l + \Gamma^i_{jk}\tfrac{\partial \xi^j}{\partial q^l}\,\dot q^k \dot q^l - \Gamma^i_{kj}\tfrac{\partial \xi^j}{\partial q^l}\,\dot q^k \dot q^l \\
&\quad + \Gamma^i_{jk}\xi^j \tfrac{\partial \dot q^k}{\partial q^l}\,\dot q^l - \Gamma^i_{kj}\xi^j \tfrac{\partial \dot q^k}{\partial q^l}\,\dot q^l + \Gamma^i_{lm}\Gamma^m_{jk}\xi^j\,\dot q^k \dot q^l \\
&\quad - \Gamma^i_{lm}\Gamma^m_{kj}\xi^j\,\dot q^k \dot q^l
\end{aligned}$$

$$\begin{aligned}
R(\xi, \dot q)\,\dot q^i &= \tfrac{\partial \Gamma^i_{lj}}{\partial q^k}\xi^k\,\dot q^l \dot q^j - \tfrac{\partial \Gamma^i_{kj}}{\partial q^l}\xi^k\,\dot q^l \dot q^j + \Gamma^m_{lj}\Gamma^i_{km}\xi^k\,\dot q^l \dot q^j \\
&\quad - \Gamma^m_{kj}\Gamma^i_{lm}\xi^k\,\dot q^l \dot q^j
\end{aligned}$$

Now by using the chain rule and combining the above to obtain the Jacobi Equation we have

$$\frac{\partial^2 \xi}{\partial t^2} + \frac{\partial \Gamma^i_{lm}}{\partial q^k} \xi^k \dot{q}^l \dot{q}^m + \Gamma^i_{lm} \frac{\partial \xi^l}{\partial t} \dot{q}^m + \Gamma^i_{lm} \frac{\partial \xi^m}{\partial t} \dot{q}^l = 0. \tag{3.3}$$

Given the geodesic equation in coordinates

$$\ddot{q}^i + \Gamma^i_{jk} \dot{q}^j \dot{q}^k = 0$$

we may write it as a matrix first-order system on TQ. Then linearizing this in the usual way using the Jacobian it is verifiable we obtain equation (3.3).

The solutions ξ of (3.3) measure variations along the geodesic c by theorem 3.8. ξ is also a solution to the linearization of $\nabla_{c'(t)} c'(t) = 0$ along a geodesic. Thus the linearization measures variations along geodesics of $\nabla_{c'(t)} c'(t) = 0$. ■

4. Affine connection control systems

Let Q be a manifold (the configuration space) and $\mathcal{Y} = \{Y_1, \ldots, Y_m\}$ be a set of input vector fields on Q. We denote an affine connection control system as a triple $\Sigma_{\text{aff}} = (Q, \nabla, \mathcal{Y})$. The governing equation for Σ_{aff} is

$$\nabla_{c'(t)} c'(t) = u^\alpha(t) Y_\alpha(c(t)).$$

It will be desirable to talk about controlled trajectories of an affine connection control systems. To do so we make the following definition.

4.1 Definition: A *controlled trajectory* for an affine connection control system is a pair (c, u) where $u\colon [0, T] \to \mathbb{R}^m$ is measurable and $c\colon [0, T] \to Q$ is a absolutely continuous curve such that $\overset{g}{\nabla}_{c'(t)} c'(t) = u^\alpha(t) Y_\alpha(c(t))$ is satisfied.

For the linearization we will need to state the control equation on TQ. It is verifiable in coordinates that the first-order equation on TQ is

$$\dot{v}(t) = Z(v(t)) + u^\alpha(t) \operatorname{vlft}(Y_\alpha)(v(t))$$

where Z is the geodesic spray of ∇. We will denote this system by $\Sigma^T_{\text{aff}} = (TQ, \{Z, \operatorname{vlft}(Y_1), \ldots, \operatorname{vlft}(Y_m)\})$

4.1. Relationships of affine connection control systems with driftless systems.

This section presents two possible methods, reducibility and kinematic controllability, for dealing with underactuated simple mechanical with zero potential energy. To do this we first need to define a driftless system and its properties.

Driftless systems. A *driftless* control system is a pair $\Sigma = (Q, \mathcal{X})$ where $\mathcal{X} = \{X_1, \ldots, X_s\}$ is a set of vector fields on Q. The control system equations are given by

$$c'(t) = u^\alpha(t) X_\alpha(c(t)). \tag{4.1}$$

There are easily checked conditions that determine controllability of a driftless system. Before this theorem can be stated we make the necessary definitions.

4.2 Definition: A *controlled trajectory* for a driftless control system is a pair (c,u) where $u\colon I \to \mathbb{R}^m$ is measurable and $c\colon I \to Q$ is a absolutely continuous curve such that equation (4.1) is satisfied.

4.3 Definition: A driftless system $\Sigma = (Q, \mathcal{X})$ is *controllable* if for $q_1, q_2 \in Q$ there exist a controlled trajectory (c,u) defined on $[0,T]$ such that $c(0) = q_1$ and $c(T) = q_2$.

To test controllability conditions we need to define a subspace, $\overline{\mathrm{Lie}}(\mathcal{X})_q$, of T_qQ. To do this let $L(\mathcal{X})$ be the smallest subalgebra of vector fields on Q such that

1. $\mathcal{X} \subset L(\mathcal{X})$;
2. $[X,Y] \subset L(\mathcal{X})\ \forall\ X,Y \in \mathcal{X}$.

We now define at each $q \in Q$,

$$\overline{\mathrm{Lie}}(\mathcal{X})_q = \{X(q) | X \in L(\mathcal{X})\}.$$

4.4 Theorem: (Chow 1939) *A driftless system $\Sigma = (Q, \mathcal{X})$ is controllable if $\overline{\mathrm{Lie}}(\mathcal{X})_q = T_qQ$ for each $q \in Q$. If the vector fields in $\mathcal{X}$ are real analytic, then this condition is also necessary.*

Reducibility. The idea of reducibility is to find an associated driftless system of the affine connection control system. The motivation is the thought that finding controls for the driftless (first-order) system may prove to be easier. We use "easier" in the sense that there are no uncontrolled dynamics; if $u = 0$ the velocities of the system are zero. Once the controls for Σ are obtained they maybe mapped to controls for Σ_{aff}. Unfortunately only a small number of underactuated systems with zero potential actually satisfy the required conditions for reducibility.

4.5 Definition: Let $\Sigma_{\mathrm{aff}} = (Q, \nabla, \mathcal{Y})$ be an affine connection control system and let $\Sigma = (Q, \mathcal{X})$ be a driftless system. Σ_{aff} is *reducible* to Σ if the following two conditions hold:

1. for each controlled trajectory $(c, \tilde{u})$ for Σ defined on $[0,T]$ with $\tilde{u}$ differentiable and piecewise C^∞, there exists a piecewise differential map $u\colon [0,T] \to \mathbb{R}^m$ so that (c,u) is a controlled trajectory for Σ_{aff};
2. for each controlled trajectory (c,u) for Σ_{aff} defined on $[0,T]$ and with $c^{'}(0) \in \mathrm{span}\{X_1, \ldots, X_s\}$, there exists a differentiable and piecewise C^∞ map $\tilde{u}\colon [0,T] \to \mathbb{R}^s$ so that $(c,\tilde{u})$ is a controlled trajectory for Σ.

