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Rigid body mechanics in Galilean spacetimes

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Abstract

An observer-independent formulation of rigid body dynamics is provided in the general setting of a Galilean spacetime. The equations governing the motion of a rigid body undergoing a rigid motion in a Galilean spacetime are derived on the basis of the principle of conservation of spatial momentum. The formulation of rigid body dynamics is then studied in the presence of an observer. It is seen that an observer defines a connection such that there exist rigid motions that are horizontal with respect to this connection that give the same physical motion of the rigid body, and for which the general equations of motion are exactly the usual Euler equations for a rigid body undergoing rigid motion.

Keywords. rigid body mechanics, Galilean spacetime.

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1. Introduction

The main aim of this paper is to understand the dynamics of a rigid body in the general framework of a Galilean spacetime. We study how the physical motion of a rigid body is related to "rigid motions"– the set of mappings belonging to the group of Galilean transformations from a Galilean spacetime to itself, called the Galilean group. We present a new formulation of rigid body dynamics that is independent of an observer. Note that the momentum associated with a particle undergoing motion in a Galilean spacetime is thought of as observer-dependent quantity in the literature (see, for example, [Artz 1981]). In this paper we take the view that momentum is an intrinsic property of a rigid body in motion, and that it is possible to define it without using any external structure. An observer merely affects the way the momentum is *measured*.

The problem of deriving equations of motion for a rigid body in a Newtonian setting has a rich history. Galileo (1564–1642) carried out the first systematic study of rigid bodies

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in motion. Newton (1643–1727) built on the foundations laid by Galileo, and came up with equations of motion for a particle in an inertial frame. Later on, Euler (1703–1783) derived the equations of motion for a rigid body fixed at a point in $\mathbb{R}^3$. A modern treatment of this subject, from a general point of view of mechanics on Lie groups, can be found in [Abraham and Marsden 1978, Arnol′d 1978] The rôle of the Galilean structure of the Newtonian spacetime has been understood in the case of the dynamics of a particle [Arnol′d 1978, Artz 1981]. A Galilean covariant formulation of the classical mechanics of a single particle has been studied in [Horzela, Kapuścik, and Kempczyński 1994], and, of course, the dynamics of a rigid body in a fixed Galilean frame has been investigated quite throughly [Arnol′d 1978, Murray, Li, and Sastry 1994]. However, to our knowledge, the rôle played by the Galilean structure has not been explained for *rigid body* mechanics. Unlike a particle or a rigid body fixed at a point in $\mathbb{R}^3$, in the general setting of a Galilean spacetime, there does not exist an exact correspondence between rigid motions and physical motions of a rigid body. We address this issue in detail and derive the "Galilean–Euler equations" for a rigid body.

We also show that an observer in a Galilean spacetime, apart from providing a reference frame for observing Newton's laws, also provides an isomorphism from the "abstract" Galilean group to the "standard" or "canonical" Galilean group which consists of rotations, spatial translations, uniform velocity boosts, and time translations. Furthermore, an observer defines a connection such that, for any given rigid motion, there exists a rigid motion that is horizontal with respect to this connection that gives the same physical motion of the body as the given rigid motion, and for which the generalized equations of motion reduce to the usual Euler equations for a rigid body.

It should be noted that notions of Galilean spacetimes more general than ours have also been studied in the literature, For example, the full machinery of affine differential geometry has been used by Rodrigues, Jr., de Souza, and Bozhkov [1995] (see also [Chamorro and Chinea 1979]) while a notion of "inertial relations" has been used in [Castrigiano 1984, Castrigiano and Süssmann 1984a, Castrigiano and Süssmann 1984b] to characterize more general Galilean spacetimes.

It is also worth noting that Souriau's approach [Souriau 1997] is different from ours. In particular, he considers a symplectic formulation and his definition of momenta are based on the "Lagrange two-form." Souriau also works with the canonical Galilean group, so obscuring the rôle of the observer.

This paper is organized as follows. In Section 2, we present the mathematical preliminaries relevant to our investigation. Several important concepts like affine spaces and subspaces, observers, Galilean spacetimes, and the Galilean group are introduced and their various properties are described. The notion of a rigid body, along with its attendant features, is introduced in Section 3. In particular, the inertia tensor of a rigid body is defined and its properties are thoroughly explained. In Section 4, the structure of the canonical, as well as the abstract, Galilean group is investigated in detail. It is shown that an observer induces an isomorphism between the Galilean group and the canonical group. Next, "canonical velocities" are defined. These are curves in the Lie algebra of the Galilean group. With this background, we first look at rigid motions in Section 5. Various quantities associated with a rigid motion, such as the body and spatial linear and angular velocities, are defined. Throughout this section, the treatment is observer-independent. The discussion then focuses on angular and spatial momenta, and finally the generalized equations of motion (called the Galilean–Euler equations) for a rigid body are derived. In Section 6, the formulation presented in Section 5 is studied in the *presence* of an observer. It is shown that, in such a case, we recover the familiar quantities associated with the classical treatment of rigid body mechanics. The Galilean–Euler equations are also studied in the presence of an observer, and the connection induced by the observer (called the Galilean connection) is defined. It is shown that, for each constant velocity boost, we recover the classical Euler equations for a rigid body.

2. Galilean spacetime

In this section, we present the mathematical background and introduce the notation to be used in the following sections. In Section 2.1, we define affine spaces and subspaces; the principle objects of interest in this paper. In Section 2.2, we introduce the notion of a Galilean spacetime and describe the affine spaces naturally associated with it. We also introduce the set of Galilean velocities. Next, we define observers in a Galilean spacetime and discuss their properties. Finally, in Section 2.5, we define the Galilean group of a Galilean spacetime and introduce the fundamental maps associated with a Galilean mapping.

2.1. Affine spaces.

In this section, we define affine spaces and subspaces, and record some of their properties. We refer to [Berger 1987] for more details.

2.1 Definition: Let V be a $\mathbb{R}$-vector space. An ***affine space*** modeled on V is a pair $(\mathscr{A}, \phi)$ where $\mathscr{A}$ is a set and $\phi : V \times \mathscr{A} \to \mathscr{A}$ is a map with the following properties:

(i) for every $x, y \in \mathscr{A}$, there exists $v \in V$ such that $y = \phi(v, x)$;

(ii) $\phi(v, x) = x$ for every $x \in \mathscr{A}$ implies that $v = 0$;

(iii) $\phi(0, x) = x$, for each $x \in \mathscr{A}$;

(iv) $\phi(u + v, x) = \phi(u, \phi(v, x))$. •

We shall now cease to use the map ϕ and instead use the more suggestive notation $\phi(v, x) = v + x$. By definition, if $x, y \in \mathscr{A}$, there exists a unique $v \in V$ such that $y = v + x$. In this case we shall denote $v = y - x$. By a slight abuse of notation, we shall denote an affine space $(\mathscr{A}, \phi)$ simply by $\mathscr{A}$. If $\mathscr{A}$ is an affine space modeled on V and we fix a point $x \in \mathscr{A}$, then $\mathscr{A}$ is isomorphic to the vector space V. We denote this vector space by $\mathscr{A}_x$.

2.2 Definition: Let $\mathscr{A}$ and $\mathscr{B}$ be affine spaces. A map $f : \mathscr{A} \to \mathscr{B}$ is an ***affine map*** if, for each $x \in \mathscr{A}$, the map f is a $\mathbb{R}$-linear map between the vector spaces $\mathscr{A}_x$ and $\mathscr{B}_{f(x)}$. •

A subset $\mathscr{B}$ of an affine space modeled on V is an ***affine subspace*** if there is a subspace U of V with the property that $\mathscr{B} = \{u + x \mid u \in U\}$ for some $x \in \mathscr{B}$. In this case, $\mathscr{B}$ is itself an affine space modeled on U.

2.2. Time and distance.

We begin by giving the basic definition of a Galilean spacetime and by providing meaning to the intuitive notions of time and distance.

2.3 Definition: A ***Galilean spacetime*** is a quadruple $\mathscr{G} = (\mathscr{E}, V, g, \tau)$ where

(i) V is a four-dimensional $\mathbb{R}$–vector space,

(ii) $\tau : V \to \mathbb{R}$ is a surjective linear map called the ***time map***,

(iii) g is an inner product on $\ker(\tau)$, and

(iv) $\mathscr{E}$ is an affine space modeled on V. •

Points in $\mathscr{E}$ are called ***events***–thus $\mathscr{E}$ is a model for the spatio-temporal world of Newtonian mechanics. With the time map, we may measure the time between two events $x_1, x_2 \in \mathscr{E}$ as $\tau(x_2 - x_1)$. Events $x_1, x_2 \in \mathscr{E}$ are called ***simultaneous*** if $\tau(x_2 - x_1) = 0$; that is, if $x_2 - x_1 \in \ker(\tau)$.

We may define the ***distance*** between simultaneous events $x_1, x_2 \in \mathscr{E}$ to be equal to $\sqrt{g(x_2 - x_1, x_2 - x_1)}$. Note that this method for defining distance does not allow us to measure distances between events that are *not* simultaneous. In particular, it doesn't make sense to talk about two non-simultaneous events as occurring in the same place.

Simultaneity is an equivalence relation on $\mathscr{E}$ and the quotient we denote by $I_{\mathscr{G}} = \mathscr{E}/\sim$, with $\sim$ denoting the relation of simultaneity. $I_{\mathscr{G}}$ is simply the collection of equivalence classes of simultaneous events. We call it the ***set of instants***. We denote by $\pi_{\mathscr{G}} : \mathscr{E} \to I_{\mathscr{G}}$ the canonical projection.

For $s \in I_{\mathscr{G}}$, we denote by $\mathscr{E}(s)$ the collection of events $x \in \mathscr{E}$ with the property that $\pi_{\mathscr{G}}(x) = s$. Thus events in $\mathscr{E}(s)$ are simultaneous. We next denote by $V_{\mathscr{G}}$ the vectors $v \in V$ for which $\tau(v) = 1$. We call vectors in $V_{\mathscr{G}}$ ***Galilean velocities***. The following result is easy to prove.

2.4 Proposition: *The following statements hold:*

(i) for each $s \in I_{\mathscr{G}}$, $\mathscr{E}(s)$ is a three-dimensional affine space modeled on $\ker(\tau)$;

(ii) $I_{\mathscr{G}}$ is a one-dimensional affine space modeled on $\mathbb{R}$;

(iii) $V_{\mathscr{G}}$ is an affine space modeled on $\ker(\tau)$.

2.3. Observers. An observer is to be thought of intuitively as someone who is present at each instant. Such an observer should be moving at a "uniform velocity." Note that, in a Galilean spacetime, the notion of "stationarity" makes no sense. We now provide our definition of an observer.

2.5 Definition: An ***observer*** in a Galilean spacetime $\mathscr{G} = (\mathscr{E}, V, g, \tau)$ is a one-dimensional affine subspace $\mathscr{O}$ of $\mathscr{E}$ with the property that $\pi_{\mathscr{G}}|\mathscr{O}$ is surjective. •

The definition thus requires that $\mathscr{O}$ not be comprised entirely of simultaneous events. As a consequence of the definition, we have the following result.

2.6 Proposition: *If $\mathscr{O}$ is an observer in a Galilean spacetime $\mathscr{G} = (\mathscr{E}, V, g, \tau)$, then, for each $s_0 \in I_{\mathscr{G}}$, there exists a unique point $x_0 \in \mathscr{O} \cap \mathscr{E}(s_0)$.*

Proof: Given $s_0 \in I_{\mathscr{G}}$, there exists $x_0 \in \mathscr{O}$ such that $\pi_{\mathscr{G}}(x_0) = s_0$. Since $\mathscr{O}$ is a one-dimensional affine space with the property that $\pi_{\mathscr{G}}|\mathscr{O}$ is surjective, there exists a one-dimensional subspace $W \subset V$ such that

$$\mathscr{O} = \{w + x_0 |\ w \in W\},$$

where W is not contained in $\ker(\tau)$. By construction, $x_0 \in \mathscr{E}(s_0) \cap \mathscr{O}$. This means that, for each $s_0 \in I_{\mathscr{G}}$, the intersection $\mathscr{E}(s_0) \cap \mathscr{O}$ is non-empty. Next, let $y_0 \in \mathscr{E}(s_0) \cap \mathscr{O}$. Now, $y_0 \in \mathscr{E}(s_0)$ implies that $y_0 - x_0 \in \ker(\tau)$. On the other hand, $y_0 \in \mathscr{O}$ implies that $y_0 - x_0 \in W$. This means that $y_0 = x_0$ which proves the uniqueness of x_0. ■

This means that an observer does exactly what it should: it resides in exactly one place at each instant. By requiring that $\mathscr{O}$ be an affine subspace, we ensure that it has a "uniform velocity" and so is an appropriate reference for observing Newton's laws. We shall denote by $\mathscr{O}_s$ the unique point in the intersection $\mathscr{O} \cap \mathscr{E}(s)$.

Since an observer $\mathscr{O}$ is a one-dimensional affine subspace, there is a unique one-dimensional subspace U of V upon which $\mathscr{O}$ is modeled. Therefore, there exists a unique vector $v_{\mathscr{O}} \in U \subset V_{\mathscr{G}}$ with the property that $\tau(v_{\mathscr{O}}) = 1$. Conversely, given $v \in V_{\mathscr{G}}$ and $x \in \mathscr{E}$, there exists a unique observer $\mathscr{O}$ such that $x \in \mathscr{O}$ and $v = v_{\mathscr{O}}$. We call $v_{\mathscr{O}}$ the ***Galilean velocity*** of the observer $\mathscr{O}$. It provides a reference velocity with which we can measure other velocities. Indeed, given an observer $\mathscr{O}$, we may define an associated map $P_{\mathscr{O}} : V \to \ker(\tau)$ by $P_{\mathscr{O}}(v) = v - (\tau(v))v_{\mathscr{O}}$. In particular, if $v \in V_{\mathscr{G}}$, we note that $v = v_{\mathscr{O}} + P_{\mathscr{O}}(v)$. Thus $P_{\mathscr{O}}$ can be thought of as giving the velocity of v relative to the observer's Galilean velocity $v_{\mathscr{O}}$. Note that such velocities always live in the three-dimensional vector space $\ker(\tau)$ that is to be thought of as the space of velocities that are familiar in mechanics. Such velocities are, however, only defined relative to an observer.

2.4. World lines. Intuitively, a world line is to be thought of as being the spatio-temporal history of something moving in the spacetime. We make the following definition.

2.7 Definition: Let $\mathscr{G} = (\mathscr{E}, V, g, \tau)$ be a Galilean spacetime. A ***world line*** in $\mathscr{G}$ is a section of $\pi_{\mathscr{G}} : \mathscr{E} \to I_{\mathscr{G}}$. •

A world line $c : I_{\mathscr{G}} \to \mathscr{E}$ is ***differentiable*** at $s_0 \in I_{\mathscr{G}}$ if the limit

$$c'(s_0) := \lim_{t\to 0} \frac{c(t + s_0) - c(s_0)}{t}$$

exists. Since c is a section of $\pi_{\mathscr{G}}$, we have $\tau(c(t + s_0) - c(s_0)) = t$ and so $c'(s_0) \in V_{\mathscr{G}}$, provided it exists. Similarly, for a differentiable world line, if the limit

$$\lim_{t\to 0} \frac{c'(t + s_0) - c'(s_0)}{t}$$

exists, we denote it by $c''(s_0)$, the acceleration of the world line at the instant s_0. Since

$$\tau(c''(s_0)) = \lim_{t\to 0} \frac{\tau(c'(t + s_0) - c'(s_0))}{t} = \lim_{t\to 0} \frac{1 - 1}{t} = 0,$$

we have $c''(s_0) \in \ker(\tau)$.

2.5. Galilean mappings. If $\mathscr{G}_i = (\mathscr{E}_i, V_i, g_i, \tau_i)$, $i = 1, 2$, are two Galilean spacetimes, a ***Galilean mapping*** from $\mathscr{G}_1$ to $\mathscr{G}_2$ is a map $\psi : \mathscr{E}_1 \to \mathscr{E}_2$ with the following properties:

(i) ψ is an affine map;

(ii) $\tau_2(\psi(x_1) - \psi(x_2)) = \tau_1(x_1 - x_2)$ for $x_1, x_2 \in \mathscr{E}_1$;

(iii) $g_2(\psi(x_1)-\psi(x_2), \psi(x_1)-\psi(x_2)) = g_1(x_1-x_2, x_1-x_2)$ for simultaneous events $x_1, x_2 \in \mathscr{E}_1$.

The set of Galilean mappings from a Galilean spacetime $\mathscr{G}$ to itself is a Lie group (under composition of Galilean mappings), and we call it the ***Galilean group*** of the Galilean spacetime $\mathscr{G}$. We shall denote this group by $\mathrm{Gal}(\mathscr{G})$ and its Lie algebra by $\mathfrak{gal}(\mathscr{G})$. If $\psi \in \mathrm{Gal}(\mathscr{G})$, then there are induced natural mappings of $V, I_{\mathscr{G}}$, and $\mathbb{R}$ as follows.

2.8 Lemma: *Let $\mathscr{G} = (\mathscr{E}, V, g, \tau)$ be a Galilean spacetime with $\psi \in \mathrm{Gal}(\mathscr{G})$. The following mappings are well defined:*

(i) the mapping $\psi_V : V \to V$ defined by $\psi_V(v) = \psi(v + x_0) - \psi(x_0)$, where $x_0 \in \mathscr{E}$;

(ii) the mapping $\psi_{I_{\mathscr{G}}} : I_{\mathscr{G}} \to I_{\mathscr{G}}$ defined by $\psi_{I_{\mathscr{G}}}(s) = \pi_{\mathscr{G}}(\psi(x))$, where $x \in \mathscr{E}(s)$;

(iii) the mapping $\psi_\tau : \mathbb{R} \to \mathbb{R}$ defined by $\psi_\tau(t) = t + \psi_{I_{\mathscr{G}}}(s) - s$, for $s \in I_{\mathscr{G}}$.

