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Controllability of a hovercraft model (and two general results)

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Abstract

Modelling and controllability studies of a hovercraft system are undertaken. The system studied is a little more complicated than some in the literature in that the inertial dynamics of the thrust fan are taken into account. The system is shown to be representative of a large class of systems that are controllable only a set described by the zeros of a nontrivial analytic function. Recent results for controllability using vector-valued quadratic forms are useful in arriving at the stated conclusions. As part of the development, two new controllability results of independent interest are proved.

Keywords. Mechanical systems, controllability, differential geometry

AMS Subject Classifications (2020). 53B05, 70Q05, 93B05, 93B27.

1. Introduction

In various papers [e.g., Bullo, Leonard, and Lewis 2000, Bullo and Lynch 2001, Lewis and Murray 1997, Manikonda and Krishnaprasad 1996] the control theory for a simplified hovercraft model is undertaken. The model consists in these papers of a planar rigid body moving on a flat frictionless surface propelled by a variable-direction thruster that applies a force to the body at a point distinct from its centre of mass. Motion planning algorithms suggested by the work of Bullo and Lynch [2001] have been implemented on a physical hovercraft at Queen's University. One of the effects not accounted for in the simplified model, but which have a noticeable effect on the physical system, is the inertial dynamics of the fan. Manoeuvres that require rapid changes of direction of the thrust fan are seen to cause significant deviations in the trajectory of the hovercraft. In this paper we model these dynamics, and illustrate how they affect the controllability analysis for the system. As we shall see, the adding of the seemingly innocuous dynamic effect of fan inertia destroys the nice controllability properties of the system as illustrated by Bullo and Lynch [2001].

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The layout of the paper is as follows. In Section 2 we cast the system as a G-invariant simple mechanical system on a principal G-bundle. General controllability results of Lewis and Murray [1997], Bullo and Lynch [2001], and Bullo and Lewis [2005] are presented in Section 3. The quadratic form results of Bullo and Lewis [2005] are particularly useful. However, it interestingly turns out that the controllability of our hovercraft model cannot be decided by using the results of Bullo and Lewis [2005] alone. Therefore, in Section 4, when we analyse the controllability of the hovercraft system, we prove two general results that characterise the controllability of a class of systems, of which the hovercraft is one. With our general results we show that the hovercraft model is only controllable from a subset of configuration space described by the intersection of the zero set of a finite collection of nontrivial analytic functions. That is to say, the system is almost never controllable.

2. Modelling

In this section we present the model for the system we study, and contrast it with the simpler model initially studied by Lewis and Murray [1997]. We adopt a modification of the approach of the approach of Bullo, Leonard, and Lewis [2000] who consider invariant systems on Lie groups. While the simpler model of Lewis and Murray [1997] *is* an invariant system on a Lie group, the more sophisticated model we present here is not. It is, however, an invariant system on a trivial principal fibre bundle, as considered by Cortés, Martínez, Ostrowski, and Zhang [2002]. A Hamiltonian setting for this sort of problem is considered by Manikonda and Krishnaprasad [1997].

2.1. The physics. The system we study is shown in Figure 1. We have a planar rigid body

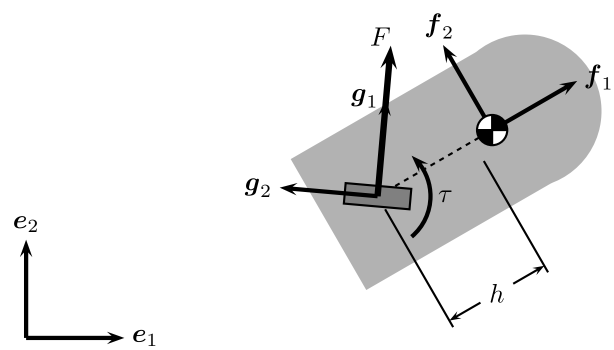

Figure 1. The mathematical model for the hovercraft

moving in a plane orthogonal to the direction of gravity. Sitting atop the rigid body is a fan which may be rotated via the torque τ, and which provides a thrust F. The frame $\{\boldsymbol{e}_1, \boldsymbol{e}_2\}$ is inertial, and we affix to the centre of mass of the body a frame $\{\boldsymbol{f}_1, \boldsymbol{f}_2\}$, choosing the $\boldsymbol{f}_1$-axis so that along it lies the point of application of the thrust force. The frame $\{\boldsymbol{g}_1, \boldsymbol{g}_2\}$ is affixed to the centre of mass of the thrust fan, which we assume to coincide with its point of rotation. We assume that the direction of the thrust force is along $\boldsymbol{g}_1$. Let us describe the configuration space by locating the frames $\{\boldsymbol{f}_1, \boldsymbol{f}_2\}$ and $\{\boldsymbol{g}_1, \boldsymbol{g}_2\}$ relative to the inertial frame. The frame $\{\boldsymbol{f}_1, \boldsymbol{f}_2\}$ is specified relative to $\{\boldsymbol{e}_1, \boldsymbol{e}_2\}$ by an element of $SE(2)$. The frame $\{\boldsymbol{g}_1, \boldsymbol{g}_2\}$ is uniquely determined by its relative orientation to $\{\boldsymbol{f}_1, \boldsymbol{f}_2\}$, i.e., by an element of $SO(2)$. Thus $Q = SE(2) \times SO(2)$. As coordinates we take (x, y, θ, ϕ) where (x, y) is the location of the origin of $\{\boldsymbol{f}_1, \boldsymbol{f}_2\}$ relative to $\{\boldsymbol{e}_1, \boldsymbol{e}_2\}$, θ is prescribed by demanding that $\left[\begin{smallmatrix}\cos\theta & -\sin\theta \\ \sin\theta & \cos\theta\end{smallmatrix}\right]$ have columns representing the components of $\{\boldsymbol{f}_1, \boldsymbol{f}_2\}$ in the basis $\{\boldsymbol{e}_1, \boldsymbol{e}_2\}$, and where ϕ is prescribed by demanding that $\left[\begin{smallmatrix}\cos\phi & -\sin\phi \\ \sin\phi & \cos\phi\end{smallmatrix}\right]$ have columns representing the components of $\{\boldsymbol{g}_1, \boldsymbol{g}_2\}$ in the basis $\{\boldsymbol{f}_1, \boldsymbol{f}_2\}$.

Now let us define the kinetic energy Lagrangian for the system. We denote the mass of the body and of the fan by m_{body} and m_{fan}, respectively, and we denote by J_{body} and J_{fan} the moment of inertia of the body about its centre of mass and the moment of inertia of the fan about its axis of rotation, respectively. With this notation the kinetic energy in the given coordinates is

$$\text{KE} = \begin{bmatrix} v_x & v_y & v_\theta & v_\phi \end{bmatrix} \begin{bmatrix} m_{\text{body}} & 0 & m_{\text{fan}} h \sin\theta & 0 \\ 0 & m_{\text{body}} + m_{\text{fan}} & -m_{\text{fan}} h \cos\theta & 0 \\ m_{\text{fan}} h \sin\theta & -m_{\text{fan}} h \cos\theta & J_{\text{body}} + J_{\text{fan}} + m_{\text{fan}} h^2 & J_{\text{fan}} \\ 0 & 0 & J_{\text{fan}} & J_{\text{fan}} \end{bmatrix} \begin{bmatrix} v_x \\ v_y \\ v_\theta \\ v_\phi \end{bmatrix}. \tag{2.1}$$

Now we define in the given set of coordinates the two forces acting on the system. We denote by F^1 the thrust force and by F^2 the torque rotating the fan. Elementary considerations give $F^1 = (\cos(\theta + \phi), \sin(\theta + \phi), -h \sin\phi)$ and $F^2 = (0, 0, 0, 1)$.