The conditions for when an affine connection control system is reducible to a driftless is a result of [Lewis 1999].

4.6 Theorem: *(Lewis [1999]) An affine connection control system $\Sigma_{\mathrm{aff}} = (Q, \nabla, \mathcal{Y})$ is reducible to a driftless system $\Sigma = (Q, \mathcal{X})$ if and only if the following conditions hold:*

1. $\mathrm{span}\{X_1, \ldots, X_s\} = \mathrm{span}\{Y_1, \ldots, Y_m\}$;
2. $\nabla_X X(q) \in \mathrm{span}\{Y_1, \ldots, Y_m\}$ *for every vector field X having the property that* $X(q) \in \mathrm{span}\{Y_1, \ldots, Y_m\}$ *for every* $q \in Q$.

If the affine connection control system in question is reducible then a correspondence between controlled trajectories of Σ and Σ_{aff} is available by Definition 4.5. This correspondence is made more precise in the follow proposition.

4.7 Proposition: *Let $\Sigma_{\text{aff}} = (Q, \nabla, \mathscr{Y})$ be an affine connection control system which is reducible to the driftless system $(Q, \mathscr{Y})$. Suppose that the vector fields $\mathscr{Y} = \{Y_1, \ldots, Y_m\}$ are linearly independent and define $\gamma^d_{ab} \colon Q \to \mathbb{R}$ $a, b, d \in \{1, \ldots, m\}$ by*

$$< Y_a : Y_b >= \gamma^d_{ab} Y_d$$

which is possible by condition 2 of Theorem 4.6. If $(c, \tilde{u})$ is a controlled trajectory for the driftless system Σ then, if we define the control u by

$$u^d(t) = \tilde{u}^a(t)\tilde{u}^b(t)(\dot{\tilde{u}}^d(t) + \frac{1}{2}\gamma^d_{ab}(c(t))), \quad d \in \{1, \ldots, m\},$$

(c, u) is a controlled trajectory for the affine connection control system Σ_{aff}.

Proof: Since $(c, \tilde{u})$ is a controlled trajectory for the driftless system Σ it must satisfy

$$c'(t) = \tilde{u}^\alpha(t) Y_\alpha(c(t)).$$

So we have,

$$\begin{aligned}
\nabla_{c'(t)} c'(t) &= \nabla_{c'(t)}(\tilde{u}^b(t) Y_b(c(t)) \\
&= \tilde{u}^b(t)\nabla_{c'(t)} Y_b(c(t)) + \dot{\tilde{u}}^b(t) Y_b(c(t)) \\
&= \tilde{u}^b(t)\nabla_{\tilde{u}^a(t)Y_a(c(t))} Y_b(c(t)) + \dot{\tilde{u}}^b(t) Y_b(c(t)) \\
&= \tilde{u}^b(t)\tilde{u}^a(t)\nabla_{Y_a(c(t))} Y_b(c(t)) + \dot{\tilde{u}}^b(t) Y_b(c(t)) \\
&= \tilde{u}^b(t)\tilde{u}^a(t)\frac{1}{2}(\nabla_{Y_a(c(t))} Y_b(c(t)) + \nabla_{Y_b(c(t))} Y_a(c(t))) + \dot{\tilde{u}}^b(t) Y_b(c(t)) \\
&= \tilde{u}^b(t)\tilde{u}^a(t)(\dot{\tilde{u}}^d(t) + \frac{1}{2}\gamma^d_{ab}(c(t)))Y_d(t).
\end{aligned}$$

To complete the proof, let $u^d(t) = \tilde{u}^b(t)\tilde{u}^a(t)(\dot{\tilde{u}}^d(t) + \frac{1}{2}\gamma^d_{ab}(c(t)))$ and we have

$$\nabla_{c'(t)} c'(t) = \tilde{u}^d(t) Y_d(c(t)).$$

∎

It should be noted that for driftless control systems (4.1) finding controls may not be all that easy. This is especially true in the case of adding a stabilizing feedback control.

4.8 Theorem: *Let $q_0 \in Q$. It is not possible to define a continuous function $u \colon Q \to \mathbb{R}^m$ with the property that the closed-loop system for (4.1) has q_0 as an asymptotically stable fixed point.*

Kinematic controllability. The second strategy is to determine whether the system is kinematically controllable. The idea is to find vector fields on Q with integral curves that can be followed by the system up to arbitrary parameterization. These vector fields are called decoupling vector fields.

4.9 Definition: A vector field $X \colon Q \to TQ$ is a *decoupling vector field* for Σ_{aff} if for every integral curve c and for every reparameterization $t \mapsto \tau(t)$ of c of X there exists a controlled trajectory $t \mapsto u(t)$ with the property that $(c \circ \tau, u)$ is a control trajectory.

4.10 Definition: Σ_{aff} is *kinematically controllable* if there exist a set of decoupling vector fields, $\mathcal{X} = \{X_1, \dots, X_s\}$ so that $\overline{\text{Lie}}(\mathcal{X})_q = T_qQ$ for each $q \in Q$.

It should be made clear that a system that is kinematically controllable can only move along decoupling vector fields and not tangent vector fields that are in their span.

In [Bullo and Lynch 2001] conditions are provided for checking whether a vector field X on Q is a decoupling vector field for Σ_{aff}. Unfortunately, it is not clear how to compute with these vector fields. In fact, it is not known how to compute a set of decoupling vector fields for a generalized system. In [Bullo and Lynch 2001] some insights are offered when dealing with a specific affine connection control system.

4.11 Proposition: (Bullo and Lynch 2001)space *A vector field is a decoupling vector field for Σ_{aff} if and only if:*

1. $X(q) \in Y_q = span_{\mathbb{R}}(Y_1(q), \dots, Y_m(q))$ *for every* $q \in Q$
2. $\nabla_X X(q) \in Y_q$ *for every* $q \in Q$

Once a set of decoupling vector field for the affine connection control system are found and it is verified that they satisfy Definition 4.10 we may find controlled trajectories of the system. In [Bullo and Lewis 2005a] results are given to calculate controls laws that move Σ_{aff} along decoupling trajectories.

4.12 Proposition: (Lewis 2002)space *Let X be a decoupling vector field for Σ_{aff}, let $t \mapsto c(t)$ be an integral curve of X and let $t \mapsto \tau(t)$ be a reparamterization for c. If $t \mapsto u(t) \in \mathbb{R}^m$ is defined by*

$$u^\alpha(t) Y_\alpha(c \circ \tau(t)) = (\tau'(t))^2 \nabla_X X(c \circ \tau(t)) + \tau''(t) X(c \circ \tau(t))$$

then $(c \circ \tau, u)$ is a controlled trajectory for Σ_{aff}.