Furthermore,

(iv) $\psi_V(v) = \psi(x_1) - \psi(x_2)$, where $x_2 - x_2 = v$, and

(v) there exists $t_\psi \in \mathbb{R}$ such that $\psi_{I_{\mathscr{G}}}(s) = s + t_\psi$ and $\psi_\tau(t) = t + t_\psi$.

Proof: (i) Let $x_0, \bar{x}_0 \in \mathscr{E}$. We have

$$\begin{aligned}\psi(v + x_0) - \psi(x_0) &= \psi((v + (x_0 - \bar{x}_0)) + \bar{x}_0) - \psi((x_0 - \bar{x}_0) + \bar{x}_0)\\ &= \psi((x_0 - \bar{x}_0) + \bar{x}_0) + \psi(v + \bar{x}_0) - \psi(\bar{x}_0) - \psi((x_0 - \bar{x}_0) + \bar{x}_0)\\ &= \psi(v + \bar{x}_0) - \psi(\bar{x}_0),\end{aligned}$$

where we have used the property

$$\psi((v_1 + v_2) + x) = (\psi(v_1 + x) + \psi(v_2 + x)) - \psi(x),$$

since ψ is an affine map. This property is readily verified using the definition of an affine map. We will now show that $\ker(\tau)$ is an invariant subspace for ψ_V. We let $x, \tilde{x} \in \mathscr{E}$ have the property that $x - \tilde{x} = u \in \ker(\tau)$. Then

$$\tau(\psi_V(u)) = \tau(\psi(x) - \psi(\tilde{x})) = \tau(u) = 0,$$

where we have used property (ii) of Galilean mappings.

(ii) Let $x, \tilde{x} \in \mathscr{E}(s)$. There exists $u \in \ker(\tau)$ such that $\tilde{x} = u + x$. Now we compute

$$\begin{aligned}\pi_{\mathscr{G}}(\psi(\tilde{x})) - \pi_{\mathscr{G}}(\psi(x)) &= \tau(\psi(\tilde{x}) - \psi(x))\\ &= \tau(\psi_V(\tilde{x} - x))\\ &= \tau(\psi_V(u)) = 0,\end{aligned}$$

using the fact that $\ker(\tau)$ is an invariant subspace for ψ_V.

(iii) We must show that the definition is independent of the choice of $s \in I_{\mathscr{G}}$. For $\tilde{s} \in I_{\mathscr{G}}$, we compute

$$\begin{aligned}t + \psi_{I_{\mathscr{G}}}(\tilde{s}) - \tilde{s} &= t + \psi_{I_{\mathscr{G}}}(s + (\tilde{s} - s)) - s + (s - \tilde{s})\\ &= t + \psi_{I_{\mathscr{G}}}(s + (\tilde{s} - s)) - \psi_{I_{\mathscr{G}}}(s) + \psi_{I_{\mathscr{G}}}(s) - s + (s - \tilde{s})\\ &= t + \psi_{I_{\mathscr{G}}}(s) - s.\end{aligned}$$

(iv) We have

$$\begin{aligned}\psi(x_1)-\psi(x_2)&=\psi((x_1-x_2)+x_2)-\psi(x_2)\\&=\psi_V(x_1-x_2),\end{aligned}$$

as desired.

(v) Let $x_0\in\mathscr{E}$ and let $t_\psi=\tau(\psi(x_0)-x_0)$. For $s\in I_{\mathscr{G}}$, let $x\in\mathscr{E}(s)$. We then have

$$\begin{aligned}\psi_{I_{\mathscr{G}}}(s)-s&=\psi_{I_{\mathscr{G}}}(s)-\pi_{\mathscr{G}}(x)\\&=\pi_{\mathscr{G}}(\psi(x))-\pi_{\mathscr{G}}(x)\\&=\tau(\psi(x)-\psi(x_0))+\tau(\psi(x_0)-x_0)+\tau(x_0-x)\\&=\tau(\psi(x_0)-x_0)=t_\psi,\end{aligned}$$

where we have used the property (ii) of Galilean mappings. This shows that the definition of t_ψ is independent of x_0, and that $\psi_{I_{\mathscr{G}}}(s)=t_\psi+s$, as desired. From (iii) it also follows that $\psi_\tau(t)=t_\psi+t$. ■

2.9 Remarks: 1. In the proof of the lemma we showed that, given $\psi\in\mathrm{Gal}(\mathscr{G})$, ψ_V leaves $\ker(\tau)$ invariant. We shall see in the next section that $\psi_V|_{\ker(\tau)}$ has a mechanical interpretation.

2. Using the definition of a Galilean mapping, it is easy to see that $V_{\mathscr{G}}$ is also invariant under ψ_V. •

Given a Galilean spacetime $\mathscr{G}$, we let $O(\ker(\tau))$ denote the g-orthogonal linear mappings of $\ker(\tau)$. The Lie algebra of $O(\ker(\tau))$ we denote by $\mathfrak{o}(\ker(\tau))$, recalling that it is the collection of g-skew symmetric linear mappings of $\ker(\tau)$. We identify $\ker(\tau)$ with $\mathfrak{o}(\ker(\tau))$ by the "hat" map (see [Murray, Li, and Sastry 1994]) given by $\omega\mapsto\hat{\omega}$. This is a generalization of the map from $\mathbb{R}^3$ to $\mathfrak{o}(3)$ defined by

$$\begin{bmatrix}\omega_1\\\omega_2\\\omega_3\end{bmatrix}\mapsto\begin{bmatrix}0&-\omega_3&\omega_2\\\omega_3&0&-\omega_1\\-\omega_2&\omega_1&0\end{bmatrix},$$

and may be explicitly defined by choosing an orthonormal basis for $\ker(\tau)$ and then applying this transformation to the components in this basis. Since the vector product in $\mathbb{R}^3$ commutes with orthogonal transformations, this definition is independent of the choice of orthonormal basis. In like manner, one can define $u_1\times u_2$ for any $u_1,u_2\in\ker(\tau)$ as the generalization of the $\mathbb{R}^3$ vector product.

3. Rigid bodies

In order to talk about momenta, we need the notion of a rigid body. In this section we provide our definition for a rigid body and provide some implications of this definition. We begin by proving, in our Galilean setting, some of the basic properties of the inertia tensor of a rigid body.

3.1. Definitions. Let $\mathscr{G} = (\mathscr{E}, V, g, \tau)$ be a Galilean spacetime. A ***rigid body*** is a pair $(\mathcal{B}, \mu)$, where $\mathcal{B} \subset \mathscr{E}(s_0)$ is a compact subset of simultaneous events, and μ is a mass-distribution on $\mathscr{E}(s_0)$ with support equal to $\mathcal{B}$. Our definition thus allows such degenerate rigid bodies as point masses, and bodies whose mass distribution is contained in a line in $\mathscr{E}(s_0)$. We denote

$$\mu(\mathcal{B}) = \int_{\mathcal{B}} \mathrm{d}\mu$$

as the ***mass*** of the body.

The ***center of mass*** of the body $(\mathcal{B}, \mu)$ is the point

$$x_c = \frac{1}{\mu(\mathcal{B})} \int_{\mathcal{B}} (x - x_0)\mathrm{d}\mu + x_0.$$

Note that the integrand is in $\ker(\tau)$ and so too will be the integral. The following lemma gives some of the basic properties of this definition. If $\mathscr{S} \subset \mathscr{A}$ is a subset of an affine space $\mathscr{A}$, we let $\mathrm{conv}(\mathscr{S})$ denote the convex hull of $\mathscr{S}$ and $\mathrm{aff}(\mathscr{S})$ denote the affine hull of $\mathscr{S}$. If X is a topological space with subsets $T \subset S \subset X$, $\mathrm{int}_S(T)$ denotes the interior of T relative to the induced topology on S.

3.1 Lemma: *Let $(\mathcal{B}, \mu)$ be a rigid body in a Galilean spacetime with $\mathcal{B} \subset \mathscr{E}(s_0)$. The following statements hold:*

(i) the expression

$$x_c = \frac{1}{\mu(\mathcal{B})} \int_{\mathcal{B}} (x - x_0)\mathrm{d}\mu + x_0$$

is independent of the choice of $x_0 \in \mathscr{E}(s_0)$;

(ii) x_c is the unique point in $\mathscr{E}(s_0)$ with the property that $\int_{\mathcal{B}} (x - x_c)\mathrm{d}\mu = 0$;

(iii) $x_c \in \mathrm{int}_{\mathrm{aff}(\mathcal{B})}(\mathrm{conv}(\mathcal{B}))$.

Proof: (i) To check that the definition of x_c is independent of $x_0 \in \mathscr{E}(s_0)$, we let $\tilde{x}_0 \in \mathscr{E}(s_0)$ and compute

$$\begin{aligned} \frac{1}{\mu(\mathcal{B})} \int_{\mathcal{B}} (x - \tilde{x}_0)\mathrm{d}\mu + \tilde{x}_0 &= \frac{1}{\mu(\mathcal{B})} \int_{\mathcal{B}} (x - x_0)\mathrm{d}\mu + \frac{1}{\mu(\mathcal{B})} \int_{\mathcal{B}} (x_0 - \tilde{x}_0)\mathrm{d}\mu \\ &\quad + (\tilde{x}_0 - x_0) + x_0 \\ &= \frac{1}{\mu(\mathcal{B})} \int_{\mathcal{B}} (x - x_0)\mathrm{d}\mu + x_0. \end{aligned}$$

(ii) By definition of x_c and by part (i), we have

$$x_c = \frac{1}{\mu(\mathcal{B})} \int_{\mathcal{B}} (x - x_c)\mathrm{d}\mu + x_c,$$

from which it follows that

$$\int_{\mathcal{B}} (x - x_c)\mathrm{d}\mu = \mu(\mathcal{B})(x_c - x_c) = 0.$$

Now suppose that $\tilde{x}_c \in \mathscr{E}(s_0)$ is an arbitrary point with the property that

$$\int_{\mathcal{B}} (x - \tilde{x}_c)\mathrm{d}\mu = 0.$$

Then, by (i),

$$x_c = \frac{1}{\mu(\mathcal{B})} \int_{\mathcal{B}} (x - \tilde{x}_c)\mathrm{d}\mu + \tilde{x}_c,$$

from which we conclude that $\tilde{x}_c = x_c$.

(iii) If x_c is on the relative boundary of $\mathrm{conv}(\mathcal{B})$ or not in $\mathcal{B}$ at all, then there exists a hyperplane P in $\mathscr{E}(s_0)$ passing through x_c such that there are points in $\mathcal{B}$ which lie on one side of P, but there are no points in $\mathcal{B}$ on the opposite side. In other words, there exists $\lambda \in \ker(\tau)^\star$ such that the set

$$\{x \in \mathcal{B} |\ \lambda(x - x_c) > 0\}$$

is non-empty, but the set

$$\{x \in \mathcal{B} |\ \lambda(x - x_c) < 0\}$$

is empty. But this would imply that

$$\int_{\mathcal{B}} \lambda(x - x_c)\mathrm{d}\mu > 0,$$

contradicting (ii). ■

3.2. The inertia tensor. The properties of a rigid body are characterized by three things: (1) its mass, (2) its center of mass, and (3) its inertia tensor. We now define the latter. Let $x_0 \in \mathscr{E}(s_0)$. The ***inertia tensor*** about x_0 of a rigid body $(\mathcal{B}, \mu)$ to be the linear map $\mathbb{I}_{x_0} : \ker(\tau) \to \ker(\tau)$ defined by

$$\mathbb{I}_{x_0}(u) = \int_{\mathcal{B}} (x - x_0) \times (u \times (x - x_0))\mathrm{d}\mu.$$

We denote the inertia tensor about the center of mass of $(\mathcal{B}, \mu)$ by $\mathbb{I}_c$. Next, we record some basic properties of the inertia tensor.

3.2 Proposition: *The inertia tensor $\mathbb{I}_{x_0}$ of a rigid body $(\mathcal{B}, \mu)$ is symmetric with respect to the inner product g.*

Proof: Using the vector identity $g(u, v \times w) = g(w, u \times v)$, we compute

$$\begin{aligned} g(\mathbb{I}_{x_0}(u_1), u_2) &= \int_{\mathcal{B}} g((x - x_0) \times (u_1 \times (x - x_0)), u_2)\mathrm{d}\mu \\ &= \int_{\mathcal{B}} g(u_1 \times (x - x_0), (u_2 \times (x - x_0))\mathrm{d}\mu \\ &= \int_{\mathcal{B}} g(u_1, (x - x_0) \times (u_2 \times (x - x_0)))\mathrm{d}\mu \\ &= g(u_1, \mathbb{I}_{x_0}(u_2)), \end{aligned}$$

which is what we wished to show. ■

3.3. Eigenvalues of the inertia tensor. Since $\mathbb{I}_{x_0}$ is symmetric, its eigenvalues are real. Furthermore, they are non-negative. The following result demonstrates this, as well as other eigenvalue related assertions.

3.3 Proposition: *Let $(\mathcal{B}, \mu)$ be a rigid body with $\mathcal{B} \in \mathscr{E}(s_0)$ and let $x_0 \in \mathscr{E}(s_0)$. Let $\mathbb{I}_{x_0}$ denote the inertia tensor of $(\mathcal{B}, \mu)$ about x_0. The following statements hold:*

(i) the eigenvalues of the inertia tensor $\mathbb{I}_{x_0}$ of a rigid body are real and nonnegative;

(ii) if $\mathbb{I}_{x_0}$ has a zero eigenvalue, then the other two eigenvalues are equal;

(iii) if $\mathbb{I}_{x_0}$ has two zero eigenvalues, then $\mathbb{I}_{x_0} = 0$.

Proof: (i) Since $\mathbb{I}_{x_0}$ is symmetric, its eigenvalues will be nonnegative if and only if the quadratic form $u \mapsto g(\mathbb{I}_{x_0}(u), u)$ is positive-semidefinite. For $u \in \ker(\tau)$, we compute

$$\begin{aligned} g(\mathbb{I}_{x_0}(u), u) &= \int_{\mathcal{B}} g(u, (x - x_0) \times (u \times (x - x_0)))\mathrm{d}\mu \\ &= \int_{\mathcal{B}} g(u \times (x - x_0), u \times (x - x_0))\mathrm{d}\mu. \end{aligned}$$

Since the integrand is nonnegative, so too will be the integral.

(ii) Let I_1 be the zero eigenvalue with v_1 a unit eigenvector. We claim that the support of the mass distribution μ must be contained in the line

$$\ell_{v_1} = \{sv_1 + x_0 |\ s \in \mathbb{R}\}.$$

To see that this must be so, suppose that the support of μ is not contained in ℓ_{v_1}. Then there exists a Borel set $S \subset \mathscr{E}(s_0) \setminus \ell_{v_1}$ such that $\mu(S) > 0$. This would imply that

$$\begin{aligned} g(\mathbb{I}_{x_0}(v_1), v_1) &= \int_{\mathcal{B}} g(v_1 \times (x - x_0), v_1 \times (x - x_0))\mathrm{d}\mu \\ &\geq \int_{S} g(v_1 \times (x - x_0), v_1 \times (x - x_0))\mathrm{d}\mu. \end{aligned}$$

Since $S \cap \ell_{v_1} = \varnothing$, it follows that, for all points $x \in S$, the vector $x - x_0$ is not collinear with v_1. Therefore

$$g(v_1 \times (x - x_0), v_1 \times (x - x_0)) > 0$$

for all $x \in S$, and this would imply that $g(\mathbb{I}_{x_0}(v_1), v_1) > 0$. But this contradicts v_1 being an eigenvector with zero eigenvalue, and so the support of $\mathcal{B}$ must be contained in the line ℓ_{v_1}.

To see that this implies that the remaining two eigenvectors are equal, we shall show that any vector that is g-orthogonal to v_1 is an eigenvector for $\mathbb{I}_{x_0}$. First write

$$x - x_0 = f^1(x)v_1 + f^2(x)v_2 + f^3(x)v_3$$

for functions $f^i : \mathscr{E}(s_0) \to \mathbb{R},\ i = 1, 2, 3$. Since the support of μ is contained in the line ℓ_{v_1}, we have

$$\int_{\mathcal{B}} (x - x_0) \times (u \times (x - x_0))\mathrm{d}\mu = v_1 \times (u \times v_1) \int_{\mathcal{B}} (f^1(x))^2 \mathrm{d}\mu$$

for all $u \in \ker(\tau)$. Now recall the property of the cross product that $v_1 \times (u \times v_1) = u$, provided that u is orthogonal to v_1 and that, v_1 has unit length. Therefore, we see that, for any u that is orthogonal to v_1, we have

$$\mathbb{I}_{x_0}(u) = \left(\int_{\mathcal{B}} (f^1(x))^2 \mathrm{d}\mu\right) u,$$

meaning that all such vectors u are eigenvectors with the same eigenvalue, which is what we wished to show.

(iii) It follows from the above arguments that, if two eigenvalues I_1 and I_2 are zero, then the support of μ must lie in the intersection of the lines ℓ_{v_1} and ℓ_{v_2} (here v_i is an eigenvector for $I_i,\ i = 1, 2$), and this intersection is a single point, that must therefore be x_0. From this and the definition of $\mathbb{I}_{x_0}$, it follows that $\mathbb{I}_{x_0} = 0$. ■

Note that, in proving the result, we have proved the following corollary.

3.4 Corollary: *Let $(\mathcal{B}, \mu)$ be a rigid body with inertia tensor $\mathbb{I}_{x_0}$. The following statements are true:*

(i) $\mathbb{I}_{x_0}$ has a zero eigenvalue if and only if $\mathcal{B}$ is contained in a line through x_0;

(ii) if $\mathbb{I}_{x_0}$ has two zero eigenvalues, then $\mathcal{B} = \{x_0\}$, i.e., $\mathcal{B}$ is a particle located at x_0;

(iii) if there is no line through x_0 that contains the support of μ, then the inertia tensor is an isomorphism.