2.2. The mathematics. Now let us put the above physical model into a mathematical framework. First we note that $Q = SE(2) \times SO(2)$ is the total space of a principal fibre bundle with structure group $G = SE(2)$ with the left group action $\Phi\colon (a_1, (a_2, \phi)) \mapsto (a_1 a_2, \phi)$. If we denote points in G as $a = (x, y, \theta)$ in correspondence to our coordinates above, multiplication takes the form

$$\begin{aligned} a_1 \cdot a_2 &= (x_1, y_1, \theta_1) \cdot (x_2, y_2, \theta_2) \\ &= (x_1 + x_2 \cos\theta_1 - y_2 \sin\theta_1, y_1 + y_2 \cos\theta_1 + x_2 \sin\theta_1, \theta_1 + \theta_2). \end{aligned}$$

Below we will interchange a and (x, y, θ) as is convenient. We denote by $\pi\colon Q \to SO(2)$ the projection onto the second factor, which corresponds in a natural way to the projection onto the set of orbits for the group action.

Let us now indicate the manner in which the our system is a G-invariant system on this principal bundle. Corresponding to the kinetic energy determined above there is a Riemannian metric g with the property that $\text{KE}(v_q) = \frac{1}{2} g(v_q, v_q)$.

2.1 Lemma: *The Riemannian metric g is G-invariant.*

Proof: To check this in coordinates, one fixes $\bar{a} = (\bar{x}, \bar{y}, \bar{\theta}) \in SE(2)$, then computes the Jacobian of the map $\Phi_{\bar{a}}\colon (a, \phi) \mapsto (\bar{a}a, \phi)$. Let us denote this Jacobian evaluated at (a, ϕ) by $\boldsymbol{D}\Phi_{\bar{a}}(a, \phi)$. Let us also denote the components of g in our coordinates (a, ϕ) by $[g(a, \phi)]$. Thus $[g(a, \phi)]$ is the matrix appearing in (2.1). With this notation, G-invariance is given by the condition

$$\boldsymbol{D}\Phi_{\bar{a}}^T(a, \phi)[g(\bar{a}a, \phi)]\boldsymbol{D}\Phi_{\bar{a}}(a, \phi) = [g(a, \phi)].$$

This can now be directly checked. ■

Recall that a G-invariant Riemannian metric gives the principal bundle $\pi\colon Q \to SO(2)$ a connection. Indeed, if we let $VQ = \ker(T\pi)$ be the subbundle of tangent spaces of the G-orbits, then $HQ = VQ^{\perp}$ (orthogonal complement taken relative to g) is a G-invariant subbundle complementary to VQ, i.e., a principal connection. This is the ***mechanical connection*** [Marsden and Ratiu 1999].

The forces are represented as one-forms on Q, and are thus given by

$$F^1 = \cos(\theta+\phi)\mathrm{d}x + \sin(\theta+\phi)\mathrm{d}y - h\sin\phi\mathrm{d}\theta, \quad F^2 = \mathrm{d}\phi.$$

These one-forms are also G-invariant.

2.2 Lemma: *The one-forms F^1 and F^2 are G-invariant.*

Proof: We use the notation $\boldsymbol{D}\Phi_{\bar{a}}(a,\phi)$ from the proof of Lemma 2.1. Here, if $[F^a(a,\phi)]$, $a \in \{1,2\}$, denotes the vector of components of F^a in our coordinate system, we should check that

$$\boldsymbol{D}\Phi_{\bar{a}}(a,\phi)^T[F(\bar{a}a,\phi)] = [F(a,\phi)].$$

This can be directly checked. ■

Therefore, from the preceding two lemmas we know that if we define vector fields $Y_a = g^{\sharp}(F^a)$, $a \in \{1,2\}$, on Q, these vector fields will be G-invariant. We also know that the Levi-Civita connection $\overset{g}{\nabla}$ for g will be G-invariant. The system is then an affine connection control system $\Sigma_{\mathrm{hc}} = (Q, \overset{g}{\nabla}, \mathscr{Y} = \{Y_1, Y_2\}, U \subset \mathbb{R}^2)$, meaning, as we shall see in Section 3, that the control equations are

$$\overset{g}{\nabla}_{\gamma'(t)}\gamma'(t) = \sum_{a=1}^{2} u_a(t) Y_a(\gamma(t)).$$

Summarising this is the following.

2.3 Proposition: *The affine connection control system $\Sigma_{\mathrm{hc}} = (Q, \overset{g}{\nabla}, \mathscr{Y}, U)$ corresponding to the system in Figure 1 is a G-invariant system on the principal fibre bundle $\pi\colon Q = SE(2) \times SO(2) \to SO(2)$.*

This structure leads to an interesting consequence concerning the input vector field Y_2. Note that the one-form F^2 annihilates the vertical bundle VQ. Therefore, the vector field Y_2 is g-orthogonal to VQ. This, along with G-invariance of Y_2, implies that Y_2 is the horizontal lift of a vector field $\tilde{Y}_2$ on $SO(2)$. Indeed, one verifies that $\tilde{Y}_2 = \frac{\partial}{\partial\phi}$. This observation turns out to yield the following.

2.4 Lemma: *The vector field $\nabla_{Y_2} Y_2$ is horizontal.*

Proof: One can simply check this directly (as we shall do in Section 4). However, let us indicate how this comes up in a more general setting for readers familiar with the symmetry picture in mechanics. The bundle HQ can be realised as the zero level set for the momentum map $J\colon TQ \to \mathfrak{g}^*$ defined by

$$\langle J(v_q); \xi\rangle = g(v_q, \xi_Q(q)),$$

where ξ_Q denotes the infinitesimal generator associated with $\xi \in \mathfrak{g}$. This means that the subbundle HQ is invariant under the unforced dynamics as a consequence of (actually, the definition of) conservation of momentum. Thus $HQ \subset TQ$ is invariant under the geodesic flow. As shown by Lewis [1998], this means that HQ has the property that $\nabla_X X$ is a section of HQ for any vector field X that is a section of HQ. From this the result follows. ■

We shall see in Section 4 that the lemma has an immediate interpretation in terms of the decoupling vector fields of Bullo and Lynch [2001].

2.5 Remark: The simpler hovercraft model considered in the papers [Bullo, Leonard, and Lewis 2000, Bullo and Lynch 2001, Lewis and Murray 1997] does not have a degree of freedom associated to the fan. The input force for the system is a force F applied directly at the position of the fan. Thus it is a system with a three-dimensional configuration space and two inputs. Bullo, Leonard, and Lewis [2000] show that the system is a left-invariant system on $SE(2)$. •

3. A review of general controllability results

In the next section we investigate the controllability properties of our hovercraft model. As we shall see, the structure is sufficiently complicated that we need to use the approach recently laid out by Bullo and Lewis [2005] (see also [Bullo and Lewis 2005, Hirschorn and Lewis 2001]). We also consider our system in the context of the work of Bullo and Lynch [2001], inasmuch as this is possible. In this section we provide a review of the necessary machinery.