Proof: We need to show the curve $c \circ \tau(t)$ satisfies,

$$\nabla_{c'(t)} c'(t) = u^\alpha(t) Y_\alpha(c(t)).$$

Since c is an integral curve of X we have by definition, $c'(t) = X(c(t))$. Now using the properties of an affine connection:

$$\begin{aligned}
\nabla_{(c\circ\tau)'(t)}(c \circ \tau)'(t) &= \nabla_{c'(\tau(t))\tau'(t)} c'(\tau(t))\tau'(t) \\
&= \tau'(t)(\tau'(t)\nabla_{c'(\tau(t))} c'(\tau(t)) + (\mathcal{L}_{c'(t)}\tau'(t))c'(t) \\
&= (\tau'(t))^2 \nabla_X X(c \circ \tau(t)) + \tau''(t) X(c \circ \tau(t))
\end{aligned}$$

Since X is a decoupling vector field,

1. $X(q) \in Y_q$ $for\ every$ $q \in Q$
2. $\nabla_X X(q) \in Y_q$ $for\ every$ $q \in Q$

thus $(\tau'(t))^2 \nabla_X X(c \circ \tau(t)) + \tau''(t) X(c \circ \tau(t)) \in Y_{c\circ\tau(t)}$ and we are done. ■

4.2. The linearized affine control system and properties of its integral curves. We saw earlier in Section 3.4 that the tangent lift of a vector field X was its linearization. With this in mind the strategy here is to tangent lift the control equation on TQ to TTQ. With the control system on TTQ we employ the Ehresmann connection to provided a splitting of $T_{X_{v_x}}TTQ$. Then, with results from [Bullo and Lewis 2005b, Chapter S4], we recover the Jacobi equation plus the linearized forcing terms that will make up a linear second-order differential equation. As one might expect, the linearized forcing terms will be familiar geometric objects.

As in Section 4 the control equation on TQ is

$$\dot{v}(t) = Z(v(t)) + u^\alpha(t)\,\mathrm{vlft}\,(Y_\alpha)(v(t))$$

where Z is the geodesic spray for ∇. The linearization is obtained using the tangent lift

$$\dot{v}(t)^T = Z^T(X_{v_q}(t)) + (u^\alpha(t)\,\mathrm{vlft}\,(Y_\alpha)(X_{v_q}(t)))^T. \tag{4.2}$$

We make a change of notation for the control u to $\bar{u}$ to prevent confusion in local coordinate expressions. Then in coordinates we have

$$(\bar{u}^\alpha(t)\,\mathrm{vlft}\,(Y_\alpha)(X_{v_q}(t)))^T = \bar{u}^\alpha(t)\left(Y^i_\alpha\frac{\partial}{\partial v^i} + \frac{\partial Y^i_\alpha}{\partial q^l}u^l\frac{\partial}{\partial w^i}\right);$$

$$\begin{aligned} Z^T = v^i\frac{\partial}{\partial q^i} - \Gamma^i_{jk}v^jv^k\frac{\partial}{\partial v^i} + w^i\frac{\partial}{\partial u^i} \\ - \left(\frac{\partial\Gamma^i_{jk}}{\partial q^l}v^jv^ku^l + \Gamma^i_{jk}w^jv^k + \Gamma^i_{kj}w^jv^k\right)\frac{\partial}{\partial w^i}. \end{aligned}$$

The tangent lift of the geodesic spray Z^T can be used to produce a second-order equation on TTQ using I_Q the canonical involution. Since I_Q is a diffeomorphism we use its pullback to obtain a second-order equation

$$\begin{aligned} I_Q^*Z^T = u^i\frac{\partial}{\partial q^i} + w^i\frac{\partial}{\partial v^i} - \Gamma^i_{jk}w^ju^k\frac{\partial}{\partial u^i} \\ - \left(\frac{\partial\Gamma^i_{jk}}{\partial q^l}u^ju^kv^l + \Gamma^i_{jk}w^ju^k + \Gamma^i_{kj}w^ju^k\right)\frac{\partial}{\partial w^i}. \end{aligned}$$

Since Z^T can be used to produce a second-order equation on TTQ, we have a connection which assigns a horizontal subspace, $HTTQ$ on $\tau_{TQ}\colon TTQ \to TQ$. This provides a splitting, [Bullo and Lewis 2005b, Chapter S4],

$$T_{X_{v_q}}TTQ \simeq T_{v_q}TQ \oplus T_{v_q}TQ$$

for $X_{v_q} \in T_{v_q}TQ$. Note that the order of the splitting has the horizontal piece first.

Now, using the geodesic spray as a second-order equation on TQ, we have a connection HTQ on $\tau_Q\colon TQ \to Q$ and a splitting $T_{v_q}TQ \simeq T_qQ \oplus T_qQ$. Thus splitting the horizontal and vertical components of the previous splitting we obtain

$$T_{X_{v_q}}TTQ \simeq T_qQ \oplus T_qQ \oplus T_qQ \oplus T_qQ.$$

Again, for the order of the splitting we have the horizontal subspace first then the vertical. Thus, the first two summands are the horizontal and vertical subspaces of the horizontal subspace of the first splitting respectively. And the second two summands are the horizontal and vertical subspaces of the vertical subspace respectively. For the main result of this report it is useful to write the linearization in a basis of the above splitting.

The horizontal subspace $HTTQ$ has $2n$ basis vectors. There are n horizontal and n vertical which and can be written respectively in coordinates as

$$\begin{aligned}\mathrm{hlft}^T\left(\frac{\partial}{\partial q^i}-\frac{1}{2}(\Gamma^j_{ik}+\Gamma^j_{ki})v^k\frac{\partial}{\partial v^j}\right)&\\ =\frac{\partial}{\partial q^i}-\frac{1}{2}(\Gamma^j_{ik}+\Gamma^j_{ki})v^k\frac{\partial}{\partial v^j}-\frac{1}{2}(\Gamma^j_{ik}+\Gamma^j_{ki})u^k\frac{\partial}{\partial u^j}&\\ -\frac{1}{2}\bigg(\frac{\partial\Gamma^j_{il}}{\partial q^k}u^lv^k+\frac{\partial\Gamma^j_{li}}{\partial q^k}u^lv^k+(\Gamma^j_{ik}+\Gamma^j_{ki})w^k&\\ -\frac{1}{2}(\Gamma^k_{il}+\Gamma^k_{li})(\Gamma^j_{km}+\Gamma^j_{mk})u^mv^l\bigg)\frac{\partial}{\partial w^j}\qquad i\in\{1\ldots n\};&\end{aligned}$$

$$\mathrm{hlft}^T\left(\frac{\partial}{\partial v^i}\right)=\frac{\partial}{\partial v^i}-\frac{1}{2}(\Gamma^j_{ik}+\Gamma^j_{ki})u^k\frac{\partial}{\partial w^j}\qquad i\in\{1\ldots n\}.$$