In coming to an understanding of the "appearance" of a rigid body, it is most convenient to refer to its inertia tensor $\mathbb{I}_c$ about its center of mass. Let $\{I_1, I_2, I_3\}$ be the eigenvalues of $\mathbb{I}_c$ that we call the ***principal inertias*** of $(\mathcal{B}, \mu)$. If $\{v_1, v_2, v_3\}$ are orthonormal eigenvectors associated with these eigenvalues, we call these the ***principal axes*** of $(\mathcal{B}, \mu)$. Related to these is the ***inertial ellipsoid*** which is the ellipsoid in $\ker(\tau)$ given by

$$E(\mathcal{B}) = \left\{x^1 v_1 + x^2 v_2 + x^3 v_3 \in \ker(\tau) \middle|\ I_1(x^1)^2 + I_2(x^2)^2 + I_3(x^3)^2 = 1\right\},$$

provided that none of the eigenvalues of $\mathbb{I}_{x_0}$ are zero. If one of the eigenvalue *does* vanish, then by Proposition 3.3, the other two eigenvalues are equal. If we suppose that $I_1 = 0$ and that $I_2 = I_3 = I$, then in the case of a single zero eigenvalue, the inertial ellipsoid is

$$E(\mathcal{B}) = \left\{x^1 v_1 + x^2 v_2 + x^3 v_3 \in \ker(\tau) \middle|\ x^2 = x^3 = 0,\ x^1 \in \left\{-\tfrac{1}{\sqrt{I}}, \tfrac{1}{\sqrt{I}}\right\}\right\}.$$

In the most degenerate case, when all eigenvalues are zero, we define $E(\mathcal{B}) = \{0\}$. These latter two inertial ellipsoids correspond to cases (ii) and (iii) in Corollary 3.4.

To relate these properties of the eigenvalues of $\mathbb{I}_c$ with the inertial ellipsoid $E(\mathcal{B})$, it is helpful to introduce the notion of an axis of symmetry for a rigid body. We let $\mathbb{I}_c$ be the inertia tensor about the center of mass, and denote by $\{I_1, I_2, I_3\}$ its eigenvalues and $\{v_1, v_2, v_3\}$ its orthogonal eigenvectors. A vector $v \in \ker(\tau) \setminus \{0\}$ is an ***axis of symmetry*** for $(\mathcal{B}, \mu)$ if, for every $R \in O(\ker(\tau))$ which fixes v, we have $R(E(\mathcal{B})) = E(\mathcal{B})$. The following result gives the relationship between axes of symmetry and the eigenvalues of $\mathbb{I}_c$.

3.5 Proposition: *Let $(\mathcal{B}, \mu)$ be a rigid body with inertia tensor $\mathbb{I}_c$ about its center of mass. Let $\{I_1, I_2, I_3\}$ be the eigenvalues of $\mathbb{I}_c$ with orthonormal eigenvectors $\{v_1, v_2, v_3\}$. If $I_1 = I_2$, then v_3 is an axis of symmetry for $(\mathcal{B}, \mu)$.*

Conversely, if $v \in \ker(\tau)$ is an axis of symmetry, then v is an eigenvector of $\mathbb{I}_c$. If I is the eigenvalue for which v is an eigenvector, then the other two eigenvalues of $\mathbb{I}_c$ are equal.

Proof: Write $I_1 = I_2 = I$. We then see that any vector $v \in \operatorname{span}_{\mathbb{R}}\{v_1, v_2\}$ will have the property that $\mathbb{I}_c(v) = Iv$. Now, let $R \in O(\ker(\tau))$ fix the vector v_3. Because R is orthogonal, if we have $v \in \operatorname{span}_{\mathbb{R}}\{v_1, v_2\}$, then $R(v) \in \operatorname{span}_{\mathbb{R}}\{v_1, v_2\}$. Also, if $v = a^1 v_1 + a^2 v_2$, then,

$$R(v) = (\cos\theta a^1 + \sin\theta a^2)v_1 + (-\sin\theta a^1 + \cos\theta a^2)v_2 \tag{3.1}$$

for some $\theta \in \mathbb{R}$, since R is simply a rotation in the plane spanned by v_1, v_2. Now let $u \in E(\mathcal{B})$. We then write $u = x^1 v_1 + x^2 v_2 + x^3 v_3$ and note that

$$I(x^1)^2 + I(x^2)^2 + I_3(x_3)^2 = 1.$$

It is now a straightforward calculation to verify that $R(u) \in E(\mathcal{B})$ using (3.1) and the fact that R fixes v_3. This shows that $R(E(\mathcal{B})) = E(\mathcal{B})$, and so v_3 is an axis of symmetry for $(\mathcal{B}, \mu)$.

For the second part of the proposition, let v be an axis of symmetry for $(\mathcal{B}, \mu)$. Denote the set of orthogonal mappings that fix v by $O(v)$. That is, let

$$O(v) = \{R \in O(\ker(\tau)) |\ R(v) = v\}.$$

Now $R \in O(v)$ has the property that $R(E(\mathcal{B})) = E(\mathcal{B})$, and thus maps principal axes of $(\mathcal{B}, \mu)$ to principal axes. Since $\{v_1, v_2, v_3\}$ form an orthonormal basis for $\ker(\tau)$, it is clear that, for every $R \in O(v)$, the set $\{R(v_1), R(v_2), R(v_3)\}$ is also an orthonormal basis. It can be seen that every vector orthogonal to v is a principal axes and thus, without loss of generality, we can take $\frac{v}{\|v\|} = v_3$. It is now clear that $\mathbb{I}_c$ acts on $v^\perp$ by scalars, and thus v is an eigenvector of $\mathbb{I}_c$. The result now follows. ■

4. The structure of the Galilean group

As defined previously, the Galilean group of a Galilean spacetime $\mathscr{G} = (\mathscr{E}, V, g, \tau)$ is the set of affine maps from $\mathscr{E}$ to itself that preserve simultaneity of events and the distance between simultaneous events. In this section, we shall examine the Galilean group and describe its properties. In Section 4.1 we study the canonical Galilean group and show that it consists of rotations, translations, velocity boosts, and temporal origin shifts. We also look at its subgroups and describe the various fundamental objects associated with it. In Section 4.2, we study the abstract Galilean group $\mathrm{Gal}(\mathscr{G})$. We show that, in the presence of an observer, the Galilean group is isomorphic to the canonical Galilean group. Finally, in Section 6.1, we introduce canonical velocities and describe their images under the isomorphism of the Lie algebras induced by the Lie group isomorphism constructed previously.

4.1. The canonical Galilean group. In this section, we study the Galilean group of a canonical Galilean spacetime, which is a generalization of the "standard" Galilean spacetime $\mathbb{R}^3 \times \mathbb{R}$. To be precise, given a Galilean spacetime $\mathscr{G} = (\mathscr{E}, V, g, \tau)$, the *canonical* spacetime of $\mathscr{G}$ is the Galilean spacetime $\mathscr{G}_{\mathrm{can}} := (\mathscr{E}_{\mathrm{can}} := \ker(\tau) \oplus \mathbb{R}, V = \ker(\tau) \oplus \mathbb{R}, g, \tau)$. We now investigate the structure of the canonical Galilean group $\mathrm{Gal}(\mathscr{G}_{\mathrm{can}})$. The next proposition shows that $\mathrm{Gal}(\mathscr{G}_{\mathrm{can}})$ decomposes into rotations, spatial translations, Galilean velocity boosts, and temporal translations.

4.1 Proposition: *The Galilean group* $\mathrm{Gal}(\mathscr{G}_{\mathrm{can}})$ *of the canonical spacetime* $\mathscr{G}_{\mathrm{can}}$ *is isomorphic to* $(O(\ker(\tau)) \ltimes \ker(\tau)) \ltimes (\ker(\tau) \times \mathbb{R})$*, where* $\ltimes$ *denotes semidirect product of groups. The group operation on* $O(\ker(\tau)) \ltimes \ker(\tau)) \ltimes (\ker(\tau) \times \mathbb{R})$ *is given by*

$$\begin{aligned}(R_1, r_1, u_1, t_1) \cdot (R_2, r_2, u_2, t_2) \\ = (R_1 \circ R_2, r_1 + R_1(r_2) + t_2 u_1, u_1 + R_1(u_2), t_1 + t_2),\end{aligned}$$

where $(R_i, r_i, u_i, t_i) \in O(\ker(\tau)) \ltimes \ker(\tau)) \ltimes (\ker(\tau) \times \mathbb{R},\ i = 1, 2.$

Proof: We first find the form of a Galilean transformation $\phi : \mathscr{E}_{\mathrm{can}} \to \mathscr{E}_{\mathrm{can}}$. Recall that, since ϕ is an affine map, it has the form $\phi(x,t) = A(x,t) + (r,\sigma)$ where $A : \ker(\tau) \oplus \mathbb{R} \to \ker(\tau)\oplus\mathbb{R}$ is $\mathbb{R}$-linear and where $(r,\sigma) \in \ker(\tau)\oplus\mathbb{R}$. Given vector spaces U and V, we denote the set of linear maps from U to V by $L(U,V)$. Let us write $A(x,t) = (A_{11}x + A_{12}t, A_{21}x + A_{22}t)$ where $A_{11} \in L(\ker(\tau), \ker(\tau)), A_{12} \in L(\mathbb{R}, \ker(\tau)), A_{21} \in L(\ker(\tau), \mathbb{R})$, and $A_{22} \in L(\mathbb{R}, \mathbb{R})$. By property (iii) of Galilean mappings, A_{11} is a g-orthogonal transformation of $\ker(\tau)$. Property (ii) of Galilean mappings implies that

$$A_{22}(t_2 - t_1) + A_{21}(x_2 - x_1) = t_2 - t_1, \quad t_1, t_2 \in \mathbb{R}, x_1, x_2 \in \ker(\tau).$$

Thus, taking $x_1 = x_2$, we see that $A_{22} = 1$. This in turn implies that $A_{21} = 0$. Gathering this information shows that a Galilean transformation has the form

$$\phi : \begin{bmatrix} x \\ t \end{bmatrix} \mapsto \begin{bmatrix} R & u \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ t \end{bmatrix} + \begin{bmatrix} r \\ \sigma \end{bmatrix},$$

where $R \in O(\ker(\tau))$, $\sigma \in \mathbb{R}$, and $r, u \in \ker(\tau)$. This proves the first part of the proposition. Now, it is easy to see that, if $\phi_i,\ i = 1, 2$, are Galilean transformations given by

$$\phi_i : \begin{bmatrix} x \\ t \end{bmatrix} \mapsto \begin{bmatrix} R_i & u_i \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ t \end{bmatrix} + \begin{bmatrix} r_i \\ \sigma_i \end{bmatrix}, \quad i = 1, 2,$$

then

$$\phi_1 \circ \phi_2 : \begin{bmatrix} x \\ t \end{bmatrix} \mapsto \begin{bmatrix} R_1 \circ R_2 & u_1 + R_1(u_2) \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ t \end{bmatrix} + \begin{bmatrix} r_1 + R_1(r_2) + \sigma_2 u_1 \\ \sigma_1 + \sigma_2 \end{bmatrix}.$$

which gives us the desired group operation. ■

4.2 Remarks: 1. It is clear from this proposition that $\mathrm{Gal}(\mathscr{G}_{\mathrm{can}})$ is a ten-dimensional group. This is not altogether obvious from the definition.

2. The meaning of the appearance of two semi-direct products in the decomposition of $\mathrm{Gal}(\mathscr{G}_{\mathrm{can}})$ should be understood correctly. They arise because $\ker(\tau) \times \mathbb{R}$ is a normal subgroup of $\mathrm{Gal}(\mathscr{G}_{\mathrm{can}})$ and the quotient group itself is a semi-direct product of $O(\ker(\tau))$ and $\ker(\tau)$.

3. A canonical Galilean transformation may now be written as a composition of one of three basic classes of transformations.

   (i) *A spatio-temporal shift of origin*:

$$\begin{bmatrix} x \\ t \end{bmatrix} \mapsto \begin{bmatrix} x \\ t \end{bmatrix} + \begin{bmatrix} r \\ \sigma \end{bmatrix},$$

   for $r \in \ker(\tau)$, $\sigma \in \mathbb{R}$.

   (ii) *A "rotation" of reference frame*:

$$\begin{bmatrix} x \\ t \end{bmatrix} \mapsto \begin{bmatrix} R & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ t \end{bmatrix},$$

   for $R \in O(\ker(\tau))$.

(iii) *A (Galilean) velocity boost*:

$$\begin{bmatrix} x \\ t \end{bmatrix} \mapsto \begin{bmatrix} \mathrm{id}_{\ker(\tau)} & u \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ t \end{bmatrix},$$

for $u \in \ker(\tau)$.

The names we have given these fundamental transformations are suggestive. A shift of the spatio-temporal origin should be thought of as moving the origin to a new position, and resetting the clock, but maintaining the same orientation in space. A rotation of reference frame means the origin stays in the same place, and uses the same clock but rotates the "point of view". The final basic transformation, a velocity boost, means that the origin maintains its orientation and uses the same clock, but now moves with a certain velocity with respect to the previous origin. •

4.2. The structure of the abstract Galilean group. In the usual presentation of Galilean invariant mechanics (e.g., [Souriau 1997]), one considers a spacetime $\mathbb{R}^3 \times \mathbb{R}$ and Galilean invariance is imposed by asking that the system admit the Galilean group as a symmetry group. In this case, the Galilean group naturally breaks down into rotations, translations, Galilean boosts (constant velocity shifts), and temporal origin shifts. In our abstract setting, the Galilean group $\mathrm{Gal}(\mathscr{G})$ does not admit such a decomposition. Note that this is similar to what one sees in an affine Euclidean space where a decomposition of an isometry into rotation and translation is not possible until one chooses an origin about which to measure rotations. However, the presence of an observer in a Galilean spacetime defines, for each instant, an isomorphism from the abstract Galilean group $\mathrm{Gal}(\mathscr{G})$ into the canonical group $\mathrm{Gal}(\mathscr{G}_{\mathrm{can}})$.

4.3 Proposition: *Let $\mathscr{G} = (\mathscr{E}, V, g, \tau)$ be a Galilean spacetime with $\mathscr{O}$ an observer. The following statements hold.*

(i) The mapping from V to $\ker(\tau) \oplus \mathbb{R}$ defined by $v \mapsto (P_{\mathscr{O}}(v), \tau(v))$ is an isomorphism.

(ii) For each $s_0 \in I_{\mathscr{G}}$, the observer at s_0, $\mathscr{O}_{s_0}$, induces a natural isomorphism $\iota_{\mathscr{O}_{s_0}}$ from $\mathrm{Gal}(\mathscr{G})$ to the group $\mathrm{Gal}(\mathscr{G}_{\mathrm{can}})$. Explicitly, if $\psi \in \mathrm{Gal}(\mathscr{G})$ with t_ψ as defined in Lemma 2.8, and if $R_\psi \in O(\ker(\tau))$ and $r_{\psi,\mathscr{O}} \in \ker(\tau)$ satisfy

$$\psi(x) = (R_\psi(x - \mathscr{O}_{s_0}) + r_{\psi,\mathscr{O}}) + \mathscr{O}_{t_\psi + s_0}, \quad x \in \mathscr{E}(s_0),$$

then

$$\iota_{\mathscr{O}_s}(\psi) = (R_\psi, r_{\psi,\mathscr{O}}, u_{\psi,\mathscr{O}}, t_\psi),$$

where $u_{\psi,\mathscr{O}} = P_{\mathscr{O}}(\psi_V(v_{\mathscr{O}}))$.

Proof: (i) It suffices to show that the mapping $v \mapsto (P_{\mathscr{O}}(v), \tau(v))$ is injective. If $\tau(v) = 0$, then $v \in \ker(\tau)$. Now, if we also have

$$P_{\mathscr{O}}(v) = v - (\tau(v))v_{\mathscr{O}} = 0,$$

we must have $v = 0$, thus the mapping is injective as desired.

