

# Geometric Jacobian linearization<sup>1</sup>

David R. Tyner<sup>2</sup>

2007/12/21

<sup>1</sup>PhD thesis, Department of Mathematics and Statistics, Queen's University

<sup>2</sup>MECHANICAL AND AEROSPACE ENGINEERING, CARLETON UNIVERSITY, OTTAWA, ON,  
CANADA, K1S 5B6

EMAIL: [DRTYNER@GMAIL.COM](mailto:DRTYNER@GMAIL.COM)

WORK PERFORMED WHILE A GRADUATE STUDENT AT QUEEN'S UNIVERSITY.

# Abstract

For control systems that evolve on Euclidean spaces, Jacobian linearization is a common technique in many control applications, analysis, and controller design methodologies. However, the standard linearization method along a non-trivial reference trajectory does not directly apply in a geometric theory where the state space is a differentiable manifold. Indeed, the standard constructions involving the Jacobian are dependent on a choice of coordinates.

The procedure of linearizing a control affine system along a non-trivial reference trajectory is studied from a differential geometric perspective. A coordinate-invariant setting for linearization is presented. With the linearization in hand, the controllability of the geometric linearization is characterized using an alternative version of the usual controllability test for time-varying linear systems. The various types of stability are defined using a metric on the fibers along the reference trajectory and Lyapunov's second method is recast for linear vector fields on tangent bundles. With the necessary background stated in a geometric framework, Kalman's theory of quadratic optimal control is understood from the perspective of the Maximum Principle. Finally, following Kalman, the resulting feedback from solving the infinite time optimal control problem is shown to uniformly asymptotically stabilize the linearization using Lyapunov's second method.

## Contents

<b>1</b>	<b>Introduction and linear systems background</b>	<b>1</b>
1.1.	Motivation . . . . .	1
1.2.	Kalman's theory of optimal control . . . . .	2
1.2.1	Time-varying linear systems. . . . .	2
1.2.2	Gramians and minimum energy controllers. . . . .	3
1.2.3	Time-varying LQR. . . . .	4
1.3.	Questions from a geometric perspective . . . . .	9
1.4.	Contribution of thesis . . . . .	10
<b>2</b>	<b>Affine systems and their linearization</b>	<b>12</b>
2.1.	Background and geometric constructions . . . . .	12
2.1.1	Time-dependent objects on a manifold. . . . .	12
2.1.2	Tangent bundle geometry. . . . .	13
2.2.	Affine systems . . . . .	17
2.3.	Linearization about a reference trajectory . . . . .	18
2.4.	Linearization about an equilibrium point . . . . .	19
2.5.	Linear controllability . . . . .	20
2.5.1	Controllability definitions. . . . .	20
2.5.2	Recasting the standard results. . . . .	21
2.5.3	Controllability of linearizations. . . . .	24
<b>3</b>	<b>LQR and the Maximum Principle</b>	<b>27</b>
3.1.	LQR problem definitions . . . . .	27
3.2.	The variational and adjoint equations . . . . .	29
3.3.	Needle variations and tangent cones . . . . .	31
3.3.1	Needle variations. . . . .	32
3.3.2	Multi-needle variations. . . . .	33
3.3.3	Fixed interval tangent cones. . . . .	34
3.4.	The Hamiltonian . . . . .	34
3.5.	Proof of Theorem 3.2 . . . . .	36
3.6.	Characterizing the finite time LQR problem . . . . .	38
3.7.	Infinite time LQR problems . . . . .	41
<b>4</b>	<b>Stability and Stabilization</b>	<b>47</b>
4.1.	Stability definitions . . . . .	47
4.2.	Lyapunov theory for linear vector fields on tangent bundles . . . . .	51
4.3.	Stabilization of the linearization . . . . .	54
<b>5</b>	<b>Conclusions and future work</b>	<b>57</b>
5.1.	Conclusions . . . . .	57
5.2.	Future work . . . . .	57
5.2.1	Open questions about stability and stabilization. . . . .	57
5.2.2	The free interval Maximum principle. . . . .	58

# Chapter 1

## Introduction and linear systems background

Jacobian linearization is a standard concept in control theory and is used to study controllability, stability, and stabilization of non-linear systems. Indeed, Jacobian linearization provides the setting for a significant number of the control algorithms implemented in practice for non-linear systems.

In the presented work, a geometric theory of linearization is developed for control-affine systems evolving on a differentiable manifold. The objective is not so much to broaden the applicability of linearization techniques, but to better understand the structure of linearization and to make explicit some of the choices that are made without mention in the standard practice of linearization. The motivation, in part, comes from examples in mechanics. Given an affine connection, what it means to linearize about a reference trajectory has a natural geometric understanding provided by the Jacobi equation of geodesic variations. In the general setup of control-affine systems, a geometric setup, thus far, is not found in the literature. However, certain ideas presented here are implicit in the paper of [Sussmann \[1997\]](#), although the present geometric framework is less abstract and so has more structure.

To emphasize the relevant geometry, it is advantageous to remove unneeded structure and data. With this in mind, the abstract setting of “affine systems” of [Hirschorn and Lewis \[2002\]](#) is the starting point for the future developments.

### 1.1. Motivation

In order to provide a point of reference for the geometric formulation of control systems and their linearizations, this section outlines the standard manner in which linearization is normally carried out for control-affine systems on  $\mathbb{R}^n$ . This standard strategy is, of course, correct but it “sweeps under the rug” various issues that must be addressed to develop a geometric theory.

Let  $\Omega \subset \mathbb{R}^n$  be an open subset and let  $f_0, f_1, \dots, f_m$  be smooth vector fields, possibly depending measurably on  $t$ , on  $\Omega$ . Consider a control system with governing equations

$$\gamma'(t) = f_0(t, \gamma(t)) + \sum_{a=1}^m u^a(t) f_a(t, \gamma(t)), \quad (1.1.1)$$

where  $\gamma: I \rightarrow \Omega$  is locally absolutely continuous and  $u: I \rightarrow \mathbb{R}^m$  is bounded and measurable for some interval  $I \subset \mathbb{R}$ . For the purposes of linearization, fix a reference trajectory  $\gamma_{\text{ref}}$  corresponding to a reference control  $u_{\text{ref}}$ , both defined on  $I \subset \mathbb{R}$ . To define the linearization, for each  $t \in I$  define  $m + 1$  smooth vector fields  $f_{a,t}$ ,  $a \in \{0, 1, \dots, m\}$ , by  $f_{a,t}(x) = f_a(t, x)$ . The linearization of this system is then defined by

$$\xi'(t) = A(t)\xi(t) + B(t)v(t), \quad (1.1.2)$$

where

$$A(t) = \mathbf{D}f_{0,t}(\gamma_{\text{ref}}(t)) + \sum_{a=1}^m u_{\text{ref}}^a(t) \mathbf{D}f_{a,t}(\gamma_{\text{ref}}(t)),$$

$$B(t) = [ f_{1,t}(\gamma_{\text{ref}}(t)) \mid \cdots \mid f_{m,t}(\gamma_{\text{ref}}(t)) ].$$

Here  $\mathbf{D}f_{a,t}$  denotes the Jacobian of the vector field  $f_{a,t}$ ,  $a \in \{0, 1, \dots, m\}$ . Once the linearization (1.1.2) is obtained its controllability properties can be investigated using the standard controllability Gramian (see Section 2.5.2 for restatements of the standard Gramian results). If the linearization is ascertained to be controllable on  $I = [0, \infty)$ , then (1.1.1) can be locally stabilized along the reference trajectory by stabilizing the linearization using linear feedback [Ikeda, Maeda and Kodama 1972, Kalman 1960]. That is, if  $L(V; W)$  denotes the set of linear maps from a vector space  $V$  to a vector space  $W$ , a map  $F: I \rightarrow L(\mathbb{R}^n; \mathbb{R}^m)$  is chosen with the property that the closed-loop system

$$\xi'(t) = (A(t) + B(t)F(t))\xi(t),$$

is uniformly asymptotically stable. If  $F^a(t) \in (\mathbb{R}^n)^*$  is the  $a$ th row of  $F(t)$ , then the non-linear closed-loop system

$$\gamma'(t) = f_0(t, \gamma(t)) + \sum_{a=1}^m (u_{\text{ref}}^a(t) + F^a(t)(\gamma(t) - \gamma_{\text{ref}}(t))) f_a(t, \gamma(t)), \quad (1.1.3)$$

is locally uniformly asymptotically stable along the trajectory  $\gamma_{\text{ref}}$ . In practice one might design  $F$  through optimal control methods using a quadratic cost, the so-called linear quadratic regulator (**LQR**).

## 1.2. Kalman's theory of optimal control

The mathematical theory behind LQR for time-varying linear systems on  $\mathbb{R}^n$  is developed in [Kalman 1960]. To provide a reference for the geometric version, an outline of Kalman's work is provided in this section. Although the results are well known, the details of the standard time-varying case as the final time tends to infinity are not pedestrian knowledge.

**1.2.1. Time-varying linear systems.** Let  $A: \mathbb{R} \rightarrow L(\mathbb{R}^n; \mathbb{R}^n)$  and  $B: \mathbb{R} \rightarrow L(\mathbb{R}^m; \mathbb{R}^n)$  be continuous maps and define a *time-varying linear system on  $\mathbb{R}^n$*  to be a pair  $(A, B)$  satisfying

$$x'(t) = A(t)x(t) + B(t)u(t). \quad (1.2.1)$$

The solution to (1.2.1) satisfying  $x(t_0) = x_0$  for  $t_0 \in I$  is given by the *variations of constants formula*

$$x(t) = \Phi(t, t_0)x_0 + \int_{t_0}^t \Phi(t, \sigma)B(\sigma)u(\sigma)d\sigma, \quad (1.2.2)$$

where  $\Phi(t, t_0)$  is the state transition matrix. That is,  $t \mapsto \Phi(t, t_0)$  is the solution to the homogeneous system  $\Phi'(t, t_0) = A(t)\Phi(t, t_0)$  with initial condition  $\Phi(t_0, t_0) = \text{id}_{\mathbb{R}^n}$ . The transition matrix has the following useful properties:

1.  $\Phi(t, \tau)\Phi(\tau, t_0) = \Phi(t, t_0)$ ;
2.  $\Phi(t, \tau)^{-1} = \Phi(\tau, t)$ .

A time-varying linear system  $(A, B)$  is **controllable at  $t_0$**  if, for each  $x_0, x_1 \in \mathbb{R}^n$ , there exists a control  $u: [t_0, t_1] \rightarrow \mathbb{R}^m$  which steers from  $x_0$  at time  $t_0$  to  $x_1$  at time  $t_1$ . If  $(A, B)$  is controllable at every time, then it is said to be **controllable**. The controllability of a time-varying linear system is determined by the **controllability Gramian**

$$W(t_0, t) = \int_{t_0}^t \Phi(t_0, \sigma)B(\sigma)B^*(\sigma)\Phi^*(t_0, \sigma)d\sigma, \quad (1.2.3)$$

where  $*$  denotes the transpose. A time-varying linear system  $(A, B)$  is controllable at  $t$ ,  $t \in [t_0, t_1]$ , if and only if  $W(t_0, t_1)$  is surjective for  $t_1 > t_0$ . Differentiating the controllability Gramian with respect to the first and second parameters provide the following differential equations:

$$\frac{d}{dt}W(t, t_1) = A(t)W(t, t_1) + W(t, t_1)A^*(t) - B(t)B^*(t) \quad W(t_1, t_1) = 0, \quad (1.2.4)$$

and

$$\frac{d}{dt}W(t_0, t) = \Phi(t_0, t)B(t)B^*(t)\Phi^*(t_0, t), \quad (1.2.5)$$

respectively [Brockett 1970]. Part of the difficulty of developing a geometric analogue of the standard setup will be understanding what replaces objects, like the controllability Gramian, that are defined by integral equations. The approach in the sequel will be to replace integral equations with the appropriate differential equations and in that sense equations (1.2.4) and (1.2.5) become more useful than (1.2.3).

**1.2.2. Gramians and minimum energy controllers.** The controllability Gramian for a time-varying linear system  $(A, B)$  provides a control,  $u_1: I \rightarrow \mathbb{R}^m$ , that drives an initial state  $x(t_0) = x_0$  at time  $t_0$  to  $x(t_1) = 0$  at time  $t_1$ . Furthermore, both the obtained controller and trajectory are linear in the initial state  $x_0$  and play a central role in proving that solutions of the infinite time LQR problem are well-defined.

If  $(A, B)$  is controllable then there exists  $\eta \in \mathbb{R}^n$  such that  $W(t_0, t_1)\eta = x_0$ . Using the variation of constants formula and properties of the transition matrix the following relations are obtained:

$$\begin{aligned} -\Phi(t_1, t_0)W(t_0, t_1)\eta &= \int_{t_0}^{t_1} \Phi(t_1, \sigma)B(\sigma)u_1(\sigma)d\sigma, \\ W(t_0, t_1)\eta &= -\int_{t_0}^{t_1} \Phi(t_0, \sigma)B(\sigma)u_1(\sigma)d\sigma. \end{aligned}$$

Then by matching the above with (1.2.3), the formula for the controllability Gramian, the desired controller is

$$u_1(\sigma) = -B^*(\sigma)\Phi^*(t_0, \sigma)\eta, \quad (1.2.6)$$

where  $\eta$  satisfies  $W(t_0, t_1)\eta = x_0$ . The trajectory corresponding to (1.2.6) is

$$\begin{aligned} x_{u_1}(\tau) &= \Phi(\tau, t_0)x_0 + \int_{t_0}^{\tau} \Phi(\tau, \sigma)B(\sigma)u(\sigma)d\sigma \\ &= \Phi(\tau, t_0) \left( x_0 - \int_{t_0}^{\tau} \Phi(t_0, \sigma)B(\sigma)B^*(\sigma)\Phi^*(t_0, \sigma)\eta d\sigma \right) \\ &= \Phi(\tau, t_0) (x_0 - W(t_0, \tau)W^{-1}(t_0, t_1)x_0). \end{aligned} \quad (1.2.7)$$

The controller given by (1.2.6) is in fact the minimum energy controller to transfer an initial state  $x_0$  to the origin [Brockett 1970]. More precisely the control given by (1.2.6) minimizes the cost

$$\int_{t_0}^{t_1} \|u(t)\|_{\mathbb{R}^m} dt. \quad (1.2.8)$$

Note that (1.2.8) makes use of the standard inner product on  $\mathbb{R}^m$ . To recast (1.2.6) for linear systems on a  $\mathbb{R}$ -vector space  $V$ , as defined in Section 2.5.2, requires a choice of an inner product.

### 1.2.3. Time-varying LQR.

**1.1 Problem:** (Finite time LQR problem) For a time-varying linear system  $(A, B)$ , find a pair  $(x(t), u(t))$  defined on  $I = [t_0, t_1]$  which minimizes the quadratic cost function

$$J(x(t_0), t_0, t_1) = \frac{1}{2}F(t_1)(x(t_1), x(t_1)) + \frac{1}{2} \int_{t_0}^{t_1} Q(t)(x(t), x(t)) + R(t)(u(t), u(t)) dt,$$

where  $F(t_1) \in L(\mathbb{R}^n; \mathbb{R}^n)$  is symmetric and positive-semidefinite,  $Q: I \rightarrow L(\mathbb{R}^n; \mathbb{R}^n)$  is symmetric and positive-semidefinite for each  $t \in I$ , and  $R: I \rightarrow L(\mathbb{R}^m; \mathbb{R}^m)$  is symmetric and positive-definite for each  $t \in I$ .

Solutions to Problem 1.1 for a finite time  $t_1$  can be obtained by variational methods or by applying the Maximum Principle [Athans and Falb 1966, Lee and Markus 1986]. The original presentation of the Maximum Principle is provided by Pontryagin, Boltyanskii, Gamkrelidze and Mishchenko [1961]. Using either method, the existence of a solution to Problem 1.1 is equivalent to the existence of a solution  $K(t, t_1)$  to the *differential Riccati equation*

$$-\frac{dK}{dt} = A^*K + KA - KBR^{-1}B^*K + Q, \quad K(t_1, t_1) = F(t_1),$$

where the time dependence has been dropped for clarity. A solution to the Riccati equation then provides an optimal linear state feedback.

**1.2 Theorem:** ([Kalman 1960]) For all finite  $t_1$  and all symmetric positive-semidefinite  $F(t_1)$ :

- (i) the differential Riccati equation has a unique solution  $\Pi(t; F(t_1), t_1)$  for all  $t \leq t_1$ ;
- (ii) the optimal cost is given by  $J_{\text{op}}(x(t_0), t_0, t_1) = x^*(t_0)\Pi(t_0; F(t_1), t_1)x(t_0)$ ;
- (iii) the optimal cost is attained if and only if the control law is given by

$$u_{\text{op}}(t) = -R^{-1}(t)B^*(t)\Pi(t; F(t_1), t_1)x(t).$$

To address the matter of asymptotic stability will require that solutions to the Riccati equation exist as  $t_1 \rightarrow \infty$ . In the study of the infinite-time problem, the terminal cost  $F(t_1)$  is considered to be zero.

**1.3 Proposition:** ([Kalman 1960]) If a time-varying linear system  $(A, B)$  is controllable and  $F(t_1) = 0$ , then

$$\lim_{t_1 \rightarrow \infty} \Pi(t; 0, t_1) = \bar{K}(t) \tag{1.2.9}$$

exists for all  $t$  and  $\bar{K}(t)$  satisfies the Riccati equation.

**Proof:** Existence of  $\bar{K}(t)$ : Since  $(A, B)$  is controllable there exists a control  $u_1(t)$  which drives  $x(t_0)$  to 0 at or before time  $t_2(x(t_0), t_0) < t_1$ . Recall that such a controller is provided by (1.2.6):

$$u_1(t) = \begin{cases} -B^*(t)\Phi^*(t_0, t)W^{-1}(t_0, t_2)x(t_0), & t \leq t_2(x(t_0), t_0), \\ 0, & t > t_2(x(t_0), t_0). \end{cases}$$

Using  $u_1(t)$ , Kalman shows that the operator norm of  $\Pi(t_0; 0, t_1)$  is bounded for all  $t_1 \geq t_0$ . Let  $J_1(x(t_0), t_0, t_1)$  be the cost corresponding to the control  $u_1(t)$ . Then

$$\begin{aligned} x^*(t_0)\Pi(t_0; 0, t_1)x(t_0) &= J_{\text{op}}(x(t_0), t_0, t_1) \leq J_1(x(t_0), t_0, t_1) \\ &= \frac{1}{2} \left( \int_{t_0}^{t_2} x_{u_1}^*(\tau)Q(\tau)x_{u_1}(\tau) + u_1^*(\tau)R(\tau)u_1(\tau) \, d\tau + \right. \\ &\quad \left. \int_{t_2}^{t_1} x_{u_1}^*(\tau)Q(\tau)x_{u_1}(\tau) + u_1^*(\tau)R(\tau)u_1(\tau) \, d\tau \right) \\ &= \frac{1}{2} \int_{t_0}^{t_2} x_{u_1}^*(\tau)Q(\tau)x_{u_1}(\tau) + u_1^*(\tau)R(\tau)u_1(\tau) \, d\tau \\ &\leq \alpha(t_0)\|x(t_0)\|_{\mathbb{R}^n}^2, \end{aligned} \tag{1.2.10}$$

where  $\alpha$  is non-negative function of  $t_0$ . The first line follows by optimality and the third line uses the fact that  $u_1(t) = 0$  and  $x_{u_1}(t) = 0$  for  $t > t_2(x(t_0), t_0)$ . The last step follows by using the variation of constants formula (1.2.2) and the fact that  $u_1(t)$  and  $x_{u_1}(t)$ , given by (1.2.7), are linear in the initial condition  $x(t_0)$ .

**1 Lemma:** ([Dunford and Schwartz 1963]) Let  $J$  be an index set. Let  $X$  and  $Y$  be Banach spaces and let  $\{T_n\}_{n \in J}$  be a sequence of bounded linear maps from  $X$  to  $Y$ . Then the following are equivalent:

- (i)  $\sup_{n \in A} \|T_n\| < \infty$ ;
- (ii)  $\sup_{n \in A} \|T_n(x)\|_Y < \infty$ ,  $x \in X$ ;



(iii)  $\sup_{n \in A} \|y^* T_n(x)\|_{\mathbb{R}} < \infty$ ,  $x \in X$ ,  $y^* \in Y^*$ .