3.1. Controllability definitions. We recall that an ***affine connection control system*** is a triple $\Sigma = (Q, \nabla, \mathscr{Y} = \{Y_1, \dots, Y_m\}, U)$ where Q is the configuration manifold, ∇ is an affine connection on Q, $\mathscr{Y}$ are vector fields on Q, and $U \subset \mathbb{R}^m$. The control set U is ***almost proper*** if $\operatorname{aff}(U) = \mathbb{R}^m$ and if $0 \in \operatorname{conv}(U)$. U is ***proper*** if $0 \in \operatorname{int}(\operatorname{conv}(U))$. Here $\operatorname{aff}(U)$ is the affine hull of U and $\operatorname{conv}(U)$ is the convex hull of U. Bullo and Lewis [2005] consider a slightly more general setup than this, but what we say here is all that is necessary. The control equations are then

$$\nabla_{\gamma'(t)}\gamma'(t) = \sum_{a=1}^{m} u_a(t) Y_a(\gamma(t)). \tag{3.1}$$

A ***controlled trajectory*** for Σ is a pair (γ, u) where $u\colon [0,T] \to U$ is measurable and $\gamma\colon [0,T] \to Q$ satisfies (3.1).

For $T > 0$ denote

$$\mathcal{R}^{\Sigma}_{TQ}(q_0, T) = \{\gamma'(T) \mid (\gamma, u) \text{ is a controlled trajectory on } [0,T] \text{ with } \gamma'(0) = 0_{q_0}\},$$

where 0_{q_0} denotes the zero vector in $T_{q_0}Q$, and denote $\mathcal{R}^{\Sigma}_{TQ}(q_0, \leq T) = \bigcup_{0 \leq t \leq T} \mathcal{R}^{\Sigma}_{TQ}(q_0, t)$. These are therefore reachable *states* in TQ starting from zero initial velocity at the configuration q_0. We also consider the reachable configurations which we denote

$$\mathcal{R}^{\Sigma}_{Q}(q_0, T) = \pi_{TQ}(\mathcal{R}^{\Sigma}_{TQ}(q_0, T)), \qquad \mathcal{R}^{\Sigma}_{Q}(q_0, \leq T) = \pi_{TQ}(\mathcal{R}^{\Sigma}_{TQ}(q_0, \leq T)),$$

where $\pi_{TQ}\colon TQ \to Q$ is the tangent bundle projection. We may now state the versions of controllability that are of interest to us in this case. More refined notions are considered by Bullo and Lewis [2005].

3.1 Definition: Let $\Sigma = (Q, \nabla, \mathscr{Y}, U)$ be an affine connection control system.

(i) Σ is ***accessible*** from $q_0 \in TQ$ if there exists $T > 0$ such that $\operatorname{int}(\mathcal{R}^{\Sigma}_{TQ}(q_0, \leq t)) \neq \varnothing$ for $t \in]0, T]$.

(ii) Σ is ***configuration accessible*** from $q_0 \in TQ$ if there exists $T > 0$ such that $\operatorname{int}(\mathcal{R}^{\Sigma}_{Q}(q_0, \leq t)) \neq \varnothing$ for $t \in]0, T]$.

(iii) Σ is ***small-time locally controllable*** (***STLC***) from q_0 if there exists $T > 0$ such that $0_{q_0} \in \operatorname{int}(\mathcal{R}^{\Sigma}_{TQ}(q_0, \leq t)) \neq \varnothing$ for $t \in]0, T]$.

(iv) Σ is ***small-time locally configuration controllable*** (***STLCC***) from q_0 if there exists $T > 0$ such that $0_{q_0} \in \operatorname{int}(\mathcal{R}^{\Sigma}_{Q}(q_0, \leq t)) \neq \varnothing$ for $t \in]0, T]$. •

3.2. General accessibility results. We now provide conditions for accessibility of affine connection control systems as given by Lewis and Murray [1997]. We let $\Sigma = (Q, \nabla, \mathscr{Y}, U)$ be an affine connection control system and define the associated ***symmetric product*** by

$$\langle X : Y \rangle = \nabla_X Y + \nabla_Y X.$$

We then let Y be the distribution generated by the vector fields $\mathscr{Y}$. We let $\mathsf{Sym}^{(0)}(\mathsf{Y}) = \mathsf{Y}$ and we inductively define

$$\begin{aligned} \mathsf{Sym}^{(1)}(\mathsf{Y})_q &= \mathsf{Y}_q + \operatorname{span}_{\mathbb{R}}(\langle Y_a : Y_b \rangle \mid\ a, b \in \{1, \dots, m\}) \\ \mathsf{Sym}^{(k)}(\mathsf{Y})_q &= \mathsf{Sym}^{(k-1)}(\mathsf{Y})_q \\ &\quad + \operatorname{span}_{\mathbb{R}}(\langle Y_a : Y_b \rangle \mid\ Y_a \in \Gamma(\mathsf{Sym}^{(k_1)}(\mathsf{Y})),\ Y_b \in \Gamma(\mathsf{Sym}^{(k_2)}(\mathsf{Y})),\ k_1 + k_2 = k + 1). \end{aligned}$$

The smallest distribution containing these distributions we denote by $\mathsf{Sym}^{(\infty)}(\mathsf{Y})$. The involutive closure of a distribution X is denoted $\mathsf{Lie}^{(\infty)}(\mathsf{Y})$. No rank assumptions are made concerning $\mathsf{Sym}^{(k)}(\mathsf{Y})$, $k \in \mathbb{Z}_+ \cup \{\infty\}$, or $\mathsf{Lie}^{(\infty)}(\mathsf{Sym}^{(\infty)}(\mathsf{Y}))$.

We now have the following result of Lewis and Murray [1997].

3.2 Theorem: *An analytic affine connection control system $\Sigma = (Q, \nabla, \mathscr{Y}, U)$ with U almost proper is*

(i) accessible from q_0 if and only if $\mathsf{Sym}^{(\infty)}(\mathsf{Y})_{q_0} = T_{q_0}Q$ and is

(ii) configuration accessible from q_0 if and only if $\mathsf{Lie}^{(\infty)}(\mathsf{Sym}^{(\infty)}(\mathsf{Y}))_{q_0} = T_{q_0}Q$

3.3. General controllability results. The controllability results of Bullo and Lewis [2005] are stated in terms of vector-valued quadratic forms, and we refer to that paper for full details. Here we will summarise the results as we need them.

For $q_0 \in Q$ we define a map $B_{\mathsf{Y}_{q_0}} : \mathsf{Y}_{q_0} \times \mathsf{Y}_{q_0} \to T_{q_0}Q/\mathsf{Y}_{q_0}$ by

$$B_{\mathsf{Y}_{q_0}}(v_1, v_2) = \pi_{\mathsf{Y}_{q_0}}(\langle V_1 : V_2 \rangle(q_0)),$$

where V_1 and V_2 are vector fields extending $v_1, v_2 \in \mathsf{Y}_{q_0}$, and where $\pi_{\mathsf{Y}_{q_0}} : T_{q_0}Q \to T_{q_0}Q/\mathsf{Y}_{q_0}$ is the canonical projection. One may verify that this definition does not depend on the extensions V_1 and V_2. By $Q_{B_{\mathsf{Y}_{q_0}}}$ we denote the quadratic $T_{q_0}Q/\mathsf{Y}_{q_0}$-valued function on Y_{q_0} defined by $Q_{B_{\mathsf{Y}_{q_0}}}(v) = B_{\mathsf{Y}_{q_0}}(v, v)$. We let $\operatorname{ann}(\mathsf{Y}_{q_0}) \subset T^*_{q_0}Q$ be the annihilator of Y_{q_0} and note that $\operatorname{ann}(\mathsf{Y}_{q_0}) \simeq (T_{q_0}Q/\mathsf{Y}_{q_0})^*$. Thus for $\lambda \in \operatorname{ann}(\mathsf{Y}_{q_0})$ we define a $\mathbb{R}$-valued symmetric bilinear map $\lambda B_{\mathsf{Y}_{q_0}}$ by $\lambda B_{\mathsf{Y}_{q_0}}(v_1, v_2) = \langle \lambda ; B_{\mathsf{Y}_{q_0}}(v_1, v_2) \rangle$.