(ii) We first assign to each $(R, r, u, t) \in \mathrm{Gal}(\mathscr{G}_{\mathrm{can}})$ a Galilean mapping ψ, and show that the construction implies that $(R, r, u, t) = (R_\psi, r_{\psi,\mathscr{O}}, u_{\psi,\mathscr{O}}, t_\psi)$, thus showing that $\iota_{\mathscr{O}_{s_0}}$ is invertible. Now, given $(R, r, u, t) \in \mathrm{Gal}_{\mathrm{can}}(\mathscr{G})$, we define a map $\psi : \mathscr{E} \to \mathscr{E}$ by

$$\psi(x) = tv_{\mathscr{O}} + (R(x - \mathscr{O}_{\pi_{\mathscr{G}}(x)}) + (\pi_{\mathscr{G}}(x) - s_0)u + r) + \mathscr{O}_{\pi_{\mathscr{G}}(x)}. \tag{4.1}$$

We now show that this mapping is Galilean. First we show that it is affine. For $v \in V$, we compute

$$\begin{aligned}\psi(v + \mathscr{O}_{s_0}) - \psi(\mathscr{O}_{s_0}) &= tv_{\mathscr{O}} + (R(v + \mathscr{O}_{s_0} - \mathscr{O}_{\tau(v)+s_0}) + ((\tau(v) + s_0) - s_0)u + r) \\ &\quad + \mathscr{O}_{\tau(v)+s} - (tv_{\mathscr{O}} + r + \mathscr{O}_{s_0}) \\ &= R(v + \mathscr{O}_{s_0} - (\tau(v)v_{\mathscr{O}} + \mathscr{O}_{s_0})) + \tau(v)(u + v_{\mathscr{O}}) \\ &= R(v - \tau(v)v_{\mathscr{O}}) + \tau(v)(u + v_{\mathscr{O}}) \\ &= R(P_{\mathscr{O}}(v)) + \tau(v)(u + v_{\mathscr{O}}).\end{aligned} \tag{4.2}$$

Thus the map $v \mapsto \psi(v + \mathscr{O}_{s_0}) - \psi(\mathscr{O}_{s_0})$ is linear, so ψ is affine. Similarly, we calculate

$$\begin{aligned}\tau(\psi(x_1) - \psi(x_2)) &= \tau(tv_{\mathscr{O}} + \mathscr{O}_{\pi_{\mathscr{G}}(x_1)}) - \tau(tv_{\mathscr{O}} + \mathscr{O}_{\pi_{\mathscr{G}}(x_2)}) \\ &= t + \pi_{\mathscr{G}}(x_1) - (t + \pi_{\mathscr{G}}(x_2)) \\ &= \tau(x_1 - x_2).\end{aligned}$$

So property (ii) of Galilean mappings is satisfied. Next, for $s_0 \in I_{\mathscr{G}}$, consider $y_1, y_2 \in \mathscr{E}(s_0)$. We compute

$$\begin{aligned}\psi(y_1) - \psi(y_2) &= tv_{\mathscr{O}} + R(y_1 - \mathscr{O}_{s_0}) + (s_0 - s_0)u + r + \mathscr{O}_{s_0} \\ &\quad - (tv_{\mathscr{O}} + R(y_2 - \mathscr{O}_{s_0}) + (s_0 - s_0)u + r + \mathscr{O}_{s_0}) \\ &= R(y_1 - y_2).\end{aligned}$$

Thus ψ satisfies property (iii) of Galilean mappings. Next we show that $(R, r, u, t) = (R_\psi, r_{\psi,\mathscr{O}}, u_{\psi,\mathscr{O}}, t_\psi)$. By restricting ψ to $\mathscr{E}(s_0)$ we get

$$\begin{aligned}(\psi|\mathscr{E}(s_0))(x) &= tv_{\mathscr{O}} + (R(x - \mathscr{O}_{s_0}) + r) + \mathscr{O}_{s_0} \\ &= (R(x - \mathscr{O}_{s_0}) + r) + \mathscr{O}_{t+s_0}.\end{aligned}$$

However, the definition of R_ψ and $r_{\psi,\mathscr{O}}$ gives

$$R(x - \mathscr{O}_{s_0}) + r = R_\psi(x - \mathscr{O}_{s_0}) + r_{\psi,\mathscr{O}},$$

for each $x \in \mathscr{E}(s_0)$. Taking $x = \mathscr{O}_{s_0}$ gives $r = r_{\psi,\mathscr{O}}$, from which it follows that $R = R_\psi$. Also, for $x \in \mathscr{E}(s_0)$, we have

$$\psi_{I_{\mathscr{G}}}(s_0) = \pi_{\mathscr{G}}(\psi(x)) = t + s_0.$$

From Lemma 2.8, it follows that $t = t_\psi$. From (4.2) we also have

$$P_{\mathscr{O}}(\psi_V(v_{\mathscr{O}})) = R(P_{\mathscr{O}}(v_{\mathscr{O}})) + \tau(v_{\mathscr{O}})u = u,$$

using the fact that $P_{\mathscr{O}}(v_{\mathscr{O}})=0$. This shows that $u=u_{\psi,\mathscr{O}}$. We have now shown that, for every $(R,r,u,t)\in\mathrm{Gal}(\mathscr{G}_{\mathrm{can}})$, there is a Galilean mapping ψ such that $\iota_{\mathscr{O}_{s_0}}(\psi)=(R,r,u,t)$. Thus we have shown that $\iota_{\mathscr{O}_{s_0}}$ is surjective. Next we show that it is injective. For this, let $\tilde{\psi}\in\mathrm{Gal}(\mathscr{G})$ be such that, for $x\in\mathscr{E}$,

$$\tilde{\psi}(x)=tv_{\mathscr{O}}+(R(x-\mathscr{O}_{s_0})+(\pi_{\mathscr{G}}(x)-s_0)u+r)+\mathscr{O}_{\pi_{\mathscr{G}}(x)}.$$

that is, suppose that

$$\iota_{\mathscr{O}_{s_0}}(\tilde{\psi})=(R,r,u,t)=\iota_{\mathscr{O}_{s_0}}(\psi).$$

We shall show that $\tilde{\psi}=\psi$. Since $\tilde{\psi}(\mathscr{O}_{s_0})=\psi(\mathscr{O}_{s_0})$, using (4.2) this will follow if we can show that $\tilde{\psi}_V=\psi_V$. As in (i), we note that $V\simeq\ker(\tau)\oplus\mathbb{R}$ and the preimage of (u,t) under this isomorphism is $u+tv_{\mathscr{O}}$. We also write

$$\psi_V(u+tv_{\mathscr{O}})=A_{11}(u)+A_{12}(t)+(A_{21}(u)+A_{22}(t))v_{\mathscr{O}},$$

for linear mappings $A_{11}:\ker(\tau)\to\ker(\tau), A_{12}:\mathbb{R}\to\ker(\tau), A_{21}:\ker(\tau)\to\mathbb{R}$, and $A_{22}:\mathbb{R}\to\mathbb{R}$. The property (ii) of Galilean mappings implies that ψ_V has $\ker(\tau)$ as an invariant subspace. Thus $A_{21}=0$. We next calculate

$$\begin{aligned}\tau(\psi_V(tv_{\mathscr{O}}))&=\tau(\psi(tv_{\mathscr{O}}+\mathscr{O}_{s_0}))-\tau(\psi(\mathscr{O}_{s_0}))\\&=\tau(tv_{\mathscr{O}}+\mathscr{O}_{s_0})-\tau(\mathscr{O}_{s_0})=t.\end{aligned}$$

This gives $A_{22}(t)=t$. With $t=0$, property (iii) of Galilean mappings implies that $A_{11}\in O(\ker(\tau))$. Thus we have

$$\psi_V(u+tv_{\mathscr{O}})=\tilde{R}(u)+t(\tilde{u}+v_{\mathscr{O}}),\tag{4.3}$$

for some $\tilde{R}\in O(\ker(\tau))$ and $\tilde{u}\in\ker(\tau)$. Since $\psi|\mathscr{E}(s)=\tilde{\psi}|\mathscr{E}(s)$ we have

$$\psi_V(u)=\tilde{\psi}(u+\mathscr{O}_{s_0})-\tilde{\psi}(\mathscr{O}_{s_0})=R_{\psi}(u),$$

giving $\tilde{R}=R$. From (4.2) we also have

$$P_{\mathscr{O}}(\psi_V(v_{\mathscr{O}})=\tilde{u},$$

from which we get $u=\tilde{u}$. This shows that $\psi_V=\tilde{\psi}_V$, thus showing that, if $\psi_1,\psi_2\in\mathrm{Gal}(\mathscr{G})$ satisfy $\iota_{\mathscr{O}_{s_0}}(\psi_1)=\iota_{\mathscr{O}_{s_0}}(\psi_2)$, then $\psi_1=\psi_2$. Therefore, $\iota_{\mathscr{O}_{s_0}}$ is injective.

Finally we show that $\iota_{\mathscr{O}_{s_0}}$ is a homomorphism. We let $\psi_1,\psi_2\in\mathrm{Gal}(\mathscr{G})$ and denote $\iota_{\mathscr{O}}(\psi_i)=(R_i,r_i,u_i,t_i),\ i=1,2$. We also let $\iota_{\mathscr{O}_{s_0}}(\psi_1\circ\psi_2)=(R_{12},r_{12},u_{12},t_{12})$. First, we compute

$$\begin{aligned}(\psi_1\circ\psi_2)_V(v)&=(\psi_1\circ\psi_2)(v+\mathscr{O}_{s_0})-(\psi_1\circ\psi_2)(\mathscr{O}_{s_0})\\&=\psi_1(\psi_2(v+\mathscr{O}_{s_0}))-\psi_1(\psi_2(\mathscr{O}_{s_0}))\\&=\psi_1(tv_{\mathscr{O}}+R_2(v+\mathscr{O}_{s_0}-\mathscr{O}_{\tau(v)+s_0})+\tau(v)u_2+r_2+\mathscr{O}_{\tau(v)+s_0})\\&\quad-\psi_1(tv_{\mathscr{O}}+r+\mathscr{O}_{s_0})\\&=\psi_1((tv_{\mathscr{O}}+r)+R_2(P_{\mathscr{O}}(v))+\tau(v)(u_2+v_{\mathscr{O}})+\mathscr{O}_{s_0})\\&\quad-\psi_1(tv_{\mathscr{O}}+r_2+\mathscr{O}_{s_0})\\&=\psi_{1,V}(R_2(P_{\mathscr{O}}(v))+\tau(v)(u_2+v_{\mathscr{O}}))\\&=\psi_{1,V}(\psi_{2,V}(v)).\end{aligned}$$

From this we deduce that

$$R_{12}(u) + t(u_{12} + v_{\mathscr{O}}) = R_1 \circ R_2(u) + t(u_1 + R_1(u_2) + v_{\mathscr{O}}),$$

for each $(u, t) \in \ker(\tau) \oplus \mathbb{R}$. Thus we have

$$R_{12} = R_1 \circ R_2, \quad u_{12} = u_1 + R_1(u_2).$$

Next we have

$$\psi_1 \circ \psi_2(\mathscr{O}_{s_0}) = r_{12} + \mathscr{O}_{t_{12}+s_0}. \tag{4.4}$$

Also,

$$\psi_2(\mathscr{O}_{s_0}) = r_2 + \mathscr{O}_{t_2+s_0}.$$

Therefore

$$\begin{aligned}\psi_1 \circ \psi_2(\mathscr{O}_{s_0}) &= \psi_1(r_2 + \mathscr{O}_{t_2+s_0}) \\ &= R_1(r_2 + \mathscr{O}_{t_2+s_0} - \mathscr{O}_{t_2+s_0}) + t_2 u_1 + r_1 + \mathscr{O}_{s_0+t_2+t_1} \\ &= R_1 r_2 + t_2 u_1 + r_1 + \mathscr{O}_{s_0+t_1+t_2}.\end{aligned}$$

Comparing this to (4.4) we get

$$r_{12} = R_1 r_2 + t_2 u_1 + r_1, \quad t_{12} = t_1 + t_2.$$

Thus we have shown that the group action defined on $\mathrm{Gal}(\mathscr{G}_{\mathrm{can}})$ agrees with that on $\mathrm{Gal}(\mathscr{G})$ under the bijection $\iota_{\mathscr{O}_{s_0}}$. ■

We end this section by listing some of the subgroups of $\mathrm{Gal}(\mathscr{G})$ that we shall have occasion to use in the sequel. The following result is easy to prove.

4.4 Proposition: *The following statements hold:*

(i) the following are subgroups of $\mathrm{Gal}(\mathscr{G})$*:*

- *(a)* $\mathrm{Gal}_0(\mathscr{G}) := \{\psi \in \mathrm{Gal}(\mathscr{G}) : \psi_{I_{\mathscr{G}}} = \mathrm{id}_{I_{\mathscr{G}}}\}$*;*
- *(b)* $N := \{\psi \in \mathrm{Gal}(\mathscr{G}) : \psi_V|_{\ker(\tau)} = \mathrm{id}_{\ker(\tau)}\}$*;*
- *(c)* $N_0 := N \cap \mathrm{Gal}_0(\mathscr{G})$*.*

(ii) the set $\mathrm{Lin}(\mathscr{G}) := \{\psi_V |\ \psi \in \mathrm{Gal}(\mathscr{G})\}$ *is a Lie group.*

5. Observer-independent formulation of rigid body mechanics

In this section we formulate rigid mechanics in an observer independent manner. All of the classical concepts in Eulerian rigid body mechanics—motions, body and spatial velocities, body and spatial momenta, and the equations of motion—are given definitions independent of an observer.

5.1. Rigid motions. A ***rigid motion*** in a Galilean spacetime $\mathscr{G} = (\mathscr{E}, V, g, \tau)$ is a smooth mapping $\Psi : \mathbb{R} \to \mathrm{Gal}(\mathscr{G})$ with the property that $(\Psi(t))_{I_{\mathscr{G}}}(s) = s + t$, for each $t \in \mathbb{R}$ (see Section 2.5). In other words, if we denote $\Psi_t := \Psi(t)$, then, a rigid motion Ψ has the property that $\Psi_t(x) \in \mathscr{E}(t + \pi_{\mathscr{G}}(x))$ for each $x \in \mathscr{E}(s)$. Thus a rigid motion maps points in $\mathscr{E}(s)$ to $\mathscr{E}(s)$ at $t = 0$, and for $t \neq 0$, the points get shifted by the affine action of $\mathbb{R}$ on $I_{\mathscr{G}}$. Let us give some of the basic properties of rigid motions. The following result is immediate.

5.1 Lemma: *Given a rigid motion Ψ, for each $x \in \mathscr{E}$ the map $I_{\mathscr{G}} \ni s \mapsto \Psi_{s-\pi_{\mathscr{G}}(x)}(x) \in \mathscr{E}$ is a world line.*

The next result shows how we can extract the "rotational component" of a rigid motion for each $t \in \mathbb{R}$.

5.2 Proposition: *Let Ψ be a rigid motion in a Galilean spacetime $\mathscr{G} = (\mathscr{E}, V, g, \tau)$. Then, for each $t \in \mathbb{R}$, we have $\Psi_{t,V}|_{\ker(\tau)} \in O(\ker(\tau))$.*

Proof: From Lemma 2.8, it is clear that $\Psi_{t,V}$ maps $\ker(\tau)$ to itself. Next for simultaneous events x_1 and x_2, we compute

$$\begin{aligned} g(\Psi_{t,V}|_{\ker(\tau)}(x_1 - x_2), \Psi_{t,V}|_{\ker(\tau)}(x_1 - x_2)) &= g(\Psi_{t,V}(x_1 - x_2), \Psi_{t,V}(x_1 - x_2)) \\ &= g(\Psi_t(x_1) - \Psi_t(x_2), \Psi_t(x_1) - \Psi_t(x_2)) \\ &= g(x_1 - x_2, x_1 - x_2), \end{aligned}$$

where we have used the properties of the rigid motion and Lemma 2.8. This shows that $\Psi_{t,V}|_{\ker(\tau)} \in O(\ker(\tau))$ as desired. ■

This proposition shows that, given a rigid motion Ψ in a Galilean spacetime, we can associate to this rigid motion a unique map from $\mathbb{R} \to O(\ker(\tau))$. We denote this map by R_Ψ.

5.2. Spatial and body velocities. In this section we define the concepts of spatial and body velocities corresponding to a rigid motion Ψ. Intuitively, the configuration of a rigid body is given by its "orientation" and "position" (with respect to an initial orientation and position). To make this precise, let us denote $Q := O(\ker(\tau)) \times \mathscr{E}$. If we choose a reference configuration, say $q_0 = (R_0, x_0) \in Q$, it is easy to see that a Galilean mapping $\phi \in \mathrm{Gal}(\mathscr{G})$ maps q_0 to another point in Q as follows:

$$\begin{aligned} &\mathrm{Gal}(\mathscr{G}) \times Q \to Q \\ &(\phi, (R_0, x_0)) \mapsto (R_\phi R_0, \phi(x_0)), \quad \phi \in \mathrm{Gal}(\mathscr{G}). \end{aligned}$$

This defines an action of $\mathrm{Gal}(\mathscr{G})$ on Q which we represent by Φ. In other words, we have $\Phi(\phi, (R_0, x_0)) = (R_\phi R_0, \phi(x_0))$. For each $q \in Q$ and $g \in \mathrm{Gal}(\mathscr{G})$, we define the maps $\Phi_q : \mathrm{Gal}(\mathscr{G}) \to Q$ and $\Phi_g : Q \to Q$ by $\Phi_q(g) := \Phi(g, q) =: \Phi_q(g)$. The action Φ also defines an action of $\mathrm{Gal}(\mathscr{G})$ on $O(\ker(\tau))$ and $\mathscr{E}$ respectively. The latter action is denoted by $\Psi^{\mathscr{E}}$. Given $\zeta \in \mathfrak{gal}(\mathscr{G})$, denote the infinitesimal generator corresponding to ζ at $(R_0, x_0) \in Q$ by $\zeta_Q(R_0, x_0)$. It is easy to see that $\zeta_Q(R_0, x_0)$ can be written as

$$\zeta_Q(R_0, x_0) = (\zeta_{O(\ker(\tau))}(R_0), \zeta_{\mathscr{E}}(x_0)),$$

where $\zeta_{O(\ker(\tau))}(R_0)$ is the infinitesimal generator at $R_0 \in O(\ker(\tau))$ corresponding to ζ of the action of $\mathrm{Gal}(\mathscr{G})$ on $O(\ker(\tau))$, and $\zeta_{\mathscr{E}}(x_0)$ is the infinitesimal generator at $x_0 \in \mathscr{E}$ corresponding to ζ of the action of $\mathrm{Gal}(\mathscr{G})$ on $\mathscr{E}$.