By Lemma 1,  $\|\Pi(t_0; 0, t_1)\|$  is bounded for all  $t_1 \geq t_0$ . That is,

$$\begin{aligned} \sup_{t_1 \in \mathbb{R}_+} |x^*(t_0)\Pi(t_0; 0, t_1)x(t_0)| &\leq \sup_{t_1 \in \mathbb{R}_+} \alpha(t_0) \|x(t_0)\|_{\mathbb{R}^n}^2 < \infty \\ \iff \sup_{t_1 \in \mathbb{R}_+} \|\Pi(t_0; 0, t_1)\| &< \infty. \end{aligned}$$

Since  $x^*(t_0)\Pi(t_0; 0, t_1)x(t_0) = J_{\text{op}}(x(t_0), t_0, t_1)$  and  $J_{\text{op}}(x(t_0), t_0, t_1) \leq J_{\text{op}}(x(t_0), t_0, t_2)$  for  $t_1 \leq t_2$ , the function  $t_1 \mapsto x^*(t_0)\Pi(t_0; 0, t_1)x(t_0)$  is non-decreasing as  $t_1 \rightarrow \infty$ . Thus the existence is proved by constructing a limit point  $\bar{K}(t_0)$  as follows. Define a quadratic form by

$$\langle K(t_0)v, v \rangle = \lim_{t_1 \rightarrow \infty} \langle \Pi(t_0; 0, t_1)v, v \rangle, \quad v \in \mathbb{R}^n. \quad (1.2.11)$$

The limit in (1.2.11) exists because it is a bounded non-decreasing function of  $t_1$ . The polarization identity,

$$4\langle \bar{K}(t_0)v, w \rangle = \langle K(t_0)(v+w), v+w \rangle - \langle K(t_0)(v-w), v-w \rangle,$$

then defines a symmetric matrix  $\bar{K}(t_0)$  for all  $v, w \in \mathbb{R}^n$ .  $\bar{K}(t)$  satisfies the Riccati equation:

$$\begin{aligned} \bar{K}(t) &= \lim_{t_2 \rightarrow \infty} \Pi(t; 0, t_2) = \lim_{t_2 \rightarrow \infty} \Pi(t; \Pi(t_1; 0; t_2), t_1) \\ &= \Pi(t; \lim_{t_2 \rightarrow \infty} \Pi(t_1; 0; t_2), t_1) = \Pi(t; \bar{K}(t_1), t_1), \end{aligned}$$

where the second step follows by the continuity of solutions to differential equations and Theorem 1.2. ■

The linear feedback defined by  $u(t) = -R^{-1}(t)B^*(t)\bar{K}(t)x(t)$  is optimal in the sense of Problem 1.4 [Kalman 1960].

**1.4 Problem:** (Infinte time LQR problem) For a time-varying linear system  $(A, B)$ , find a pair  $(x(t), u(t))$  defined on  $I = [t_0, \infty)$  which minimizes the quadratic cost function

$$J(x(t_0), t_0, \infty) = \frac{1}{2} \int_{t_0}^{\infty} Q(t)(x(t), x(t)) + R(t)(u(t), u(t)) dt,$$

where  $Q: I \rightarrow L(\mathbb{R}^n; \mathbb{R}^n)$  is symmetric and positive-semidefinite for each  $t \in I$ , and  $R: I \rightarrow L(\mathbb{R}^m; \mathbb{R}^m)$  is symmetric and positive-definite for each  $t \in I$ .

The asymptotic stability of the closed-loop system  $(A(t) - B(t)R^{-1}(t)B^*(t)\bar{K}(t), 0)$  is proved by showing that  $\bar{K}(t)$  is a suitable Lyapunov function. This requires uniformity conditions on the LQR and linear system data.

**1.5 Definition:** A time-varying linear system  $(A, B)$  is *uniformly controllable* if

- (i)  $(A, B)$  is controllable and
- (ii) there exists a constant  $\sigma$  such that for all  $t \in I$  the following hold:
  - (a)  $0 < \alpha_0(\sigma) \text{id}_{\mathbb{R}^n} \leq W(t, t + \sigma) \leq \alpha_1(\sigma) \text{id}_{\mathbb{R}^n}$ ;
  - (b)  $0 < \alpha_2(\sigma) \text{id}_{\mathbb{R}^n} \leq \Phi(t + \sigma, t)W(t, t + \sigma)\Phi^*(t + \sigma, t) \leq \alpha_3(\sigma) \text{id}_{\mathbb{R}^n}$ .

The idea of uniform controllability, to this author's knowledge, was first stated in [Kalman 1960]. A geometric interpretation in terms of the transition matrix is also provided by [Kalman 1960].

**1.6 Lemma:** *For a uniformly controllable time-varying linear system  $(A, B)$  and a fixed constant  $\sigma$ , the following are equivalent:*

(i)

$$\begin{aligned} 0 < \alpha_0(\sigma) \text{id}_{\mathbb{R}^n} &\leq W(t, t + \sigma) \leq \alpha_1(\sigma) \text{id}_{\mathbb{R}^n} \\ 0 < \alpha_2(\sigma) \text{id}_{\mathbb{R}^n} &\leq \Phi(t + \sigma, t)W(t, t + \sigma)\Phi^*(t + \sigma, t) \leq \alpha_3(\sigma) \text{id}_{\mathbb{R}^n} \end{aligned}$$

(ii)  $\|\Phi(t, \tau)\| \leq \alpha_4(|t - \tau|)$ ,  $\tau \leq t - \sigma$ ,

where  $\alpha_i$ ,  $i \in \{0, \dots, 4\}$ , are strictly increasing functions with  $\alpha_i(0) = 0$ .

**Proof:** Since  $(A, B)$  is controllable the controllability Gramian is invertible. Using this, together with the properties of the transition matrix, we obtain

$$0 < \alpha_2(\sigma)\Phi^*(t, t + \sigma)W^{-1}(t, t + \sigma) \leq \Phi(t + \sigma, t) \leq \alpha_3(\sigma)\Phi^*(t, t + \sigma)W^{-1}(t, t + \sigma),$$

for all  $t \in I$ . Then, multiplying on the right by  $\Phi^*(t + \sigma, t)$  and using part (a) of Definition 1.5, we obtain

$$0 < \sqrt{\frac{\alpha_2(\sigma)}{\alpha_1(\sigma)}} \leq \|\Phi(t + \sigma, t)\| \leq \sqrt{\frac{\alpha_3(\sigma)}{\alpha_0(\sigma)}},$$

or equivalently

$$0 < \sqrt{\frac{\alpha_0(\sigma)}{\alpha_3(\sigma)}} \leq \|\Phi(t, t + \sigma)\| \leq \sqrt{\frac{\alpha_1(\sigma)}{\alpha_2(\sigma)}}.$$

Now, using (1.2.3) and the previous calculations, it can be verified that (i) holds for any  $\sigma' \geq \sigma$ . Let  $t \geq \tau + \sigma$  to obtain the desired result:

$$\|\Phi(t, \tau)\| = \|\Phi(t, \tau + \sigma)\Phi(\tau + \sigma, \tau)\| \leq \|\Phi(t, \tau + \sigma)\| \sqrt{\frac{\alpha_3(\sigma)}{\alpha_0(\sigma)}} = \sqrt{\frac{\alpha_3(|t - \tau|)}{\alpha_0(|t - \tau|)}}. \quad \blacksquare$$

**1.7 Theorem:** ([Kalman 1960]) *If a time-varying linear system  $(A, B)$  is uniformly controllable and uniformly observable with*

$$\begin{aligned} 0 < \alpha_1 \text{id}_{\mathbb{R}^n} &\leq Q(t) \leq \alpha_2 \text{id}_{\mathbb{R}^n} \\ 0 < \alpha_2 \text{id}_{\mathbb{R}^m} &\leq R(t) \leq \alpha_4 \text{id}_{\mathbb{R}^m}, \end{aligned}$$

then the closed-loop system  $(A(t) - B(t)R^{-1}(t)B^*(t)\bar{K}(t), 0)$  is uniformly asymptotically stable.

**Proof:** The proof of Theorem 1.7 follows by showing that  $\bar{K}(t)$  is a “suitable” Lyapunov function to ensure uniform asymptotic stability. More precisely, referring to standard texts such as [Khalil 1996, Vidyasagar 2002], “suitable” implies that, there exists positive constants  $c_1$  and  $c_2$  such that:

1.  $\bar{K}(t)$  is bounded above independently of  $t$ :  $x^*(t)\bar{K}(t)x(t) \leq c_1\|x(t)\|_{\mathbb{R}^n}^2$ ;
2.  $\bar{K}(t)$  is bounded below independently of  $t$ :  $c_2\|x(t)\|_{\mathbb{R}^n}^2 \leq x^*(t)\bar{K}(t)x(t)$ ;

3. the derivative of  $\bar{K}(t)$  is negative along the trajectories of the closed-loop system  $(A(t) - B(t)R^{-1}(t)B^*(t)\bar{K}(t), 0)$ .

The upper bound is obtained using (1.2.6) in a similar fashion as in the proof of Proposition 1.3. Let

$$u_1(t) = \begin{cases} -B^*(t)\Phi^*(t_0, t)W^{-1}(t_0, t_2)x(t_0), & t \leq t_2(x(t_0), t_0), \\ 0, & t > t_2(x(t_0), t_0), \end{cases}$$

with  $t_2 = t_0 + \sigma$ , where  $\sigma$  is the constant prescribed by uniform controllability. Now, following the procedure in the proof of Proposition 1.3 with the added hypotheses on the cost data, we obtain

$$\begin{aligned} x^*(t_0)\bar{K}(t_0)x(t_0) &= J_{\text{op}}(x(t_0), t_0, t_1) \leq J_1(x(t_0), t_0, t_1) \\ &= \frac{1}{2} \int_{t_0}^{t_2} x_{u_1}^*(\tau)Q(\tau)x_{u_1}(\tau) + u_1^*(\tau)R(\tau)u_1(\tau) \, d\tau \\ &\leq \frac{1}{2} \int_{t_0}^{t_2} \alpha_2 \|x_{u_1}(\tau)\|^2 + \alpha_4 \|u_1(\tau)\|^2 \, d\tau. \end{aligned} \quad (1.2.12)$$

To obtain a bound on the trajectory consider the following calculation:

$$\begin{aligned} \|x_{u_1}(\tau)\|^2 &= \|\Phi(\tau, t_0)(I - W(t_0, \tau)W^{-1}(t_0, t_2))x_0\|^2 \\ &= \|\Phi(\tau, t_2)\Phi(t_2, t_0)((W(t, t_2) - W(t_0, \tau))W^{-1}(t_0, t_2)x_0)\|^2 \\ &\leq \alpha_5(|t_2 - t_0|)\|\Phi(\tau, t_2)\|^2\|(W(t, t_2) - W(t_0, \tau))W^{-1}(t_0, t_2)x_0\|^2 \\ &\leq \alpha_5(\sigma)\|\Phi(\tau, t_2)\|^2\|x_0\|^2 \leq \alpha_5 \text{tr}(\Phi^*(\tau, t_2)\Phi(\tau, t_2))\|x_0\|^2, \end{aligned}$$

where

1. in line three, Lemma 1.6 is used,
2. in line four,  $\|(W(t, t_2) - W(t_0, \tau))W^{-1}(t_0, t_2)\|^2 \leq 1$  follows from the simultaneous diagonalization of the symmetric positive-definite matrix  $W^{-1}(t_0, t_2)$  and the non-negative definite symmetric matrix  $(W(t, t_2) - W(t_0, \tau))$  [Bellman 1960], and
3. in line four, the property  $\|Mx\|^2 \leq \|M\|^2\|x\|^2 \leq \text{tr}(M^*M)\|x\|^2$  for a matrix  $M$  is used.

Similar calculations for the control yield

$$\begin{aligned} \|u_1(\tau)\|^2 &= \|-B^*(\tau)\Phi^*(t_0, \tau)W^{-1}(t_0, t_2)x(t_0)\|^2 \\ &\leq \|-B^*(\tau)\Phi^*(t_0, \tau)\|^2\|W^{-1}(t_0, t_2)x(t_0)\|^2 \\ &\leq \text{tr}(\Phi(t_0, \tau)B(\tau)B^*(\tau)\Phi^*(t_0, \tau))\|W^{-1}(t_0, t_2)x(t_0)\|^2. \end{aligned}$$

The above computations and (1.2.12) imply

$$\begin{aligned} x^*(t_0)\bar{K}(t_0)x(t_0) &\leq \frac{1}{2}\|x_0\|^2 \int_{t_0}^{t_2} \alpha_2 \alpha_5 \text{tr}(\Phi^*(\tau, t_2)\Phi(\tau, t_2)) + \\ &\quad \alpha_4 \text{tr}(\Phi(t_0, \tau)B(\tau)B^*(\tau)\Phi^*(t_0, \tau))\|W^{-1}(t_0, t_2)\| \, d\tau \\ &\leq \frac{1}{2}\|x_0\|^2(\alpha_2 \alpha_5 \text{tr}(W^*(t_2, t_0)) + \alpha_4 \alpha_6 \text{tr}(W(t_2, t_0))) \leq c_1 \|x_0\|^2, \end{aligned}$$

where  $W^*(t_2, t_0)$  denotes the controllability Gramian of the dual system  $(-A^*, \text{id}_{\mathbb{R}^n})$  and the last step follows by uniform controllability of  $(A, B)$  and  $(-A^*, \text{id}_{\mathbb{R}^n})$  since  $t_2 = t_0 + \sigma$ .

The lower bound is obtained in [Ikeda, Maeda and Kodama 1972] by formulating the “dual” differential Riccati equation. It is verifiable that, if  $P(t)$  is the inverse of  $\bar{K}(t)$ , then  $P(t)$  satisfies

$$\dot{P} - PA^* - AP - PQP + BR^{-1}B^* = 0. \quad (1.2.13)$$

The “dual” differential Riccati equation corresponds to the optimal cost

$$J_{\text{op}}(\eta(t_0), t_0, \infty) = \frac{1}{2} \int_{t_0}^{\infty} R^{-1}(t)(B^*(t)\eta(t), B^*(t)\eta(t)) + Q^{-1}(t)(v(t), v(t)) dt$$

for the fully actuated system  $(-A^*, \text{id}_{\mathbb{R}^n})$  with dual state variable  $\eta$ . The minimizing controller is  $v(t) = -Q(t)P(t)\eta(t)$ . By assumption  $(-A^*, \text{id}_{\mathbb{R}^n})$  is uniformly controllable so an upper bound for  $P(t)$  can be constructed independently of  $t$  as was done for  $\bar{K}(t)$ . ■

### 1.3. Questions from a geometric perspective

The above procedures by which (1.1.2), the linear system, is obtained and stabilized poses some problems when the Euclidean state space of the non-linear system (1.1.1) is replaced with a differentiable manifold. At first glance the most obvious issues include the following.

1. The two families of linear maps  $\{A(t)\}$  and  $\{B(t)\}$  depend on the choice coordinate chart since the Jacobian of the vector fields  $f_{a,t}$ ,  $a \in \{0, 1, \dots, m\}$  does not have a geometric analogue on a manifold.
2. Since the control system given by (1.1.1) has a state space that is an open subset of  $\mathbb{R}^n$ , there are several natural identifications that can be (unknowingly) made. The fact that the state space is naturally identified with each tangent space implies that (1.1.2) lives in a vector space. Where does the geometric version of (1.1.2) live?
3. By virtue of (1.1.2) living in a vector space, its controllability can be checked using the controllability Gramian which makes use of the standard inner product and the coordinate-dependent family of maps  $\{B(t)\}$ . What does it mean for the geometric version of (1.1.2) to be controllable, and how can it be checked whether such a system has this property?
4. What is a linear state feedback for the geometric version of (1.1.2)?
5. What does the geometric version of the linear quadratic regulator problem look like? What is the analogue of the Riccati equation?
6. Again, since the state space is naturally identified with each tangent space, the stability of both the non-linear system and its linearization are measured with respect to the standard Euclidean norm. What are the appropriate norms in a geometric setting?
7. After stabilizing the linearisation, how can the stabilizing linear state feedback be implemented for the non-linear system?

The objective of this thesis is to present a geometric framework for Jacobian linearization that addresses the above questions.

### 1.4. Contribution of thesis

This thesis is a systematic investigation of Jacobian linearization and Kalman’s theory of optimal control from a differential geometric perspective. In the course of this geometric investigation, the questions of Section 1.3 are answered. The layout and contribution of this thesis is as follows.

**Chapter 2:** Section 2.1 provides notation and a review of the necessary geometric constructions required to formulate a geometric theory of Jacobian linearization. As will be seen, the various ways to lift vector fields, tangent, cotangent, and vertical are fundamental in this thesis.

In Section 2.2, time-dependent affine systems on  $M$  are introduced. An affine system provides a quite general formulation of a control-affine system and will serve as the base object on which a geometric theory for Jacobian linearization is built. As part of the formulation, the control is effectively eliminated from the setup and thus one talks only of trajectories.

In Section 2.3, what it means to linearize an affine system along a reference trajectory is understood in terms of the tangent lift. In that sense, answering question 1, the tangent lift plays the role of the Jacobian in Section 1.1. Answering question 2, the geometry dictates that the linearization is an affine system on  $TM$ . Thus, in contrast to the standard case, an affine system and its linearization live on different manifolds. In Section 2.4, the complexity of the above development reduces significantly when the reference trajectory is chosen to be an equilibrium point.

In Section 2.5, question 3 is addressed when the controllability of the linearization is considered. To begin, the standard controllability results for (1.1.2) are re-characterized in Section 2.5.2. These re-characterizations have the feature that they may be applied directly to the geometric setting of the linearization and this is done in Section 2.5.3.

**Chapter 3:** In this chapter the main geometric structure for Kalman’s theory of optimal control is presented. In Section 3.1 the geometric versions of both the finite and infinite time LQR problems are formulated for the linearization of an affine system. The bulk of this chapter involves characterizing solutions to the finite time LQR problem using the Maximum Principle. Given the geometric setup of the linearization along a reference trajectory as an affine system on  $TM$ , the regular Maximum Principle statements do not directly apply without reverting to working in a set of coordinates. Thus, a new Maximum Principle statement is provided by Theorem 3.2 and proved in Section 3.5.

In Sections 3.2, 3.3, and 3.4, the key ingredients to prove the Maximum Principle are introduced. These include the variational and adjoint equations, needle variations, tangent cones, and, of course, the Hamiltonian. For readers familiar with the Hamiltonian in the standard setup, the Hamiltonian presented in Section 3.4 will look “different.” However, it maintains the required maximization properties—see Lemmata 3.17 and 3.18—required to prove the Maximum Principle.

In Section 3.6, the Maximum Principle is used, answering question 5, to characterize solutions to the finite time LQR problem. In this characterization the geometric version of the Riccati equation is given.

In Section 3.7, the infinite time LQR problem is addressed. In particular, solutions to the Riccati equation are shown to exist as the final time in the LQR problem tends to infinity. To prove their existence, the geometric analogue of the minimum energy control from Section 1.2.2 is developed. Finally, the trajectory corresponding to the solution of the Riccati equation, as the final time tends to infinity, is shown to be optimal in the sense of the infinite time LQR problem.

**Chapter 4:** In this chapter, stability and stabilization by LQR methods of the linearization are formulated to complete the geometric picture of Kalman’s theory of optimal control.

In Section 4.1, the stability definitions are provided for a fixed reference vector field  $X_{\text{ref}}$  and for linear vector fields over  $X_{\text{ref}}$ . For example, the linearisation  $X_{\text{ref}}^T$  is a linear vector field over  $X_{\text{ref}}$ . These definitions are made using both a metric on  $M$  and a metric on the fibres of  $TM$  over  $\text{image}(\gamma_{\text{ref}})$ . Such metrics are naturally induced by choosing a Riemannian metric,  $\mathbf{G}$ , on  $M$ . This answers question 6 and contrasts with the standard setup of Section 1.1 where standard Euclidean metric on  $\mathbb{R}^n$  is used for both the non-linear system and its linearization. It is noted that a metric  $\mathbf{G}$  on the fibres over  $\text{image}(\gamma_{\text{ref}})$ , unlike the Euclidean norm, will in general be time-dependent. As a consequence, any stability definitions made in terms of  $\mathbf{G}$ , will be dependent on the choice of metric unless the state manifold is compact.

In Section 4.2, Lyapunov’s direct method for linear vector fields on tangent bundles is introduced. As in the standard setup, the stability of the linear vector field is inferred from the properties of a Lyapunov candidate and its derivative along integral curves of the linear vector field. The derivative of the Lyapunov candidate along an integral curve is defined using Sussmann’s Lie derivative operator of Section 2.1.

In Section 4.3, question 4 is addressed. After making geometric sense of the terms “linear state-feedback” and “closed-loop system,” it is proved that the linearization of an affine system is uniformly asymptotically stabilized using the linear state-feedback provided by the infinite time LQR problem. The proof follows by showing that the solution to the Riccati equation, as the final time tends to infinity, is a suitable Lyapunov function. Finally, in Chapter 5, a rough answer to question 7 is posed as future work.