Similarly $VTTQ$ has $2n$ basis vector that can be expressed in coordinates as

$$\begin{aligned}\mathrm{vlft}^T\left(\frac{\partial}{\partial q^i}-\frac{1}{2}(\Gamma^j_{ik}+\Gamma^j_{ki})v^k\frac{\partial}{\partial v^j}\right)&\\ =\frac{\partial}{\partial u^i}-\frac{1}{2}(\Gamma^j_{ik}+\Gamma^j_{ki})v^k\frac{\partial}{\partial w^j}\qquad i\in\{1\ldots n\};&\end{aligned}$$

$$\mathrm{vlft}^T\left(\frac{\partial}{\partial v^i}\right)=\frac{\partial}{\partial w^i}\qquad i\in\{1\ldots n\}.$$

4.13 Proposition: $(\bar{u}^\alpha(t)\,\mathrm{vlft}\,(Y_\alpha)(X_{v_x}(t)))^T$ *written in terms of the basis vectors for* $HTTQ$ *and* $VTTQ$ *is given by*

$$\begin{aligned}(\bar{u}^\alpha(t)\,\mathrm{vlft}(Y_\alpha)(X_{v_q}(t)))^T&=\bar{u}^\alpha(t)\big(Y^i_\alpha\,\mathrm{hlft}^T\left(\frac{\partial}{\partial v^i}\right)\\ &\quad+[\frac{1}{2}T(Y_\alpha,u)+\nabla_uY_\alpha]^j\,\mathrm{vlft}^T\left(\frac{\partial}{\partial v^j}\right)\big)\end{aligned}$$

Proof: Using the basis vectors for $HTTQ$ and $VTTQ$ we may write

$$\begin{aligned}(\bar{u}^\alpha(t)\,\mathrm{vlft}(Y_\alpha)(X_{v_q}(t)))^T&=\bar{u}^\alpha(t)\big(Y^i_\alpha\,\mathrm{hlft}^T\left(\frac{\partial}{\partial v^i}\right)+\frac{\partial Y^j_\alpha}{\partial q^l}u^l\,\mathrm{vlft}^T\left(\frac{\partial}{\partial v^j}\right)\\ &\quad+\frac{1}{2}(\Gamma^j_{ik}+\Gamma^j_{ki})u^kY^i_\alpha\,\mathrm{vlft}^T\left(\frac{\partial}{\partial v^j}\right)\big).\end{aligned}$$

Now by re-writing

$$\frac{1}{2}(\Gamma^j_{ik}+\Gamma^j_{ki})u^k Y^i_\alpha \frac{\partial}{\partial w^j} = (\Gamma^j_{ik}+\Gamma^j_{ki})u^k Y^i_\alpha \frac{\partial}{\partial w^j} - \frac{1}{2}(\Gamma^j_{ik}+\Gamma^j_{ki})u^k Y^i_\alpha \frac{\partial}{\partial w^j}$$

we have in coordinates

$$\begin{aligned}&(\bar{u}^\alpha(t)\,\mathrm{vlft}\,(Y_\alpha)(X_{v_q}(t)))^T \\ &\qquad = \bar{u}^\alpha(t)\Big(Y^i_\alpha \frac{\partial}{\partial v^i} - \frac{1}{2}(\Gamma^j_{ik}+\Gamma^j_{ki})u^k Y^i_\alpha \frac{\partial}{\partial w^j} + \frac{\partial Y^j_\alpha}{\partial q^l}u^l \frac{\partial}{\partial w^j} \\ &\qquad + (\Gamma^j_{ik}+\Gamma^j_{ki})u^k Y^i_\alpha \frac{\partial}{\partial w^j} - \frac{1}{2}(\Gamma^j_{ik}+\Gamma^j_{ki})u^k Y^i_\alpha \frac{\partial}{\partial w^j}\Big).\end{aligned}$$

With a little rearrangement of the Christoffel symbols,

$$\begin{aligned}(\bar{u}^\alpha(t)\,\mathrm{vlft}\,(Y_\alpha)(X_{v_q}(t)))^T &= \bar{u}^\alpha(t)\Big(Y^i_\alpha \frac{\partial}{\partial v^i} - \frac{1}{2}(\Gamma^j_{ik}+\Gamma^j_{ki})u^k Y^i_\alpha \frac{\partial}{\partial w^j} \\ &\qquad + [\frac{1}{2}(\Gamma^j_{ik}-\Gamma^j_{ki})u^k Y^i_\alpha + \frac{\partial Y^j_\alpha}{\partial q^l}u^l + \Gamma^j_{lm}u^l Y^m_\alpha]\frac{\partial}{\partial w^j}\Big)\end{aligned}$$

and we have our desired result. ■

4.14 Proposition: *The linearization of an affine connection control system written as the direct sum of its horizontal and vertical parts is,*

$$\begin{aligned}&Z^T + (\bar{u}^\alpha(t)\,\mathrm{vlft}\,(Y_\alpha))^T)(u_{v_q}\oplus w_{v_q}) \\ &\qquad = v_q \oplus Y_\alpha \oplus w_{v_q} \oplus \frac{1}{2}T(Y_\alpha, u_{v_q}) + \nabla_{u_{v_q}} Y_\alpha - \\ &\qquad \mathrm{vlft}_{v_q}(R(u_q,v_q)v_q - \frac{1}{2}(\nabla_{u_q}T)(v_q,v_q) \\ &\qquad + \frac{1}{2}(\nabla_{v_q}T)(u_q,v_q) - \frac{1}{4}T(T(u_q,v_q),v_q) \\ &\qquad - T(T(v_q,v_q),u_q)).\end{aligned}$$

4.15 Lemma: (Lewis 2000)space *Let Y be a time dependent vector field on Q and suppose that $c\colon I \to Q$ is a curve satisfying $\nabla_{c'(t)}c'(t) = Y(t, c(t))$, and denote by $\sigma\colon I \to TQ$ the tangent vector field of c (i.e. $c'(t) = \sigma(t)$). Let $X\colon I \to TTQ$ be vector field along σ, and denote $X(t) = X_1(t) \oplus X_2(t) \in T_{c(t)}Q \oplus T_{c(t)}Q \simeq T_{\sigma(t)}TQ$. Then the tangent vector to the curve $t \mapsto X(t)$ is give by $c'(t) \oplus Y(t, c(t)) \oplus \tilde{X}_1(t) \oplus \tilde{X}_2(t)$ where $\tilde{X}_1(t) = \nabla_{c'(t)}X_1(t) + \frac{1}{2}T(X_1(t), c'(t))$ and $\tilde{X}_2(t) = \nabla_{c'(t)}X_2(t) + \frac{1}{2}T(X_2(t), c'(t))$.*

Proof: Given the curve $X(t) = X_1(t) \oplus X_2(t)$ we can re-write it with the induced basis vectors for the vertical and horizontal parts.

$$\begin{aligned}X(t) &= X_1(t)\,\mathrm{hlft}\left(\frac{\partial}{\partial q^k}\right) + X_2(t)\,\mathrm{vlft}\left(\frac{\partial}{\partial q^j}\right) \\ &= X^k_1(t)\frac{\partial}{\partial q^k} - \frac{1}{2}(\Gamma^j_{kl}+\Gamma^j_{lk})\dot{q}^l\frac{\partial}{\partial v^j} + X^j_2(t)\frac{\partial}{\partial v^j}.\end{aligned}$$