The following definition gives some useful properties for $B_{\mathsf{Y}_{q_0}}$. For a discussion of the ideas represented here we refer to Bullo, Cortés, Lewis, and Martínez [2004].

3.3 Definition: $B_{\mathsf{Y}_{q_0}}$ is

(i) ***definite*** if there exists $\lambda \in \operatorname{ann}(\mathsf{Y}_{q_0})$ so that $\lambda B_{\mathsf{Y}_{q_0}}$ is positive-definite;

(ii) ***semidefinite*** if there exists $\lambda \in \operatorname{ann}(\mathsf{Y}_{q_0})$ so that $\lambda B_{\mathsf{Y}_{q_0}}$ is strictly positive-semidefinite (i.e., positive-semidefinite and nonzero);

(iii) ***indefinite*** if for each $\lambda \in \operatorname{ann}(\mathsf{Y}_{q_0})$, $\lambda B_{\mathsf{Y}_{q_0}}$ is neither positive nor negative-semidefinite. •

A simplified version of the results of Bullo and Lewis [2005] are the following.

3.4 Theorem: *Let $\Sigma = (Q, \nabla, \mathscr{Y}, U)$ be an analytic affine connection control system. The following statements hold:*

(i) if

(a) U is proper,

(b) $\mathsf{Sym}^{(\infty)}(\mathsf{Y})_{q_0} = \mathsf{Sym}^{(2)}(\mathsf{Y})_{q_0}$, and if

(c) $B_{\mathsf{Y}_{q_0}}$ is indefinite

then Σ is STLC from q_0 if it is accessible from q_0 and is STLCC from q_0 if it is configuration accessible from q_0;

(ii) if

(a) U is compact,

(b) q_0 is a regular point for Y, and

(c) $B_{\mathsf{Y}_{q_0}}$ is definite

then Σ is not STLCC from q_0.

3.5 Remark: Theorem 3.4 is a "first-order" result, meaning that its hypotheses require knowledge of the first derivatives of the vector fields $\mathscr{Y}$. The case where $B_{\mathsf{Y}_{q_0}}$ is semidefinite will generally be the boundary situation where one will have to use higher-order conditions to determine controllability. However, as we shall see in Theorem 4.3, for some systems one can determine controllability in the semidefinite case without resorting to higher-order conditions. Indeed, our new result stated as Theorem 4.2 is a generalisation of the second part of Theorem 3.4, and allows $B_{\mathsf{Y}_{q_0}}$ to be semidefinite. •

3.4. Kinematic reductions. Let $\Sigma = (Q, \nabla, \mathscr{Y}, U = \mathbb{R}^m)$ be an affine connection control system. Bullo and Lynch [2001] consider a class of vector fields with interesting properties. For an interval $[0, T]$ a ***reparameterisation*** is a differentiable map $\tau\colon [0, \tilde{T}] \to [0, T]$ with the properties

1. $\tau(0) = 0$,
2. $\tau(\tilde{T}) = T$, and
3. $\tau'(t) > 0$ for $t \in]0, \tilde{T}[$.

The following definition combines a definition of Bullo and Lynch [2001] and its generalisation by Bullo and Lewis [2005].

3.6 Definition: Let $\Sigma = (Q, \nabla, \mathscr{Y}, U = \mathbb{R}^m)$ be an affine connection control system.

(i) A vector field X on Q is a ***decoupling vector field*** if for every integral curve $\gamma\colon [0,T] \to Q$ for X and for every reparameterisation $\tau\colon [0,\tilde{T}] \to [0,T]$, there exists a control $u\colon [0,\tilde{T}] \to \mathbb{R}^m$ such that $(\gamma \circ \tau, u)$ is a controlled trajectory for Σ.

(ii) A constant rank distribution X is a ***kinematic reduction*** of Σ if every X-valued vector field is a decoupling vector field.

(iii) Σ is ***kinematically controllable*** if there exists kinematic reductions $\mathsf{X}_1, \dots, \mathsf{X}_k$ for Σ with
$$\mathsf{Lie}^{(\infty)}(\mathsf{X}_1 + \cdots + \mathsf{X}_k) = TQ. \qquad \bullet$$

To provide connections between controllability and kinematic controllability, we extend the definition of the vector-valued quadratic form $B_{\mathsf{Y}_{q_0}}$ to all of Q. Thus we define a map $B_{\mathsf{Y}}\colon \mathsf{Y} \times \mathsf{Y} \to TQ/\mathsf{Y}$ by
$$B_{\mathsf{Y}}(X,Y)(q) = B_{\mathsf{Y}_q}(X(q), Y(q))$$
for Y-valued vector fields X and Y. The notation $Q_{B_{\mathsf{Y}}}$ and λB_{Y} can then be defined in the obvious manner. Kinematic reductions are then interestingly characterised by the following result of Bullo and Lewis [2005].

3.7 Theorem: *Let $\Sigma = (Q, \nabla, \mathscr{Y} = \{Y_1, \dots, Y_m\}, \mathbb{R}^m)$ be an affine connection control system with Y of constant rank and let X be a distribution of constant rank. The following statements are equivalent:*

(i) X *is a kinematic reduction of* Σ*;*

(ii) $\mathsf{Sym}^{(1)}(\mathsf{X}) \subset \mathsf{Y}$*;*

(iii) $\mathsf{X} \subset \mathsf{Y}$ *and* $Q_{B_{\mathsf{Y}}}|\mathsf{X} = 0$.

4. Controllability for the hovercraft model

We now apply the results reviewed in Section 3 to the system of Section 2. We first characterise the accessibility of the system, since this is straightforward. Next we turn to controllability. To give a little context to the controllability of the hovercraft, we first prove two general controllability results for affine connection control systems. One result extends Theorem 3.4 to a situation where $B_{\mathsf{Y}_{q_0}}$ is semidefinite. The other result is a structural result for two-input systems with one decoupling vector field. This is an interesting application of the vector-valued quadratic form technology. Then we look at the controllability of the hovercraft. As we see, the system Σ_{hc} has quite a complicated structure as concerns its controllability. To summarise, we shall show that the system is accessible from all configurations and that the only configurations from which the system is STLC are those shown in Figure 2, and any $SE(2)$-translation of the configurations shown.