## Chapter 2

# Affine systems and their linearization

In this chapter, Section 2.1 provides basic notation and the necessary geometric background material. The geometric ideas that are used frequently in the current and future chapters include the various ways to lift vector fields and Sussmann's Lie derivative operator. In Section 2.2, time-dependent affine systems on  $M$  are introduced. In Section 2.3 the linearization of an affine system on  $M$  along a non-trivial reference trajectory is obtained using the tangent lift. The resulting linearization has the structure of an affine system on  $TM$ . In Section 2.4, it is seen that the complexity of the above development reduces significantly at an equilibrium point. In Section 2.5, the controllability of the linearization is considered and a re-characterization of the standard controllability notions is provided in Section 2.5.3.

### 2.1. Background and geometric constructions

Let  $M$  be an  $n$ -dimensional Hausdorff manifold with a  $C^\infty$  differentiable structure. The letter  $I$  will always denote an interval in  $\mathbb{R}$ . The set of class  $C^r$  functions on  $M$  is denoted by  $C^r(M)$ . The tangent bundle of  $M$  is denoted by  $\tau_M: TM \rightarrow M$  and the cotangent bundle by  $\pi_M: T^*M \rightarrow M$ . If  $\phi: M \rightarrow N$  is a differentiable map between manifolds, its derivative is denoted  $T\phi: TM \rightarrow TN$ . For a vector bundle  $\pi: E \rightarrow M$ ,  $\Gamma^r(E)$  denotes the sections of  $E$  that are of class  $C^r$ . The subbundle  $VE \triangleq \ker(T\pi) \subset TE$  is the **vertical bundle** of  $E$ .

Let  $V$  and  $W$  be  $\mathbb{R}$ -vector spaces. The notation  $L(V; W)$  denotes the set of linear maps from  $V$  to  $W$ . The **dual space** to  $V$  is defined by  $V^* = L(V; \mathbb{R})$ . For any nonempty set  $U \subset V$  the **annihilator** of  $U$  is a subspace of  $V^*$  defined by  $\text{ann}(U) = \{\alpha \in V^* \mid \alpha(v) = 0, v \in U\}$ . Similarly, for any nonempty set  $S \subset V^*$  the **coannihilator** of  $S$  is a subspace of  $V$  defined by  $\text{coann}(S) = \{v \in V \mid \alpha(v) = 0, \alpha \in S\}$ . For a bilinear map  $T: V \times V \rightarrow \mathbb{R}$  the **flat map**  $T^\flat: V \rightarrow V^*$  is defined by  $\langle T^\flat(v); u \rangle = T(u, v)$  for all  $u \in V$ . If  $T^\flat$  is invertible then its inverse, the **sharp map**, is denoted by  $T^\sharp: V^* \rightarrow V$ .

**2.1.1. Time-dependent objects on a manifold.** To define time-dependent vector fields on manifolds in a general way, following Sussmann [1997, §3] it is convenient to first introduce

time-dependent functions. A **Carathéodory function** on  $M$  is a map  $\phi: I \times M \rightarrow \mathbb{R}$  with the property that  $\phi^t \triangleq \phi(t, \cdot)$  is continuous for each  $t \in I$ , and  $\phi_x \triangleq \phi(\cdot, x)$  is Lebesgue measurable for each  $x \in M$ . A Carathéodory function  $\phi$  is **locally integrally bounded (LIB)** if, for each compact subset  $K \subset M$ , there exists a positive locally integrable function  $\psi_K: I \rightarrow \mathbb{R}$  such that  $|\phi(t, x)| \leq \psi_K(t)$  for each  $x \in K$ . A Carathéodory function  $\phi: I \times M \rightarrow \mathbb{R}$  is of **class  $C^r$**  if  $\phi^t$  is of class  $C^r$  for each  $t \in I$  and is **locally integrally of class  $C^r$  (LIC $^r$ )** if it is of class  $C^r$  and if  $X_1 \cdots X_r \phi^t$  is LIB for all  $t \in I$  and  $X_1, \dots, X_r \in \Gamma^\infty(TM)$ .

A **Carathéodory vector field** on  $M$  is a map  $X: I \times M \rightarrow TM$  with the property that  $X(t, x) \in T_x M$  and with the property that the function  $\alpha \cdot X: (t, x) \mapsto \alpha(x) \cdot X(t, x)$  is a Carathéodory function for each  $\alpha \in \Gamma^\infty(T^*M)$ . For a Carathéodory vector field  $X$  on  $M$ , denote by  $X_t: M \rightarrow TM$  the map  $X_t(x) = X(t, x)$ . A Carathéodory vector field  $X$  on  $M$  is **locally integrally of class  $C^r$  (LIC $^r$ )** if  $\alpha \cdot X_t$  is LIC $^r$  for every  $\alpha \in \Gamma^\infty(T^*M)$  and for every  $t \in I$ . The set of LIC $^r$  vector fields on  $M$  is denoted by LIC $^r(TM)$ .

The classical theory of time-dependent vector fields with measurable time dependence gives the existence of locally absolutely continuous integral curves for LIC $^\infty$  vector fields [Sontag 1998, Appendix C]. An integral curve  $\gamma: I \rightarrow M$  is **locally absolutely continuous (LAC)** if, for any  $\phi \in C^\infty(M)$ , the map  $t \mapsto \phi \circ \gamma(t)$  is locally absolutely continuous. Let  $\gamma'(t)$  denote the tangent vector to  $\gamma$  at  $t \in I$ , noting that this is defined for almost every  $t \in I$ . The flow of  $X \in \text{LIC}^\infty(TM)$  is denoted by  $\Phi_{t_0, t}^X$  and the curve  $\gamma: t \mapsto \Phi_{t_0, t}^X(x_0)$  is the integral curve for  $X$  with initial condition  $\gamma(t_0) = x_0$ .

Let  $\gamma: I \rightarrow M$  be an LAC curve. A **vector field along  $\gamma$**  is a map  $\xi: I \rightarrow TM$  with the property that  $\xi(t) \in T_{\gamma(t)}M$ . A vector field  $\xi$  along  $\gamma$  is **locally absolutely continuous (LAC)** if it is LAC as a curve in  $TM$ . A weaker notion than that of an LAC vector field along  $\gamma$  is that of a **locally integrable (LI)** vector field along  $\gamma$ , which is a vector field  $\xi$  along  $\gamma$  having the property that the function  $t \mapsto \alpha(\gamma(t)) \cdot \xi(t)$  is locally integrable for every  $\alpha \in \Gamma^\infty(T^*M)$ .

Let  $X \in \text{LIC}^\infty(TM)$  and let  $\gamma: I \rightarrow M$  be an integral curve for  $X$ . As described by Sussmann [1997], there is a naturally defined Lie derivative operator along  $\gamma$  that maps LAC sections of  $TM$  along  $\gamma$  to LI sections of  $TM$  along  $\gamma$ . This operator, denoted by  $\mathcal{L}^{X, \gamma}$ , is defined by

$$\mathcal{L}^{X, \gamma}(V_\gamma)(t) = [X_t, V](\gamma(t)), \quad \text{a.e. } t \in I,$$

where  $V \in \Gamma^1(TM)$  and  $V_\gamma$  is the LAC section of  $TM$  along  $\gamma$  defined by  $V_\gamma(t) = V(\gamma(t))$ . One easily verifies in coordinates that, for an LAC vector field  $\xi$  along  $\gamma$ ,  $\mathcal{L}^{X, \gamma}(\xi)$  is given in coordinates  $(x^1, \dots, x^n)$  by

$$\mathcal{L}^{X, \gamma}(\xi)(t) = \left( \frac{d\xi^i}{dt}(t) - \frac{\partial X_t}{\partial x^j}(\gamma(t)) \xi^j(t) \right) \frac{\partial}{\partial x^i}, \quad \text{a.e. } t \in I, \quad (2.1.1)$$

where a summation over  $i \in \{1, \dots, n\}$  is implied. This operation plays an important role in future developments, particularly in Sections 2.1.2 and 2.5.3.

**2.1.2. Tangent bundle geometry.** Let  $\pi: E \rightarrow M$  be a vector bundle. An LIC $^\infty$  vector field  $X$  on  $E$  is **linear** if

1.  $X$  is  $\pi$ -projectable (denote the resulting vector field on  $M$  by  $\pi X$ ) and



2.  $X$  is a linear morphism of vector bundles relative to the following diagram:

$$\begin{array}{ccc} E & \xrightarrow{X} & TE \\ \pi \downarrow & & \downarrow T\pi \\ M & \xrightarrow{\pi X} & TM \end{array}$$

That is, the induced mapping from  $\pi^{-1}(x)$  to  $T\pi^{-1}(\pi X(x))$  is a linear mapping of  $\mathbb{R}$ -vector spaces.

The flow of a linear vector field has the property that  $\Phi_{t_0,t}^X|_{E_x}: E_x \rightarrow E_{\Phi_{\pi_0,t}^X}$  is a linear transformation.

A linear vector field on a vector bundle generalizes the notion of a time-varying differential equation in the following manner. Let  $V$  be a finite-dimensional  $\mathbb{R}$ -vector space and consider on  $V$  a linear differential equation

$$\xi'(t) = A(t)(\xi(t)),$$

where  $A: \mathbb{R} \rightarrow L(V;V)$  is locally integrable. Now define an  $LIC^\infty$  linear vector field on the trivial bundle  $\text{pr}_1: \mathbb{R} \times V \rightarrow \mathbb{R}$ , where  $\text{pr}_1$  is the projection onto the first factor, by  $X_A(\tau, v) = ((\tau, v), (1, A(\tau)(v)))$ . Here the projected vector field on the base space is simply  $\pi X_A = \frac{\partial}{\partial \tau}$ . This special case of a linear vector field has the feature that the vector bundle admits a natural global trivialization. The lack of this feature in general accounts for some of the additional complexity in our development.

Now consider the case when  $E$  is the tangent bundle. Let  $\tilde{X} \in \Gamma^\infty(TM)$  and define the **tangent lift** of  $\tilde{X}$  as the vector field  $\tilde{X}^T \in \Gamma^\infty(TTM)$  obtained by

$$\tilde{X}^T(v_x) = \left. \frac{d}{ds} \right|_{s=0} T_x \Phi_{0,s}^{\tilde{X}}(v_x).$$

The definition of the tangent lift can be extended to time-varying vector fields as follows. For  $X \in LIC^\infty(TM)$ , the **tangent lift** of  $X$  is the vector field  $X^T \in LIC^\infty(TTM)$  defined by  $X^T(t, v_x) = X_t^T(v_x)$ . In natural coordinates  $(x^1, \dots, x^n, v^1, \dots, v^n)$  for  $TM$ , the vector field  $X^T(t, v_x)$  is given by the coordinate expression

$$X^T(t, v_x) = X^i(t, x) \frac{\partial}{\partial x^i} + \frac{\partial X^i}{\partial x^j}(t, x) v^j \frac{\partial}{\partial v^i}. \quad (2.1.2)$$

To provide an interpretation of the tangent lift, let  $\gamma: I \rightarrow M$  be an integral curve of  $X \in LIC^\infty(TM)$ . A **variation of  $X$  along  $\gamma$**  is a map  $\sigma: I \times J \rightarrow M$  satisfying

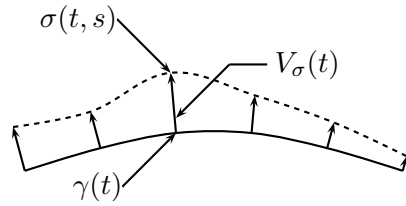
1.  $J \subset \mathbb{R}$  is an interval for which  $0 \in \text{int}(J)$ ,
2.  $\sigma$  is continuous,
3. the map  $I \ni t \mapsto \sigma_s(t) \triangleq \sigma(t, s) \in M$  is an integral curve for  $X$  for each  $s \in J$ ,
4. the map  $J \ni s \mapsto \sigma^t(s) \triangleq \sigma(t, s) \in M$  is LAC for each  $t \in I$ ,
5. the map  $I \ni t \mapsto \left. \frac{d}{ds} \right|_{s=0} \sigma^t(s) \in TM$  is LAC, and

6.  $\sigma_0 = \gamma$ .

Corresponding to a variation  $\sigma$  of  $X$  along  $\gamma$  is an LAC vector field  $V_\sigma$  along  $\gamma$  defined by

$$V_\sigma(t) = \left. \frac{d}{ds} \right|_{s=0} \sigma^t(s).$$

The idea of a variation  $\sigma$  and corresponding vector field  $V_\sigma$  along  $\gamma$  is depicted in Figure 2.1. With this notation, the following result records some useful properties of the tangent lift.



**Figure 2.1.** A pictorial representation of a variation  $\sigma$  and corresponding vector field  $V_\sigma$  along  $\gamma$

**2.1 Proposition:** Let  $X: I \times M \rightarrow TM$  be an  $\text{LIC}^\infty$  vector field, let  $v_{x_0} \in T_{x_0}M$ , let  $t_0 \in I$ , and let  $\gamma: I \rightarrow M$  be the integral curve of  $X$  satisfying  $\gamma(t_0) = x_0$ . For a vector field  $\Upsilon$  along  $\gamma$  satisfying  $\Upsilon(t_0) = v_{x_0}$ , the following statements are equivalent:

- (i)  $\Upsilon$  is an integral curve for  $X^T$ ;
- (ii) there exists a variation  $\sigma$  of  $X$  along  $\gamma$  such that  $V_\sigma = \Upsilon$ ;
- (iii)  $\mathcal{L}^{X,\gamma}(\Upsilon) = 0$ .

**Proof:** The equivalence of (i) and (ii) will follow from the more general Proposition 2.3 below. Thus only the equivalence of (ii) and (iii) needs to be proved. This, however, follows directly from the coordinate expressions (2.1.1) and (2.1.2).  $\blacksquare$

**2.2 Corollary:** For  $X \in \text{LIC}^\infty(TM)$ ,  $X^T$  is a linear vector field on  $\tau_M: TM \rightarrow M$  and  $\pi X^T = X$ .

The cotangent version of  $X^T$ , used in Chapter 3, is defined in a similar manner. For  $X \in \text{LIC}^\infty(TM)$ , the **cotangent lift** of  $X$  is the vector field  $X^{T*} \in \text{LIC}^\infty(TT^*M)$  defined by

$$X^{T*}(t, \alpha_x) = \left. \frac{d}{ds} \right|_{s=0} T_x^* \Phi_{t,-s}^X(\alpha_x).$$

In natural coordinates  $(x^1, \dots, x^n, p_1, \dots, p_n)$  for  $T^*M$ , the vector field  $X^{T*}(t, \alpha_x)$  is given by the coordinate expression

$$X^{T*}(t, \alpha_x) = X^i(t, x) \frac{\partial}{\partial x^i} - \frac{\partial X^j}{\partial x^i}(t, x) p_j \frac{\partial}{\partial p_i}. \quad (2.1.3)$$

The  $\text{LIC}^\infty$  vector fields  $X^T$  and  $X^{T*}$  define an  $\text{LIC}^\infty$  vector field  $X^T \times X^{T*}$  on  $TM \times T^*M$  by

$$X^T \times X^{T*}(t, v, \alpha) = (X^T(t, v), X^{T*}(t, \alpha)).$$

The Whitney sum  $TM \oplus T^*M$  is an embedded submanifold of  $TM \times T^*M$  with embedding  $v_x \oplus \alpha_x \mapsto (v_x, \alpha_x)$ . Since the  $\text{LIC}^\infty$  vector field  $X^T \times X^{T*}$  is tangent to  $TM \oplus T^*M$  its restriction to  $TM \oplus T^*M$ , denoted by  $X^T \oplus X^{T*}$ , is well-defined. An interesting joint property of  $X^T$  and  $X^{T*}$  is that the  $\text{LIC}^\infty$  vector field  $X^T \oplus X^{T*}$  leaves invariant the function  $v_x \oplus \alpha_x \mapsto v_x \cdot \alpha_x$  on  $TM \oplus T^*M$ .

Corresponding to  $X \in \text{LIC}^\infty(TM)$  there is also a natural vertical vector field  $\text{vlft}(X)$  on  $\tau_M: TM \rightarrow M$  defined by

$$\text{vlft}(X)(t, v_x) = \left. \frac{d}{ds} \right|_{s=0} (v_x + sX(t, x)).$$

In natural coordinates  $(x^1, \dots, x^n, v^1, \dots, v^n)$  for  $TM$ , the coordinate expression for the vertical lift is

$$\text{vlft}(X)(t, v_x) = X^i(t, x) \frac{\partial}{\partial v^i}.$$

## 2.2. Affine systems

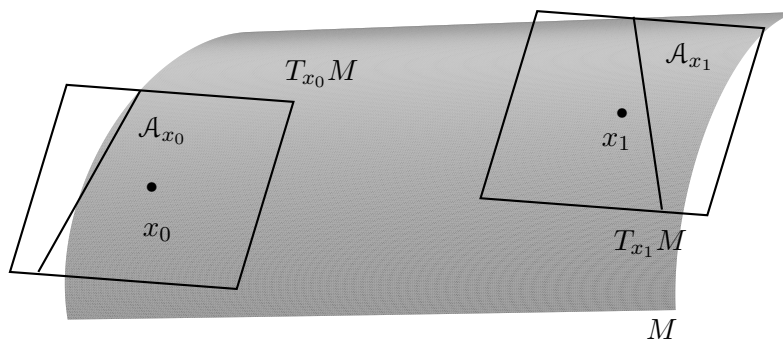
A *time-dependent distribution* on  $M$  is a subset  $\mathcal{D} \subset \mathbb{R} \times TM$  with the property that, for each  $x_0 \in M$ , there exist a neighbourhood  $\mathcal{N}$  and  $\text{LIC}^\infty$  vector fields  $\mathcal{Z} = \{X_1, \dots, X_k\}$  on  $\mathcal{N}$  such that

$$\mathcal{D}_{(t,x)} \triangleq \mathcal{D} \cap (\{t\} \times T_x M) = \left\{ \sum_{j=1}^k u^j X_j(t, x) \mid u \in \mathbb{R}^k \right\}.$$

The vector fields  $\mathcal{Z}$  are called *generators* for  $\mathcal{D}$ . A *time-dependent affine subbundle* on  $M$  is a subset  $\mathcal{A} \subset \mathbb{R} \times TM$  with the property that, for each  $x_0 \in M$ , there exists a neighbourhood  $\mathcal{N}$  and  $\text{LIC}^\infty$  vector fields  $\mathcal{Z} = \{X_0, X_1, \dots, X_k\}$  on  $\mathcal{N}$  such that

$$\mathcal{A}_{(t,x)} \triangleq \mathcal{A} \cap (\{t\} \times T_x M) = \left\{ X_0(t, x) + \sum_{j=1}^k u^j X_j(t, x) \mid u \in \mathbb{R}^k \right\}.$$

The vector fields  $\mathcal{Z}$  are called *generators* for  $\mathcal{A}$ . A pictorial interpretation of an affine subbundle—dropping the time dependency—is provided by Figure 2.2. The *linear part*



**Figure 2.2.** For each  $x \in M$ , an affine subbundle  $\mathcal{A}$  on  $M$  specifies an affine subspace  $\mathcal{A}_x \subset T_x M$ .

of a time-dependent affine subbundle is the time-dependent distribution  $L(\mathcal{A})$  defined by  $L(\mathcal{A})_{(t,x)}$  being the subspace of  $T_x M$  upon which the affine subspace  $\mathcal{A}_{(t,x)}$  is modelled. If  $\mathcal{Z}$  are generators for  $\mathcal{A}$  as above, then the vector fields  $\{X_1, \dots, X_k\}$  are *linear generators* for  $L(\mathcal{A})$ . In the setting of [Hirschorn and Lewis 2002], the next step is to define an “affine system” in  $\mathcal{A}$  to be an assignment to each  $(t, x) \in \mathbb{R} \times M$  a subset  $\mathcal{S}(t, x)$  of  $\mathcal{A}_{(t,x)}$ . This amounts to specifying the control set for the system. However, in order to focus

on the geometry associated with an affine system and its linearization, it is assumed that  $\mathcal{A}_{(t,x)} = \mathcal{A}_{(t,x)}$ . This essentially means that the controls are unrestricted. Accepting a slight abuse of notation, a time-dependent affine subbundle  $\mathcal{A}$  will be called a **time-dependent affine system**. A **trajectory** for  $\mathcal{A}$  is then an LAC curve  $\gamma: I \rightarrow M$  with the property that  $\gamma'(t) \in \mathcal{A}_{(t,\gamma(t))}$ .

Note that the specification of an affine system does not provide a natural notion of a drift vector field and control vector fields. As seen in [Hirschorn and Lewis \[2002\]](#) the basic properties like controllability can depend on the choice of a drift vector field. For the geometric development of the linearization, this is a non-issue since it is natural to assume the presence of a reference vector field, cf. the discussion of [Section 1.1](#). To be formal about this, a **reference vector field** for an affine system  $\mathcal{A}$  is an  $\text{LIC}^\infty$  vector field  $X_{\text{ref}} \in \text{LIC}^\infty(TM)$  with the property that  $X_{\text{ref}}(t, x) \in \mathcal{A}_{(t,x)}$  for all  $x \in M$  and almost every  $t \in \mathbb{R}$ . Of course, integral curves of  $X_{\text{ref}}$  are trajectories for  $\mathcal{A}$ . If  $\gamma: I \rightarrow M$  is a trajectory for  $\mathcal{A}$ , then there exists a reference vector field  $X_{\text{ref}}$  for  $\mathcal{A}$  for which  $\gamma$  is an integral curve [[Sussmann 1997](#), Proposition 4.1].