This gives the curve $t \mapsto X(t)$ in coordinates of the form:

$$(q^n(t), \dot{q}^m(t), X_1^k(t), X_2^j(t) - \frac{1}{2}(\Gamma^j_{kl} + \Gamma^j_{lk})\dot{q}^l X_1^k(t)).$$

Then the tangent curve to this is given a.e. by

$$\begin{aligned}
&\dot{q}^i(t)\frac{\partial}{\partial q^i} + (Y^i - \Gamma^i_{jk}\dot{q}^j\dot{q}^k)\frac{\partial}{\partial v^i} + \dot{X}_1^i(t)\frac{\partial}{\partial u^i} + \Big(\dot{X}_2^i(t) \\
&- \frac{1}{2}\frac{\partial \Gamma^i_{jk}}{\partial q^l}\dot{q}^k\dot{q}^l X_1^j - \frac{1}{2}\frac{\partial \Gamma^i_{kj}}{\partial q^l}\dot{q}^k\dot{q}^l X_1^j - \frac{1}{2}(\Gamma^i_{jk} + \Gamma^i_{kj})(Y^k - \Gamma^k_{lm}\dot{q}^l\dot{q}^m)X_1^j \\
&- \frac{1}{2}(\Gamma^i_{jk} + \Gamma^i_{kj})\dot{q}^k\dot{X}_1^k\Big)\frac{\partial}{\partial w^i}.
\end{aligned}$$

Now by using the $2n$ basis vectors for $HTTQ$ and the $2n$ basis vectors for $VTTQ$ we have:

$$\begin{aligned}
c'(t) \oplus Y(t, c(t)) \oplus (\nabla_{c'(t)}X_1(t) &+ \frac{1}{2}T(X_1(t), c'(t))) \\
&\oplus (\nabla_{c'(t)}X_2(t) + \frac{1}{2}T(X_2(t), c'(t))).
\end{aligned}$$

■

We are now ready to state the main result of this paper.

4.16 Proposition: *Let Z be the geodesic spray for ∇, an affine connection on Q. Let $c\colon I \to Q$ be a geodesic and $\sigma\colon I \to TQ$ such that $c'(t) = \sigma(t)$ is the integral curve of $Z + \bar{u}^\alpha(t)\,\mathrm{vlft}\,(Y_\alpha)$. Let $a \in I$ and $u, w \in T_{c(a)}Q$. Define vector fields ξ and ζ along c with $t \mapsto \xi(t) \oplus \zeta(t) \in T_{c(t)}Q \oplus T_{c(t)}Q \simeq T_{\sigma(t)}TQ$ an integral curve of $Z^T + (\bar{u}^\alpha(t)\,\mathrm{vlft}\,(Y_\alpha))^T$ with initial conditions $u \oplus w \in T_{c(a)}Q \oplus T_{c(a)}Q \simeq T_{\sigma(a)}TQ$. Then ξ and ζ have the following properties:*

1. *ξ satisfies*

$$\begin{aligned}
\nabla^2_{c'(t)}\xi + R(\xi, c'(t))c'(t) &+ \nabla_{c'(t)}T(\xi, c'(t)) \\
&- \frac{1}{2}T(\bar{u}^\alpha Y_\alpha, \xi) - \nabla_\xi(\bar{u}^\alpha Y_\alpha) = 0;
\end{aligned}$$

2. *$\zeta = \nabla_{c'(t)}\xi + \frac{1}{2}T(\xi, c'(t))$.*

Proof: The tangent vector to the curve $t \mapsto \xi(t) \oplus \zeta(t)$ must be equal to $(Z^T + (U^\alpha(t)\,\mathrm{vlft}\,(Y_\alpha))^T)(\xi(t) \oplus \zeta(t))$ at time t by definition of an integral curve. Then by Lemma (4.15) we have

1. $\nabla_{c'(t)}\zeta + \frac{1}{2}T(\zeta, c'(t)) = -R(\xi, c'(t))c'(t) - \frac{1}{2}(\nabla_{c'(t)}T)(\xi, c'(t)) + \frac{1}{4}T(T(\xi, c'(t)), c'(t)) + \frac{1}{2}T(\bar{u}^\alpha Y_\alpha, \xi) + \nabla_\xi(\bar{u}^\alpha Y_\alpha)$
2. $\nabla_{c'(t)}\xi + \frac{1}{2}T(\xi, c'(t)) = \zeta$.

Equation (2) proves 2. To finish the proof we take the covariant derivative of (2) and substitute it into equation (1).

$$\begin{aligned}\nabla^2_{c'(t)}\xi + \nabla_{c'(t)}\tfrac{1}{2}T(\xi, c'(t)) \quad = \quad & -\tfrac{1}{2}T(\zeta, c'(t)) + \tfrac{1}{2}T(\bar{u}^\alpha Y_\alpha, \xi) + \nabla_\xi(\bar{u}^\alpha Y_\alpha) \\ & -R(\xi, c'(t))c'(t) - \tfrac{1}{2}(\nabla_{c'(t)}T)(\xi, c'(t)) \\ & +\tfrac{1}{4}T(T(\xi, c'(t)), c'(t)).\end{aligned}$$

Now writing the expression in terms of ξ and bring everything to the left hand side

$$\nabla^2_{c'(t)}\xi + R(\xi, c'(t))c'(t) + \nabla_{c'(t)}T(\xi, c'(t)) - \frac{1}{2}T(\bar{u}^\alpha Y_\alpha, \xi) - \nabla_\xi(\bar{u}^\alpha Y_\alpha) = 0 \tag{4.3}$$

as desired. ■

The solutions of the linear second-order differential equation (4.3) measures the variations of an affine connection control system with zero potential energy along its reference trajectories. We can use this as a means to adding feedback for these systems. We want a stabilizing feedback control that will drive the solutions of (4.3) towards zero. To do this we state (4.3) as a linear time varying control system (4.4).

$$\begin{aligned}\nabla^2_{c'(t)}\xi \quad + \quad & R(\xi, c'(t))c'(t) + \nabla_{c'(t)}T(\xi, c'(t)) \\ - \quad & \frac{1}{2}T(\bar{u}^\alpha Y_\alpha, \xi) - \nabla_\xi(\bar{u}^\alpha Y_\alpha) = u^\alpha(t)Y_\alpha(c(t))\end{aligned} \tag{4.4}$$

Before finding a control $u(t)$ that will force the variations of a given affine connection control system towards zero, the transition matrix ,equation (2.3) is required. The transition matrix for the linear system is needed to check its controllability using Definition (4.4). Once the systems controllability is verified then a feedback control can be designed. To obtain this stabilizing feedback control, a linear quadratic regulator formulation of Section 2.3 is applied to (4.4). The Riccati equation is solved backwards from specified final conditions to obtain the initial conditions as described in Section 2.4. With the initial conditions in hand the Riccati equation is integrated in the forwards direction to give $u_{opt}(t)$ at each $t \in [0, T]$, equation (2.5).