4.1. Accessibility of the hovercraft model. Let us first provide some explicit expressions for the vector fields Y_1, Y_2, and some of their symmetric products so that we can see how the analysis might proceed. The $SE(2)$-invariance of the system is useful here since we may without loss of generality evaluate all symmetric products at $(x,y,\theta,\phi) = (0,0,0,\phi)$,

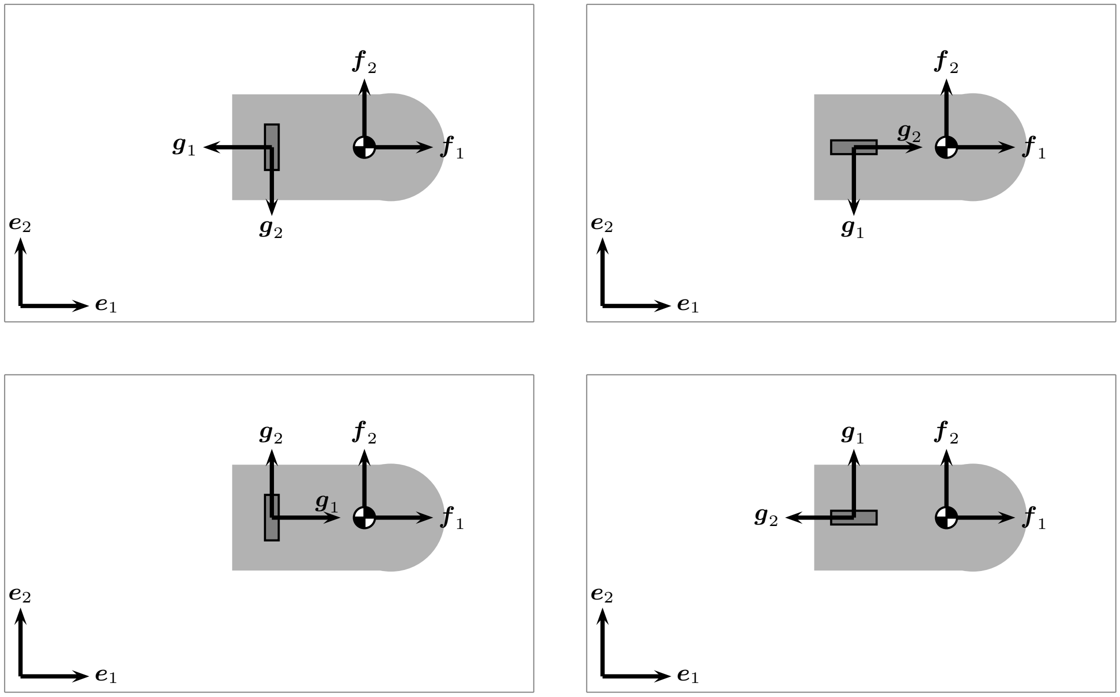

Figure 2. The configurations (up to $SE(2)$-translation) from which the hovercraft system is controllable

essentially meaning we evaluate at the group identity. Thus in the expressions immediately below this simplification is tacitly made. We introduce the constants

$$C_1 = m_{\text{body}} m_{\text{fan}} h^2 + J_{\text{body}}(m_{\text{body}} + m_{\text{fan}}), \quad M = m_{\text{body}} + m_{\text{fan}},$$

and compute

$$\begin{aligned}
Y_1 &= \frac{\cos\phi}{M}\frac{\partial}{\partial x} + \frac{J_{\text{body}}\sin\phi}{C_1}\frac{\partial}{\partial y} - \frac{m_{\text{body}} h \sin\phi}{C_1}\frac{\partial}{\partial\theta} + \frac{m_{\text{body}} h\sin\phi}{C_1}\frac{\partial}{\partial\phi}\\
Y_2 &= -\frac{m_{\text{fan}} h}{C_1}\frac{\partial}{\partial y} - \frac{M}{C_1}\frac{\partial}{\partial\theta} + \frac{C_1 + J_{\text{fan}} M}{J_{\text{fan}} C_1}\frac{\partial}{\partial\phi}\\
\langle Y_1 : Y_1\rangle &= -\frac{m_{\text{body}}^2 m_{\text{fan}} h^3 \sin(2\phi)}{M C_1^2}\frac{\partial}{\partial y} - \frac{m_{\text{body}}^2 h^2 \sin(2\phi)}{C_1^2}\frac{\partial}{\partial\theta} + \frac{m_{\text{body}}^2 h^2 \sin(2\phi)}{C_1^2}\frac{\partial}{\partial\phi}\\
\langle Y_1 : Y_2\rangle &= -\frac{\sin\phi}{J_{\text{fan}} M}\frac{\partial}{\partial x} + \frac{(J_{\text{body}} C_1 - J_{\text{fan}} m_{\text{body}} m_{\text{fan}} h^2)\cos\phi}{J_{\text{fan}} C_1^2}\frac{\partial}{\partial y}\\
&\qquad - \frac{m_{\text{body}} h (C_1 + J_{\text{fan}} M)\cos\phi}{J_{\text{fan}} C_1^2}\frac{\partial}{\partial\theta} + \frac{m_{\text{body}} h (C_1 + J_{\text{fan}} M)\cos\phi}{J_{\text{fan}} C_1^2}\frac{\partial}{\partial\phi}\\
\langle Y_2 : Y_2\rangle &= 0\\
\langle Y_1 : \langle Y_1 : Y_2\rangle\rangle &= -\frac{m_{\text{body}}^2 m_{\text{fan}} h^3 (C_1 + J_{\text{fan}} M)\cos(2\phi)}{J_{\text{fan}} M C_1^3}\frac{\partial}{\partial y}
\end{aligned}$$

$$
\begin{aligned}
&\quad - \frac{m_{\text{body}}^2 h^2 (C_1 + J_{\text{fan}} M)\cos(2\phi)}{J_{\text{fan}} C_1^3} \frac{\partial}{\partial \theta} + \frac{m_{\text{body}}^2 h^2 (C_1 + J_{\text{fan}} M)\cos(2\phi)}{J_{\text{fan}} C_1^3} \frac{\partial}{\partial \phi} \\
\langle Y_2 : \langle Y_1 : Y_2 \rangle\rangle &= -\frac{\cos\phi}{J_{\text{fan}}^2 M} \frac{\partial}{\partial x} \\
&\quad - \frac{((J_{\text{body}} C_1 - J_{\text{fan}} m_{\text{body}} m_{\text{fan}} h^2)^2 - C_1 J_{\text{fan}}^2 m_{\text{body}} m_{\text{fan}} h^2)\sin\phi}{J_{\text{body}} J_{\text{fan}}^2 C_1^3} \frac{\partial}{\partial y} \\
&\quad + \frac{m_{\text{body}} h (C_1 + J_{\text{fan}} M)^2 \sin\phi}{J_{\text{fan}}^2 C_1^3} \frac{\partial}{\partial \theta} - \frac{m_{\text{body}} h (C_1 + J_{\text{fan}} M)^2 \sin\phi}{J_{\text{fan}}^2 C_1^3} \frac{\partial}{\partial \phi}
\end{aligned}
$$

With these computations we have the following result.

4.1 Proposition: *The affine connection control system Σ_{hc} defined in Section 2 is accessible from every $q_0 \in Q$.*

Proof: Provided that $\cos(2\phi) \neq 0$ one can check that the vector fields $\{Y_1, Y_2, \langle Y_1 : Y_2\rangle, \langle Y_1 : \langle Y_1 : Y_2\rangle\rangle\}$ are linearly independent at q_0, and provided that $\sin\phi \neq 0$ one can show that the vector fields $\{Y_1, Y_2, \langle Y_1 : Y_2\rangle, \langle Y_2 : \langle Y_1 : Y_2\rangle\rangle\}$ are linearly independent at q_0. Since the set of points where both $\cos(2\phi)$ and $\sin\phi$ vanish is empty, the result follows. ■

4.2. Two new controllability results. To investigate the controllability of our hovercraft model it is interesting to state a slightly more general result that characterises systems like Σ_{hc}. Part of this is the following general result which gives a version of the necessary condition of Theorem 3.4 when the quadratic form is only semidefinite.