### 2.3. Linearization about a reference trajectory

Let  $\mathcal{A}$  be a time-dependent affine system and let  $X_{\text{ref}}$  be a reference vector field with corresponding LAC reference trajectory  $\gamma_{\text{ref}}: I \rightarrow M$ . The embedding of  $\gamma_{\text{ref}}$  as an integral curve of a reference vector field gives additional useful structure and corresponds more naturally to the standard case, cf. [Section 1.1](#).

What it means to linearize about  $\gamma_{\text{ref}}$  is captured in the following definition. An  **$\mathcal{A}$ -variation** of  $\gamma_{\text{ref}}$  is a map  $\sigma: I \times J \rightarrow M$  with the following properties:

1.  $J \subset \mathbb{R}$  is an interval for which  $0 \in \text{int}(J)$ ;
2.  $\sigma$  is continuous;
3. the map  $I \ni t \mapsto \sigma_s(t) \triangleq \sigma(t, s) \in M$  is a trajectory of  $\mathcal{A}$  for each  $s \in J$ ;
4. the map  $J \ni s \mapsto \sigma^t(s) \triangleq \sigma(t, s) \in M$  is LAC for each  $t \in I$ ;
5. the map  $I \ni t \mapsto \left. \frac{d}{ds} \right|_{s=0} \sigma^t(s) \in TM$  is LAC;
6.  $\sigma_0 = \gamma_{\text{ref}}$ .

Given an  $\mathcal{A}$ -variation  $\sigma$  of  $\gamma_{\text{ref}}$ , a vector field  $V_\sigma$  along  $\gamma_{\text{ref}}$  is defined by

$$V_\sigma(t) = \left. \frac{d}{ds} \right|_{s=0} \sigma^t(s). \quad (2.3.1)$$

The vector field  $V_\sigma$  should be thought of as being the result of linearizing in the direction of the  $\mathcal{A}$ -variation  $\sigma$ . Using the geometric constructions of [Section 2.1.2](#), these vector fields along  $\gamma_{\text{ref}}$  arise as trajectories for a time-dependent affine system on  $TM$ . Such a time-dependent affine subbundle  $\mathcal{A}_{\text{ref}}^T$  on  $TM$  is defined as follows. Let  $(t, v_x) \in \mathbb{R} \times TM$  and define

$$\mathcal{A}_{\text{ref},(t,v_x)}^T = \{X_{\text{ref}}^T(t, v_x) + \text{vlft}(X) \mid X \in L(\mathcal{A})_{(t,x)}\}.$$

This is a time-dependent affine subbundle since it possesses generators  $\{X_{\text{ref}}^T, \text{vlft}(X_1), \dots, \text{vlft}(X_k)\}$  where  $\{X_{\text{ref}}, X_1, \dots, X_k\}$  are generators for  $\mathcal{A}$ .

**2.3 Proposition:** *Let  $\mathcal{A}$  be a time-dependent affine system, and let  $X_{\text{ref}}$  be a reference vector field with differentiable reference trajectory  $\gamma_{\text{ref}}$ , as above. For a vector field  $\Upsilon$  along  $\gamma_{\text{ref}}$ , the following statements are equivalent:*

- (i)  $\Upsilon$  is a trajectory for  $\mathcal{A}_{\text{ref}}^T$ ;
- (ii) there exists an  $\mathcal{A}$ -variation  $\sigma$  of  $\gamma_{\text{ref}}$  such that  $V_\sigma = \Upsilon$ .

**Proof:** Let  $\sigma$  be an  $\mathcal{A}$ -variation giving rise to the vector field  $V_\sigma$  along  $\gamma_{\text{ref}}$ . Using a set of generators  $\{X_{\text{ref}}, X_1, \dots, X_k\}$  for  $\mathcal{A}$ ,

$$\sigma'_s(t) = X_{\text{ref}}(t, \sigma_s(t)) + \sum_{j=1}^k u^j(t) X_j(t, \sigma_s(t)),$$

since  $\sigma_s$  is a trajectory for  $\mathcal{A}$ . Differentiating with respect to  $s$  at  $s = 0$  gives

$$V'_\sigma(t) = X_{\text{ref}}^T(t, \gamma'_{\text{ref}}(t)) + \sum_{j=1}^k \left( v^j(t) X_j(t, \gamma_{\text{ref}}(t)) + u^j(0, t) \frac{d}{ds} \Big|_{s=0} X_j(t, \sigma_s(t)) \right),$$

where  $v^j(t) = \frac{\partial u^j}{\partial s}(0, t)$ . Since  $\sigma_0 = \gamma_{\text{ref}}$  it follows that  $u^j(0, t) = 0$ , and so it follows that  $V'_\sigma(t) \in \mathcal{A}_{\text{ref},(t, \gamma_{\text{ref}}(t))}^T$ , as desired.

Let  $\sigma_{j,s}$  be the  $\mathcal{A}$ -variation of  $\gamma_{\text{ref}}$  satisfying

$$\frac{d}{dt} \sigma_{j,s}(t) = X_{\text{ref}}(t, \sigma_{j,s}(t)) + s X_j(t, \sigma_{j,s}(t)),$$

noting that the corresponding infinitesimal variation is

$$\frac{d}{dt} V_{\sigma_j}(t) = X_{\text{ref}}^T(t, \gamma'_{\text{ref}}(t)) + \text{vlft}(X_j(t))(\gamma'_{\text{ref}}(t)).$$

The convexity of the set of variations of a given order (see [Bianchini and Stefani 1993]) now ensures the existence of a variation for any trajectory  $\Upsilon$  that covers  $\gamma_{\text{ref}}$ .  $\blacksquare$

## 2.4. Linearization about an equilibrium point

The above developments concerning linearization about a reference trajectory simplify significantly when dealing with an equilibrium point. Here the development looks a lot more like the standard non-geometric setup.

Let  $\mathcal{A}$  be a time-dependent affine subbundle on  $M$  and let  $X_{\text{ref}}: I \times M \rightarrow TM$  be a reference vector field for  $\mathcal{A}$ . A point  $x_0 \in M$  is an **equilibrium point** for  $X_{\text{ref}}$  if  $X_{\text{ref}}(t, x_0) = 0_{x_0}$  for each  $t \in I$ . Thus the curve  $I \ni t \mapsto x_0 \in M$  is an integral curve for  $X_{\text{ref}}$ . The tangent lift  $X_{\text{ref}}^T$  at an equilibrium point for  $X_{\text{ref}}$  has the following properties.

**2.4 Proposition:** *If  $x_0 \in M$  is an equilibrium point for an  $\text{LIC}^\infty$  vector field  $X_{\text{ref}}: I \times M \rightarrow TM$ , then  $X_{\text{ref}}^T(t, v_{x_0})$  is vertical for each  $v_{x_0} \in T_{x_0}M$ . Furthermore, for each  $t \in I$  there exists a unique  $A(t) \in \text{L}(T_{x_0}M; T_{x_0}M)$  such that  $X_{\text{ref}}^T(t, v_{x_0}) = \text{vlft}(A(t)(v_{x_0}))$ , and the map  $I \ni t \mapsto A(t) \in \text{L}(T_{x_0}M; T_{x_0}M)$  is Lebesgue measurable.*

**Proof:** This follows directly from the coordinate representation (2.1.2) for the tangent lift.  $\blacksquare$

Thus the tangent lift is vertical-valued on  $T_{x_0}M$ . Since  $V_{v_{x_0}}TM \simeq T_{x_0}M$  this means that the linearization is a time-dependent linear affine system on  $T_{x_0}M$  of the form

$$\mathcal{A}_{\text{ref}}^T(t, v_{x_0}) = \{A(t)(v_{x_0}) + b \mid b \in L(\mathcal{A})_{(t, x_0)}\}.$$

Trajectories  $\xi: I \rightarrow T_{x_0}M$  of the linearization then satisfy

$$\xi'(t) = A(t)(\xi(t)) + b(t), \quad (2.4.1)$$

for some measurable curve  $b: I \rightarrow T_{x_0}M$  having the property that  $b(t) \in L(\mathcal{A})_{(t, x_0)}$ . To make this look more like the usual notion of a time-varying linear system, for each  $t \in I$  let  $U$  be a finite-dimensional  $\mathbb{R}$ -vector space and let  $B(t) \in L(U; T_{x_0}M)$  have the property that  $\text{image}(B(t)) = L(\mathcal{A})_{(t, x_0)}$ . Then the equation governing trajectories become

$$\xi'(t) = A(t)(\xi(t)) + B(t)(u(t)),$$

for a measurable curve  $u: I \rightarrow U$ . This then recovers the usual notion of a time-dependent linear system.

## 2.5. Linear controllability

In Section 2.2, a time-dependent affine subbundle  $\mathcal{A}_{\text{ref}}^T$  on  $TM$  was constructed by linearizing a time-dependent affine subbundle  $\mathcal{A}$  on  $M$  along a reference trajectory  $\gamma_{\text{ref}}$ . In this section notions of the reachable set for both  $\mathcal{A}$  and  $\mathcal{A}_{\text{ref}}^T$  are defined, as well as the associated versions of controllability along  $\gamma_{\text{ref}}$ . In Section 2.5.2 the standard results for a time-varying linear system are cast in a geometric manner. In Section 2.5.3 this geometric characterization is used to state a completely geometric result that describes the reachable sets for the linearization.

**2.5.1. Controllability definitions.** In this section the reachable sets for both  $\mathcal{A}$  and  $\mathcal{A}_{\text{ref}}^T$  are defined and, for the latter, two equivalent characterizations of the reachable set are provided. Then controllability along a reference trajectory is defined for  $\mathcal{A}$  and  $\mathcal{A}_{\text{ref}}^T$ . Recall that a trajectory for a time-dependent affine system  $\mathcal{A}$  is an LAC curve  $\gamma: I \rightarrow M$  such that  $\gamma'(t) \in \mathcal{A}_{(t, \gamma(t))}$ . The set of trajectories defined on  $[t_0, t_1]$  is denoted by  $\text{Traj}(\mathcal{A}, t_1, t_0)$  and  $\text{Traj}(\mathcal{A}, t_0) = \bigcup_{t_1 \geq t_0} \text{Traj}(\mathcal{A}, t_1, t_0)$ . For  $x_0 \in M$  and  $t \geq t_0$ , the **reachable set** of  $\mathcal{A}$  from  $x_0$  is defined by

$$\mathcal{R}_{\mathcal{A}}(x_0, t, t_0) = \{\gamma(t) \mid \gamma \in \text{Traj}(\mathcal{A}, t_0), \gamma(t_0) = x_0\}.$$

Similarly, a **trajectory** for the linearized time-dependent affine system  $\mathcal{A}_{\text{ref}}^T$  is an LAC curve  $\Upsilon: I \rightarrow TM$  such that  $\Upsilon'(t) \in \mathcal{A}_{\text{ref}, (t, \Upsilon(t))}^T$ . Then the set of trajectories defined on  $[t_0, t_1]$  is denoted by  $\text{Traj}(\mathcal{A}_{\text{ref}}^T, t_1, t_0)$  and  $\text{Traj}(\mathcal{A}_{\text{ref}}^T, t_0) = \bigcup_{t_1 \geq t_0} \text{Traj}(\mathcal{A}_{\text{ref}}^T, t_1, t_0)$ . For  $v_{x_0} \in T_{x_0}M$  and  $t \geq t_0$ , the **reachable set** from  $v_{x_0}$  is defined by

$$\mathcal{R}_{\mathcal{A}_{\text{ref}}^T}(v_{x_0}, t, t_0) = \{\Upsilon(t) \mid \Upsilon \in \text{Traj}(\mathcal{A}_{\text{ref}}^T, t_0), \Upsilon(t_0) = v_{x_0}\}.$$

With these notions of the reachable sets, the controllability of each system is defined as follows.

**2.5 Definition:** Let  $X_{\text{ref}}$  be a reference vector field for  $\mathcal{A}$  and let  $\gamma_{\text{ref}}: I \rightarrow M$  be a reference trajectory. Let  $x_0 \in M$  and  $\gamma_{\text{ref}}(t_0) = x_0$ . The system  $\mathcal{A}$  is

- (i) **controllable at  $t_0$**  along  $\gamma_{\text{ref}}$  if  $\gamma_{\text{ref}}(t) \in \text{int } \mathcal{R}_{\mathcal{A}}(x_0, t, t_0)$  for each  $t > t_0$  and is
- (ii) **linearly controllable at  $t_0$**  along  $\gamma_{\text{ref}}$  if  $\mathcal{R}_{\mathcal{A}_{\text{ref}}}^T(0_{x_0}, t, t_0) = T_{\gamma_{\text{ref}}(t)}M$  for each  $t > t_0$ .

**2.5.2. Recasting the standard results.** In a step toward a geometric theory of Jacobian linearization, the standard setup of Section 1.2.1 is recast on general  $\mathbb{R}$ -vector spaces. In doing so, the extra structure available with Euclidean spaces, in particular the standard inner product, is removed. Let  $U$  and  $V$  be  $\mathbb{R}$ -vector spaces with  $\dim(U) = m$  and  $\dim(V) = n$ . Let  $A: \mathbb{R} \rightarrow L(V; V)$  and  $B: \mathbb{R} \rightarrow L(U; V)$  be continuous and define a time-varying affine subbundle  $\mathcal{A}_{(A,B)}$  on  $V$  by

$$\mathcal{A}_{(A,B),(t,v)} = \{A(t)v + B(t)u \mid u \in U\}.$$

A trajectory  $\xi$  of  $\mathcal{A}_{(A,B)}$  satisfies

$$\xi'(t) = A(t)\xi(t) + B(t)u(t). \quad (2.5.1)$$

The solution to (2.5.1) satisfying  $\xi(t_0) = \xi_0$  for  $t_0 \in I$  is given by,

$$\xi(t) = \Phi(t, t_0)\xi_0 + \int_{t_0}^t \Phi(t, \sigma)B(\sigma)u(\sigma)d\sigma, \quad (2.5.2)$$

where  $\Phi(t, t_0)$  is the state transition matrix. That is,  $t \mapsto \Phi(t, t_0)$  is the solution to the homogeneous system  $\Phi'(t, t_0) = A(t)\Phi(t, t_0)$  with initial condition  $\Phi(t_0, t_0) = \text{id}_V$ . The transition matrix then has the following properties:

1.  $\Phi(t, \tau)\Phi(\tau, t_0) = \Phi(t, t_0)$ ;
2.  $\Phi(t, \tau)^{-1} = \Phi(\tau, t)$ .

The subbundle  $\mathcal{A}_{(A,B)}$  is **controllable at  $t_0$**  if, for each  $\xi_0, \xi_1 \in V$ , there exists a control  $u: [t_0, t_1] \rightarrow U$  which steers from  $\xi_0$  at time  $t_0$  to  $\xi_1$  at time  $t_1$ . In the standard case—see Section 1.2.1—the controllability of a time-varying linear system is determined by the controllability Gramian,

$$W(t_0, t) = \int_{t_0}^t \Phi(t_0, \sigma)B(\sigma)B^*(\sigma)\Phi^*(t_0, \sigma)d\sigma. \quad (2.5.3)$$

This definition makes use of the standard inner product on  $\mathbb{R}^m$  to identify  $\mathbb{R}^m$  with  $(\mathbb{R}^m)^*$ . Indeed, the domain of  $B: \mathbb{R} \rightarrow L(\mathbb{R}^m; \mathbb{R}^n)$  and image of  $B^*: \mathbb{R} \rightarrow L((\mathbb{R}^n)^*; (\mathbb{R}^m)^*)$  for each  $t$  are different spaces. Inducing an inner product on  $U$  by a symmetric map  $R: I \rightarrow L(U; U^*)$  which is positive-definite for each  $t \in I$  yields a Gramian of the form

$$W(t_0, t) = \int_{t_0}^t \Phi(t_0, \sigma)B(\sigma)R^\sharp(t)B^*(\sigma)\Phi^*(t_0, \sigma)d\sigma. \quad (2.5.4)$$

The derivation of (2.5.4) follows directly from the standard case in [Brockett 1970] and the time-varying affine subbundle  $\mathcal{A}_{(A,B)}$  is controllable at  $t_0$  if and only if  $W(t_0, t)$  is surjective



for  $t > t_0$ . Later, in Section 3.7, the quadratic cost in the LQR problem provides a natural choice of an inner product.

The notion of a controllability Gramian does not make sense in the geometric framework of Section 2.2. There is no natural way to construct the analogue of  $W(t_0, t)$  for the linearization of a reference vector field  $X_{\text{ref}}$  along a reference trajectory  $\gamma_{\text{ref}}$  since (2.5.4) is an integral of maps  $\{t \mapsto B(t)\}$  that depend on a choice of coordinates. Therefore, an alternative characterization of controllability that can be applied in the geometric setting is needed. The following result gives one such characterization.

**2.6 Theorem:** *Let  $V$ ,  $U$ ,  $A$ , and  $B$  be as above. Then*

$$\text{image}(W(t_0, t)) = \text{span}_{\mathbb{R}} \left( \bigcup_{\substack{\tau \in [t_0, t] \\ b_\tau \in \text{image}(B(\tau))}} \Phi(t_0, \tau) b_\tau \right).$$

**Proof:** For notational convenience define

$$\mathcal{S}_{\mathcal{A}_{(A,B)}}(t_0, t) = \text{span}_{\mathbb{R}} \left( \bigcup_{\substack{\tau \in [t_0, t] \\ b_\tau \in \text{image}(B(\tau))}} \Phi(t_0, \tau) b_\tau \right).$$

Let  $v \in \text{image}(W(t_0, t))$ . Then there exists a continuous control  $u: [t_0, t] \rightarrow U$  such that

$$v = \int_{t_0}^t \Phi(t_0, \sigma) B(\sigma) u(\sigma) d\sigma$$

[Brockett 1970]. Since  $A$ ,  $B$ , and  $u$  are continuous, there exists a sequence of partitions  $P_i = \{t_0 = t_{1,i}, \dots, t_{k_i,i} = t\}$  of  $[t_0, t]$  such that if

$$v_i = \sum_{j=2}^{k_i} \Phi(t_0, t_{j,i}) B(t_{j,i}) u(t_{j,i}) (t_{j,i} - t_{j-1,i}),$$

then  $\lim_{i \rightarrow \infty} v_i = v$ . It is clear that, for each  $i \in \mathbb{N}$ ,  $v_i \in \mathcal{S}_{\mathcal{A}_{(A,B)}}(t_0, t)$ . Since  $\mathcal{S}_{\mathcal{A}_{(A,B)}}(t_0, t)$  is closed it follows that  $v \in \mathcal{S}_{\mathcal{A}_{(A,B)}}(t_0, t)$ .

Now assume that  $v \in \mathcal{S}_{\mathcal{A}_{(A,B)}}(t_0, t)$ . Choose  $t_1, \dots, t_k \in [t_0, t]$  and  $b_{t_j} \in \text{image}(B(t_j))$ ,  $j \in \{1, \dots, k\}$ , such that

$$\mathcal{S}_{\mathcal{A}_{(A,B)}}(t_0, t) = \text{span}_{\mathbb{R}}(\Phi(t_0, t_1) b_{t_1}, \dots, \Phi(t_0, t_k) b_{t_k}).$$

Then  $v$  can be written as

$$v = \sum_{j=1}^k c_j \Phi(t_0, t_j) b_{t_j}.$$

A useful characterization of points in  $\text{image}(W(t_0, t))$  is provided by Lemma 1.

**1 Lemma:**  $\text{image}(W(t_0, t)) = \{\Phi(t_0, t)\tilde{v} \in V \mid \exists u: [t_0, t] \rightarrow U \text{ steering zero to } \tilde{v}\}$ .