5. Planar rigid body - The hovercraft

The hovercraft is an underactuated simple mechanical system with zero potential energy, Figure 1. Thus its linearization about the system's equilibrium points will not be controllable. However, perhaps the hovercraft is controllable about a nontrivial reference trajectory, which in turn will allow us to add feedback to the system. After using the main results from Section 4.2 we shall see it is indeed controllable. This chapter is divided in two parts. In the first we cover background definitions and the open-loop controller results for the hovercraft. In the second the linear second-order differential equation (4.3) is calculated for the two types of attainable trajectories of the hovercraft. The controllability of the two formed linear systems is verified and a feedback control is simulated.

5.1. The hovercraft system. The hovercraft is a simple mechanical system (Q, g, V) with:

1. $Q = \mathbb{R}^2 \times \mathbb{S}^1$;

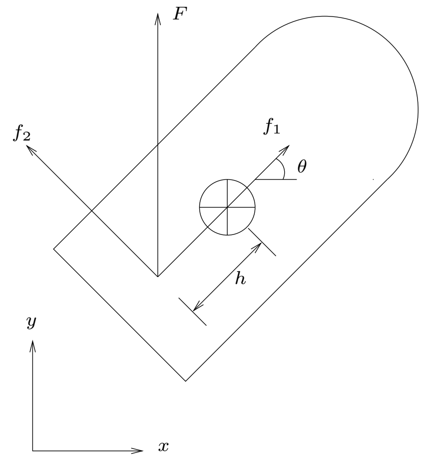

Figure 1. The hovercraft

2. $g = m(dx \otimes dx) + m(dy \otimes dy) + J(d\theta \otimes d\theta)$;
3. $V = 0$.

We may write the kinetic energy using the Riemannian metric g and choosing coordinates (x, y, θ) as

$$KE = \frac{1}{2} g(v, v) = \frac{1}{2} m(v_x^2 + v_y^2) + \frac{1}{2} J v_\theta^2.$$

In this case the potential energy is zero and thus the Lagrangian is just the kinetic energy. By employing the Euler-Lagrange equations, the equations of motion are determined to be:

$$\begin{aligned} \ddot{x} &= \frac{f_1 \cos(\theta)}{m} - \frac{f_2 \sin(\theta)}{m}; \\ \ddot{y} &= \frac{f_1 \sin(\theta)}{m} + \frac{f_2 \cos(\theta)}{m}; \\ \ddot{\theta} &= -\frac{f_2 h}{J}. \end{aligned}$$

From Figure 1 we split the force F into components f_1 and f_2 along the body axis of the hovercraft. These are then written as one-forms on Q.

$$f_1 = \cos(\theta) dx + \sin(\theta) dy, \;\; f_2 = -\sin(\theta) dx + \cos(\theta) dy - h d\theta$$

Now that we have defined our system, we will rewrite it as a affine connection control system.

The hovercraft is an affine connection control system, $\Sigma_{\mathrm{HC}} = (\mathbb{R}^2 \times \mathbb{S}^1, \overset{g}{\nabla}, \mathscr{Y})$ with governing equations

$$\overset{g}{\nabla}_{c'(t)} c'(t) = \bar{u}^\alpha(t) Y_\alpha(c(t)), \text{ where } Y_\alpha = g^\sharp(f_\alpha).$$

Thus

$$
\begin{aligned}
Y_1 &= \frac{\cos(\theta)}{m}\frac{\partial}{\partial x} + \frac{\sin(\theta)}{m}\frac{\partial}{\partial y} \\
Y_2 &= \frac{-\sin(\theta)}{m}\frac{\partial}{\partial x} + \frac{\cos(\theta)}{m}\frac{\partial}{\partial y} - \frac{h}{J}\frac{\partial}{\partial \theta}.
\end{aligned}
$$

The hovercraft also has two decoupling vector fields. The existence of these decoupling vector fields allows us to check the kinematic controllability of the system. By verifying the spanning condition, $\overline{\mathrm{Lie}}(\mathcal{X})_q = T_qQ$, for each $q \in Q$ the hovercraft is indeed kinematically controllable.

5.1 lemma: *X_1 and X_2 are decoupling vector fields for Σ_{HC}.*

$$
\begin{aligned}
X_1 &= \cos(\theta)\frac{\partial}{\partial x} + \sin(\theta)\frac{\partial}{\partial y} \\
X_2 &= -\sin(\theta)\frac{\partial}{\partial x} + \cos(\theta)\frac{\partial}{\partial y} - \frac{hm}{J}\frac{\partial}{\partial \theta}
\end{aligned}
$$

Proof: To show that X_1 and X_2 are decoupling vector fields for the hovercraft system we verify they satisfy Proposition 4.11. Property one is satisfied by inspection.

$$X_1 = mY_1 \in Y_q,\ \ X_2 = mY_2 \in Y_q$$

Property two may be satisfied by a simple calculation.

$$\nabla_{X_1}X_1(q) = 0 \in Y_q,\ \ \nabla_{X_2}X_2(q) = \frac{mh}{J}Y_1 \in Y_q$$

■

5.2 Proposition: *Σ_{HC} is kinematically controllable.*

Proof: Again the proof is purely computational. We compute $L(X)$ by taking iterative Lie brackets until we stop producing new directions. In this case we require only one first-order bracket.

$$L(X) = \{X_1, X_2, [X_1, X_2]\},\ \text{where } [X_1, X_2] = \frac{mh}{J}\sin(\theta)\frac{\partial}{\partial x} - \frac{mh}{J}\cos(\theta)\frac{\partial}{\partial y}$$

Since X_1, X_2, and $[X_1, X_2]$ are linearly independent,

$$\overline{\mathrm{Lie}}(\mathcal{X})_q = T_qQ \ for\ each\ q \in Q.$$

■

With the hovercraft being kinematically controllable we may follow concatenation of integrals corresponding to the decoupling vector fields. When piecing these curves together it is necessary to start and end at zero velocity. An instantaneous velocity jump would correspond to a infinite acceleration. The controls to accomplish this can be computed using Proposition 4.12.

The corresponding integral curves to X_1 and X_2 with initial condition $(x(0), y(0), \theta(0)) = (x_0, y_0, \theta_0)$ are respectively:

$$\begin{aligned} c_{X_1}(t) =& (t\cos(\theta_0) + x_0, t\sin(\theta_0) + y_0, \theta_0) \\ c_{X_2}(t) =& \Bigg(x_0 + \frac{J}{mh}\Bigg(\cos(\theta_0) - \cos\left(\theta_0 - \frac{mh}{J}t\right)\Bigg), \\ & y_0 + \frac{J}{mh}\Bigg(\sin(\theta_0) + \sin\left(\frac{mh}{J}t - \theta_0\right)\Bigg), \theta_0 - \frac{mh}{J}t\Bigg) \end{aligned}$$

Then choosing $\tau(t) = \frac{T}{2}(1 - \cos(\frac{2\pi t}{T}))$ as a reparameterization to make sure we start and stop at zero velocity the controls to move along vector field X_1 and X_2 respectively are:

$$u_{11} = \frac{2m\pi^2\cos(\frac{2\pi t}{T})}{T}; \tag{5.1}$$

$$u_{12} = 0; \tag{5.2}$$

$$u_{21} = \frac{hm^2\pi^2\sin^2(\frac{2\pi t}{T})}{J}; \tag{5.3}$$

$$u_{22} = \frac{2m\pi^2\cos(\frac{2\pi t}{T})}{T}. \tag{5.4}$$

The controls (5.2) and (5.2) correspond to a straight line movement, whereas (5.4) and (5.4) give the hovercraft a circular motion about a point $\frac{J}{mh}$ from the center of mass. These moves are depicted in Figure 2.