4.2 Theorem: *Let $\Sigma = (Q, \nabla, \mathscr{Y} = \{Y_1, \dots, Y_m\}, U)$ be an affine connection control system and let $q_0 \in Q$ have the following properties:*

(i) q_0 is a regular point for Y;

(ii) the distribution Y_1 generated by the vector fields $\{Y_1, \dots, Y_k\}$ has q_0 as a regular point;

(iii) the distribution Y_2 generated by the vector fields $\{Y_{k+1}, \dots, Y_m\}$ is a kinematic reduction for Σ;

(iv) with Y_1 and Y_2 as above, $\mathsf{Y} = \mathsf{Y}_1 \oplus \mathsf{Y}_2$;

(v) U is compact;

(vi) $B_{\mathsf{Y}_{q_0}} | \mathsf{Y}_{1,q_0}$ is definite.

Then Σ is not STLCC from q_0.

Proof: We work locally. Therefore we may assume that the sets of vector fields $\{Y_1, \dots, Y_k\}$ and $\{Y_{k+1}, \dots, Y_m\}$ are linearly independent in a neighbourhood of q_0. First we show that the system is not weakly STLC from q_0 using calculations of Bullo and Lewis [2005]. We will not provide here a self-contained justification for all of our computations since they take considerable space, but we refer to the paper [Bullo and Lewis 2005]. The calculation uses the Chen-Fliess-Sussmann series [Chen 1957, Fliess 1981, Sussmann 1983]. For an analytic control-affine system

$$
\xi'(t) = f_0(\xi(t)) + \sum_{a=1}^{m} u_a(t) f_a(\xi(t)), \qquad \xi(t) \in M,
$$

on a manifold M with a compact control set, and for an analytic function ϕ,[1] the Chen-

[1] Our use of ϕ for something other than the fan angle will be restricted to this proof.

Fliess-Sussmann series gives the following formula for the value of ϕ along a controlled trajectory (ξ, u):

$$\phi(\xi(t)) = \sum_I \Big(\int_0^t u_I\Big) f_I \phi(\xi(0)).$$

The sum is over multi-indices $I = (a_1, \dots, a_k)$ in $\{0, 1, \dots, m\}$,

$$\int_0^t u_I = \int_0^t \int_0^{\tau_k} \int_0^{\tau_{k-1}} \cdots \int_0^{\tau_2} u_{a_k}(\tau_k) u_{a_{k-1}}(\tau_{k-1}) \cdots u_{a_2}(\tau_2) u_{a_1}(\tau_1)\, \mathrm{d}\tau_1\, \mathrm{d}\tau_2 \cdots \mathrm{d}\tau_k,$$

and

$$f_I \phi = f_{a_1} f_{a_2} \cdots f_{a_k} \phi.$$

We adopt the convention that $u_0 = 1$. If we regard (3.1) as a control-affine system, this means taking $M = TQ$, $f_0 = Z$ (the geodesic spray for ∇), and $f_a = \mathrm{vlft}(Y_a)$ (the vertical lift). We refer to [Lewis and Murray 1997] for details on how affine connection control systems are control-affine systems.

The function we evaluate is defined as follows. We let λ be an analytic one-form defined in a neighbourhood of q_0 with the following properties:

1. λ annihilates the distribution Y;
2. $\lambda(q_0) B_{\mathsf{Y}_{q_0}} | \mathsf{Y}_{1,q_0}$ is negative-definite.

By a linear input transformation one can ensure that the input vector fields diagonalise $\lambda(q_0) B_{\mathsf{Y}_{q_0}}$, with the nonzero diagonal entries (i.e., those corresponding to $\{Y_1, \dots, Y_k\}$) being -1. We assume this input transformation to have been made. We then define a function ϕ_λ on TQ by $\phi_\lambda(v_q) = \lambda(q) \cdot v_q$, and we also define

$$\Phi_\lambda^+ = \{v_q \in TQ \mid \phi_\lambda(v_q) > 0\}, \quad \Phi_\lambda^- = \{v_q \in TQ \mid \phi_\lambda(v_q) < 0\}.$$

Note that in any neighbourhood V of 0_{q_0} in Q the sets $V \cap \Phi_\lambda^-$ and $V \cap \Phi_\lambda^+$ will be nonempty since ϕ_λ is linear on the fibres of TQ. Therefore, we can show that Σ is not STLC from q_0 by showing that ϕ_λ has constant sign along any controlled trajectory of Σ. One may directly verify that ϕ_λ has the following properties:

1. $f_a \phi_\lambda$, $a \in \{1, \dots, m\}$, is zero in a neighbourhood of 0_{q_0};
2. $\mathrm{ad}_{f_0}^k f_a \phi_\lambda(0_{q_0}) = 0$, $a \in \{1, \dots, m\}$, $k \in \mathbb{N}$;
3. $[f_a, [f_0, f_a]] \phi_\lambda(0_{q_0}) = -1$, $a \in \{1, \dots, k\}$ (this and the next two facts use the formula $[f_a, [f_0, f_b]] = \mathrm{vlft}(\langle Y_a : Y_b \rangle)$, $a, b \in \{1, \dots, m\}$);
4. $[f_a, [f_0, f_a]] \phi_\lambda(0_{q_0}) = 0$, $b \in \{k+1, \dots, m\}$;
5. $[f_a, [f_0, f_b]] \phi_\lambda(0_{q_0}) = 0$, $a, b \in \{1, \dots, m\}$, $a \neq b$.

For an input $u\colon [0, T] \to U$ let us define

$$\|u\|_{2,t}^1 = \max_{a \in \{1, \dots, k\}} \Big(\int_0^t |u_a(t)|^2\Big)^{1/2}.$$

The calculations of Bullo and Lewis [2005] now immediately give the following inequality for $\phi_\lambda(\gamma'(t))$ along a controlled trajectory (γ, u) for an affine connection control system like that under consideration here:

$$\phi_\lambda(\gamma'(t)) \geq \tfrac{1}{2}(\|u\|^1_{2,t})^2 - |E(t)|.$$

Here $E(t)$ is a function of t that Bullo and Lewis [2005] show to satisfy a bound

$$|E(t)| \leq tE_0(\|u\|^1_{2,t})^2$$

for some $E_0 > 0$. For t sufficiently small this shows that $\phi_\lambda(\gamma'(t))$ has constant sign, provided that $u_1, \dots, u_k$ are not zero a.e. If $u_1, \dots, u_k$ *are* zero a.e., then since $Y_{k+1}, \dots, Y_m$ are decoupling, it follows that $\phi_\lambda(\gamma'(t)) = 0$ along the corresponding controlled trajectory. This shows that Σ is not STLC from q_0.