Now, given a rigid motion Ψ, define curves $\xi_\Psi(t)$ and $\eta_\Psi(t) \in \mathfrak{gal}(\mathscr{G})$ as follows:

$$\eta_\Psi(t) = \Psi_t^{-1}\dot{\Psi}_t, \quad \xi_\Psi(t) = \dot{\Psi}_t\Psi_t^{-1}.$$

Notice that $\xi_\Psi(t) = \mathrm{Ad}_{\Psi_t}\eta_\Psi(t)$. These curves $\xi_\Psi(t)$ and $\eta_\Psi(t)$ are called "spatial velocity" and "body velocity," respectively, in the literature (see, for example, [Murray, Li, and Sastry 1994]) and the interpretation for defining them in this way is somewhat unintuitive. In the sequel, given a rigid motion Ψ, we think of velocities at a point $q_0 = (R_0, x_0) \in Q$ as tangent vectors defined by the infinitesimal generators corresponding to the curves $\eta_\Psi(t)$ and $\xi_\Psi(t)$, respectively, at q_0. The idea is that a rigid motion generates a curve in Q (starting at q_0) given by $\Psi_t(q_0) := (R_\Psi(t)R_0, \Psi_t(x_0))$, and the tangent vector to this curve at a point corresponds to the velocity at that point. We now give our definitions for spatial and body velocities respectively. First, given a rigid motion Ψ, we define maps $\Omega, \omega : \mathbb{R} \to \mathfrak{o}(\ker(\tau))$ by $\hat{\Omega}_\Psi(t) := R_\Psi^{-1}(t)\dot{R}_\Psi(t)$ and $\hat{\omega}_\Psi(t) := \dot{R}_\Psi(t)R_\Psi^{-1}(t)$, respectively. We also represent by $\tau_{\mathrm{ang}} : TQ \to T(O(\ker(\tau)))$ and $\tau_{\mathrm{lin}} : TQ \to T\mathscr{E}$ the respective projections. Denote by $\Theta_{\mathrm{ang}} : T(O(\ker(\tau))) \to O(\ker(\tau)) \times \mathfrak{o}(\ker(\tau))$ the right-trivialisation of $T(O(\ker(\tau)))$. Thus $\Theta_{\mathrm{ang}}(v_g) = (g, T_gR_{g^{-1}}v_g)$ for $v_g \in T_g(O(\ker(\tau)))$ and $g \in O(\ker(\tau))$. Similarly, denote by $\Theta_{\mathrm{lin}} : T\mathscr{E} \to \mathscr{E} \times V$ the natural trivialisation of $T\mathscr{E}$. Let's us denote by pr_2 the projection onto the second components of $O(\ker(\tau)) \times \mathfrak{o}(\ker(\tau))$ and $\mathscr{E} \times V$, respectively. With an abuse of notation, we represent the maps $\mathrm{pr}_2 \circ \Theta_{\mathrm{ang}}$ and $\mathrm{pr}_2 \circ \Theta_{\mathrm{lin}}$ by Θ_{ang} and Θ_{lin} respectively.

5.3 Definition: Let Ψ be a rigid motion in a Galilean spacetime $\mathscr{G}$.

(i) The ***body velocity*** is the map $V_\Psi^b : Q \times \mathbb{R} \to TQ$ given by

$$V_\Psi^b(R, x, t) = (\eta_\Psi(t))_Q(R, x).$$

The maps $V_{\Psi,\mathrm{ang}}^b := \Theta_{\mathrm{ang}} \circ \tau_{\mathrm{ang}} \circ V_\Psi^b$ and $V_{\Psi,\mathrm{lin}}^b := \Theta_{\mathrm{lin}} \circ \tau_{\mathrm{lin}} \circ V_\Psi^b$ are called ***body angular velocity*** and ***body linear velocity***, respectively.

(ii) The ***spatial velocity*** is the map $V_\Psi^s : Q \times \mathbb{R} \to TQ$ defined by

$$V_\Psi^s(R, x, t) = (\xi_\Psi(t))_Q(R, x).$$

The maps $V_{\Psi,\mathrm{ang}}^s := \Theta_{\mathrm{ang}} \circ \tau_{\mathrm{ang}} \circ V_\Psi^s$ and $V_{\Psi,\mathrm{lin}}^s := \Theta_{\mathrm{lin}} \circ \tau_{\mathrm{lin}} \circ V_\Psi^s$ are called ***spatial angular velocity*** and ***spatial linear velocity***, respectively. •

Let us make a few comments about these definitions.

5.4 Remarks: 1. We identify the TQ-valued velocities with the corresponding $Q \times (\mathfrak{o}(\ker(\tau)) \times V)$-valued trivialisation. Let us provide some intuition for these definitions. Notice that, for $q \in Q$, we have

$$\frac{d}{dt}(\Psi_t(q)) = T_q\Phi_{\Psi_t}(\xi_\Psi(t)^{-1}\dot{\xi}_\Psi(t))_Q(q) = T_q\Phi_{\Psi_t}(V_\Psi^b(q, t)).$$

Therefore, $V_\Psi^b(q, t) = T_q\Phi_{\Psi_t^{-1}}(\frac{d}{dt}(\Psi_t(q)))$. The body velocity can therefore be thought of as the velocity of the curve $\Psi_t(q)$ at $t \in \mathbb{R}$ as seen in the "frame" fixed at q.

2. The definition of spatial velocity is less intuitive. It will become clearer once we prove Proposition 5.6.

3. It is not clear at this stage how our definitions are consistent with the existing ones. We shall see, in Section 6.2, that, in the presence of an observer, the body and spatial velocities correspond to "canonical" velocities. •

The definitions lead to the following relationship between body and spatial velocities, which we shall have occasion to use.

5.5 Lemma: $V^s_\Psi(R_\Psi(t)R, \Psi_t(x), t) = T_{(R,x)}\Phi_{\Psi_t} V^b_\Psi(R, x, t)$.

Proof: Given an action $\Phi : G \times Q \to Q$ of a Lie group G on a manifold Q, we have, for $\zeta \in \mathfrak{g}$ and $g \in G$, $(\mathrm{Ad}_g\zeta)_Q(g \cdot q) = T_q\Phi_g\zeta_Q(q)$. The result follows directly from this equality. ■

Next, we write down the expressions for linear and angular velocities.

5.6 Proposition: *Let Ψ be a motion in a Galilean spacetime $\mathscr{G}$. Then*

(i) $V^b_{\Psi,\mathrm{ang}}(R, x, t) = \hat{\Omega}_\Psi(t)$,

(ii) $V^b_{\Psi,\mathrm{lin}}(R, x, t) = \Psi^{-1}_{t,V}(\frac{d}{ds}\Psi_t(x))$,

(iii) $V^s_{\Psi,\mathrm{ang}}(R, x, t) = \hat{\omega}_\Psi(t)$, *and*

(iv) $V^s_{\Psi,\mathrm{lin}}(R, x, t) = -\Psi_{t,V}(\frac{d}{ds}\Psi^{-1}_t(x))$.

Proof: (i) The projection $\tau_{\mathrm{ang}}(V^b_{\Psi,\mathrm{ang}}(R, x, t))$ is actually the infinitesimal generator corresponding to $(\Psi_{t,V}|^{-1}_{\ker(\tau)}\dot{\Psi}_{t,V}|_{\ker(\tau)}) = \hat{\Omega}_\Psi(t)$. We compute

$$V^b_{\Psi,\mathrm{ang}}(R, x, t) = \frac{\mathrm{d}}{\mathrm{d}s}\exp(s\hat{\Omega}_\Psi(t))R\bigg|_{s=0} = \hat{\Omega}_\Psi(t)R.$$

Thus $V^b_{\Psi,\mathrm{ang}}(R, x, t) = \hat{\Omega}_\Psi(t)$.

(ii) The body linear velocity $V^b_{\Psi,\mathrm{lin}}(R, x, t)$ is the infinitesimal generator corresponding to $\eta_\Psi(t)$ at $x \in \mathscr{E}$ of the action of $\mathrm{Gal}(\mathscr{G})$ on $\mathscr{E}$. We compute

$$\begin{aligned} V^b_{\Psi,\mathrm{lin}}(R, x, t) &= T_e\Phi^{\mathscr{E}}_x(T_{\Psi_t}L_{\Psi^{-1}_t}\dot{\Psi}_t) = T_{\Psi_t}(\Phi_{\Psi^{-1}_t}\Phi^{\mathscr{E}}_x)(\dot{\Psi}_t) \\ &= \frac{\mathrm{d}}{\mathrm{d}s}\Phi_{\Psi^{-1}_t}\Phi^{\mathscr{E}}_x \circ (\Psi_{t+s})\bigg|_{s=0} = \frac{\mathrm{d}}{\mathrm{d}s}\Phi_{\Psi^{-1}_t}(\Psi_{t+s}(x))\bigg|_{s=0} \\ &= T_{\Psi_t(x)}\Phi_{\Psi^{-1}_t}\frac{\mathrm{d}}{\mathrm{d}s}(\Psi_t(x)) = \Psi^{-1}_{t,V}\frac{\mathrm{d}}{\mathrm{d}s}(\Psi_t(x)). \end{aligned}$$

This is what we wanted to prove.

(iii) This is identical to the proof of part (i).

(iv) The spatial linear velocity $V^s_{\Psi,\mathrm{lin}}(R, x, t)$ is the infinitesimal generator corresponding to $\xi_\Psi(t)$ at $x \in \mathscr{E}$ of the action of $\mathrm{Gal}(\mathscr{G})$ on $\mathscr{E}$. Notice that, by differentiating $\Psi_t\Psi^{-1}_t = \mathrm{id}_{\mathscr{E}}$, we get

$$T_{\Psi_t}R_{\Psi^{-1}_t}\dot{\Psi}_t = -T_{\Psi^{-1}_t}L_{\Psi_t}\dot{\Psi}^{-1}_t.$$

We compute

$$\begin{aligned} V^s_{\Psi,\mathrm{lin}}(R, x, t) &= T_e\Phi^{\mathscr{E}}_x(T_{\Psi_t}R_{\Psi^{-1}_t}\dot{\Psi}_t) = -T_e\Phi^{\mathscr{E}}_x(T_{\Psi_t}L_{\Psi_t}\dot{\Psi}^{-1}_t) \\ &= -\frac{\mathrm{d}}{\mathrm{d}s}\Phi_{\Psi_t}\Phi^{\mathscr{E}}_x(\Psi^{-1}_{t+s})\bigg|_{s=0} = -\frac{\mathrm{d}}{\mathrm{d}s}\Phi_{\Psi_t}(\Psi^{-1}_{t+s}(x))\bigg|_{s=0} \\ &= -T_{\Psi^{-1}_t(x)}\Phi_{\Psi_t}\frac{\mathrm{d}}{\mathrm{d}s}(\Psi^{-1}_t(x)) = -\Psi_{t,V}\frac{\mathrm{d}}{\mathrm{d}s}(\Psi^{-1}_t(x)), \end{aligned}$$

as desired. ■

Thus, the spatial velocity at $(q_0, t) \in Q \times \mathbb{R}$ is obtained by taking the tangent vector to the curve $\Psi^{-1}_t(q_0)$, and then "pushing" this vector by the map $(-T_{\Psi^{-1}_t(x)}\Phi_{\Psi_t})$. In other words, $V^s_\Psi(q_0, t)$ can be thought of as the velocity of a point in Q traveling thorough q_0 at time t. This is exactly the interpretation of spatial velocity given in [Murray, Li, and Sastry 1994].

5.3. Spatial and body momenta. In this section we define the spatial and body momenta for a rigid body. We identify $\mathfrak{o}(\ker(\tau))$ with $\ker(\tau)$ by the inverse of the $\hat{}$ map defined earlier and denote the angular velocities thought of as taking values in $\ker(\tau)$ by $V^b_{\Psi,\text{ang}}$ and $V^s_{\Psi,\text{ang}}$. Given a rigid body $(\mathcal{B},\mu)$, a rigid motion Ψ, and a curve $u:\mathbb{R}\to\ker(\tau)$, we define the ***instantaneous inertia tensor*** $\mathbb{I}_c(t):\ker(\tau)\to\ker(\tau)$ by

$$\mathbb{I}_c(t)(u(t)) = \int_{\mathcal{B}(t)} (\Psi_t(x)-\Psi_t(x_c))\times(u(t)\times(\Psi_t(x)-\Psi_t(x_c))\mathrm{d}\mu(t),$$

where $\mathcal{B}(t)=\Psi_t(\mathcal{B})$ and $u(t)\in\ker(\tau)$. Notice that, since the integrand is in $\ker(\tau)$, so too will be the integral. The following result shows what the spatial angular momentum looks like in terms of the inertia tensor of the body about its center of mass.

5.7 Lemma: $\mathbb{I}_c(t)(\omega_\Psi(t)) = R_\Psi(t)\mathbb{I}_c(R^{-1}_\Psi(t)\omega_\Psi(t))$.

Proof: We represent by $\mathcal{B}(t)$ the rigid body after it has undergone the transformation Ψ_t and the corresponding mass distribution by $\mathrm{d}\mu(t)$. We compute

$$\begin{aligned}
\mathbb{I}_c(t)(\omega_\Psi(t)) &= \int_{\mathcal{B}(t)} (\Psi_t(x)-\Psi_t(x_c))\times(\omega_\Psi(t)\times(\Psi_t(x)-\Psi_t(x_c)))\,\mathrm{d}\mu(t)\\
&= \int_{\mathcal{B}(t)} (\Psi_{t,V}(x-x_c))\times(\omega_\Psi(t)\times(\Psi_{t,V}(x-x_c)))\,\mathrm{d}\mu(t)\\
&= \int_{\mathcal{B}(t)} (R_\Psi(t)(x-x_c))\times(\omega_\Psi(t)\times(R_\Psi(t)(x-x_c)))\,\mathrm{d}\mu(t)\\
&= R_\Psi(t)\int_{\mathcal{B}} (x-x_c)\times\big(R^{-1}_\Psi(t)\omega_\Psi(t)\times(x-x_c)\big)\,\mathrm{d}\mu\\
&= R_\Psi(t)\mathbb{I}_c(R^{-1}_\Psi(t)\omega_\Psi(t)),
\end{aligned}$$

where we have used the fact that $x-x_c\in\ker(\tau)$, and therefore $\Psi_{t,V}(x-x_c)=R_\Psi(t)(x-x_c)$. ■

We can now define spatial and body momenta.

5.8 Definition: Let $(\mathcal{B},\mu)$ be a rigid body in a Galilean spacetime $\mathscr{G}$ and let Ψ be a rigid motion.

(i) The ***spatial momentum*** is a map $p_{\Psi,\mathcal{B}}:\mathbb{R}\to\ker(\tau)\times V$ given by

$$p_{\Psi,\mathcal{B}}(t) = \big(\mathbb{I}_c(t)V^s_{\Psi,\text{ang}},\ \mu(\mathcal{B})V^s_{\Psi,\text{lin}}\big)\,(R_\Psi(t),\Psi_t(x_c),t).$$

(ii) The ***body momentum*** is the map $P_{\Psi,\mathcal{B}}:\mathbb{R}\to\ker(\tau)\times V$ given by

$$p_{\Psi,\mathcal{B}}(t) = \Big(\mathbb{I}_c(t)V^b_{\Psi,\text{ang}},\ \mu(\mathcal{B})V^b_{\Psi,\text{lin}}\Big)\,(\mathrm{id}_{\ker(\tau)},x_c,t).$$ •

The following result can be readily proved using Lemma 5.5.

5.9 Proposition: *Let $(\mathcal{B}, \mu)$ be a rigid body in a Galilean spacetime $\mathscr{G}$ and let Ψ be a rigid motion. Then*

(i) $p_{\Psi,\mathcal{B}}(t) = (\mathbb{I}_c(t)\omega_\Psi(t), \mu(\mathcal{B})\frac{d}{ds}\Psi_t(x_c))$,

(ii) $P_{\Psi,\mathcal{B}}(t) = (R_\Psi(t)^{-1}\mathbb{I}_c(t)\omega_\Psi(t), \mu(\mathcal{B})\Psi_{t,V}^{-1}\frac{d}{ds}\Psi_t(x_c))$.

Given a rigid body $(\mathcal{B}, \mu)$, define an equivalence class of Galilean mappings as follows. Two mappings ϕ and $\psi \in \mathrm{Gal}(\mathscr{G})$ are called ***$\mathcal{B}$-equivalent*** if $\phi(\mathcal{B}) = \psi(\mathcal{B})$. It is easy to see that this is an equivalence relation. In such a case, we denote the equivalence class containing ϕ by $[\phi]_\mathcal{B}$. In other words, any two mappings in the equivalence class $[\phi]_\mathcal{B}$ map the rigid body $\mathcal{B}$ to the same set of points in $\mathscr{E}$. The following result is readily verified.

5.10 Proposition: *$\phi \in [\psi]_\mathcal{B}$ if and only if $\phi(x_c) = \psi(x_c)$ and $R_\phi = R_\psi$.*

Proof: If $\phi \in [\psi]_\mathcal{B}$ then, for each $x \in \mathcal{B}$, we have $\phi(x) = \psi(x)$. Since $\mathcal{B}$ is a subset of $\mathscr{E}(s_0)$ (which is an affine space modeled on $\ker(\tau)$), we can write $x \in \mathcal{B}$ as $x = x_c + w$ for some $w \in \ker(\tau)$. We thus have

$$\phi(x_c + w) = \psi(x_c + w),$$

which implies that

$$\phi(x_c + w) - \phi(x_c) = \psi(x_c + w) - \psi(x_c).$$

We thus have $\phi_V(w) = \psi_V(w)$ and thus $R_\phi w = R_\psi w$. Conversely, assume that ψ is such that $R_\psi = R_\phi$ and $\psi(x_c) = \phi(x_c)$. By reversing the argument above, it is easy to show that $\phi(x) = \psi(x)$ for all $x \in \mathcal{B}$. ■

Given a rigid body $(\mathcal{B}, \mu)$ and a rigid motion Ψ, a rigid motion $\tilde{\Psi}$ is $\mathcal{B}$-*equivalent* to Ψ if $\tilde{\Psi}_t \in [\Psi_t]_\mathcal{B}$ for each $t \in \mathbb{R}$. The following result is immediate.