Proof: By (2.5.2), the set of points reachable from  $0 \in V$  in time  $t$  from  $t_0$  is

$$\left\{ \int_{t_0}^t \Phi(t, \sigma)B(\sigma)u(\sigma)d\sigma \mid u: [t_0, t] \rightarrow U \text{ continuous} \right\}.$$

Using the composition property of the transition matrix, apply  $\Phi(t_0, t)$  to any point in this set:

$$\Phi(t_0, t) \int_{t_0}^t \Phi(t, \sigma)B(\sigma)u(\sigma)d\sigma = \int_{t_0}^t \Phi(t_0, \sigma)B(\sigma)u(\sigma)d\sigma.$$

The lemma now follows by comparison with (2.5.3).  $\blacktriangledown$

If the system can be steered from 0 to  $\Phi(t, t_0)v$ , this part of the theorem will follow from Lemma 1. Let  $\mu_j \in U$  have the property that  $B(t_j)\mu_j = b_{t_j}$   $j \in \{1, \dots, k\}$ . Now consider the distributional control  $u = \sum_{j=1}^k c_j \delta_{t_j} \mu_j$ , where  $\delta_{t_j}$  is the delta-distribution with support  $\{t_j\}$ . Applying this control, by (2.5.2), yields

$$\int_{t_0}^t \Phi(t, \sigma)B(\sigma)u(\sigma) d\sigma = \sum_{j=1}^k c_j \Phi(t, t_j) b_{t_j} = \Phi(t, t_0)v. \quad (2.5.5)$$

Thus the distributional control  $u$  steers from 0 to  $\Phi(t, t_0)v$ , as desired. To show the distributional control  $u$  can be replaced with a sequence of piecewise continuous controls, consider the following lemma.

**2 Lemma:** *There exists a sequence of controls  $\{u_i\}_{i \in \mathbb{N}}$  such that*

$$\lim_{i \rightarrow \infty} \int_{t_0}^t \Phi(t, \sigma)B(\sigma)u_i(\sigma)d\sigma = \sum_{j=1}^k c_j \Phi(t, t_j) b_{t_j}.$$

Proof: For  $j \in \{1, \dots, k\}$  and  $i \in \mathbb{N}$  define

$$u_{j,i}(t) = \begin{cases} i c_j \mu_j, & t \in [t_j, t_j + \frac{1}{i}], \\ 0, & \text{otherwise.} \end{cases}$$

Now note that, using the Peano-Baker series,

$$\begin{aligned} \int_{t_0}^t \Phi(t, \sigma)B(\sigma)u_{j,i}(\sigma)d\sigma &= \Phi(t, t_j + \frac{1}{i}) \int_{t_0}^t \Phi(t_j + \frac{1}{i}, \sigma)B(\sigma)u_{j,i}(\sigma)d\sigma \\ &= \Phi(t, t_j + \frac{1}{i}) \int_{t_j}^{t_j + \frac{1}{i}} \Phi(t_j + \frac{1}{i}, \sigma)B(\sigma)u_{j,i}(\sigma)d\sigma \\ &= i c_j \Phi(t, t_j + \frac{1}{i}) \int_{t_j}^{t_j + \frac{1}{i}} \left( \text{id}_V + \int_{\sigma}^{t_j + \frac{1}{i}} A(\sigma_1)d\sigma_1 \right. \\ &\quad \left. + \int_{\sigma}^{t_j + \frac{1}{i}} A(\sigma_1) \int_{\sigma}^{\sigma_1} A(\sigma_2)d\sigma_2 d\sigma_1 + \dots \right) B(\sigma)\mu_j d\sigma. \end{aligned}$$

Because  $A$  is continuous, all terms in the Peano-Baker series go to zero at least as fast as  $(\frac{1}{i})^2$ . Thus only the first term remains in the limit, giving

$$\lim_{i \rightarrow \infty} \int_{t_0}^t \Phi(t, \sigma) B(\sigma) u_{j,i}(\sigma) d\sigma = c_j \Phi(t, t_j) b_{t_j}.$$

The result now follows by taking  $u_i = \sum_{j=1}^k u_{j,i}$ . ▼

Let  $\{u_i\}_{i \in \mathbb{N}}$  be a sequence of controls defined by Lemma 2. For each  $i \in \mathbb{N}$ ,

$$\Phi(t_0, t) \int_{t_0}^t \Phi(t, \sigma) B(\sigma) u_i(\sigma) d\sigma \in \text{image}(W(t_0, t))$$

by Lemma 1. Therefore, the limit as  $i \rightarrow \infty$  is also in  $\text{image}(W(t_0, t))$ . But, by (2.5.5),

$$\lim_{i \rightarrow \infty} \Phi(t_0, t) \int_{t_0}^t \Phi(t, \sigma) B(\sigma) u_i(\sigma) d\sigma = v,$$

giving the result. ■

**2.5.3. Controllability of linearizations.** To provide a geometric characterization, using Theorem 2.6, of the reachable set for the linearization of a reference vector field  $X_{\text{ref}}$  along a reference trajectory  $\gamma_{\text{ref}}$  requires the following definition. For an LAC curve  $\gamma: I \rightarrow M$ , a **distribution along  $\gamma$**  is a subset  $D \subset TM|_{\text{image}(\gamma)}$  with the property that, for each  $t_0 \in I$ , there exists a neighbourhood  $J \subset I$  of  $t_0$  and LAC vector fields  $\xi_1, \dots, \xi_k$  along  $\gamma|_J$  such that  $D_{\gamma(t)} = \text{span}_{\mathbb{R}}(\xi_1(t), \dots, \xi_k(t))$  for each  $t \in J$ . Let  $t_0 \in \text{int}(I)$  and denote  $T = \sup I$ , allowing that  $T = \infty$ . Let  $I_{t_0} = [t_0, T)$ . Denote by  $\gamma_{t_0}$  the restriction of  $\gamma_{\text{ref}}$  to  $I_{t_0}$ . Recall from Section 2.1.1 the definition of  $\mathcal{L}^{X_{\text{ref}}, \gamma_{t_0}}$  and denote by  $\langle \mathcal{L}^{X_{\text{ref}}, \gamma_{t_0}}, L(\mathcal{A})_{t_0} \rangle$  the smallest  $\mathcal{L}^{X_{\text{ref}}, \gamma_{t_0}}$ -invariant distribution along  $\gamma_{t_0}$  that agrees with  $L(\mathcal{A})$  at  $\gamma_{\text{ref}}(t_0)$ .

**2.7 Theorem:** *Let  $\mathcal{A}$  be a time-dependent affine system on  $M$  with  $X_{\text{ref}}$  a reference vector field and  $\gamma_{\text{ref}}: I \rightarrow M$  a differentiable reference trajectory. For  $t_0 \in I$  and  $t > t_0$  the following sets are equal:*

- (i)  $\mathcal{R}_{\mathcal{A}_{\text{ref}}}^{X_{\text{ref}}}(0_{x_0}, t, t_0)$ ;
- (ii)  $\text{span}_{\mathbb{R}} \left( \bigcup_{\substack{\tau \in [t_0, t] \\ v_\tau \in L(\mathcal{A})_{\tau, \gamma_{\text{ref}}(\tau)}}} \Phi_{\tau, t}^{X_{\text{ref}}}(v_\tau) \right)$ ;
- (iii)  $\langle \mathcal{L}^{X_{\text{ref}}, \gamma_{t_0}}, L(\mathcal{A})_{t_0} \rangle_{\gamma_{\text{ref}}(t)}$ .

**Proof:** Since  $\gamma_{\text{ref}}$  is differentiable the pull-back bundle  $\gamma_{\text{ref}}^* \tau_M: \gamma_{\text{ref}}^* TM \rightarrow I$  is well defined. Recall that

$$\gamma_{\text{ref}}^* TM = \{(t, v) \mid \gamma_{\text{ref}}(t) = \tau_M(v)\},$$

where  $\gamma_{\text{ref}}^* \tau_M(t, v) = t$ . Thus  $\gamma_{\text{ref}}^* TM$  is a vector bundle over  $I$  with fibre over  $t \in I$  being  $T_{\gamma_{\text{ref}}(t)} M$ . This bundle may be trivialized since  $I$  is contractible. Denote by  $\rho: \gamma_{\text{ref}}^* TM \rightarrow I \times V$  a particular trivialization with a commuting diagram

$$\begin{array}{ccc} \gamma_{\text{ref}}^* TM & \xrightarrow{\rho} & I \times V \\ & \searrow \gamma_{\text{ref}}^* \tau_M & \swarrow \text{pr}_1 \\ & & I \end{array}$$

where  $\text{pr}_1$  the projection onto the first factor.

The following lemma records some useful properties of the representation of trajectories of  $\mathcal{A}_{\text{ref}}^T$ .

**1 Lemma:** *The following statements hold:*

- (i) *there exists a vector bundle endomorphism  $A: I \times V \rightarrow I \times V$  over  $\text{id}_I$  with the property that  $T_{(t, \gamma_{\text{ref}}(t))} \rho(1, X_{\text{ref}}^T(t, v_{\gamma_{\text{ref}}(t)})) = (1, A(t) \cdot \rho(v_{\gamma_{\text{ref}}(t)}))$ ;*
- (ii) *if  $X_{\text{ref}} \in \Gamma^\infty(TM)$  then there exists a section  $\xi_{X_{\text{ref}}}$  of  $\text{pr}_1: I \times V \rightarrow I$  such that  $T_{(t, \gamma_{\text{ref}}(t))} \rho(0, \text{vlft}(X_{\text{ref}})(\gamma_{\text{ref}}(t))) = \xi_X(t)$ .*

**Proof:** The first assertion follows since  $X_{\text{ref}}^T$  is a vector bundle mapping over  $X_{\text{ref}}$ . The second part of the lemma is merely the definition of  $\xi_{X_{\text{ref}}}$ .  $\blacktriangledown$

The lemma says that, if  $v(t) = T\rho(\Upsilon(t))$  for a trajectory  $\Upsilon$  for  $\mathcal{A}_{\text{ref}}^T$ , then we have

$$v'(t) = A(t)v(t) + b(t),$$

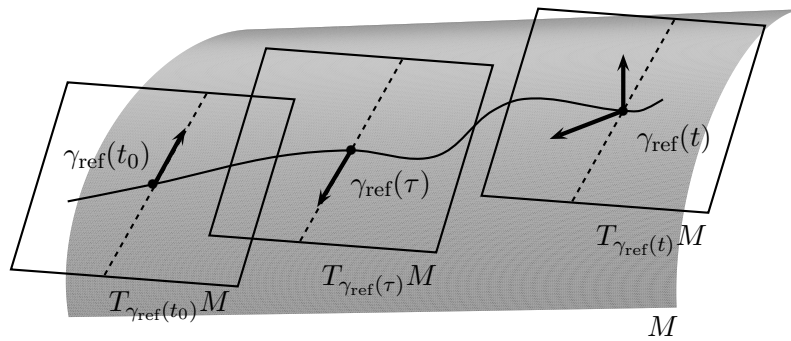
where  $b(t) \in \text{image}(\rho(\text{vlft}(\mathcal{A}_{(t, \gamma_{\text{ref}}(t))}))$ . Therefore, the equality of the sets (i) and (ii) follows from Theorem 2.6.

From the definition of the set in (ii), the equivalence of (ii) and (iii) will follow if it can be shown that the notion of a distribution along  $\gamma_{t_0}$  being invariant under  $\mathcal{L}^{X_{\text{ref}}, \gamma_{t_0}}$  is equivalent to the notion of being invariant under the flow of  $X_{\text{ref}}^T$  along  $\gamma_{t_0}$ . Let  $\mathcal{D}$  be a distribution along  $\gamma_{t_0}$ . The distribution  $\mathcal{D}$  is invariant under the flow of  $X_{\text{ref}}^T$  if and only if it is invariant under  $\mathcal{L}^{X_{\text{ref}}, \gamma_{t_0}}$ . This claim follows from the characterization of the flow of  $X_{\text{ref}}$  in Proposition 2.1 in terms of  $\mathcal{L}^{X_{\text{ref}}, \gamma_{t_0}}$ . That result states that the flow of  $X_{\text{ref}}^T$  through  $v \in \mathcal{D}_{\gamma_{t_0}(t_0)}$  is obtained by transporting  $v$  along  $\gamma_{t_0}$ .  $\blacksquare$

The set described in part (iii) of Theorem 2.7 should be thought of as the analogue of “the smallest  $A$ -invariant subspace containing  $\text{image}(B)$ ” in the time-invariant linear theory and Figure 2.3 captures the intuition behind the set in part (ii). Finally, Theorem 2.7 immediately gives the following corollary, the second part of which follows from the variational cone results of [Bianchini and Stefani 1993].

**2.8 Corollary:** *Let  $\mathcal{A}$ ,  $X_{\text{ref}}$ , and  $\gamma_{\text{ref}}$  be as in Theorem 2.7. Then the following statements hold for  $t_0 \in I$ :*

- (i)  *$\mathcal{A}$  is linearly controllable at  $t_0$  along  $\gamma_{\text{ref}}$  if the smallest  $\mathcal{L}^{X_{\text{ref}}, \gamma_{\text{ref}}}$ -invariant distribution along  $\gamma_{\text{ref}}$  containing  $L(\mathcal{A})|_{\text{image}(\gamma_{\text{ref}})}$  is equal to  $TM|_{\text{image}(\gamma_{\text{ref}})}$ ;*
- (ii) *if  $\mathcal{A}$  is linearly controllable at  $t_0$  along  $\gamma_{\text{ref}}$  then it is controllable at  $t_0$  along  $\gamma_{\text{ref}}$ .*



**Figure 2.3.** A pictorial representation of flowing forward elements of  $L(\mathcal{A})$ , at each time  $\tau \in [t_0, t]$  to time  $t > t_0$ , by the linearization of the reference flow to construct the set in part (ii) of Theorem 2.7. The dashed lines correspond to  $L(\mathcal{A})$  at each point along  $\gamma_{\text{ref}}$ .

## Chapter 3

# LQR and the Maximum Principle

In this chapter the main geometric structure for Kalman’s theory of optimal control is presented by characterizing solutions to the finite time LQR problem using the Maximum Principle as stated in Theorem 3.2. Although Theorem 3.2 is a new formulation of the Maximum Principle, the ideas required to prove it come from the existing formulations of the Maximum Principle. Thus, many of the technicalities follow from the standard versions of the Maximum Principle found in [Lee and Markus 1986].

After providing the geometric versions of both the finite and infinite time LQR problems in Section 3.1, the bulk of this chapter builds the tools to prove Theorem 3.2 in Section 3.5. In Sections 3.2, 3.3, and 3.4, the variational and adjoint equations, needle variations, tangent cones, and, of course, the Hamiltonian are defined. Again, it is noted that the Hamiltonian presented in Section 3.4 will look “different” from the standard case but maintains the required maximization properties—see Lemmata 3.17 and 3.18—required to prove the Maximum Principle. In Section 3.6, the Maximum Principle is used to characterize solutions to the finite time LQR problem and the geometric version of the Riccati equation is given. In Section 3.7, solutions to the Riccati equation are shown to exist as the final time in the LQR problem tends to infinity. In arriving at this result, the geometric analogue of the minimum energy control from Section 1.2.2 is defined. Finally, the trajectory corresponding to the solution of the Riccati equation, as the final time tends to infinity, is shown to be optimal in the sense of the infinite time LQR problem.

### 3.1. LQR problem definitions

Let  $X_{\text{ref}}$  be a reference vector field, with a reference trajectory  $\gamma_{\text{ref}}$  defined on  $I$ , for an affine system  $\mathcal{A}$ . The linearization, a time-dependent affine system  $\mathcal{A}_{\text{ref}}^T$  on  $TM$ , is defined as in Section 2.2:

$$\mathcal{A}_{\text{ref},(t,v_x)}^T = \{X_{\text{ref}}^T(t, v_x) + \text{vlft}(X) \mid X \in L(\mathcal{A})_{(t,x)}\}.$$

To formulate an LQR problem for the linearization  $\mathcal{A}_{\text{ref}}^T$  requires the following constructions. For a vector bundle  $\pi: E \rightarrow B$ , we denote by  $\Sigma_2(E)$  be the bundle of symmetric  $(0, 2)$ -tensors on  $B$ . To define the cost function along the reference trajectory, let  $Q$  be an LI section of  $\Sigma_2(TM|_{\text{image}(\gamma_{\text{ref}})})$  with the property that  $Q(t)$  is positive-semidefinite for each  $t \in I$ . Also, let  $R$  be an LI section of  $\Sigma_2(L(\mathcal{A})|_{\text{image}(\gamma_{\text{ref}})})$  with the property that  $R(t)$  is positive-definite for each  $t \in I$ .

**3.1 Problem:** (i) (Finite time, fixed interval, free endpoint problem) Find a vector field  $\Upsilon: [t_0, t_1] \rightarrow TM$  along  $\gamma_{\text{ref}}$  such that  $\Upsilon \in \text{Traj}(\mathcal{A}_{\text{ref}}^T)$  and is a minimum of the cost

$$J(\Upsilon(t_0), t_0, t_1) = \frac{1}{2}F(t_1)(\Upsilon(t_1), \Upsilon(t_1)) + \int_{t_0}^{t_1} \frac{1}{2}Q(t)(\Upsilon(t), \Upsilon(t)) + \frac{1}{2}R(t)(X(t), X(t))dt,$$

where  $F(t_1)$  is a symmetric positive-semidefinite  $(0, 2)$ -tensor on  $T_{\gamma_{\text{ref}}(t_1)}M$ .

(ii) (Infinite time, free endpoint problem) Find a vector field  $\Upsilon: [t_0, \infty) \rightarrow TM$  along  $\gamma_{\text{ref}}$  such that  $\Upsilon \in \text{Traj}(\mathcal{A}_{\text{ref}}^T)$  and is a minimum of the cost

$$J(\Upsilon(t_0), t_0, \infty) = \int_{t_0}^{\infty} \frac{1}{2}Q(t)(\Upsilon(t), \Upsilon(t)) + \frac{1}{2}R(t)(X(t), X(t))dt.$$

The **Lagrangian** associated with Problem 3.1 is the map  $L: TM \times L(\mathcal{A}) \rightarrow \mathbb{R}$  defined by

$$L(\Upsilon(t), X(t)) = \frac{1}{2}Q(t)(\Upsilon(t), \Upsilon(t)) + \frac{1}{2}R(t)(X(t), X(t)). \quad (3.1.1)$$

Let  $\bar{L}: TM \rightarrow \mathbb{R}$  be a smooth map and define the **fiber derivative** as the map  $F\bar{L}: TM \rightarrow T^*M$  given by

$$\langle F\bar{L}(v_x); w_x \rangle = \left. \frac{d}{ds} \right|_{s=0} \bar{L}(v_x + sw_x).$$

In the natural coordinates for  $TM$  and  $T^*M$ , the local representative of the fibre derivative is given by

$$((x^1, \dots, x^n), (v^1, \dots, v^n)) \mapsto \left( (x^1, \dots, x^n), \left( \frac{\partial \bar{L}}{\partial v^1}, \dots, \frac{\partial \bar{L}}{\partial v^n} \right) \right).$$

For a fixed section  $X$  of  $L(\mathcal{A})$ , the restriction of the fiber derivative along  $\gamma_{\text{ref}}$  applied to (3.1.1) yields  $FL(\Upsilon(t)) = Q^b(t)\Upsilon(t)$ . In this chapter, Sections 3.2, 3.3, and 3.4 build the necessary background to prove the following theorem in Section 3.5. Let  $\iota(t): L(\mathcal{A})_{\gamma_{\text{ref}}(t)} \rightarrow T_{\gamma_{\text{ref}}(t)}M$  and  $\iota^*(t): T_{\gamma_{\text{ref}}(t)}^*M \rightarrow L(\mathcal{A})_{\gamma_{\text{ref}}(t)}^*$  be the natural inclusion maps.

**3.2 Theorem:** *Let  $\mathcal{A}$  be an affine system with reference vector field  $X_{\text{ref}}$  and a controllable linearization  $\mathcal{A}_{\text{ref}}^T$  along  $\gamma_{\text{ref}}$ . Let  $t_0, t_1 \in \mathbb{R}$  satisfy  $t_0 < t_1$  and let  $v_{\gamma_{\text{ref}}(t_0)} \in T_{\gamma_{\text{ref}}(t_0)}M$ . Suppose that  $(\Upsilon_*, X_*)$  solves Problem 3.1(i). Then there exists a covector field  $\Lambda_*: [t_0, t_1] \rightarrow T^*M$  along  $\gamma_{\text{ref}}$  such that the following equations on  $TM \oplus T^*M$  hold:*

$$\begin{aligned} \dot{\Upsilon}_*(t) &= X_{\text{ref}}^T(t, \Upsilon_*(t)) + \text{vlft}(R^\sharp(t)\iota^*(t)\Lambda_*(t))(\Upsilon(t)), \\ \dot{\Lambda}_*(t) &= (X_{\text{ref}}^{T*})(t, \Lambda_*(t)) - \text{vlft}(FL(\Upsilon_*(t)))(\Lambda_*(t)), \end{aligned}$$

where  $\Upsilon_*(t_0) = v_{\gamma_{\text{ref}}(t_0)}$  and  $\Lambda_*(t_1) = F^b(t_1)\Upsilon(t_1)$ .

In Section 3.6, Theorem 3.2 is used to characterize solutions to Problem 3.1(i). The infinite time problem, Problem 3.1(ii), is addressed in Section 3.7.