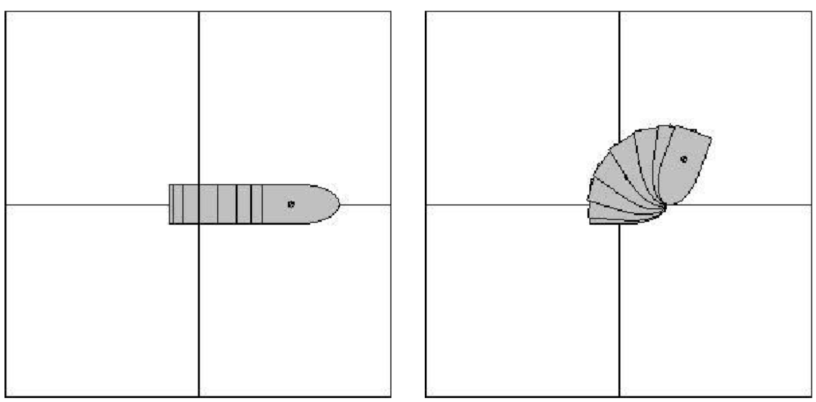

Figure 2. (A)A straight line movement along the integral curve of decoupling vector field X_1. (B)An arc movement along the integral curve of decoupling vector field X_2.

5.2. Linearization of the hovercraft system. To linearize the hovercraft along its decoupling trajectories we will use Proposition 4.16. Notice that for the hovercraft all the Christoffel symbols are zero. In coordinates the linear equation becomes:

$$\frac{\partial^2\xi^i}{\partial t^2} - \left(\frac{\partial}{\partial q^l}[\bar{u}^\alpha Y_\alpha](c_{X_\beta}(t))\right)\xi^l = u^\alpha Y_\alpha(c_{X_\beta}(t)).$$

We write this in matrix form so that we may apply definitions from Section 2. We will first consider the straight line movement of decoupling vector field X_1. The system's coeffiecent matrices are

$$\boldsymbol{A}(t) = \begin{pmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & -\frac{1}{2}\pi^2 \sin(\theta_0)\cos(\pi t) & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{2}\pi^2 \cos(\theta_0)\cos(\pi t) & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \text{ and } \boldsymbol{B}(t) = \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ \cos(\theta_0) & -\sin(\theta_0) \\ \sin(\theta_0) & \cos(\theta_0) \\ 0 & -\frac{mh}{J} \end{pmatrix}.$$

The states of the system correspond to $(x, y, \theta, \dot{x}, \dot{y}, \dot{\theta})$. With the system formed we would like to check its controllability. The transition matrix for the system is computed as

$$\boldsymbol{\Phi}(t, t_0) = \begin{pmatrix} 1 & 0 & \frac{1}{2}\sin(\theta_0)\cos(\pi t) + \frac{1}{2}\sin(\theta_0) & t & 0 & -\frac{1}{2\pi}\sin(\theta_0)(-2\sin(\pi t) + \cos(\pi t)\pi t + \pi t) \\ 0 & 1 & -\frac{1}{2}\cos(\theta_0)\cos(\pi t) + \frac{1}{2}\cos(\theta_0) & 0 & t & -\frac{1}{2\pi}\cos(\theta_0)(-2\sin(\pi t) + \cos(\pi t)\pi t + \pi t) \\ 0 & 0 & 1 & 0 & 0 & t \\ 0 & 0 & \frac{1}{2}\sin\theta_0)\pi\sin(\pi t)t & 1 & 0 & \frac{1}{2}\sin(\theta_0)\cos(\pi t) + \frac{1}{2}\sin(\theta_0)\pi t\sin(\pi t) - \frac{1}{2}\sin(\theta_0) \\ 0 & 0 & \frac{1}{2}\cos(\theta_0)\pi\sin(\pi t) & 0 & 1 & \frac{1}{2}\cos(\theta_0)\cos(\pi t) + \frac{1}{2}\cos(\theta_0)\pi t\sin(\pi t) - \frac{1}{2}\cos(\theta_0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

The transition matrix then defines the controllability Gramian,

$$W(t_0, t_1) = \int_{t_0}^{t_1} \boldsymbol{\Phi}(t_0, \sigma)\boldsymbol{B}(\sigma)\boldsymbol{B}^*(\sigma)\boldsymbol{\Phi}^*(t_0, \sigma)d\sigma$$

which is computed using MapleV. It is however, too large to include here. Nevertheless, one verifies the resulting matrix is of full rank and thus the linear time varying system is controllable.

Similar calculations can be done along the arc motion of decoupling vector field X_2 but the expressions are too lengthy to display here. However, for completeness sake we will include the system coefficients

$$\boldsymbol{A}(t) = \begin{pmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & -\frac{1}{4J}hm^2\pi^2\sin^2(\pi t)\sin(-\theta_0 + \frac{mht}{J}) - \frac{1}{2}m\pi^2\cos(\pi t)\cos(-\theta_0 + \frac{mht}{J}) & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{4J}hm^2\pi^2\sin^2(\pi t)\cos(-\theta_0 + \frac{mht}{J}) + \frac{1}{2}m\pi^2\cos(\pi t)\sin(-\theta_0 + \frac{mht}{J}) & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

and

$$\boldsymbol{B}(t) = \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ \cos(-\theta_0 + \frac{mht}{J}) & -\sin(-\theta_0 + \frac{mht}{J}) \\ \sin(-\theta_0 + \frac{mht}{J}) & \cos(-\theta_0 + \frac{mht}{J}) \\ 0 & -\frac{mh}{J} \end{pmatrix}.$$

5.3. Adding a feedback control to the hovercraft system. The solutions, $\boldsymbol{\xi}(t)$, to the linear equations derived for the hovercraft measure the deviations from the desired reference trajectory. Thus we wish to find a control that minimizes $\boldsymbol{\xi}(t) = \boldsymbol{x}_{actual} - \boldsymbol{x}_{ref}$. The desired path, $\boldsymbol{x}_{ref}$, is the integral curve of the decoupling vector field that we wish the system to follow. To accomplish this we use a linear quadratic regulator formulation of the problem and minimize the error by finding the optimal control,

$$\boldsymbol{u}_{opt}(t) = -\boldsymbol{R}^{-1}\boldsymbol{B}^*(t)\boldsymbol{K}(t)(\boldsymbol{x}_{actual} - \boldsymbol{x}_{ref}).$$

The optimal control is used in conjunction with the reference controls (open-loop controls) for the hovercraft to give:

$$\begin{aligned} u_1 &= u_{1,opt} + u_{1,ref}; \\ u_2 &= u_{2,opt} + u_{2,ref}, \end{aligned} \tag{5.5}$$

which are the controls used in the simulations. The simulation is run using Dlxsim. Dlxsim numerically integrates the equations of motions with the controls (5.5) to produce plots of position. The hovercraft can move to any position in at most three moves. The total trajectory consists of an arc movement followed by a straight line movement and finish with another arc movement. For the simulation plotted in figure 3 the hovercraft starts at an initial condition $(0, 0, 0)$ and moves to $(.128, -.212, \frac{\pi}{2})$. This is the first move of three required to get from $(0, 0, 0)$ to $(1, 1, \pi)$. The reference controls to do this are equations (5.7) and (5.7) along the trajectory (5.8). The controls and curves were computed using MapleV code with the inertia, mass, and length h set to (0.05274,1.576,0.14) respectively.