5.11 Lemma: *Let $(\mathcal{B}, \mu)$ be a rigid body in a Galilean spacetime and let Ψ be a motion. If a rigid motion $\tilde{\Psi}$ is $\mathcal{B}$-equivalent to Ψ, then $p_{\tilde{\Psi},\mathcal{B}} = p_{\Psi,\mathcal{B}}$*

5.4. Galilean–Euler equations. In this section we derive the equations of motion for a rigid body in our general framework. As remarked in the Introduction, the problem of finding the equations of motion for rigid bodies has a rich history. The key observation of Newton and Euler is that the free motion of a rigid body is completely determined by imposing conservation of spatial linear and angular momenta. The observation that this approach generalizes to other physical settings such as hydrodynamics, was first made by Arnol′d [1966] (see also, [Abraham and Marsden 1978]). Using his method, the Euler equations for an incompressible fluid can be written as geodesic equations on a certain infinite-dimensional Lie group. We note that the Galilean group $\mathrm{Gal}(\mathscr{G})$ of a Galilean spacetime $\mathscr{G}$ does not have a natural invariant metric, and thus we cannot use Arnol'd's method in this setup. Intuitively speaking, $\mathrm{Gal}(\mathscr{G})$ is "too big" to uniquely determine the physical motion of the body. We shall have more to say on this matter in Section 6.5. We use the principle of conservation of spatial momentum to derive differential equations in terms of "spatial" as well as "body" quantities that describe the physical motion of the body. We also show that, if a rigid motion Ψ satisfies these equations for the body $(\mathcal{B}, \mu)$, then every rigid motion $\mathcal{B}$-equivalent to Ψ also satisfies the equations. From the definitions given in the previous section, it can be seen that $p_{\Psi,\mathcal{B}} = (\ell_{\Psi,\mathcal{B}}(t), m_{\Psi,\mathcal{B}}(t)) = (R_\Psi(t)L_{\Psi,\mathcal{B}}(t), \Psi_{t,V}M_{\Psi,\mathcal{B}}(t))$.

5.12 Proposition: (Galilean–Euler equations) *Let $(\mathcal{B}, \mu)$ be a rigid body in a Galilean spacetime and let Ψ be a rigid motion. The following statements are equivalent:*

(i) the spatial momentum $p_{\Psi,\mathcal{B}}$ is conserved;

(ii) the motion of the body satisfies the ***spatial Galilean–Euler equations***

$$\begin{aligned}\mathbb{I}_c(\dot{\omega}_\Psi(t)) &= \mathbb{I}_c(\omega_\Psi(t)) \times \omega_\Psi(t)\\ \ddot{x}_c(t) &= 0,\end{aligned}$$

where $\ddot{x}_c(t) = \frac{d^2}{dt^2}(\Psi_t(x_c))$;

(iii) the motion of the body satisfies the ***body Galilean–Euler equations***

$$\begin{aligned}\dot{L}_{\Psi,\mathcal{B}}(t) &= L_{\Psi,\mathcal{B}}(t) \times \Omega_\Psi(t)\\ \dot{M}_{\Psi,\mathcal{B}}(t) &= -(\eta_V(t))_V(M_{\Psi,\mathcal{B}}(t)),\end{aligned}$$

where $(\eta_V(t))_V(M_{\Psi,\mathcal{B}}(t))$ is the infinitesimal generator corresponding to $\eta_V(t) = \Psi_{t,V}^{-1}\dot{\Psi}_{t,V}$ of the action of $\mathrm{Lin}(\mathscr{G})$ on V.

Furthermore,

(iv) if $\tilde{\Psi}$ is a rigid motion $\mathcal{B}$-equivalent to Ψ, then Ψ can be replaced with $\tilde{\Psi}$ in the above statements.

Proof: Conservation of spatial momentum implies that $\dot{p}_{\Psi,\mathcal{B}}(t) = 0$. The equation $\ddot{x}_c(t) = 0$ immediately follows. To derive the first equation, we note that

$$\mathbb{I}_c(t)\omega_\Psi(t) = R_\Psi(t)\mathbb{I}_c(R_\Psi^{-1}\omega_\Psi(t)).$$

Therefore,

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}s}R_\Psi(t)\mathbb{I}_c(R_\Psi^{-1}\omega_\Psi(t)) &= \dot{R}_\Psi(t)\mathbb{I}_c(R_\Psi^{-1}\omega_\Psi(t) + R_\Psi(t)\mathbb{I}_c(R_\Psi^{-1}\dot{\omega}_\Psi(t))\\ &= \hat{\omega}_\Psi(t)R_\Psi(t)\mathbb{I}_c(R_\Psi^{-1}\omega_\Psi(t)) + R_\Psi(t)\mathbb{I}_c(R_\Psi^{-1}\dot{\omega}_\Psi(t))\\ &= \hat{\omega}_\Psi(t)\mathbb{I}_c(\omega_\Psi(t)) + \mathbb{I}_c(t)\dot{\omega}_\Psi(t)\\ &= 0,\end{aligned}$$

by conservation of spatial momentum. Therefore

$$\mathbb{I}_c(\dot{\omega}_\Psi(t)) = -\hat{\omega}_\Psi(t)\mathbb{I}_c(\omega_\Psi(t)) = \mathbb{I}_c(\omega_\Psi) \times \omega_\Psi(t)).$$

Next, we write spatial momentum in terms of the body momentum. That is,

$$p_{\Psi,\mathcal{B}}(t) = (R_\Psi(t)L_{\Psi,\mathcal{B}}, \Psi_{t,V}M_{\Psi,\mathcal{B}}(t)).$$

Conservation of spatial momentum implies that

$$\frac{\mathrm{d}}{\mathrm{d}s}(R_\Psi(t)L_{\Psi,\mathcal{B}}(t)) = 0 \quad \text{and} \quad \frac{\mathrm{d}}{\mathrm{d}s}(\Psi_{t,V}M_{\Psi,\mathcal{B}}(t)) = 0.$$

The first equation gives

$$\begin{aligned}0 &= (\dot{R}_\Psi(t)L_{\Psi,\mathcal{B}}(t) + (R_\Psi(t)\dot{L}_{\Psi,\mathcal{B}}(t)\\ &= R_\Psi(t)\hat{\Omega}_\Psi(t)L_{\Psi,\mathcal{B}}(t) + R_\Psi(t)\dot{L}_{\Psi,\mathcal{B}}(t).\end{aligned}$$

We therefore get

$$\dot{L}_{\Psi,\mathcal{B}}(t) = -\hat{\Omega}_\Psi(t) L_{\Psi,\mathcal{B}}(t) = L_{\Psi,\mathcal{B}}(t) \times \Omega_\Psi(t).$$

Next, consider the second equation. Written appropriately in terms of the action Φ^V of $\mathrm{Lin}(\mathscr{G})$ on V, the equation becomes

$$\frac{\mathrm{d}}{\mathrm{d}s} \Phi^V_{\Psi_{t,V}}(M_{\Psi,\mathcal{B}}(t)) = 0.$$

We compute

$$\frac{\mathrm{d}}{\mathrm{d}s} \Phi^V_{\Psi_{t,V}}(M_{\Psi,\mathcal{B}}(t)) = T_{M_{\Psi,\mathcal{B}}(t)} \Phi^V_{\Psi_{t,V}} \dot{M}_{\Psi,\mathcal{B}}(t) + T_{M_{\Psi,\mathcal{B}}} \Phi^V_{\Psi_{t,V}} (\eta_V(t))_V(M)) = 0,$$

which gives us the requisite equation. The final part of the proposition follows directly from Lemma 5.11. ■

5.13 Remarks: 1. Proposition 5.12 shows that, if a motion Ψ satisfies the Galilean–Euler equations for a rigid body, so does every motion $\mathcal{B}$-equivalent to Ψ. In other words, the Galilean–Euler equations hold for an equivalence class of motions specified by the rigid body.

2. The Galilean–Euler equations are very general because they have been derived in the setting of an abstract Galilean spacetime without requiring an observer. However, the generality of the treatment makes certain things less obvious. In particular, it is not clear how the classical Euler equations fit into this setup and, if they do, whether or not there is a geometrical explanation for it. We shall see, in the next section, that the presence of an observer allows us to answer these questions. •

6. Dynamics of rigid bodies in the presence of an observer

In Section 5, we formulated rigid body dynamics in an observer independent way. In this section, we shall explore the effect of introducing an observer in this formulation. In Section 6.1 we introduce canonical velocities associated with a rigid motion in the presence of an observer. In Section 6.2 we show that, in the presence of an observer, the body and spatial velocities defined in Section 5 project to the corresponding canonical velocities. In the next section, we show that the momenta also project to the well known quantities in the presence of the observer. Finally, in Section 6.4, we illustrate how an observer enables us to recover the classical Euler equations for a rigid body.

6.1. Canonical velocities. Consider a rigid motion Ψ in a Galilean spacetime $\mathscr{G} = (\mathscr{E}, V, g, \tau)$. In Section 5.2, we introduced the curves $\eta_\Psi(t)$ and $\xi_\Psi(t) \in \mathfrak{gal}(\mathscr{G})$ corresponding to Ψ. The following result provides a decomposition of these curves in presence of an observer.

6.1 Proposition: *Let $\mathscr{G} = (\mathscr{E}, V, g, \tau)$ be a Galilean spacetime with $\mathscr{O}$ an observer and Ψ a rigid motion. For $s_0 \in I_{\mathscr{G}}$, let $\iota_{\mathscr{O}_{s_0}}(\Psi_t) = (R_\Psi(t), r_{\Psi,\mathscr{O}}(t), u_{\Psi,\mathscr{O}}(t), t)$. Then, the following statements hold:*

(i) the image of $\eta_\Psi(t) \in \mathfrak{gal}(\mathscr{G})$ under the isomorphism of the Lie algebras induced by $\iota_{\mathscr{O}_{s_0}}$ is

$$(\hat{\Omega}_\Psi(t), V_{\Psi,\mathscr{O}}(t) - R_\Psi^{-1}(t)u_{\Psi,\mathscr{O}}(t), R_\Psi^{-1}(t)\dot{u}_{\Psi,\mathscr{O}}(t), 1) \in \mathfrak{gal}(\mathscr{G}_{\mathrm{can}});$$

(ii) the image of $\xi_\Psi(t) \in \mathfrak{gal}(\mathscr{G})$ under the isomorphism of Lie algebra induced by $\iota_{\mathscr{O}_{s_0}}$ is

$$(\hat{\omega}_\Psi(t), v_{\Psi,\mathscr{O}}(t) - t(\dot{u}_{\Psi,\mathscr{O}}(t) + u_{\Psi,\mathscr{O}}(t) \times \omega_\Psi(t)), \dot{u}_{\Psi,\mathscr{O}}(t) + u_{\Psi,\mathscr{O}}(t) \times \omega_\Psi(t), 1) \in \mathfrak{gal}(\mathscr{G}_{\mathrm{can}}),$$

where $V_{\Psi,\mathscr{O}}(t) = R_\Psi^{-1}(t)\dot{r}_{\Psi,\mathscr{O}}(t)$, and $v_{\Psi,\mathscr{O}}(t) = \dot{r}_{\Psi,\mathscr{O}}(t) + r_{\Psi,\mathscr{O}}(t) \times \omega_\Psi(t)$.

Proof: We start by faithfully representing $\mathrm{Gal}(\mathscr{G}_{\mathrm{can}})$ in a vector space. We let $W = \ker(\tau) \oplus \mathbb{R} \oplus \mathbb{R}$ and, for $g = (R, r, u, t) \in \mathrm{Gal}(\mathscr{G}_{\mathrm{can}})$, define an isomorphism ρ_g of W by

$$(\mu, \sigma, \xi) \mapsto (R(\mu) + \sigma u + \xi r, \sigma + \xi t, \xi).$$

One readily verifies that the map $\rho : \mathrm{Gal}(\mathscr{G}_{\mathrm{can}}) \to GL(W)$ defined by $\rho(g) = \rho_g$ is a homomorphism. To see that the representation is faithful, suppose that

$$(\mu, \sigma, \xi) \mapsto (R(\mu) + \sigma u + \xi r, \sigma + \xi t, \xi) = (\mu, \sigma, \xi)$$

for all $(\mu, \sigma, \xi) \in W$. Then we must have $\sigma + \xi t = 0$, for all $\sigma, \xi \in \mathbb{R}$, implying that $t = 0$. Similarly, $R(\mu) + \sigma u + \xi r = \mu$ for all $(\mu, \sigma, \xi) \in W$ implies that $r = 0, u = 0$, and $R = \mathrm{id}_{\ker(\tau)}$. Thus the representation is faithful. In block matrix form, the representation of (R, r, u, t) on W is

$$\begin{bmatrix} R & u & r \\ 0 & 1 & t \\ 0 & 0 & 1 \end{bmatrix} \in GL(W).$$

We then compute

$$\begin{bmatrix} R & u & r \\ 0 & 1 & t \\ 0 & 0 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} R^{-1} & -R^{-1}u & R^{-1}(tu - r) \\ 0 & 1 & -t \\ 0 & 0 & 1 \end{bmatrix}.$$

With this expression, both parts follow from direct computation. ■

We call the first two components in (i) as, respectively, the ***canonical body angular velocity***, the ***canonical body linear velocity***, and denote them by $\Omega_\Psi^{\mathrm{can}}$ and $V_{\Psi,\mathscr{O}}^{\mathrm{can}}$, respectively. Similarly, we call the first two components in (ii) as, respectively, the ***canonical spatial angular velocity***, the ***canonical spatial linear velocity***, and represent them by $\omega_\Psi^{\mathrm{can}}$ and $v_{\Psi,\mathscr{O}}^{\mathrm{can}}$, respectively.

6.2. Linear and angular velocities. Recall that, given a rigid motion Ψ in a Galilean spacetime, the body linear velocity is the map $V^b_{\Psi,\mathrm{lin}}\colon Q\times\mathbb{R}\to V$ given by

$$V^b_{\Psi,\mathrm{lin}}(R,x,t)=\Psi^{-1}_{t,V}(\frac{\mathrm{d}}{\mathrm{d}t}(\Psi_t(x)),$$

and the spatial linear velocity is the map $V^s_{\Psi,\mathrm{lin}}\colon Q\times\mathbb{R}\to V_{\mathscr{G}}$ given by

$$V^s_{\Psi,\mathrm{lin}}(R,x,t)=-\Psi_{t,V}(\frac{\mathrm{d}}{\mathrm{d}t}(\Psi^{-1}_t(x))).$$

Let's see what these velocities look like in the presence of an observer. We look at body linear velocity first.

6.2 Proposition: *Let Ψ be a rigid motion in a Galilean spacetime $\mathscr{G}=(\mathscr{E},V,g,\tau)$ and let $\mathscr{O}$ be an observer. Then $P_{\mathscr{O}}(V^b_{\Psi,\mathrm{lin}}(R,x,t))=V^{\mathrm{can}}_{\Psi,\mathscr{O}}(t)$.*

Proof: We know that, for each instant $s_0\in I_{\mathscr{G}}$, there exists an isomorphism $\iota_{\mathscr{O}_{s_0}}$ such that, for a motion Ψ, we have

$$\iota_{\mathscr{O}_{s_0}}(\Psi_t)=(R_\Psi(t),r_{\Psi,\mathscr{O}}(t),u_{\Psi,\mathscr{O}}(t),t).$$

Also,

$$\Psi_t(x)=R_\Psi(t)(x-\mathscr{O}_{\pi_{\mathscr{G}}(x)})+(\pi_{\mathscr{G}}(x)-s_0)u_{\Psi,\mathscr{O}}(t)+r_{\Psi,\mathscr{O}}(t)+\mathscr{O}_{\pi_{\mathscr{G}}(x)+t}.$$

Now, for $x\in\mathscr{O}$, we have

$$\Psi_t(x)=r_{\Psi,\mathscr{O}}(t)+\mathscr{O}_{\pi_{\mathscr{G}}(x)+t},$$

so we have

$$\frac{\mathrm{d}}{\mathrm{d}t}(\Psi_t(x))=\dot r_{\Psi,\mathscr{O}}(t)+v_{\mathscr{O}}.$$

Next,

$$\begin{aligned}\Psi^{-1}_{t,V}\Big(\frac{\mathrm{d}}{\mathrm{d}t}\Psi_t(x))\Big)&=\Psi^{-1}_{t,V}(\dot r_{\Psi,\mathscr{O}}(t)+v_{\mathscr{O}})\\&=\Psi^{-1}_t(\dot r_{\Psi,\mathscr{O}}(t)+v_{\mathscr{O}}+\mathscr{O}_{s_0})-\Psi^{-1}_t(\mathscr{O}_{s_0})\\&=R^{-1}_\Psi(t)(\dot r_{\Psi,\mathscr{O}}(t)+v_{\mathscr{O}}+\mathscr{O}_{s_0}-\mathscr{O}_{1+s_0})-R^{-1}_\Psi(t)u_{\Psi,\mathscr{O}}(t)\\&\quad+R^{-1}_\Psi(t)(tu_{\Psi,\mathscr{O}}(t)-r_{\Psi,\mathscr{O}}(t))+(1-t)v_{\mathscr{O}}+\mathscr{O}_{s_0}\\&\quad-R^{-1}_\Psi(t)(tu_{\Psi,\mathscr{O}}(t)-r_{\Psi,\mathscr{O}}(t))+tv_{\mathscr{O}}-\mathscr{O}_{s_0}\\&=R^{-1}_\Psi(t)\dot r_{\Psi,\mathscr{O}}(t)-R^{-1}_\Psi(t)u_{\Psi,\mathscr{O}}(t)+v_{\mathscr{O}}.\end{aligned}$$

From this, the result follows. ■

Let us look at the spatial linear velocity now.