### 3.2. The variational and adjoint equations

In the standard theory of optimal control for nonlinear controls systems on a manifold  $M$ , the variational equations are given by the linearization of the dynamics. A trajectory of the variational equations is interpreted as an infinitesimal variation arising from varying the initial conditions of a fixed trajectory on  $M$ . In the present geometric framework, the varying of initial conditions for a trajectory for  $\mathcal{A}_{\text{ref}}^T$  corresponds to a variation in the fiber over  $\gamma_{\text{ref}}$  at the initial time. In other words, the trajectories of the variational equation will be vertical.

**3.3 Definition:** Let  $\mathcal{A}$  be an affine system with linearization  $\mathcal{A}_{\text{ref}}^T$  along  $\gamma_{\text{ref}}$  and reference vector field  $X_{\text{ref}}$ . The *variational equation* for  $\mathcal{A}_{\text{ref}}^T$  is

$$\begin{aligned}\dot{\Upsilon}(t) &= X_{\text{ref}}^T(t, \Upsilon(t)) + \text{vlft}(X(t))(\Upsilon(t)), \\ \dot{\Xi}(t) &= (X_{\text{ref}}^T)^T(t, \Xi(t)),\end{aligned}$$

where  $\Xi(t)$  is a vertical vector field along  $\Upsilon(t)$ .

The trajectories of the variational equations can be interpreted as infinitesimal variations in the following way. Fix a section  $X$  of  $L(\mathcal{A})$  along  $\gamma_{\text{ref}}$  and consider the variations of the resulting trajectory  $\Upsilon$ . An  $\mathcal{A}_{\text{ref}}^T$ -*variation* of  $\Upsilon$  is a map  $\Sigma: J \times I \rightarrow TM$  with the following properties:

1.  $J \subset \mathbb{R}$  for which  $0 \in \text{int}(J)$ ;
2.  $\Sigma$  is continuous;
3.  $\Sigma(s, t)$  is a trajectory of  $\mathcal{A}_{\text{ref}}^T$  for each  $s \in J$ ;
4.  $\Sigma(0, t) = \Upsilon(t)$ ;
5.  $\pi_{TM}\Sigma(s, t) = \gamma_{\text{ref}}$ ; and
6. the map  $I \ni t \mapsto \frac{d}{ds}\Big|_{s=0} \Sigma(s, t) \in TTM$  is LAC.

For an  $\mathcal{A}_{\text{ref}}^T$ -variation  $\Sigma$  of  $\Upsilon$ , a vector field  $V_\Sigma$  along  $\Upsilon$  is defined by

$$V_\Sigma(t) = \frac{d}{ds}\Big|_{s=0} \Sigma(s, t). \quad (3.2.1)$$

As a consequence of property 5, the vector field  $V_\Sigma(t)$  along  $\Upsilon$  is vertical. In the natural coordinates  $(x, v, u, w)$  for  $TTM$ ,  $V_\Sigma(t)$  is given by

$$\mathbf{V}_\Sigma(t) = (\gamma_{\text{ref}}(t), \Upsilon(t), 0, V_\Sigma(t)).$$



**3.4 Proposition:** *Let  $\mathcal{A}$  be an affine system on  $M$  with reference vector field  $X_{\text{ref}}$  and differentiable reference trajectory  $\gamma_{\text{ref}}$ . Let  $\Upsilon$  be a trajectory of  $\mathcal{A}_{\text{ref}}^T$ . For a vertical vector field  $\Xi$  along  $\Upsilon$ , the following statements are equivalent:*

- (i)  $\Xi$  is a integral curve of  $(X_{\text{ref}}^T)^T$ ;
- (ii) there exists an  $\mathcal{A}_{\text{ref}}^T$ -variation  $\Sigma$  of  $\Upsilon$  such that  $V_\Sigma = \Xi$ .

**Proof:** (ii)  $\Rightarrow$  (i) Let  $\Sigma$  be an  $\mathcal{A}_{\text{ref}}^T$ -variation  $\Sigma$  of  $\Upsilon$  for which  $V_\Sigma = \Xi$ . Since  $\Sigma(s, t)$  is a trajectory of  $\mathcal{A}_{\text{ref}}^T$  for each  $s \in J$ ,

$$\frac{d}{dt}\Sigma(s, t) = X_{\text{ref}}^T(t, \Sigma(s, t)) + \text{vlft}(X(t))(\Sigma(s, t)), \quad (3.2.2)$$

using a set of generators  $\{X_{\text{ref}}^T, \text{vlft}(X_1), \dots, \text{vlft}(X_k)\}$  for  $\mathcal{A}_{\text{ref}}^T$  where  $\{X_{\text{ref}}, X_1, \dots, X_k\}$  are generators for  $\mathcal{A}$ . Differentiating (3.2.2) in coordinates with respect to  $s$  at  $s = 0$  yields

$$\begin{aligned} \frac{d}{dt}\mathbf{V}_\Sigma(t) &= \left. \frac{d}{ds} \right|_{s=0} X_{\text{ref}}^T(t, \Sigma(s, t)) \\ &= (X_{\text{ref}}(t, \gamma_{\text{ref}}(t)), \mathbf{D}X_{\text{ref}}(t, \gamma_{\text{ref}}(t))(\Upsilon(t)), 0, \mathbf{D}X_{\text{ref}}(t, \gamma_{\text{ref}}(t))(V_\Sigma(t))). \end{aligned}$$

For  $w_{v_x} \in T_{v_x}T M$  the principle part of  $(X_{\text{ref}}^T)^T(t, w_{v_x})$  in the natural coordinates for  $T M$  is  $(X_{\text{ref}}(t, x), \mathbf{D}X_{\text{ref}}(t, x)v, \mathbf{D}X_{\text{ref}}(t, x)u, \mathbf{D}X_{\text{ref}}(t, x)w)$  which gives  $\frac{d}{dt}V_\Sigma(t) = (X_{\text{ref}}^T)^T(t, V_\Sigma(t))$  as desired.

(i)  $\Rightarrow$  (ii) This follows by choosing  $\Sigma$  to be the  $\mathcal{A}_{\text{ref}}^T$ -variation of  $\Upsilon$  such that

$$\left. \frac{d}{ds} \right|_{s=0} \Sigma(s, t_0) = \Xi(t_0). \quad \blacksquare$$

By making the natural identification of the fibers of the vertical subbundle of  $T M$  with the fibers of the tangent bundle of  $M$ , it is possible to view trajectories of the variational equation as vector fields along  $\gamma_{\text{ref}}$ . The end effect is that the curves  $t \mapsto \Upsilon(t) \oplus \Xi(t)$  in the Whitney sum  $T M \oplus T^* M$ , a vector bundle over  $M$ , are used to formulate the optimal control problem. The refined variational definitions are as follows.

**3.5 Definition:** The *variational equation* for  $\mathcal{A}_{\text{ref}}^T$  is

$$\begin{aligned} \dot{\Upsilon}(t) &= X_{\text{ref}}^T(t, \Upsilon(t)) + \text{vlft}(X(t))(\Upsilon(t)), \\ \dot{\Xi}(t) &= (X_{\text{ref}}^T)^T(t, \Xi(t)), \end{aligned}$$

where  $\Xi$  is a vector field along  $\gamma_{\text{ref}}$ .

Note that Definition 3.5 agrees with the statement ‘‘the linearization of a linear system is the original linear system’’. The upshot is that the adjoint equation will evolve on  $T M \oplus T^* M$ , which allows for the effect of the cost to be incorporated into the adjoint equations for the extended system; see Definition 3.10.

**3.6 Definition:** The *adjoint equation* for  $\mathcal{A}_{\text{ref}}^T$  is

$$\begin{aligned}\dot{\Upsilon}(t) &= X_{\text{ref}}^T(t, \Upsilon(t)) + \text{vlft}(X(t))(\Upsilon(t)), \\ \dot{\Lambda}(t) &= (X_{\text{ref}}^{T*})(t, \Lambda(t)),\end{aligned}$$

where  $\Lambda$  is a covector field along  $\gamma_{\text{ref}}$ .

The adjoint equations will play an important role in the statement and proof of the Maximum Principle. The relationship between the adjoint equations and variational equations is provided by Proposition 3.7.

**3.7 Proposition:** *If the curves  $t \mapsto \Upsilon(t) \oplus \Lambda(t)$  (resp.  $t \mapsto \Upsilon(t) \oplus \Xi(t)$ ) satisfies the adjoint equations (resp. variational equations), then the pairing  $\langle \Lambda(t); \Xi(t) \rangle$  is constant along  $\gamma_{\text{ref}}$  (i.e.  $\langle \Lambda(t); \Xi(t) \rangle = \langle \Lambda(t_0); \Xi(t_0) \rangle$ ).*

**Proof:** This follows from a direct computation of  $\frac{d}{dt} \langle \Lambda(t); \Xi(t) \rangle$  using the coordinate versions of the adjoint and variational equations.  $\blacksquare$

A geometric interpretation of the adjoint equations is that they describe the evolution of a hyperplane in  $TM$  along  $\gamma_{\text{ref}}$ .

**3.8 Corollary:** *Let  $\Lambda(t_0) \in T_{\gamma_{\text{ref}}(t_0)}^*M$  and let  $P_{t_0} = \text{coann}(\Lambda(t_0))$ . For each  $t \in [t_0, t_1]$  define  $P_t \subset T_{\gamma_{\text{ref}}(t)}M$  and  $\Lambda(t) \in T_{\gamma_{\text{ref}}(t)}^*M$  by asking that*

$$P_t = \{\Xi(t) \mid t \mapsto \Upsilon(t) \oplus \Xi(t) \text{ satisfies the variation equation with } \Xi(t_0) \in P_{t_0}\}$$

*and that  $t \mapsto \Upsilon(t) \oplus \Lambda(t)$  is a solution of the adjoint equations with initial condition  $\Lambda(t_0) \in T_{\gamma_{\text{ref}}(t_0)}^*M$ . Then  $P_t = \text{coann}(\Lambda(t))$  for every  $t \in [t_0, t_1]$ .*

### 3.3. Needle variations and tangent cones

Roughly speaking the tangent cone is constructed by pushing forward needle variations. Its properties are instrumental in proving the Maximum Principle. The key property of the tangent cone is convexity. The main role of the tangent cone is to approximate the reachable set and it is interpreted as the set of “directions” from which a trajectory can start. In the case of a linear system, both the reachable set  $\mathcal{R}(v_{\gamma_{\text{ref}}(t_0)}, t, t_0)$  and the tangent cone at time  $t$  are contained in the tangent space  $T_{\gamma_{\text{ref}}(t)}M$ . In fact, they are equal [Lee and Markus 1986]. However, since the proof of the Maximum Principle makes use of the extended system in Definition 3.9, which is not linear because of the cost being quadratic, this means that the general setup to construct the tangent cone is still required.

To prove the Maximum Principle, it is advantageous to include the cost as a state variable by defining the extended system.

**3.9 Definition:** The *extended system*, denoted by  $\hat{\mathcal{A}}_{\text{ref}}^T \subset TM \times \mathbb{R}$ , is defined by asking that a trajectory  $\hat{\Upsilon} = (\Upsilon, \Upsilon^0)$  satisfies the differential equations

$$\begin{aligned}\dot{\Upsilon}(t) &= X_{\text{ref}}^T(t, \Upsilon(t)) + \text{vlft}(X(t))(\Upsilon(t)), \\ \dot{\Upsilon}^0(t) &= L(\Upsilon(t), X(t)).\end{aligned}$$

The adjoint and variational equations can be obtained as before from the linearization of the extended system along a trajectory that projects to the reference trajectory. The effect of the cost enters the adjoint and variational equations using the fiber derivative of the Lagrangian.

**3.10 Definition:**

- (i) The
- extended adjoint equation*
- is defined by

$$\begin{aligned}\dot{\Upsilon}(t) &= X_{\text{ref}}^T(t, \Upsilon(t)) + \text{vlft}(X(t))(\Upsilon(t)), \\ \dot{\Upsilon}^0(t) &= L(\Upsilon(t), X(t)), \\ \dot{\Lambda}(t) &= (X_{\text{ref}}^{T*})(t, \Lambda(t)) + \Lambda^0(t) \text{vlft}(FL(\Upsilon(t)))\Lambda(t), \\ \dot{\Lambda}^0(t) &= 0,\end{aligned}$$

where  $\Lambda(t)$  is a covector field along  $\gamma_{\text{ref}}$ .

- (ii) The
- extended variational equation*
- is defined by

$$\begin{aligned}\dot{\Upsilon}(t) &= X_{\text{ref}}^T(t, \Upsilon(t)) + \text{vlft}(X(t))(\Upsilon(t)), \\ \dot{\Upsilon}^0(t) &= L(\Upsilon(t), X(t)), \\ \dot{\Xi}(t) &= (X_{\text{ref}}^T)(t, \Xi(t)), \\ \dot{\Xi}^0(t) &= FL(\Upsilon(t))\Xi(t),\end{aligned}$$

where  $\Xi$  is a vector field along  $\gamma_{\text{ref}}$ .

**3.3.1. Needle variations.** In this section needle variations for the extended system, Definition 3.9, are defined. The motivation for using needle variations versus some other variety of variations is that the constructions involving needle variations are enough to prove the Maximum Principle.

**3.11 Definition:** Let  $t_0, t_1 \in \mathbb{R}$  satisfy  $t_0 < t_1$ . Let  $\hat{\mathcal{A}}_{\text{ref}}^T$  be an extended system with initial conditions  $\hat{\Upsilon}(t_0)$  and  $X$  a section of  $L(\mathcal{A})$  along  $\gamma_{\text{ref}}$ .

- (i)
- Fixed interval needle variation data*
- is a triple
- $\theta = (\tau_\theta, \ell_\theta, Z_\theta)$
- , where

- (a)  $\tau_\theta \in (t_0, t_1]$ ,
- (b)  $\ell_\theta \in \mathbb{R}_{\geq 0}$ , and
- (c)  $Z_\theta$  is a section of  $L(\mathcal{A})$ .

- (ii) The
- variation of  $X$*
- associated to the fixed interval needle variation data
- $\theta = (\tau_\theta, \ell_\theta, Z_\theta)$
- is a map
- $X_\theta: J \times [t_0, t_1] \rightarrow L(\mathcal{A})$
- defined by

$$X_\theta(s, t) = \begin{cases} Z_\theta(t), & t \in [\tau_\theta - s\ell_\theta, \tau_\theta], \\ X(t), & \text{otherwise,} \end{cases} \quad (3.3.1)$$

where  $J = [0, s_0]$  is an interval sufficiently small such that  $X_\theta(s, \cdot): t \rightarrow X_\theta(s, t)$  is a section of  $L(\mathcal{A})$  along  $\gamma_{\text{ref}}$  for each  $s \in J$ .

- (iii) Let
- $t \mapsto \Sigma(X_\theta(s, t), \hat{\Upsilon}(t_0), t_0, t)$
- be the trajectory of
- $\hat{\mathcal{A}}_{\text{ref}}^T$
- corresponding to
- $X_\theta(s, \cdot)$
- with the fixed interval needle variation data
- $\theta = (\tau_\theta, \ell_\theta, Z_\theta)$
- . The
- fixed interval needle variation*
- associated with
- $X$
- is defined by

$$v_\theta = \left. \frac{d}{ds} \right|_{s=0} \Sigma(X_\theta(s, \cdot), \hat{\Upsilon}(t_0), t_0, \cdot) \quad (3.3.2)$$

and is a vertical curve in  $VTM \times \mathbb{R}$  which projects to  $\gamma_{\text{ref}}$ .

The existence of (3.3.2) is guaranteed when  $\tau_\theta$  is a Lebesgue point [Lee and Markus 1986, §4.1].

**3.12 Definition:** Let  $X_{\text{ref}}$  be an  $\text{LIC}^\infty$  reference vector field for  $\mathcal{A}$  with reference trajectory  $\gamma_{\text{ref}}$  and let  $X$  be an LI-section of  $L(\mathcal{A})$  along  $\gamma_{\text{ref}}$ . A point  $\tau \in I$  is a **Lebesgue point of  $\hat{\mathcal{A}}_{\text{ref}}^T$**  if in local coordinates

$$\lim_{s \rightarrow 0} \frac{1}{s} \int_{\tau-s}^{\tau} \|X^i(t) - X^i(\tau)\| dt = 0,$$

where  $i \in \{1, \dots, n\}$ .

It is noted that almost every  $\tau_\theta \in (t_0, t_1]$  is a Lebesgue point. For the fixed interval needle variation data  $\theta = (\tau_\theta, \ell_\theta, Z_\theta)$ , where  $\tau_\theta$  is a Lebesgue point, the fixed interval needle variation  $v_\theta$  has the form,

$$v_\theta(\tau_\theta) = \ell_\theta(\text{vlft}(Z_\theta(\tau_\theta) - X(\tau_\theta), L(\Upsilon(\tau_\theta), Z_\theta(\tau_\theta) - X(\tau_\theta)))), \quad (3.3.3)$$

[Lee and Markus 1986, §4.1]. In light of (3.3.3), fixed interval needle variations will be considered as elements of  $L(\mathcal{A}) \times \mathbb{R}$ . The set of fixed interval needle variations at Lebesgue points form a cone in  $L(\mathcal{A}) \times \mathbb{R}$ . More precisely, if  $v_\theta$  is a fixed interval needle variation with data  $\theta = (\tau_\theta, \ell_\theta, Z_\theta)$  and  $\lambda \in \mathbb{R}_{\geq 0}$ , then  $\lambda v_\theta$  is a fixed interval needle variation with data  $(\tau_\theta, \lambda \ell_\theta, Z_\theta)$ . Assigning the notation  $\lambda\theta = (\tau_\theta, \lambda \ell_\theta, Z_\theta)$  implies the relation  $v_{\lambda\theta} = \lambda v_\theta$ .

**3.3.2. Multi-needle variations.** The constructions of the previous section are now extended to allow for multiple variations of  $X$  to contribute to corresponding fixed interval needle variations.

**3.13 Definition:** Let  $t_0, t_1 \in \mathbb{R}$  satisfy  $t_0 < t_1$ . Let  $\hat{\mathcal{A}}_{\text{ref}}^T$  be an extended system with initial conditions  $\hat{\Upsilon}(t_0)$  and  $X$  a section of  $L(\mathcal{A})$ .

- (i) **Fixed interval multi-needle variation data** is a collection  $\Theta = \{\theta_1, \dots, \theta_k\}$  of fixed interval needle variation data  $\theta_j = (\tau_j, \ell_j, Z_j)$ ,  $j \in \{1, \dots, k\}$ , such that the times  $\tau_1, \dots, \tau_k$  are distinct Lebesgue points.
- (ii) The **variation of  $X$**  associated to the fixed interval multi-needle variation data  $\Theta = \{\theta_1, \dots, \theta_k\}$  is a map  $X_\Theta: J \times [t_0, t_1] \rightarrow L(\mathcal{A})$  defined by

$$X_\Theta(s, t) = \begin{cases} Z_j(t), & t \in [\tau_j - s\ell_j, \tau_j], j \in \{1, \dots, k\}, \\ X(t), & \text{otherwise,} \end{cases} \quad (3.3.4)$$

where  $J = [0, s_0]$  is an interval sufficiently small that  $X_\Theta(s, \cdot): t \rightarrow X_\Theta(s, t)$  is a section of  $L(\mathcal{A})$  along  $\gamma_{\text{ref}}$  for each  $s \in J$ .

- (iii) Let  $\Sigma(X_\Theta(s, \cdot), \hat{\Upsilon}(t_0), t_0, \cdot)$  be the trajectory corresponding to  $X_\Theta(s, \cdot)$  with the fixed interval multi-needle variation data  $\Theta = \{\theta_1, \dots, \theta_k\}$ . The **fixed interval multi-needle variation** associated with  $X$  is defined by

$$v_\Theta = \left. \frac{d}{ds} \right|_{s=0} \Sigma(X_\Theta(s, \cdot), \hat{\Upsilon}(t_0), t_0, \cdot) \quad (3.3.5)$$

and projects to  $\gamma_{\text{ref}}$ .