The MapleV code lengthens the time for the hovercraft to traverse the curve. This is done to keep the required forces small enough so that the they are in the range of the current fans on the hovercraft model. It does this by is picking a new reparameterization

$$\tau(t) = \frac{T}{2}\left(1 - \cos\left(\frac{\pi J t}{10mhT}\right)\right).$$

This gives a new final time of $\frac{10mhT}{J}$ where T is calculated to be 0.261 seconds.

$$u_{1,ref} = .009295052447 \sin(.2882845276t)^2 \tag{5.6}$$

$$u_{2,ref} = -.01705898949 \cos(.2882845276t) \tag{5.7}$$

$$\boldsymbol{x}_{ref} = \begin{pmatrix} x_0 + J(\cos(\theta_0) - \cos(mh(-.1302430086 + .1302430086\cos(.09176381517t\pi))/J - \theta_0))/(mh) \\ y_0 + J(\sin(\theta_0) + \sin(mh(-.1302430086 + .1302430086\cos(.09176381517t\pi))/J - \theta_0))/(mh) \\ \theta_0 - mh(-.1302430086 + .1302430086\cos(.09176381517t\pi))/J \\ -.1195159537e - 1\sin(mh(-.1302430086 + .1302430086\cos(.09176381517t\pi))/J - \theta_0)\sin(.09176381517t\pi)\pi \\ -.01195159537\cos(mh(-.1302430086 + .1302430086\cos(.09176381517t\pi))/J - \theta_0) * \sin(.09176381517t\pi)\pi \\ .01195159537mh\sin(.09176381517t\pi)\pi/J \end{pmatrix} \tag{5.8}$$

To construct the optimal control we assume we have knowledge of all the states. The following cost matrices are used to compute the optimal control.

$$\boldsymbol{Q} = .0018 * \boldsymbol{I}_{6\times 6}$$
$$\boldsymbol{F} = 1000 * \boldsymbol{I}_{6\times 6}$$
$$\boldsymbol{R} = \boldsymbol{I}_{2\times 2}$$

When producing the forward time solution of the Riccati Equation the initial conditions are

$$K(0) = \begin{pmatrix} 1.744453 & 0.950904 & 0.219701 & 1.810858 & 17.702559 & 4.233238 \\ 0.950904 & 0.833569 & 0.200179 & 0.635275 & 12.481588 & 3.001413 \\ 0.219701 & 0.200179 & 0.057152 & 0.152089 & 2.960709 & 0.729445 \\ 1.810858 & 0.635275 & 0.152089 & 2.885844 & 14.550763 & 3.452372 \\ 17.70255 & 12.48158 & 2.960709 & 14.55076 & 206.78614 & 49.642845 \\ 4.23323 & 3.001413 & 0.729445 & 3.452372 & 49.642845 & 11.991678 \end{pmatrix}.$$

These initial conditions are obtained from solving the Riccati equation backwards in time from $K(T) = 1000 * \boldsymbol{I}_{6\times 6}$

6. Conclusions

6.1. Conclusion. In this report the linearization of underactuated affine connection control systems with zero potential energy is investigated along system trajectories. A second-order linear differential equation is developed that measures the variations of the affine connection control system along the specified trajectory. It is shown that for the planar rigid body this set of equations formed a controllable time varying system and thus gave us a means

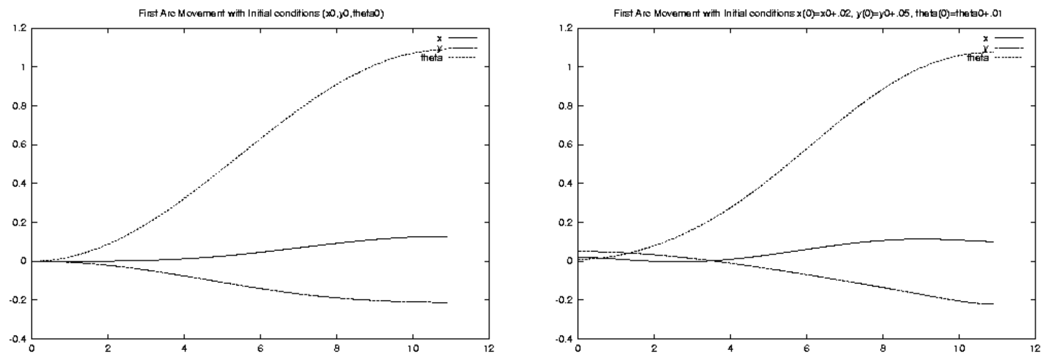

Figure 3. The feedback control rejecting an initial condition disturbance during a movement along the integral curve of decoupling vector field X_2.

for constructing a feedback control. A optimal feedback control is computed using a LQR formulation. This optimal control drives the error, $\boldsymbol{e}(t) = \boldsymbol{x}_{actual} - \boldsymbol{x}_{ref}$, towards zero. The theory is illustrated in simulation and produced the expected results of rejecting small disturbances.

The solution of the Riccati equation was not always attainable from the numerical integration; the solution would become unbound. The solving of the differential Riccati equation using a numerical scheme does not seem to be as well behaved as the algebraic version and was sensitive to initial condition changes. It took may iterations of the initial condition to obtain the solution used in the simulation.

6.2. Future Work. Stemming from this project are several areas of future work. The problems we encountered solving the Riccati brings into question the practicality of implementing a LQR optimal control for a continue time, time-varying system. As well, computing the solution to the differential Riccati equation in real time for use with the actual Hovercraft system does not appear to be possible. Before we can implement a feedback control on the Hovercraft we need to address this issue. Perhaps, at trade off between optimality and computation time can be made.

Another area of future work includes finding a geometric formulation for a linear quadratic regulator along a reference trajectory. This is not for reasons of better implementation but for an improved understanding of the topic.

It is also desirable to find a set of condition for an affine connection control system with zero potential energy that will imply its associated time varying linear system will be controllable. If the conditions are related to the geometry of the affine connection control system then perhaps this would eliminate the need to find the controllability Gramian for the linear system.

Acknowledgements

I would like to thank Dr. Andrew Lewis my supervisor for his constant help and patience throughout the term. As well, I thank Dr. Ron Hirschorn for clarifying aspects of LQR design.

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