6.3 Proposition: *Let Ψ be a motion in a Galilean spacetime $\mathscr{G} = (\mathscr{E}, V, g, \tau)$ and let $\mathscr{O}$ be an observer. Then $P_{\mathscr{O}}(V^s_{\Psi,\mathrm{lin}}(R, x, t)) = v^{\mathrm{can}}_{\Psi,\mathscr{O}}(t)$.*

Proof: For $x \in \mathscr{O}$, we compute,

$$\Psi_t^{-1}(x) = R_\Psi^{-1}(t)(tu_{\Psi,\mathscr{O}}(t) - r_{\Psi,\mathscr{O}}(t)) - tv_{\mathscr{O}} + \mathscr{O}_{s_0}.$$

Therefore,

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}(\Psi_t^{-1}(x)) &= R_\Psi^{-1}(t)(u_{\Psi,\mathscr{O}}(t) + t\dot{u}_{\Psi,\mathscr{O}}(t) - \dot{r}_{\Psi,\mathscr{O}}(t)) \\
&\quad + (-R_\Psi^{-1}(t)\dot{R}_\Psi(t)R_\Psi^{-1}(t))(tu_{\Psi,\mathscr{O}}(t) - r_{\Psi,\mathscr{O}}(t))) - v_{\mathscr{O}} \\
&= R_\Psi^{-1}(t)(u_{\Psi,\mathscr{O}}(t) + t\dot{u}_{\Psi,\mathscr{O}}(t) - \dot{r}_{\Psi,\mathscr{O}}(t)) \\
&\quad - R_\Psi^{-1}(t)\hat{\omega}_\Psi(t)(tu_{\Psi,\mathscr{O}}(t) - r_{\Psi,\mathscr{O}}(t)) - v_{\mathscr{O}} \\
&= R_\Psi^{-1}(t)u_{\Psi,\mathscr{O}}(t) + R_\Psi^{-1}(t)[t(\dot{u}_{\Psi,\mathscr{O}}(t) + u_\Psi(t) \times \omega_\Psi(t))] \\
&\quad - R_\Psi^{-1}(t)(\dot{r}_{\Psi,\mathscr{O}}(t) + r_{\Psi,\mathscr{O}}(t) \times \omega_\Psi(t)) - v_{\mathscr{O}} \\
&= R_\Psi^{-1}(t)u_\Psi(t) - R_\Psi^{-1}(t)(v_{\Psi,\mathscr{O}}(t) \\
&\quad - t(\dot{u}_{\Psi,\mathscr{O}}(t) + u_{\Psi,\mathscr{O}}(t) \times \omega_\Psi(t))) - v_{\mathscr{O}}.
\end{aligned}$$

Let's call the last expression as $\dot{\Psi}_t^{-1}(x)$. Now, we compute

$$\begin{aligned}
-\Psi_{t,V}(\frac{\mathrm{d}}{\mathrm{d}t}(\Psi_t^{-1}(x)) &= \Psi_t(-\dot{\Psi}_t^{-1}(x) + \mathscr{O}_{s_0}) - \Psi_t(\mathscr{O}_{s_0}) \\
&= R_\Psi(t)(-\dot{\Psi}_t^{-1}(x) + \mathscr{O}_{s_0} - \mathscr{O}_{1+s_0}) + u_{\Psi,\mathscr{O}}(t) + r_{\Psi,\mathscr{O}}(t) + tv_{\mathscr{O}} \\
&\quad + \mathscr{O}_{1+s_0} - r_{\Psi,\mathscr{O}}(t) - tv_{\mathscr{O}} - \mathscr{O}_{s_0} \\
&= R_\Psi(t)(-\dot{\Psi}_t^{-1}(x) - v_{\mathscr{O}}) + u_{\Psi,\mathscr{O}}(t) + v_{\mathscr{O}} \\
&= R_\Psi(t)\left(R_\Psi^{-1}(t)(v_{\Psi,\mathscr{O}}(t) - t(\dot{u}_{\Psi,\mathscr{O}}(t) + u_{\Psi,\mathscr{O}}(t) \times \omega_\Psi(t))\right) \\
&\quad - u_{\Psi,\mathscr{O}}(t) + u_{\Psi,\mathscr{O}}(t) + v_{\mathscr{O}} \\
&= v_{\Psi,\mathscr{O}}(t) - t(\dot{u}_{\Psi,\mathscr{O}}(t) + u_{\Psi,\mathscr{O}}(t) \times \omega_\Psi(t)) + v_{\mathscr{O}}.
\end{aligned}$$

From this the result follows. ■

We notice that, in the presence of an observer, the spatial linear and body linear velocities project onto the canonical spatial linear and canonical body linear velocities, respectively.

6.3. Spatial and body momenta. We let $\mathscr{G} = (\mathscr{E}, V, g, \tau)$ be a Galilean spacetime with Ψ a rigid motion, $\mathscr{O}$ an observer and $(\mathcal{B}, \mu)$ a rigid body with $\mathcal{B} \in \mathscr{E}(s_0)$. We next see how our definitions of spatial and body momenta look when we have an observer $\mathscr{O}$. In such a case, we have the following result.

6.4 Proposition: *Let $\mathscr{G} = (\mathscr{E}, V, g, \tau)$ be a Galilean spacetime with Ψ a rigid motion, $\mathscr{O}$ an observer, and $(\mathcal{B}, \mu)$ a rigid body with $\mathcal{B} \in \mathscr{E}(s_0)$, and let $m_{\Psi,\mathcal{B}}$, $\ell_{\Psi,\mathcal{B}}$, $M_{\Psi,\mathcal{B}}$, and $L_{\Psi,\mathcal{B}}$ be as defined in Section 5.3. For $s_0 \in I_{\mathscr{G}}$, let*

$$\iota_{\mathscr{O}_{s_0}}(\Psi_t) = (R_\Psi(t), r_{\Psi,\mathscr{O}}(t), u_{\Psi,\mathscr{O}}(t), t).$$

Then the following statements hold:

(i) $P_{\mathscr{O}}(m_{\Psi,\mathcal{B}}(t)) = \mu(\mathcal{B})\dot{r}_{\Psi,\mathscr{O}}(t)$;

(ii) $P_{\mathscr{O}}(\ell_{\Psi,\mathcal{B}}(t)) = R_\Psi(t)\mathbb{I}_c(R_\Psi^{-1}(t)\omega_\Psi(t))$;

(iii) $P_{\mathscr{O}}(M_{\Psi,\mathcal{B}}(t)) = \mu(\mathcal{B})V^{\mathrm{can}}_{\Psi,\mathscr{O}}(t)$;

(iv) $P_{\mathscr{O}}(L_{\Psi,\mathcal{B}}(t)) = \mathbb{I}_c(R_\Psi^{-1}(t)\omega_\Psi(t))$.

Proof: (i) We compute

$$\begin{aligned}P_{\mathscr{O}}(m_{\Psi,\mathcal{B}}(t)) &= \mu(\mathcal{B})P_{\mathscr{O}}(\frac{\mathrm{d}}{\mathrm{d}t}(\Psi_t(x_c)) = \mu(\mathcal{B})P_{\mathscr{O}}(\dot{r}_{\Psi,\mathscr{O}}(t) + v_{\mathscr{O}})\\ &= \mu(\mathcal{B})\dot{r}_{\Psi,\mathscr{O}}(t),\end{aligned}$$

where we have used the computations carried out in Proposition 6.2.

Parts (ii) and (iv) are easily seen to be true since both $\ell_{\Psi,\mathcal{B}}$ and $L_{\Psi,\mathcal{B}}$ take their values in $\ker(\tau)$ and the projection $P_{\mathscr{O}}$ is the identity map on $\ker(\tau)$.

To obtain (iii), we compute

$$\begin{aligned}P_{\mathscr{O}}(M_{\Psi,\mathcal{B}}(t)) &= \mu(\mathcal{B})P_{\mathscr{O}}(\Psi^{-1}_{t,V}(\frac{\mathrm{d}}{\mathrm{d}t}(\Psi_t(x_c)))\\ &= \mu(\mathcal{B})P_{\mathscr{O}}(\Psi^{-1}_{t,V}(\dot{r}_{\Psi,\mathscr{O}}(t) + v_{\mathscr{O}}))\\ &= \mu(\mathcal{B})P_{\mathscr{O}}(R_\Psi^{-1}(t)\dot{r}_{\Psi,\mathscr{O}}(t) - R_\Psi^{-1}(t)u_{\Psi,\mathscr{O}}(t) + v_{\mathscr{O}})\\ &= \mu(\mathcal{B})V^{\mathrm{can}}_{\Psi,\mathscr{O}}(t),\end{aligned}$$

as desired. ■

It is worth pointing out that the classical definition of spatial angular momentum requires an observer, and is different from ours. Given an observer $\mathscr{O}$ and a rigid motion Ψ, the ***classical spatial angular momentum*** $\ell^{\mathrm{cl}}_{\Psi,\mathcal{B}}$ for a rigid body $(\mathcal{B},\mu)$ about its center of mass x_c is defined as

$$\ell^{\mathrm{cl}}_{\Psi,\mathcal{B}}(t) = \int_{\mathcal{B}} P_{\mathscr{O}}(\Psi_t(x) - x_c) \times P_{\mathscr{O}}(\frac{\mathrm{d}}{\mathrm{d}t}(\Psi_t(x) - x_c))\mathrm{d}\mu.$$

One can motivate this definition of spatial angular momentum by recalling how it might be defined for a particle of mass m (see, for example, [Arnol′d 1978]). If a particle is moving in $\mathbb{R}^3$ following a curve $t \mapsto x(t)$, then we would define the spatial angular momentum at time t to be $mx(t) \times \dot{x}(t)$. This is exactly the intuition behind the definition of $\ell^{\mathrm{cl}}_{\Psi,\mathscr{O}}$. Our definition of body angular momentum agrees with the classical one, and therefore we do not need to consider it separately. We shall see in the next section that the equations of motion derived on the basis of the conservation of classical spatial angular momentum are equivalent to the general spatial Euler equations. We have the following result.

6.5 Proposition: *Let $(\mathcal{B},\mu)$ be a rigid body in a Galilean spacetime $\mathscr{G}$ undergoing a rigid motion Ψ, and let $\mathscr{O}$ be an observer with the property that $x_c \in \mathscr{O}$. Then $\ell^{\mathrm{cl}}_{\Psi,\mathcal{B}}(t) = R_\Psi(t)\mathbb{I}_c(R_\Psi(t)^{-1}\omega_\Psi(t)) + \mu(\mathcal{B})r_{\Psi,\mathscr{O}}(t) \times \dot{r}_{\Psi,\mathscr{O}}(t)$.*

Proof: We let $s_0 = \pi_{\mathscr{G}}(x_c)$, and use the isomorphism $\iota_{\mathscr{O}_{s_0}}$ between $\mathrm{Gal}(\mathscr{G})$ and $\mathrm{Gal}(\mathscr{G}_{\mathrm{can}})$ to get

$$\Psi_t(x) = R_\Psi(t)(x - \mathscr{O}_{\pi_{\mathscr{G}}(x)}) + r_{\Psi,\mathscr{O}}(t) + tv_{\mathscr{O}} + \mathscr{O}_{\pi_{\mathscr{G}}(x)}, \quad x \in \mathcal{B},$$

which implies that $\Psi_t(x) - x_c = R_\Psi(t)(x - x_c) + r_{\Psi,\mathscr{O}}(t)$. The result is readily verified by using part (ii) of Lemma 3.1 in the computation of the integral. ■

6.4. Euler equations of a rigid body. In this section we look at the Galilean–Euler equations, as derived in Section 5.4, in the presence of an observer. Since we consider the abstract Galilean group in our analysis, derivatives of velocity boosts also appear in the equations. We first write down the general Galilean–Euler equations in the presence of an observer.

6.6 Proposition: *Let $(\mathcal{B}, \mu)$ be a rigid body, Ψ a rigid motion, and let $\mathscr{O}$ be an observer. For $s_0 \in I_{\mathscr{G}}$, let*

$$\iota_{\mathscr{O}_{s_0}}(\Psi_t) = (R_\Psi(t), r_{\Psi,\mathscr{O}}(t), u_{\Psi,\mathscr{O}}(t), t)$$

for each $t \in \mathbb{R}$. The following statements hold:

(i) the spatial Galilean–Euler equations for Ψ are equivalent to

$$\begin{aligned}\mathbb{I}_c(\dot{\omega}_\Psi(t)) &= \mathbb{I}_c(\omega_\Psi(t)) \times \omega_\Psi(t)\\ \ddot{r}_{\Psi,\mathscr{O}}(t) &= -\ddot{R}_\Psi(t)(x_c - \mathscr{O}_{\pi_{\mathscr{G}}(x_c)}) - (\pi_{\mathscr{G}}(x_c) - s_0)\ddot{u}_{\Psi,\mathscr{O}}(t);\end{aligned}$$

(ii) the body Galilean–Euler equations for Ψ are equivalent to

$$\begin{aligned}\dot{L}_{\Psi,\mathcal{B}}(t) &= L_{\Psi,\mathcal{B}}(t) \times \Omega_\Psi(t)\\ \dot{M}_{\Psi,\mathcal{B}}(t) &= P_{\mathscr{O}}(M_{\Psi,\mathcal{B}}(t)) \times \Omega_\Psi(t) - R_\Psi^{-1}(t)\dot{u}_{\Psi,\mathscr{O}}(t).\end{aligned}$$

Proof: (i) The first spatial Galilean–Euler equation evolves on $\ker(\tau)$ and thus remains the same under the projection $P_{\mathscr{O}}$. For the second equation, we compute

$$\begin{aligned}0 = \ddot{x}_c(t) &= \frac{\mathrm{d}}{\mathrm{d}t}\Big(\dot{R}_\Psi(t)(x_c - \mathscr{O}_{\pi_{\mathscr{G}}}(x_c)) + (\pi_{\mathscr{G}}(x_c) - s_0)\dot{u}_{\Psi,\mathscr{O}}(t) + \dot{r}_{\Psi,\mathscr{O}}(t) + v_{\mathscr{O}}\Big)\\ &= \ddot{R}_\Psi(t)(x_c - \mathscr{O}_{\pi_{\mathscr{G}}}(x_c)) + \ddot{r}_{\Psi,\mathscr{O}}(t) + (\pi_{\mathscr{G}}(x_c) - s_0)\ddot{u}_{\Psi,\mathscr{O}}(t),\end{aligned}$$

from which we get the required equation.

(ii) Similarly, the first body Galilean–Euler equation remains unchanged under the projection onto $\ker(\tau)$. To get the second equation we use the definition of $M_{\Psi,\mathscr{O}}$ and the relation $\Psi_V(v) = R_\Psi P_{\mathscr{O}}(v) + \tau(v)(u_{\Psi,\mathscr{O}} + v_{\mathscr{O}})$, and compute

$$\begin{aligned}0 = \frac{\mathrm{d}}{\mathrm{d}t}m_{\Psi,\mathcal{B}}(t) &= \frac{\mathrm{d}}{\mathrm{d}t}\Psi_{t,V}(M_{\Psi,\mathscr{O}}(t))\\ &= \frac{\mathrm{d}}{\mathrm{d}t}\big(R_\Psi(t)P_{\mathscr{O}}(M_{\Psi,\mathscr{O}}(t))) + \tau(M_{\Psi,\mathscr{O}}(t))(u_{\Psi,\mathscr{O}}(t) + v_{\mathscr{O}})\big)\\ &= \dot{R}_\Psi(t)P_{\mathscr{O}}(M_{\Psi,\mathscr{O}}(t)) + R_\Psi(t)\frac{\mathrm{d}}{\mathrm{d}t}(P_{\mathscr{O}}(M_{\Psi,\mathscr{O}}(t))) + \dot{u}_{\Psi,\mathscr{O}}(t)\\ &= R_\Psi(t)\hat{\Omega}P_{\mathscr{O}}(M_{\Psi,\mathscr{O}}(t)) + R_\Psi(t)(\dot{M}_{\Psi,\mathscr{O}}(t))) + \dot{u}_{\Psi,\mathscr{O}}(t),\end{aligned}$$

since $M_{\Psi,\mathscr{O}}(t) \in V_{\mathscr{G}}$ and thus $\dot{M}_{\Psi,\mathscr{O}}(t) = \frac{\mathrm{d}}{\mathrm{d}t}P_{\mathscr{O}}(M_{\Psi,\mathscr{O}}(t))$. The result now follows. ■

Let us now show that we get the same equations of motion if we use the classical spatial angular momentum $\ell^{\mathrm{cl}}_{\Psi,\mathcal{B}}$ instead of $\ell_{\Psi,\mathcal{B}}$. Let us write $p^{\mathrm{cl}}_{\Psi,\mathcal{B}} = (\ell^{\mathrm{cl}}_{\Psi,\mathcal{B}}, m_{\Psi,\mathcal{B}})$ to denote the classical spatial momentum. We also call the equations of motion derived on the basis of the conservation of classical spatial momentum the ***classical spatial Euler equations***. We have the following result.

6.7 Proposition: *Let $(\mathcal{B},\mu)$ be a rigid body undergoing a rigid motion Ψ in a Galilean spacetime $\mathscr{G}$. Let $\mathscr{O}$ be an observer with the property that $x_c \in \mathscr{O}$. The following statements are equivalent:*

(i) the classical spatial momentum $\mathrm{p}^{\mathrm{cl}}_{\Psi,\mathcal{B}}$ is conserved;

(ii) the motion of the body satisfies the classical spatial Euler equations

$$\begin{aligned}\mathbb{I}_c(\dot{\omega}_\Psi(t)) &= \mathbb{I}_c(\omega_\Psi(t)) \times \omega_\Psi(t)\\ \ddot{r}_{\Psi,\mathscr{O}}(t) &= 0.\end{aligned}$$

Proof: As before, we let $s_0 = \pi_{\mathscr{G}}(x_c)$ and consider the isomorphism $\iota_{\mathscr{O}_{s_0}}$, using which, it is easy to see that the conservation of spatial linear momentum $m_{\Psi,\mathcal{B}}$ implies that $\ddot{r}_{\Psi,\mathscr{O}}(t) = 0$. It is a simple computation to show that this also implies that

$$\frac{\mathrm{d}}{\mathrm{d}t}(\ell^{\mathrm{cl}}_{\Psi,\mathcal{B}}(t)) = \frac{\mathrm{d}}{\mathrm{d}t}(R_\Psi(t)\mathbb{I}_c(R_\Psi(t)^{-1}\omega_\Psi(t))) = \frac{\mathrm{d}}{\mathrm{d}t}(l_{\Psi,\mathcal{B}}(t)).$$

The result now follows. ■

6.5. The Galilean connection. In Proposition 6.6, we wrote down the general form of the Galilean–Euler equations in the presence of an observer. Since we have considered the abstract Galilean group in our analysis, we have imposed no restrictions on the velocity boost (the "$u_{\Psi,\mathscr{O}}$" component) corresponding to a rigid motion. This is the reason why the derivatives of these velocity boosts appear in the equations given in Proposition 6.6. Recall that the classical Euler equations for a rigid body do not include these derivative terms because the velocity boosts are assumed to be "uniform". In this section, we explain how, in our general setting, an observer allows us to recover the classical equations of motion for a rigid body by defining a special geometric structure (namely a principal connection) on $\mathrm{Gal}(\mathscr{G})$. We refer to [Kobayashi and Nomizu 1963] for the definitions and properties of principal connections.