If  $\tau_j < t$ ,  $j \in \{1, \dots, k\}$ , associated with the fixed interval multi-needle variation data  $\Theta = \{\theta_1, \dots, \theta_k\}$ , then  $v_\Theta$  exists and is given by

$$v_\Theta(t) = \sum_{j=1}^k \Phi_{\tau_j, t} v_{\theta_j} \quad (3.3.6)$$

where  $v_{\theta_j}$  is the fixed interval needle variation for  $\theta_j$  and where  $\Phi_{\tau_j, t}$  is the flow, from  $\tau_j$  to  $t$ , of the linear part of the extended variational equation, Definition 3.10(ii), with fixed section  $X(t)$ , [Lee and Markus 1986, §4.1]. That is,  $\Phi_{\tau_j, t}$  is the flow corresponding to

$$\begin{aligned} \dot{\Xi}(t) &= (X_{\text{ref}}^T)(t, \Xi(t)), \\ \dot{\Xi}^0(t) &= FL(\Upsilon(t))\Xi(t). \end{aligned}$$

If  $\Theta = \{\theta_1, \dots, \theta_k\}$  is fixed interval multi-needle variation data and if  $\lambda = \{\lambda_1, \dots, \lambda_k\}$ ,  $\lambda_j \in \mathbb{R}_{\geq 0}$ , then denote  $\lambda\Theta = \{\lambda_1\theta_1, \dots, \lambda_k\theta_k\}$ . With this notation, a coned convex combination of fixed interval multi-needle variations takes the form

$$v_{\lambda\Theta}(t) = \sum_{j=1}^k \Phi_{\tau_j, t} \lambda_j v_{\theta_j}. \quad (3.3.7)$$

Furthermore, if  $\sum_{j=1}^k \lambda_j = 1$ , then the limit  $\frac{d}{ds}\big|_{s=0} \Sigma(X_{\lambda\Theta}(s, \cdot), \hat{\Upsilon}(t_0), t_0, \cdot)$  exists uniformly in  $\lambda$ .

### 3.3.3. Fixed interval tangent cones.

**3.14 Definition:** Let  $t_0, t_1 \in \mathbb{R}$  satisfy  $t_0 < t_1$ . Let  $\hat{\mathcal{A}}_{\text{ref}}^T$  be an extended system with initial conditions  $\hat{\Upsilon}(t_0)$  and  $X$  a section of  $L(\mathcal{A})$ . For  $t \in [t_0, t_1]$  define the **fixed interval tangent cone at  $t$** , denoted by  $K(X, \hat{\Upsilon}(t_0), t_0, t)$ , as the closure of the coned convex hull of the set

$$\bigcup \{ \Phi_{\tau, t} v \mid v \text{ is a fixed interval needle variation at a Lebesgue point } \tau \}. \quad (3.3.8)$$

The next lemma tells us that points in the interior of the fixed interval tangent cone are in the reachable set.

**3.15 Lemma:** ([Lee and Markus 1986, §4.1, Lemma 2]) Let  $t_0, t_1 \in \mathbb{R}$  satisfy  $t_0 < t_1$ . Let  $\hat{\mathcal{A}}_{\text{ref}}^T$  be an extended system with initial conditions  $\hat{\Upsilon}(t_0)$  and  $X$  a section of  $L(\mathcal{A})$ . If  $v_0 \in \text{int}(K(X, \hat{\Upsilon}(t_0), t_0, t))$  for  $t \in [t_0, t_1]$ , then there exists a cone  $K \subset K(X, \hat{\Upsilon}(t_0), t_0, t)$  such that

- (i)  $v_0 \in \text{int}(K)$  and
- (ii)  $\{ \Sigma(X(t), \hat{\Upsilon}(t_0), t_0, t) + \epsilon v \mid v \in K, \} \subset \mathcal{R}_{\hat{\mathcal{A}}_{\text{ref}}^T}(\hat{\Upsilon}(t_0), t, t_0)$  where  $\epsilon \geq 0$  is sufficiently small.

## 3.4. The Hamiltonian

The next required ingredient is a Hamiltonian. The Hamiltonian,  $H_{\hat{\mathcal{A}}_{\text{ref}}^T}$ , defined below differs from that of the general setup because of the previous identification of the fibers

of the vertical subbundle of  $TTM$  with the fibers of  $TM$ . Note that since there is not a natural notion of exterior differentiation along a curve, this Hamiltonian does not give rise to Hamilton's equations. However, the following Hamiltonian does maintain the required maximization properties, see Lemmata 3.17 and 3.18, required to prove the Maximum Principle. Also, the Hamilton–Jacobi equations, using the analogue of  $H_{\mathcal{A}_{\text{ref}}^T, \Lambda^0 L}$  in the standard setup, provide the desired value/form of the optimal cost and its relationship with the Riccati equation.

**3.16 Definition:** Let  $\mathcal{A}$  be an affine system with linearization  $\mathcal{A}_{\text{ref}}^T$  along  $\gamma_{\text{ref}}$ . Let  $L: TM \times L(\mathcal{A}) \rightarrow \mathbb{R}$  be the Lagrangian defined by (3.1.1) and let  $\hat{\mathcal{A}}_{\text{ref}}^T$  be the extended system. Let  $\iota^*: T^*M \rightarrow L(\mathcal{A})^*$  be the dual of the inclusion map of  $L(\mathcal{A})$  into  $TM$ .

(i) The **Hamiltonian** is a function  $H_{\mathcal{A}_{\text{ref}}^T}: (TM \oplus T^*M) \times L(\mathcal{A}) \rightarrow \mathbb{R}$  given by

$$H_{\mathcal{A}_{\text{ref}}^T}(\Upsilon(t) \oplus \Lambda(t), X(t)) = \langle \iota^*(t)\Lambda(t); X(t) \rangle.$$

(ii) The **maximum Hamiltonian** is a function  $H_{\mathcal{A}_{\text{ref}}^T}^{\max}: TM \oplus T^*M \rightarrow \mathbb{R}$  given by

$$H_{\mathcal{A}_{\text{ref}}^T}^{\max}(\Upsilon(t) \oplus \Lambda(t)) = \sup\{H_{\mathcal{A}_{\text{ref}}^T}(\Upsilon(t) \oplus \Lambda(t), X(t)) \mid X(t) \in L(\mathcal{A})_{t, \gamma_{\text{ref}}(t)}\}.$$

(iii) The **extended Hamiltonian** is a function  $H_{\mathcal{A}_{\text{ref}}^T, \Lambda^0 L}: ((TM \times \mathbb{R}) \oplus (T^*M \times \mathbb{R})) \times L(\mathcal{A}) \rightarrow \mathbb{R}$  defined by

$$H_{\mathcal{A}_{\text{ref}}^T, \Lambda^0 L}(\hat{\Upsilon}(t) \oplus \hat{\Lambda}(t), X(t)) = \langle \iota^*(t)\Lambda(t); X(t) \rangle + \Lambda^0(t)L(\Upsilon(t), X(t)),$$

where  $\hat{\Lambda} = (\Lambda, \Lambda^0)$  satisfies the extended adjoint equations.

(iv) The **extended maximum Hamiltonian** is a function  $H_{\mathcal{A}_{\text{ref}}^T, \Lambda^0 L}^{\max}: (TM \times \mathbb{R}) \oplus (T^*M \times \mathbb{R}) \rightarrow \mathbb{R}$  defined by

$$H_{\mathcal{A}_{\text{ref}}^T, \Lambda^0 L}^{\max}(\hat{\Upsilon}(t) \oplus \hat{\Lambda}(t)) = \sup\{H_{\mathcal{A}_{\text{ref}}^T, \Lambda^0 L}(\hat{\Upsilon}(t) \oplus \hat{\Lambda}(t), X(t)) \mid X(t) \in L(\mathcal{A})_{t, \gamma_{\text{ref}}(t)}\},$$

where  $\hat{\Lambda} = (\Lambda, \Lambda^0)$  satisfies the extended adjoint equations.

The following lemmata provide a relationship between the Hamiltonian and the tangent cones of Section 3.3. It is interesting to note that these maximization statements only involve properties of tangent cones and do not rely on the optimal control problem data and, although they are stated for the extended system, they hold for general non-linear systems.

**3.17 Lemma: (Hamiltonian maximization property)** *Let  $\mathcal{A}$  be an affine system with linearization  $\mathcal{A}_{\text{ref}}^T$  along  $\gamma_{\text{ref}}$  and extended system  $\hat{\mathcal{A}}_{\text{ref}}^T$ . Let  $t_0, t_1 \in \mathbb{R}$  satisfy  $t_0 < t_1$  and let  $(\hat{\Upsilon}(t) \oplus \hat{\Lambda}(t), \bar{X}(t)) \in ((TM \times \mathbb{R}) \oplus (T^*M \times \mathbb{R})) \times L(\mathcal{A})$ . Then  $H_{\hat{\mathcal{A}}_{\text{ref}}^T}(\hat{\Upsilon}(t) \oplus \hat{\Lambda}(t), \bar{X}(t)) = H_{\mathcal{A}_{\text{ref}}^T}^{\max}(\hat{\Upsilon}(t) \oplus \hat{\Lambda}(t))$  if and only if,*

$$\langle (\iota^*(t)\Lambda(t), \Lambda^0(t)); v(t) \rangle \leq 0_{\gamma_{\text{ref}}(t)},$$

where  $v(t) \in \{(X(t) - \bar{X}(t), L(\Upsilon(t), X(t) - \bar{X}(t))) \mid X(t) \in L(\mathcal{A})_{t, \gamma_{\text{ref}}(t)}\}$ .

**Proof:** Let  $X$  be a section of  $L(\mathcal{A})$  along  $\gamma_{\text{ref}}$ .

$$\begin{aligned}
H_{\hat{\mathcal{A}}_{\text{ref}}^T}(\hat{\Upsilon}(t) \oplus \hat{\Lambda}(t), \bar{X}(t)) &= H_{\hat{\mathcal{A}}_{\text{ref}}^T}^{\max}(\hat{\Upsilon}(t) \oplus \hat{\Lambda}(t)) \\
\iff H_{\hat{\mathcal{A}}_{\text{ref}}^T}(\hat{\Upsilon}(t) \oplus \hat{\Lambda}(t), \bar{X}(t)) &\geq H_{\hat{\mathcal{A}}_{\text{ref}}^T}(\hat{\Upsilon}(t) \oplus \hat{\Lambda}(t), X(t)), \\
\iff H_{\hat{\mathcal{A}}_{\text{ref}}^T}(\hat{\Upsilon}(t) \oplus \hat{\Lambda}(t), \bar{X}(t)) - H_{\hat{\mathcal{A}}_{\text{ref}}^T}(\hat{\Upsilon}(t) \oplus \hat{\Lambda}(t), X(t)) &\geq 0_{\gamma_{\text{ref}}(t)}, \\
\iff \langle \iota^*(t)\Lambda(t); X(t) - \bar{X}(t) \rangle + \Lambda^0(t)L(\Upsilon(t), X(t) - \bar{X}(t)) &\leq 0_{\gamma_{\text{ref}}(t)}, \\
\iff \langle (\iota^*(t)\Lambda(t), \Lambda^0(t)); v(t) \rangle &\leq 0_{\gamma_{\text{ref}}(t)},
\end{aligned}$$

where  $v(t) \in \{(X(t) - \bar{X}(t), L(\Upsilon(t), X(t) - \bar{X}(t))) \mid X(t) \in L(\mathcal{A})_{t, \gamma_{\text{ref}}(t)}\}$ .  $\blacksquare$

### 3.18 Lemma: (Hamiltonian maximization and the fixed interval tangent cone)

Let  $\mathcal{A}$  be an affine system with linearization  $\mathcal{A}_{\text{ref}}^T$  along  $\gamma_{\text{ref}}$  and extended system  $\hat{\mathcal{A}}_{\text{ref}}^T$ . Let  $t_0, t_1 \in \mathbb{R}$  satisfy  $t_0 < t_1$  and let  $X(t) \in L(\mathcal{A})_{t, \gamma_{\text{ref}}(t)}$ . For each  $t \in [t_0, t_1]$  let  $\kappa_t$  be a convex cone in  $L(\mathcal{A})_{t, \gamma_{\text{ref}}(t)}$  containing  $K(X(t), \hat{\Upsilon}(t_0), t_0, t)$  and suppose, for some time  $\tau \in [t_0, t_1]$ , that there exists a covector  $(\Lambda(\tau), \Lambda^0(\tau)) \in T_{\gamma_{\text{ref}}(\tau)}^*M \times \mathbb{R}$  such that

$$\langle (\iota^*(t)\Lambda(\tau), \Lambda^0(\tau)); v_\tau \rangle \leq 0_{\gamma_{\text{ref}}(\tau)}, \quad v_\tau \in \kappa_\tau.$$

Let  $t \mapsto \hat{\Upsilon}(t) \oplus \hat{\Lambda}(t)$  be a solution to the extended adjoint equation for  $\hat{\mathcal{A}}_{\text{ref}}^T$  along  $\gamma_{\text{ref}}$  with the above property at time  $\tau$ . Then, for almost every  $t \in [t_0, \tau]$ ,

$$H_{\hat{\mathcal{A}}_{\text{ref}}^T}(\hat{\Upsilon}(t) \oplus \hat{\Lambda}(t), X(t)) = H_{\hat{\mathcal{A}}_{\text{ref}}^T}^{\max}(\hat{\Upsilon}(t) \oplus \hat{\Lambda}(t)).$$

**Proof:** Let  $\chi_t \in L(\mathcal{A})_{\gamma_{\text{ref}}(t), t}$ . Then, by definition of the fixed interval tangent cone,

$$\Phi_{t, \tau}(\chi_t - X(t), L(\Upsilon, \chi_t - X(t))) \in K(X(\tau), \hat{\Upsilon}(t_0), t_0, \tau) \subset \kappa_\tau.$$

By hypothesis,

$$\begin{aligned}
&\langle (\iota^*(t)\Lambda(\tau), \Lambda^0(\tau)); \Phi_{t, \tau}(\chi_t, L(\Upsilon(t), \chi_t)) \rangle - \\
&\quad \langle (\iota^*(t)\Lambda(\tau), \Lambda^0(\tau)); \Phi_{t, \tau}(X(t), L(\Upsilon(t), X(t))) \rangle \leq 0.
\end{aligned}$$

Now use the definition of the adjoint equations to obtain

$$\langle (\iota^*(t)\Lambda(t), \Lambda^0(t)); (\chi_t - X(t), L(\Upsilon(t), \chi_t - X(t))) \rangle \leq 0.$$

Since this holds for every  $\chi_t \in L(\mathcal{A})_{t, \gamma_{\text{ref}}(t)}$ , the lemma follows by Lemma 3.17.  $\blacksquare$

## 3.5. Proof of Theorem 3.2

With the constructions of Sections 3.2, 3.3, and 3.4 for the extend system, the proof of the Maximum Principle is as follows. Note that  $\hat{\mathcal{R}}_{\hat{\mathcal{A}}_{\text{ref}}^T}(v_x, t_0, t_1)$  denotes the reachable set from  $v_x$  for the extend system and is defined in a similar manner to  $\mathcal{R}_{\mathcal{A}_{\text{ref}}^T}(v_x, t_0, t_1)$ .

**Theorem 3.2:** *Let  $\mathcal{A}$  be an affine system with a controllable linearisation  $\mathcal{A}_{\text{ref}}^T$  along  $\gamma_{\text{ref}}$  and reference vector field  $X_{\text{ref}}$ . Let  $t_0, t_1 \in \mathbb{R}$  satisfy  $t_0 < t_1$  and let  $v_{\gamma_{\text{ref}}(t_0)} \in T_{\gamma_{\text{ref}}(t_0)}M$ . Suppose that  $(\Upsilon_*, X_*)$  solves Problem 3.1(i). Then there exists a covector field  $\Lambda_*: [t_0, t_1] \rightarrow T^*M$  along  $\gamma_{\text{ref}}$  such that the following equations on  $TM \oplus T^*M$  hold:*

$$\begin{aligned}\dot{\Upsilon}_*(t) &= X_{\text{ref}}^T(t, \Upsilon_*(t)) + \text{vlft}(R^\sharp(t)\iota^*(t)\Lambda_*(t))(\Upsilon(t)), \\ \dot{\Lambda}_*(t) &= (X_{\text{ref}}^{T*})(t, \Lambda_*(t)) - \text{vlft}(FL(\Upsilon_*(t)))\Lambda_*(t),\end{aligned}$$

where  $\Upsilon_*(t_0) = v_{\gamma_{\text{ref}}(t_0)}$  and  $\Lambda_*(t_1) = F^\flat(t_1)\Upsilon(t_1)$ .

**Proof:** Let  $(\Upsilon_*, X_*)$  be an optimal trajectory for Problem 3.1(i). The proof relies on the construction of a hyperplane which is used to define the final condition for the extended adjoint equations.

**1 Lemma:**  $(0_{\gamma_{\text{ref}}(t_1)}, -1) \in T_{\gamma_{\text{ref}}(t_1)}M \times \mathbb{R}$  is not in the interior of  $K(X_*, \hat{\Upsilon}(t_0), t_0, t_1)$ .

**Proof:** At each time  $t \in [t_0, t_1]$  the reachable set at time  $t$  for the extended system is contained in  $T_{\gamma_{\text{ref}}(t)}M \times \mathbb{R}$ , as is the fixed interval tangent cone. To prove the lemma, the important non-trivial fact that the fixed interval tangent cone is contained in the reachable set is utilized, Lemma 3.15. Suppose that  $(0_{\gamma_{\text{ref}}(t_1)}, -1) \in \text{int}(K(X_*, \hat{\Upsilon}(t_0), t_0, t_1))$ . Then  $(0_{\gamma_{\text{ref}}(t_1)}, -1) \in \hat{\mathcal{R}}_{\hat{\mathcal{A}}_{\text{ref}}^T}(v_{\gamma_{\text{ref}}(t_0)}, t_0, t_1)$ . This implies there are points in  $\hat{\mathcal{R}}_{\hat{\mathcal{A}}_{\text{ref}}^T}(v_{\gamma_{\text{ref}}(t_0)}, t_0, t_1)$  whose final cost are lower than  $\Upsilon_*^0(t_1)$  which contradicts the hypothesis that  $(\Upsilon_*, X_*)$  is optimal.  $\blacktriangledown$

Since  $K(X_*, \hat{\Upsilon}(t_0), t_0, t_1)$  is convex and  $(0_{\gamma_{\text{ref}}(t_1)}, -1) \notin \text{int}(K(X_*, \hat{\Upsilon}(t_0), t_0, t_1))$ , there exist a hyperplane  $\hat{P}(t_1)$  separating  $K(X_*, \hat{\Upsilon}(t_0), t_0, t_1)$  and  $(0_{\gamma_{\text{ref}}(t_1)}, -1)$  [Lee and Markus 1986]. Let  $\hat{\Lambda}_*(t_1) \in \text{ann}(\hat{P}(t_1))$  and note that

$$\begin{aligned}\langle \hat{\Lambda}_*(t_1); (0_{\gamma_{\text{ref}}(t_1)}, -1) \rangle &\geq 0, \\ \langle \hat{\Lambda}_*(t_1); \hat{v}_{\gamma_{\text{ref}}(t_1)} \rangle &\leq 0, \quad \hat{v}_{\gamma_{\text{ref}}(t_1)} \in K(X_*, \hat{v}_{\gamma_{\text{ref}}(t_0)}, t_0, t_1).\end{aligned}$$

The first expression implies that  $\Lambda_*^0(t_1) \leq 0$ . Let  $\hat{\Lambda}_*$  be the adjoint response with final condition  $\hat{\Lambda}_*(t_1)$  which implies that  $\Lambda_*^0(t)$  is also constant. If  $\Lambda_*^0 \neq 0$  then, without loss of generality, set  $\Lambda_*^0 = -1$  by redefining  $\hat{\Lambda}_*$  as  $-(\Lambda_*^0)^{-1}\hat{\Lambda}_*$ . The linearity of the adjoint equation and a non-zero initial condition implies that  $\hat{\Lambda}_* \neq (0_{\gamma_{\text{ref}}(t)}, 0)$  for every  $t \in [t_0, t_1]$  which implies that either  $\Lambda_*^0 = -1$  or  $\Lambda_*(t_0) \neq 0_{\gamma_{\text{ref}}(t_0)}$ .

The next step is to prove by contradiction, using the controllability assumption, that  $\Lambda_*^0 \neq 0$ . Suppose that  $\Lambda_*^0 = 0$ . Then  $(\Lambda_*, 0)$  satisfies the adjoint equations:

$$\begin{aligned}\dot{\Upsilon}(t) &= X_{\text{ref}}^T(t, \Upsilon(t)) + \text{vlft}(X(t))(\Upsilon(t)), \\ \dot{\Upsilon}^0(t) &= L(\Upsilon(t), X(t)), \\ \dot{\Lambda}(t) &= (X_{\text{ref}}^{T*})(t, \Lambda(t)), \\ \dot{\Lambda}^0(t) &= 0,\end{aligned}$$

where  $\Lambda(t)$  is a covector field along  $\gamma_{\text{ref}}$ . The Hamiltonian for the extended system becomes

$$H_{\hat{\mathcal{A}}_{\text{ref}}^T, 0L}(\hat{\Upsilon}_*(t) \oplus \hat{\Lambda}_*(t), X_*(t)) = \langle \iota^*(t)\Lambda_*(t); X_*(t) \rangle.$$



The function  $H_{\hat{A}_{\text{ref}}^T, 0L}$  is a linear function of  $X_*$  and is maximized if  $\iota^*(t)\Lambda_*(t) = 0$ . Thus  $\Lambda_*(t) \in \text{ann}(L(\mathcal{A})_{t, \gamma_{\text{ref}}(t)})$  for all  $t \in [t_0, t_1]$ . The controllability hypothesis then implies that  $\Lambda_*(t) = 0_{\gamma_{\text{ref}}(t)}$  for all  $t > t_0$ . Since  $\Lambda_*(t)$  satisfies the linear differential equation  $\dot{\Lambda}(t) = (X_{\text{ref}}^{T*})(t, \Lambda(t))$ , it follows that  $\Lambda_*(t_0) = 0_{\gamma_{\text{ref}}(t_0)}$ , which contradicts the non-triviality condition that either  $\Lambda_*^0 = -1$  or  $\Lambda_*(t_0) \neq 0_{\gamma_{\text{ref}}(t_0)}$ .