Now let us show that our above construction also precludes Σ from being STLCC. Choose a coordinate chart (A, χ) for Q around q_0 with the following properties: (1) $\chi(q_0) = \mathbf{0}$ and (2) $\mathrm{d}q^n = \lambda(q_0)$. Let us define a function ψ_λ on the coordinate domain U by $\psi_\lambda(q) = q^n$. We define $\psi_\lambda\colon Q \to \mathbb{R}$ by $\psi_\lambda(q) = q^n$ so that the sets

$$\Psi^+_\lambda = \{q \in Q \mid \ \psi_\lambda(q) > 0\}, \quad \Psi^-_\lambda = \{q \in Q \mid \ \psi_\lambda(q) < 0\}$$

each intersect any neighbourhood of $q_0 \in Q$. Along any nonstationary trajectory $t \mapsto \gamma(t)$ we have

$$\left.\frac{\mathrm{d}\psi_\lambda(\gamma(t))}{\mathrm{d}t}\right|_{t=0} = \mathrm{d}\psi_\lambda(\gamma'(0)) = \phi_\lambda(\gamma'(0)) < 0,$$

Since $\phi_\lambda(q_0) = 0$, this means that for sufficiently small t, $\phi_\lambda(\gamma(t)) < 0$, and this shows that the points in Ψ^+_λ are not reachable in small time, and so Σ is not STLCC. ■

Now we use the preceding result to characterise a class of systems, of which our hovercraft model is one. It is helpful to have some terminology. Let us say that Σ is ***weakly STLC*** from q_0 if q_0 is in the interior of $\mathcal{R}^\Sigma_{TQ}(q_0, \leq T)$, relative to the orbit topology on the integral manifold through 0_{q_0} of the accessibility distribution. Thus weakly controllable means, roughly, that the system is "as controllable as it can be." In like manner we have the notion of Σ being ***weakly STLCC***. An ***analytic subset*** of Q is a subset $S \subset Q$ with the property that for each $q \in Q$ there exists a neighbourhood N of q for which $N \cap S$ is the common zero set of a finite number of analytic functions on N.

With this language, we have the following result.

4.3 Theorem: *Let $\Sigma = (Q, \nabla, \mathscr{Y}, U)$ be an affine connection control system for which*

(i) Y has constant rank 2,

(ii) there exists a rank 1 kinematic reduction X for Σ, and

(iii) U is compact.

Then for any $q_0 \in Q$, $B_{\mathsf{Y}_{q_0}}$ is not definite. Furthermore, we have the following dichotomy: either

(iv) there exists two rank 1 kinematic reductions $\mathsf{X}_1 = \mathsf{X}$ and X_2; or

(v) there exists a proper analytic subset S of Q with the property that if $q_0 \in Q$ then Σ is weakly STLC from q_0, and if $q_0 \in Q \setminus S$ then Q is not STLCC from q_0.

Proof: We work locally. Thus we let $\mathscr{Y} = \{Y_1, Y_2\}$ with Y_1 and Y_2 linearly independent and with Y_2 a decoupling vector field. We let $X_1, \dots, X_{n-2}$ be vector fields having the property that $\{Y_1, Y_2, X_1, \dots, X_{n-2}\}$ are linearly independent (here $n = \dim(Q)$). We then write

$$\langle Y_a : Y_b \rangle = \beta^1_{ab} Y_1 + \beta^2_{ab} Y_2 + B^1_{ab} X_1 + \cdots + B^{n-2}_{ab} X_{n-2}, \qquad a, b \in \{1, 2\},$$

for analytic functions β^c_{ab}, B^d_{ab}, $a, b, c \in \{1, 2\}$, $d \in \{1, \dots, n-2\}$. Since Y_2 is a decoupling vector field, the functions B^d_{22}, $d \in \{1, \dots, n-2\}$, are identically zero. We then define $n-2$ symmetric matrix functions on Q by

$$B^d = \begin{bmatrix} B^d_{11} & B^d_{12} \\ B^d_{12} & 0 \end{bmatrix}, \qquad a \in \{1, \dots, n-2\}.$$

To check the definiteness or indefiniteness of $B_{\mathsf{Y}_\mathsf{q}}$ we use the following observations, valid since we are considering $\dim(\mathsf{Y}) = 2$ and since Y_2 is decoupling:

1. B_{Y_q} is indefinite if and only if for every $(\lambda_1, \dots, \lambda_{n-2}) \in \mathbb{R}^{n-2}$ the symmetric matrix $\lambda_1 B^1(q) + \cdots + \lambda_{n-2} B^{n-2}(q)$ is either identically zero or has strictly negative determinant;
2. B_{Y_q} is definite if and only if there exists $(\lambda_1, \dots, \lambda_{n-2}) \in \mathbb{R}^{n-2}$ so that the symmetric matrix $\lambda_1 B^1(q) + \cdots + \lambda_{n-2} B^{n-2}(q)$ has strictly positive determinant.

Note that

$$\lambda_1 B^1 + \cdots + \lambda_{n-2} B^{n-2} = \begin{bmatrix} \lambda_1 B^1_{11} + \cdots + \lambda_{n-2} B^{n-2}_{11} & \lambda_1 B^1_{12} + \cdots + \lambda_{n-2} B^{n-2}_{12} \\ \lambda_1 B^1_{12} + \cdots + \lambda_{n-2} B^{n-2}_{12} & 0 \end{bmatrix},$$

so that

$$\det(\lambda_1 B^1 + \cdots + \lambda_{n-2} B^{n-2}) = -(\lambda_1 B^1_{12} + \cdots + \lambda_{n-2} B^{n-2}_{12})^2.$$

This shows that B_{Y_q} is never definite. One can then check for indefiniteness. To do so it is convenient to introduce two linear maps $L_{11}(q)$ and $L_{12}(q)$ from $\mathbb{R}^{n-2}$ to $\mathbb{R}$ defined by

$$\begin{aligned} L_{11}(q)(\lambda_1, \dots, \lambda_{n-2}) &= \lambda_1 B^1_{11}(q) + \cdots + \lambda_{n-2} B^{n-2}_{11}(q) \\ L_{12}(q)(\lambda_1, \dots, \lambda_{n-2}) &= \lambda_1 B^1_{12}(q) + \cdots + \lambda_{n-2} B^{n-2}_{12}(q). \end{aligned}$$

This leads to the following cases and subcases.

1. $\dim(\ker(L_{12}(q))) = n-2$:
   (a) $\dim(\ker(L_{11}(q))) = n-2$: B_{Y_q} is indefinite;
   (b) $\dim(\ker(L_{11}(q))) = n-3$: B_{Y_q} is semidefinite.
2. $\dim(\ker(L_{12}(q))) = n-3$:
   (a) $\ker(L_{12}(q)) \cap \ker(L_{11}(q)) \neq \{(0,0)\}$: B_{Y_q} is indefinite;
   (b) $\ker(L_{12}(q)) \cap \ker(L_{11}(q)) = \{(0,0)\}$: B_{Y_q} is semidefinite.

There are therefore two cases when B_{Y} is indefinite.

1. $L_{12}(q) = L_{11}(q) = 0$: Since L_{11} and L_{12} are analytic functions of q this will happen on the set of coincident zeros of the analytic functions B^d_{11}, B^d_{12}, $d \in \{1, \dots, n-2\}$.
2. $\dim(\ker(L_{12}(q)) \cap \ker(L_{11}(q))) = 1$: This will happen on the set of zeros of the analytic functions

$$f_1(q) = L_{11}(q)(\lambda_1(q), \dots, \lambda_{n-2}(q)), \quad f_2(q) = L_{12}(q)(\lambda_1(q), \dots, \lambda_{n-2}(q)),$$

where $(\lambda_1(q), \dots, \lambda_{n-2}(q))$ is a basis for $\ker(L_{12}(q)) \cap \ker(L_{11}(q))$.