Let $Q = O(\ker(\tau)) \times \mathscr{E}$ and $(\mathcal{B},\mu)$ be a rigid body. For the center of mass $x_c \in \mathscr{E}$ of the rigid body, consider the map

$$\begin{aligned}\pi_c : \mathrm{Gal}(\mathscr{G}) &\to Q\\ \psi &\mapsto (R_\psi, \psi(x_c)).\end{aligned}$$

An observer $\mathscr{O}$ defines a map $\pi_{c,\mathscr{O}} : \mathrm{Gal}(\mathscr{G}) \to O(\ker(\tau)) \times \ker(\tau) \times \mathbb{R}$ given by

$$\psi \mapsto (R_\psi, P_{\mathscr{O}}(\psi(x_c) - x_c), \tau(\psi(x_c) - x_c)).$$

We also know that, for $s_0 \in I_{\mathscr{G}}$, there is an isomorphism $\iota_{\mathscr{O}_{s_0}}$ from $\mathrm{Gal}(\mathscr{G})$ to $O(\ker(\tau)) \times \ker(\tau) \times \ker(\tau) \times \mathbb{R}$ given by

$$\psi \mapsto (R_\psi, r_{\psi,\mathscr{O}}, u_{\psi,\mathscr{O}}, t_\psi).$$

For $x_c \in \mathscr{O}$, we can write

$$\psi(x_c) = R_\psi(x_c - x_c) + (\pi_{\mathscr{G}} - s_0)u_{\psi,\mathscr{O}} + r_{\psi,\mathscr{O}} + t_\psi v_{\mathscr{O}} + x_c,$$

and thus we have

$$P_{\mathscr{O}}(\psi(x_c) - x_c) = (\pi_{\mathscr{G}}(x_c) - s_0)u_{\psi,\mathscr{O}} + r_{\psi,\mathscr{O}}.$$

Also, $\tau(\psi(x_c) - x_c) = t_\psi$, so the map $\pi_{c,\mathscr{O}}$ induces a map

$$\begin{aligned}\pi^{\mathrm{can}}_{c,\mathscr{O}} : \mathrm{Gal}(\mathscr{G}_{\mathrm{can}}) &\to O(\ker(\tau)) \times \ker(\tau) \times \mathbb{R} = \mathrm{Gal}(\mathscr{G}_{\mathrm{can}})/\ker(\tau)\\ (R_\psi, r_{\psi,\mathscr{O}}, u_{\psi,\mathscr{O}}, t_\psi) &\mapsto (R_\psi, (\pi_{\mathscr{G}} - s_0)u_{\psi,\mathscr{O}} + r_{\psi,\mathscr{O}}, t_\psi),\end{aligned}$$

where the quotient $\mathrm{Gal}(\mathscr{G}_{\mathrm{can}})/\ker(\tau)$ corresponds to the following action of $\ker(\tau)$ on $\mathrm{Gal}(\mathscr{G}_{\mathrm{can}})$.

$$\begin{aligned}\ker(\tau) \times \mathrm{Gal}(\mathscr{G}_{\mathrm{can}}) &\to \mathrm{Gal}(\mathscr{G}_{\mathrm{can}})\\ (\mu, (R, r, u, s)) &\mapsto (R, r - (\pi_{\mathscr{G}}(x_c) - s_0)\mu, u + \mu, s).\end{aligned}$$

Thus, $\ker(\tau)$ acts on $\mathrm{Gal}(\mathscr{G}_{\mathrm{can}})$ by appropriately changing the $r_{\psi,\mathscr{O}}$ and $u_{\psi,\mathscr{O}}$ components of a given $\psi \in \mathrm{Gal}(\mathscr{G})$ such that the resulting mapping gives the same physical motion of the body as ψ, that is, it lies in $[\psi]_{\mathcal{B}}$. We also write $\pi := \pi^{\mathrm{can}}_{c,\mathscr{O}} \circ \iota_{\mathscr{O}_{s_0}} : \mathrm{Gal}(\mathscr{G}) \to \mathrm{Gal}(\mathscr{G}_{\mathrm{can}})/\ker(\tau)$. It is clear that, given $\psi \in \mathrm{Gal}(\mathscr{G})$, a Galilean mapping $\phi \in [\psi]_{\mathcal{B}}$ if and only if $\pi(\phi) = \pi(\psi)$. We have the following result.

6.8 Proposition: *Let $(\mathcal{B}, \mu)$ be a rigid body in a Galilean spacetime and $\mathscr{O}$ be an observer. For fixed $s_0 \in I_{\mathscr{G}}$, the map $\omega_{\mathrm{can}} : T\mathrm{Gal}(\mathscr{G}_{\mathrm{can}}) \to \ker(\tau)$ given by*

$$\omega_{\mathrm{can}}(X_R, X_r, X_u, X_t) = X_u, \quad (X_R, X_r, X_u, X_t) \in T_{(R,r,u,t)}\mathrm{Gal}(\mathscr{G}_{\mathrm{can}}),$$

is a principal connection 1-form in the bundle $\pi^{\mathrm{can}}_{c,\mathscr{O}} : \mathrm{Gal}(\mathscr{G}_{\mathrm{can}}) \to \mathrm{Gal}(\mathscr{G}_{\mathrm{can}})/\ker(\tau)$.

Proof: Given $X = (X_R, X_r, X_u, X_t) \in T_{(R,r,u,t)}\mathrm{Gal}(\mathscr{G}_{\mathrm{can}})$, it is easy to see that

$$T\pi^{\mathrm{can}}_{c,\mathscr{O}}(X_R, X_R, X_u, X_t) = (X_R, X_r + (\pi_{\mathscr{G}}(x_c) - s_0)X_u, X_t).$$

The observer $\mathscr{O}$ allows us to decompose X into its *vertical* and *horizontal* components as follows. We write

$$X = \mathrm{hor}(X) + \mathrm{ver}(X), \tag{6.1}$$

where

$$\begin{aligned}\mathrm{hor}(X) &= (X_R, X_r + (\pi_{\mathscr{G}}(x_c) - s_0)X_u, 0, X_t),\\ \mathrm{ver}(X) &= (0, -(\pi_{\mathscr{G}}(x_c) - s_0)X_u, X_u, 0).\end{aligned}$$

It can be seen that $T\pi^{\mathrm{can}}_{c,\mathscr{O}}(\mathrm{ver}(X)) = 0$, and $\omega_{\mathrm{can}}(\mathrm{hor}(X)) = 0$. Thus (6.1) defines an Ehresmann connection in $\pi^{\mathrm{can}}_{c,\mathscr{O}} : \mathrm{Gal}(\mathscr{G}_{\mathrm{can}}) \to O(\ker(\tau)) \times \ker(\tau) \times \mathbb{R}$. Next, the infinitesimal generator $\zeta_{\mathrm{Gal}(\mathscr{G})}$ corresponding to $\zeta \in \ker(\tau)$ for the action of $\ker(\tau)$ on $\mathrm{Gal}(\mathscr{G}_{\mathrm{can}})$ is given by

$$\begin{aligned}\zeta_{\mathrm{Gal}(\mathscr{G})}(R, r, u, s) &= \left.\frac{d}{dt}\right|_{t=0} (R, r - (\pi_{\mathscr{G}}(x_c) - s_0)\exp(\zeta t), u + \exp(\zeta t), s)\\ &= (0, -(\pi_{\mathscr{G}}(x_c) - s_0)\zeta, \zeta, 0),\end{aligned}$$

and thus, by definition,

$$\omega_{\mathrm{can}}(\zeta_{\mathrm{Gal}(\mathscr{G})}(R,r,u,s)) = \zeta.$$

Next, given $h \in \ker(\tau)$ and $X = (X_R, X_r, X_u, X_t) \in T_{(R,r,u,t)}\mathrm{Gal}(\mathscr{G}_{\mathrm{can}})$, it is easy to see that $\omega_{\mathrm{can}}(T_{(R,r,u,t)}\Phi_h X) = \mathrm{ad}(h)\cdot\omega_{\mathrm{can}}(X)$, where $\Phi_h : \mathrm{Gal}(\mathscr{G}_{\mathrm{can}}) \to \mathrm{Gal}(\mathscr{G}_{\mathrm{can}})$ is the action of $\ker(\tau)$ on $\mathrm{Gal}(\mathscr{G}_{\mathrm{can}})$ and $\mathrm{ad}_h : \mathfrak{gal}(\mathscr{G}_{\mathrm{can}}) \to \mathfrak{gal}(\mathscr{G}_{\mathrm{can}})$ is defined by $\mathrm{ad}_h(\beta) = T_e L_h R_{h^{-1}}(\beta)$, $\beta \in \mathfrak{gal}(\mathscr{G}_{\mathrm{can}})$. So ω_{can} is indeed a connection one-form. ■

Now, it is easy to see that $\ker(\tau)$ also acts on $\mathrm{Gal}(\mathscr{G})$ as follows:

$$\begin{aligned} \ker(\tau) \times \mathrm{Gal}(\mathscr{G}) &\to \mathrm{Gal}(\mathscr{G}) \\ (\mu, \psi) &\mapsto \psi_\mu \circ \psi, \end{aligned}$$

where ψ_μ is such that $\iota_{\mathscr{O}_{s_0}}(\psi_\mu) = (\mathrm{id}_{O(\ker(\tau))}, -(\pi_{\mathscr{G}}(x_c) - s_0)\mu, \mu, 0) \in \mathrm{Gal}(\mathscr{G}_{\mathrm{can}})$. In other words, $\ker(\tau)$ acts on $\mathrm{Gal}(\mathscr{G})$ as a subgroup of N_0 that fixes x_c. It can be seen that, for any $x \in \mathscr{E}$, we have $\psi_\mu(x) = x + (\pi_{\mathscr{G}}(x) - \pi_{\mathscr{G}}(x_c))\mu$. As a direct consequence of Proposition 6.8, we have the following corollary.

6.9 Corollary: *The* $\ker(\tau)$*-valued one-form on* $\mathrm{Gal}(\mathscr{G})$ *defined by* $\omega_{\mathscr{O}} = (\iota_{\mathscr{O}_{s_0}})^*\omega_{\mathrm{can}}$ *is a connection one-form in the bundle* $\mathrm{Gal}(\mathscr{G}) \to \mathrm{Gal}(\mathscr{G})/\ker(\tau)$*. We call* $\omega_{\mathscr{O}}$ *the* ***Galilean connection induced by*** $\mathscr{O}$*.*

Thus, the Galilean connection $\omega_{\mathscr{O}}$ induced by $\mathscr{O}$ is the pull-back of ω_{can} to $\mathrm{Gal}(\mathscr{G})$ by $\iota_{\mathscr{O}_{s_0}}$. It allows us to recover the classical Euler equations for a rigid body. It may be recalled that these equations do not contain derivatives of velocity boosts (that is, the "$\dot{u}_{\Psi,\mathscr{O}}$" terms) corresponding to the given rigid motion. The next proposition shows that, given a rigid motion, the Galilean connection allows us to choose a rigid motion that gives the same physical motion of the rigid body as the given rigid motion, and such that the corresponding Galilean–Euler equations do not contain the "$\dot{u}_{\Psi,\mathscr{O}}$" terms. This is made precise in the following proposition.

6.10 Proposition: *Let* $(\mathcal{B}, \mu)$ *be a rigid body in a Galilean spacetime,* $\mathscr{O}$ *be an observer such that* $x_c \in \mathscr{O}$*, and* $s_0 \in I_{\mathscr{G}}$*. Then, for every rigid motion* Ψ*, there exists a rigid motion* Φ *with the following properties:*

(i) Φ *is* $\mathcal{B}$*-equivalent to* Ψ*;*

(ii) Φ_t *is horizontal with respect to* $\omega_{\mathscr{O}}$*;*

(iii) The Galilean–Euler equations for Φ *are equivalent to*

$$\begin{aligned} \mathbb{I}_c(t)(\dot{\omega}_\Phi(t)) &= \mathbb{I}_c(t)(\omega_\Phi(t)) \times \omega_\Phi(t) \\ \ddot{r}_{\Phi,\mathscr{O}}(t) &= 0 \\ \dot{L}_{\Phi,\mathcal{B}}(t) &= L_{\Phi,\mathcal{B}}(t) \times \Omega_\Phi(t) \\ \dot{M}_{\Phi,\mathcal{B}}(t) &= P_{\mathscr{O}}(M_{\Phi,\mathcal{B}}(t)) \times \Omega_\Phi(t). \end{aligned}$$

Moreover, given $C_0 \in \ker(\tau)$*, the rigid motion* Φ *can be uniquely chosen such that* $u_{\Phi,\mathscr{O}}(t) = C_0$ *for every* $t \in \mathbb{R}$*. In particular, if* $x(t) = \pi(\Psi_t) = (R_\Psi(t), a_{\Psi,\mathscr{O}}(t), t)$*, then,* Φ_t *is the horizontal lift of* $x(t)$ *passing through* $\iota^{-1}_{\mathscr{O}_{s_0}}(R_\Psi(t_0), a_{\Psi,\mathscr{O}}(t_0) - (\pi_{\mathscr{G}}(x_c) - s_0)C_0, C_0, t_0)$ *for some (and therefore every)* $t_0 \in \mathbb{R}$*.*

Proof: For $x(t) = \pi(\Psi_t) = (R_\Psi(t), a_{\Psi,\mathscr{O}}(t), t) \in O(\ker(\tau)) \times \ker(\tau) \times \mathbb{R}$, we have, for each $t \in \mathbb{R}$,

$$\begin{aligned}(\pi^{\mathrm{can}}_{c,\mathscr{O}})^{-1}&(R_\Psi(t), a_{\Psi,\mathscr{O}}(t), t)\\ &=\{(R_\Psi(t), a_{\Psi,\mathscr{O}}(t) - (\pi_{\mathscr{G}}(x_c) - s_0)\tilde{u}(t),\ \tilde{u}(t),\ t) \in \mathrm{Gal}(\mathscr{G}_{\mathrm{can}}) \mid \tilde{u}(t) \in \ker(\tau)\}.\end{aligned}$$

Thus, all rigid motions Φ for which $\iota_{\mathscr{O}_{s_0}}(\Phi_t) \in (\pi^{\mathrm{can}}_{c,\mathscr{O}})^{-1}(R_\Psi(t), a_{\Psi,\mathscr{O}}(t), t)$ for each $t \in \mathbb{R}$, have the property that $\pi(\Psi_t) = \pi(\Phi_t)$, $t \in \mathbb{R}$, and map the rigid body $(\mathcal{B}, \mu)$ to the same set of points. Therefore, every such Φ is $\mathcal{B}$-equivalent to Ψ. Now, given $C_0 \in \ker(\tau)$, define a motion Φ by

$$\Phi_t = \iota^{-1}_{\mathscr{O}_{s_0}}\left(R_\Psi(t), a_{\Psi,\mathscr{O}}(t) - (\pi_{\mathscr{G}}(x_c) - s_0)C_0,\ C_0, t)\right).$$

Clearly, Φ_t is horizontal with respect to $\omega_{\mathscr{O}}$ and $\Phi_t \in (\pi^{\mathrm{can}}_{c,\mathscr{O}})^{-1}(R_\Psi(t), a_{\Psi,\mathscr{O}}(t), t)$, for each $t \in \mathbb{R}$. It can be directly verified that the curve Φ_t passes through the point $\iota^{-1}_{\mathscr{O}_{s_0}}(R_\Psi(t_0), a_{\Psi,\mathscr{O}}(t_0) - (\pi_{\mathscr{G}}(x_c) - s_0)C_0, C_0, t_0)$ at $t = t_0$, for each $t_0 \in \mathbb{R}$, and thus corresponds to the unique rigid motion Φ with the property that $u_{\Phi,\mathscr{O}}(t) = C_0$, for all $t \in \mathbb{R}$. From Proposition 6.6 we can see that, for $x_c \in \mathscr{O}$, the Galilean–Euler equations for the rigid motion Φ are equivalent to

$$\begin{aligned}\mathbb{I}_c(t)(\dot{\omega}_\Phi(t)) &= \mathbb{I}_c(t)(\omega_\Phi(t)) \times \omega_\Phi(t)\\ \ddot{r}_{\Phi,\mathscr{O}}(t) &= 0\\ \dot{L}_{\Phi,\mathcal{B}}(t) &= L_{\Phi,\mathcal{B}}(t) \times \Omega_\Phi(t)\\ \dot{M}_{\Phi,\mathcal{B}}(t) &= P_{\mathscr{O}}(M_{\Phi,\mathcal{B}}(t)) \times \Omega_\Phi(t).\end{aligned}$$

The result now follows. ■

Proposition 6.10 finally explains how the presence of an observer enables us to provide a geometric explanation of how one can start with the general setup of a rigid body in a Galilean spacetime and recover the classical Euler equations of motion for a rigid body. Thus, for a rigid motion, one can restrict oneself to only the horizontal rigid motions without losing any physical motions of the body. Note that many different motions are $\mathcal{B}$-equivalent, due to the fact that the Galilean group is "too big" to give a one-to-one correspondence between rigid motions and motions of the body. Horizontality with respect to the Galilean connection induced by the observer specifies a relation in N_0 that ensures that, with respect to the observer, the "velocity boost" component of Ψ_t is constant, and therefore does not appear in the Galilean–Euler equations.

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