This proves that we have indefiniteness of B_{Y_q} either for all $q \in Q$ or for q lying in a proper analytic subset S of Q. Furthermore, when B_{Y_q} is indefinite for all $q \in Q$ we have $B^d_{11}(q) = 0$ for all $q \in Q$ and $d \in \{1, \dots, n-2\}$. This means that Y_1 is decoupling, and so gives the existence of another rank 1 kinematic reduction. It remains to show that when B_{Y_q} is indefinite only on a proper analytic subset S, then from $q_0 \in Q \setminus S$ the system is not STLCC. This, however, follows from Theorem 4.2. ■

4.4 Remarks: The basic idea of the theorem is that for systems of the type under consideration, either the system is “nice” (it essentially possesses two decoupling vector fields), or it is “not nice” (it is uncontrollable except on a proper analytic subset). Note that the “nice” case is not the same as the system being kinematically controllable, since one would also have to check that the two decoupling vector fields have a maximal involutive closure. •

4.3. Controllability of the hovercraft model. To use the controllability conditions of Theorem 3.4 we need to compute $B_{\mathsf{Y}_{q_0}}$. To do so we need a model for the quotient $T_{q_0}Q/\mathsf{Y}_{q_0}$. It turns out to be convenient to use the g-orthogonal complement for Y_{q_0}, and one can check that the vector fields

$$X_1 = \sin\phi \frac{\partial}{\partial x} - \cos\phi \frac{\partial}{\partial y}, \quad X_2 = h\cos\phi\sin\phi \frac{\partial}{\partial x} + h\sin^2\phi \frac{\partial}{\partial y} + \frac{\partial}{\partial \theta}$$

give a basis for this orthogonal complement at $q_0 = (0, 0, 0, \phi)$ for every ϕ. Pushing these forward by left-translation will then give a global basis for the orthogonal complement to Y, although its value at $(0, 0, 0, \phi)$ is all we need. To compute $B_{\mathsf{Y}_{q_0}}$ using the basis of vector fields $\{Y_1, Y_2, X_1, X_2\}$, we write

$$\langle Y_a : Y_b \rangle = \beta^1_{ab} Y_1 + \beta^2_{ab} Y_2 + B^1_{ab} X_1 + B^2_{ab} X_2, \qquad a, b \in \{1, 2\},$$

and then the matrices

$$B^1 = \begin{bmatrix} B^1_{11} & B^1_{12} \\ B^1_{12} & B^1_{22} \end{bmatrix}, \quad B^2 = \begin{bmatrix} B^2_{11} & B^2_{12} \\ B^2_{12} & B^1_{22} \end{bmatrix}$$

may then be used to characterise $B_{\mathsf{Y}_{q_0}}$. It turns out that when one does this computation, in all terms appears the positive denominator

$$C_2 = J_{\text{fan}} C_1 M (J_{\text{fan}} M + C_1 + m^2_{\text{body}} h^2 \sin^2\phi).$$

Pulling this out of all matrices will not change their relationship to the definiteness or indefiniteness of $B_{\mathrm{Y}_{q_0}}$, so we do this to simplify the analysis. Doing the computations gives

$$B^1 = \begin{bmatrix} J_{\mathrm{fan}} m_{\mathrm{body}}^2 m_{\mathrm{fan}} h^3 \cos\phi \sin(2\phi) & C_3 \\ C_3 & 0 \end{bmatrix}$$
$$B^2 = \begin{bmatrix} -J_{\mathrm{fan}} m_{\mathrm{body}}^2 M h^2 \sin(2\phi) & -m_{\mathrm{body}} M (C_1 + J_{\mathrm{fan}} M)\cos\phi \\ -m_{\mathrm{body}} M (C_1 + J_{\mathrm{fan}} M)\cos\phi & 0 \end{bmatrix},$$

where

$$C_3 = -\frac{1}{2} M (2(h^2 m_{\mathrm{body}} (J_{\mathrm{fan}} + h^2 m_{\mathrm{body}}) m_{\mathrm{fan}} + J_{\mathrm{body}}^2 M) \sin^2\phi \\ + 2 J_{\mathrm{body}} J_{\mathrm{fan}} M + J_{\mathrm{body}}^2 M + J_{\mathrm{body}} (h^2 m_{\mathrm{body}} (m_{\mathrm{body}} + 3 m_{\mathrm{fan}})))$$

We wish now to apply Theorem 4.3. First we make the following observation.

4.5 Lemma: *Y_2 is a decoupling vector field for Σ_{hc}.*

4.6 Remark: One can ascertain that Y_2 is decoupling since we have computed $\nabla_{Y_2} Y_2 = 0$. However, this observation is also consistent, and may indeed be derived from, Lemma 2.4. What's more, Lemma 2.4 gives the nice interpretation of the integral curves of Y_2 as being zero $SE(2)$-momentum moves for the system. •

Thus we are indeed in the case described by Theorem 4.3. It turns out that we are in case (v) of the theorem, and that it is possible to explicitly describe the analytic subset S. The following result summarises this.

4.7 Proposition: *Let $q_0 = (0, 0, 0, \phi)$. The following statements hold:*

(i) Σ_{hc} is STLC from q_0 if $\sin(2\phi) = 0$;

(ii) Σ_{hc} is not STLCC from q_0 if $\sin(2\phi) \neq 0$.

Thus the system is STLC only from the configurations shown in Figure 2.

Proof: We refer to the notation used in the proof of Theorem 4.3. Also, throughout the proof we take points in Q of the form $q = (0, 0, 0, \phi)$, this being without loss of generality by $SE(2)$-invariance. With $\{Y_1, Y_2, X_1, X_2\}$ as described above we note that $\dim(\ker(L_{12}(q))) = 1$ for all $q \in Q$ since $B_{12}^1 = C_3 \neq 0$. Now let $(\lambda_1(q), \lambda_2(q)) = (-B_{12}^2(q), B_{12}^1(q))$, and note that $(\lambda_1(q), \lambda_2(q)) \in \ker(L_{12}(q))$ for all $q \in Q$. Therefore

$$\lambda_1(q) B^1(q) + \lambda_2(q) B^2(q) = \begin{bmatrix} \alpha(q) & 0 \\ 0 & 0 \end{bmatrix}$$

where we compute

$$\alpha(\phi) = -h^2 J_{\mathrm{fan}} m_{\mathrm{body}}^2 (m_{\mathrm{body}} + m_{\mathrm{fan}})(h^2 m_{\mathrm{body}} m_{\mathrm{fan}} + J_{\mathrm{body}}(m_{\mathrm{body}} + m_{\mathrm{fan}})) \\ \times (J_{\mathrm{body}} m_{\mathrm{body}} + J_{\mathrm{fan}} m_{\mathrm{body}} + J_{\mathrm{body}} m_{\mathrm{fan}} + J_{\mathrm{fan}} m_{\mathrm{fan}} \\ + h^2 m_{\mathrm{body}} m_{\mathrm{fan}} + 2 h^2 m_{\mathrm{body}}^2 \sin^2\phi) \sin(2\phi)$$

This function is zero only when $\sin(2\phi) = 0$, directly giving the result via the analysis of Theorem 4.3. ■

4.8 Remarks: 1. The system has the interesting property of possibly being STLC at a point (points where $\sin(2\phi) = 0$), but of not being STLCC at points in a neighbourhood at the point. A similar, but simpler, example was examined by Shen, Sanyal, and McClamroch [2002].

2. It is interesting to contrast our hovercraft model with the simpler ones considered in, for example, [Bullo, Leonard, and Lewis 2000, Bullo and Lynch 2001, Lewis and Murray 1997], and described briefly in Remark 2.5. In this case, the system is STLC from every point in Q [Lewis and Murray 1997], and is furthermore kinematically controllable [Bullo and Lynch 2001]. The addition of inertial properties to the fan destroys these controllability properties. •

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