Assume  $\Lambda_*^0 = -1$ . By Lemma 3.18,  $\hat{\Lambda}_*$  maximizes the Hamiltonian  $H_{\hat{A}_{\text{ref}}^T, \Lambda^0 L}$ . Since  $H_{\hat{A}_{\text{ref}}^T, \Lambda^0 L}$  is a quadratic function of  $X$  and  $R(t)$  is positive-definite, the Hamiltonian is maximized if

$$\iota^*(t)\Lambda_*(t) + \Lambda_*^0 R^\flat(t)X_*(t) = 0.$$

The above equation can be solved for the optimal control in terms of the costate variable to obtain

$$X_*(t) = -\frac{1}{\Lambda_*^0} R^\sharp(t)\iota^*(t)\Lambda_*(t).$$

The proof now follows from the form of the extended adjoint equations.  $\blacksquare$

### 3.6. Characterizing the finite time LQR problem

The next theorem characterizes solutions to Problem 3.1(i) using Theorem 3.2 and introduces the geometric version of the differential Riccati equation.

**3.19 Theorem:** *The following are equivalent:*

- (i) *there exists a unique solution  $K$ , a symmetric  $(0, 2)$ -tensor along  $\gamma_{\text{ref}}$  such that  $K(t)$  is positive-definite for each  $t \in [t_0, t_1]$ , to the **Riccati equation***

$$\mathcal{L}^{X_{\text{ref}}, \gamma_{\text{ref}}} K(t) - K^\flat(t)\iota(t)R^\sharp(t)\iota^*(t)K^\flat(t) + Q(t) = 0, \quad K(t_1) = F(t_1),$$

for each  $t \in [t_0, t_1]$ ;

- (ii) *there exists a solution to Problem 3.1(i);*

- (iii) *the pair  $(\Upsilon(t), \Lambda(t))$  satisfy*

$$\begin{aligned} \dot{\Upsilon}(t) &= X_{\text{ref}}^T(t, \Upsilon(t)) + \text{vlft}(R^\sharp(t)\iota^*(t)\Lambda(t))(\Upsilon(t)), \\ \dot{\Lambda}(t) &= (X_{\text{ref}}^{T*})(t, \Lambda(t)) - \text{vlft}(FL(\Upsilon(t)))\Lambda(t), \end{aligned}$$

where  $\Upsilon(t_0) = v_{\gamma_{\text{ref}}(t_0)}$  and  $\Lambda(t_1) = F^\flat(t_1)\Upsilon(t_1)$ .

**Proof:** (i)  $\Rightarrow$  (ii): Let  $\Upsilon \in \text{Traj}(A_{\text{ref}}^T, t_1, t_0)$  and let  $K$  be a symmetric  $(0, 2)$ -tensor field along  $\gamma_{\text{ref}}$  such that  $K(t)$  is positive-definite for each  $t \in [t_0, t_1]$  and that  $K$  satisfies the Riccati equation with final condition  $K(t_1) = F(t_1)$ . Note that  $K^\flat(t): T_{\gamma_{\text{ref}}(t)}M \rightarrow T_{\gamma_{\text{ref}}(t)}^*M$  is a linear map. Now integrate both sides of

$$\frac{1}{2} \frac{d}{dt} \langle K^\flat(t)\Upsilon(t); \Upsilon(t) \rangle = \frac{1}{2} \left( \langle \mathcal{L}^{X_{\text{ref}}, \gamma_{\text{ref}}}(K^\flat(t)\Upsilon(t)); \Upsilon(t) \rangle + \langle K^\flat(t)\Upsilon(t); \mathcal{L}^{X_{\text{ref}}, \gamma_{\text{ref}}}(\Upsilon(t)) \rangle \right),$$

over the interval  $[t_0, t_1]$  and add the result to the cost to obtain

$$\begin{aligned} J(\Upsilon(t_0), t_0, t_1) + \frac{1}{2} \langle K^b(t_1) \Upsilon(t_1); \Upsilon(t_1) \rangle - \frac{1}{2} \langle K^b(t_0) \Upsilon(t_0); \Upsilon(t_0) \rangle = \\ \frac{1}{2} \int_{t_0}^{t_1} Q(t)(\Upsilon(t), \Upsilon(t)) + R(t)(X, X) + \langle \mathcal{L}^{X_{\text{ref}}, \gamma_{\text{ref}}}(K^b(t) \Upsilon(t)); \Upsilon(t) \rangle + \\ \langle K^b(t) \Upsilon(t); \mathcal{L}^{X_{\text{ref}}, \gamma_{\text{ref}}}(\Upsilon(t)) \rangle dt + \frac{1}{2} F(t_1)(\Upsilon(t_1), \Upsilon(t_1)). \end{aligned}$$

Now in local coordinates—for clarity the time dependence is no longer indicated—the right hand side is

$$\begin{aligned} & \frac{1}{2} \int_{t_0}^{t_1} Q_{ij} \Upsilon^i \Upsilon^j + R_{ij} X^i X^j + \left( \frac{d}{dt} (K_{ij} \Upsilon^i) + \frac{\partial X_{\text{ref}}^\ell}{\partial x^j} K_{i\ell} \Upsilon^i \right) \Upsilon^j + K_{ij} \Upsilon^i \iota(t) X^j dt \\ &= \frac{1}{2} \int_{t_0}^{t_1} Q_{ij} \Upsilon^i \Upsilon^j + R_{ij} X^i X^j + \frac{dK_{ij}}{dt} \Upsilon^i \Upsilon^j + K_{ij} \frac{\partial X_{\text{ref}}^i}{\partial x^\ell} \Upsilon^\ell \Upsilon^j + \\ & \quad K_{ij} \iota(t) X^i \Upsilon^j \frac{\partial X_{\text{ref}}^\ell}{\partial x^j} K_{i\ell} \Upsilon^i \Upsilon^j + K_{ij} \Upsilon^i \iota(t) X^j dt \\ &= \frac{1}{2} \int_{t_0}^{t_1} R_{ij} (X^i + R^{is} \iota^*(t) K_{s\ell} \Upsilon^\ell) (X^j + R^{js} \iota^*(t) K_{s\ell} \Upsilon^\ell) + \\ & \quad \left( \frac{dK_{ij}}{dt} + K_{sj} \frac{\partial X_{\text{ref}}^s}{\partial x^i} + \frac{\partial X_{\text{ref}}^\ell}{\partial x^j} K_{i\ell} - K_{\ell i} \iota(t) R^{\ell s} \iota^*(t) K_{sj} + Q_{ij} \right) \Upsilon^i \Upsilon^j dt. \end{aligned}$$

Using the hypothesis, the cost becomes

$$J(\Upsilon(t_0), t_0, t_1) = \frac{1}{2} \langle K^b(t_0) \Upsilon(t_0); \Upsilon(t_0) \rangle + \frac{1}{2} \int_{t_0}^{t_1} \|X(t) + R^\sharp(t) \iota^*(t) K^b(t) \Upsilon(t)\|_{R(t)}^2 dt$$

and the cost is then minimized by choosing a trajectory  $\Upsilon$  such that

$$X(t) = -R^\sharp(t) \iota^*(t) K^b(t) \Upsilon(t). \quad (3.6.1)$$

(ii)  $\Rightarrow$  (iii): By the Maximum Principle, Theorem 3.2, this follows.

(iii)  $\Rightarrow$  (i): The following lemma will aid in the construction of a solution to the Riccati equation.

**1 Lemma:** *Let  $\Upsilon: I \rightarrow TM$  and  $\Lambda: I \rightarrow T^*M$  be vector and covector fields along  $\gamma_{\text{ref}}$ , respectively. Consider the following statements:*

(i) *the pair  $(\Upsilon, \Lambda)$  satisfy*

$$\begin{aligned} \dot{\Upsilon}(t) &= X_{\text{ref}}^T(t, \Upsilon(t)) + \text{vlft}(R^\sharp(t) \iota^*(t) \Lambda(t))(\Upsilon(t)), \\ \dot{\Lambda}(t) &= (X_{\text{ref}}^{T*})(t, \Lambda(t)) - \text{vlft}(FL(\Upsilon(t)))\Lambda(t), \end{aligned}$$

where  $\Upsilon(t_0) = v_{\gamma_{\text{ref}}(t_0)}$  and  $\Lambda(t_1) = F^b(t_1) \Upsilon(t_1)$ ;

(ii)  $\Upsilon(t) = K_1(t) \Phi_{t_1, t}^{X_{\text{ref}}^T}(\Upsilon(t_1))$ ,  $\Lambda(t) = K_2(t) \Phi_{t_1, t}^{X_{\text{ref}}^T}(\Upsilon(t_1))$  where  $K_1(t): T_{\gamma_{\text{ref}}(t)} M \rightarrow T_{\gamma_{\text{ref}}(t)} M$  and  $K_2(t): T_{\gamma_{\text{ref}}(t)} M \rightarrow T_{\gamma_{\text{ref}}(t)}^* M$  are linear maps for each  $t \in [t_0, t_1]$  and satisfy

$$\begin{aligned} \mathcal{L}^{X_{\text{ref}}, \gamma_{\text{ref}}} K_1(t) &= \iota(t) R^\sharp(t) \iota^*(t) K_2(t), \\ \mathcal{L}^{X_{\text{ref}}, \gamma_{\text{ref}}} K_2(t) &= Q^b(t) K_1(t), \end{aligned}$$

where  $K_1$  is invertible,  $K_1(t_1) = Id_{TM}$ , and  $K_2(t_1) = F(t_1)$ .

Then (i)  $\Rightarrow$  (ii).

Proof: Note that  $FL(\Upsilon(t)) = Q^b(t)\Upsilon(t)$ . Assuming (i), let  $(\Upsilon, \Lambda)$  be the pair satisfying

$$\begin{aligned}\mathcal{L}^{X_{\text{ref}}, \gamma_{\text{ref}}}(\Upsilon(t)) &= \iota(t)R^\sharp(t)\iota^*(t)\Lambda(t), \\ \mathcal{L}^{X_{\text{ref}}, \gamma_{\text{ref}}}(\Lambda(t)) &= -Q^b(t)\Upsilon(t),\end{aligned}$$

with initial conditions  $\Upsilon(t_0) = v_{\gamma_{\text{ref}}(t_0)}$  and  $\Lambda(t_1) = F^b(t_1)\Upsilon(t_1)$ . Let  $\eta(t) = \Phi_{t_1, t}^{X_{\text{ref}}^T}(\Upsilon(t_1))$ . That is,  $\eta$  is the integral curve of  $X_{\text{ref}}^T$  such that  $\eta(t_1) = \Upsilon(t_1)$ . Next define the pair  $(\hat{\Upsilon}(t), \hat{\Lambda}(t))$  by  $\hat{\Upsilon}(t) = K_1(t)\eta(t)$  and  $\hat{\Lambda}(t) = K_2(t)\eta(t)$ , respectively. The coordinate calculations

$$\begin{aligned}(\mathcal{L}^{X_{\text{ref}}, \gamma_{\text{ref}}}(\hat{\Upsilon}(t)))^i &= (\mathcal{L}^{X_{\text{ref}}, \gamma_{\text{ref}}}(K_1(t)\eta(t)))^i \\ &= \left( \frac{d}{dt}K_{1,\ell}^i(t) \right) \eta^\ell(t) + K_{1,s}^i(t) \frac{\partial X_{\text{ref}}^s}{\partial x^\ell}(\eta(t)) \eta^\ell(t) - \frac{\partial X_{\text{ref}}^i}{\partial x^j}(\eta(t)) K_{1,\ell}^j(t) \eta^\ell(t) \\ &= \frac{\partial K_{1,\ell}^i(t)}{\partial x^s} X_{\text{ref}}^s(\eta(t)) \eta^\ell(t) + K_{1,s}^i(t) \frac{\partial X_{\text{ref}}^s}{\partial x^\ell}(\eta(t)) \eta^\ell(t) - \\ &\quad \frac{\partial X_{\text{ref}}^i}{\partial x^j}(\eta(t)) K_{1,\ell}^j(t) \eta^\ell(t) \\ &= (\mathcal{L}^{X_{\text{ref}}, \gamma_{\text{ref}}}(K_1(t)\eta(t)))^i = (\iota(t)R^\sharp(t)\iota^*(t)\hat{\Lambda}(t))^i,\end{aligned}$$

and similarly

$$\begin{aligned}(\mathcal{L}^{X_{\text{ref}}, \gamma_{\text{ref}}}(\hat{\Lambda}(t)))^i &= (\mathcal{L}^{X_{\text{ref}}, \gamma_{\text{ref}}}(K_2(t)\eta(t)))^i \\ &= (\mathcal{L}^{X_{\text{ref}}, \gamma_{\text{ref}}}(K_2(t)\eta(t)))^i = -(Q^b(t)\hat{\Upsilon}(t))^i,\end{aligned}$$

show that  $(\hat{\Upsilon}(t), \hat{\Lambda}(t))$  and  $(\Upsilon(t), \Lambda(t))$  satisfy the same differential equation. Since  $K_1(t_1) = \text{id}_{TM}$ , and  $K_2(t_1) = F^b(t_1)$ , it follows that  $(\hat{\Upsilon}(t_1), \hat{\Lambda}(t_1)) = (\Upsilon(t_1), F^b(t_1)\Upsilon(t_1))$ , and the lemma follows by the uniqueness of solutions to differential equations.  $\blacktriangledown$

It is now shown that, given any  $w_{\gamma_{\text{ref}}(\tau)} \in T_{\gamma_{\text{ref}}(\tau)}M$ ,  $w_{\gamma_{\text{ref}}(\tau)} \in \text{image}(K_1(\tau))$ . Let  $(\Upsilon_*, \Lambda_*)$  satisfy (iii) with  $\Upsilon(\tau) = w_{\gamma_{\text{ref}}(\tau)}$  and  $\Lambda_*(t_1) = F^b(t_1)\Upsilon(t_1)$ . By Lemma 1 it follows that

$$w_{\gamma_{\text{ref}}(\tau)} = \Upsilon_*(\tau) = K_1(\tau)\Phi_{t_1, \tau}^{X_{\text{ref}}^T}(\Upsilon_*(t_1))$$

which implies that  $K_1(\tau)$  is invertible.

If  $(\Upsilon, \Lambda)$  satisfy (iii) and  $K_1(t)$  is invertible for all  $t \in [t_0, t_1]$ , then by Lemma 1 the following linear relationship holds:

$$\Lambda(t) = K_2(t)K_1^{-1}(t)\Upsilon(t).$$

To show that  $K(t) = K_2(t)K_1^{-1}(t)$  satisfies the Riccati equation in statement (i), it is first observed that  $K(t_1) = F^b(t_1)$  by construction. Using the linear relationship  $\Lambda(t) = K(t)\Upsilon(t)$ , the costate equation is

$$\langle Q^b(t)\Upsilon(t); \Upsilon(t) \rangle + \langle \mathcal{L}^{X_{\text{ref}}, \gamma_{\text{ref}}}(K^b(t)\Upsilon(t)); \Upsilon(t) \rangle = 0.$$

In coordinates the above equation becomes

$$\begin{aligned} \dot{K}_{ij}(t)\Upsilon^i(t)\Upsilon^j(t) + K_{ij}(t) \left( \frac{\partial X_{\text{ref}}^i}{\partial x^\ell}(\gamma_{\text{ref}}(t)) - \iota(t)R^{si}(t)\iota^*(t)\Lambda_s(t) \right) \Upsilon^j(t) + \\ \frac{\partial X_{\text{ref}}^\ell}{\partial x^j}(\gamma_{\text{ref}}(t))K_{i\ell}(t)\Upsilon^i(t)\Upsilon^j(t) + Q_{ij}(t)\Upsilon^i(t)\Upsilon^j(t) = 0. \end{aligned}$$

Again, using the linear relationship of state and costate implies that

$$\begin{aligned} \dot{K}_{ij}(t) + K_{\ell j}(t) \frac{\partial X_{\text{ref}}^\ell}{\partial x^i}(\gamma_{\text{ref}}(t)) + \frac{\partial X_{\text{ref}}^\ell}{\partial x^j}(\gamma_{\text{ref}}(t))K_{i\ell}(t) - \\ K_{\ell j}(t)\iota(t)R^{s\ell}(t)\iota^*(t)K_{is}(t) + Q_{ij}(t) = 0. \end{aligned}$$

Thus  $K(t)$  satisfies

$$\mathcal{L}^{X_{\text{ref}}, \gamma_{\text{ref}}} K(t) - K^\flat(t)\iota(t)R^\sharp(t)\iota^*(t)K^\flat(t) + Q(t) = 0. \quad \blacksquare$$

**3.20 Remark:** The coordinate expression

$$\begin{aligned} \dot{K}_{ij}(t) + K_{\ell j}(t) \frac{\partial X_{\text{ref}}^\ell}{\partial x^i}(\gamma_{\text{ref}}(t)) + \frac{\partial X_{\text{ref}}^\ell}{\partial x^j}(\gamma_{\text{ref}}(t))K_{i\ell}(t) - \\ K_{\ell j}(t)\iota(t)R^{s\ell}(t)\iota^*(t)K_{is}(t) + Q_{ij}(t) = 0, \end{aligned}$$

recovers the standard Riccati equation

$$\dot{K}(t) + K(t)A(t) + A^T(t)K(t) - K(t)B(t)R^{-1}(t)B^T(t)K(t) + Q(t) = 0.$$

### 3.7. Infinite time LQR problems

In this section the solution to Problem 3.1(ii) is constructed by extending the ideas of Theorem 3.19 to the infinite time case. This will require various uniformity bounds on the problem data and ensuring the existence of solutions to the Riccati equation as the final time tends to infinity.

Define  $W(t, t_1)$ , a  $(2, 0)$ -tensor on  $T_{\gamma_{\text{ref}}(t)}M$ , as the solution to

$$\mathcal{L}^{X_{\text{ref}}, \gamma_{\text{ref}}} W^\flat(t, t_1) = \iota(t)R(t)\iota^*(t). \quad (3.7.1)$$

The differential equation (3.7.1) should be thought of as the geometric analogue of (1.2.4). The idea is to construct the analogue of the minimum energy controller of Section 1.2.2 in the present geometric framework. It is useful to think of (3.7.1) as differentiation of  $W(t, t_1)$  with respect to the first parameter  $t$ . The rule for differentiating with respect to second parameter is provided by the following lemma.

**3.21 Lemma:** *If  $W(t, t_1)$  is a  $(2, 0)$ -tensor on  $T_{\gamma_{\text{ref}}(t)}M$  satisfying  $\mathcal{L}^{X_{\text{ref}}, \gamma_{\text{ref}}}W^b(t, \tau) = \iota(t)R(t)\sharp\iota^*(t)$ , then*

$$\frac{d}{d\tau}W(t, \tau) = \Phi_{\tau, t}^{X_{\text{ref}}^T} \iota(\tau)R^\sharp(\tau)\iota^*(\tau)\Phi_{t, \tau}^{X_{\text{ref}}^{T*}}.$$

**Proof:** In coordinates  $\mathcal{L}^{X_{\text{ref}}, \gamma_{\text{ref}}}W^b(t, \tau) = \iota(t)R(t)\sharp\iota^*(t)$  is given by

$$\begin{aligned} & \mathcal{L}^{X_{\text{ref}}, \gamma_{\text{ref}}}W^{ij}(t, \tau) - R^{ij}(t) = 0 \\ \Rightarrow & X_{\text{ref}}^k(\gamma_{\text{ref}}(t)) \frac{\partial W^{ij}}{\partial x^k}(t, \tau) - W^{kj}(t, \tau) \frac{\partial X_{\text{ref}}^i}{\partial x^k} - W^{ik}(t, \tau) \frac{\partial X_{\text{ref}}^j}{\partial x^k} - R^{ij}(t) = 0 \\ \Rightarrow & \frac{d}{dt} \left( \frac{d}{d\tau} W^{ij}(t, \tau) \right) - \frac{d}{d\tau} \left( W^{kj}(t, \tau) \right) \frac{\partial X_{\text{ref}}^i}{\partial x^k} - \frac{d}{d\tau} \left( W^{ik}(t, \tau) \right) \frac{\partial X_{\text{ref}}^j}{\partial x^k} = 0. \end{aligned} \quad (3.7.2)$$

To complete the proof it is shown that  $\Phi_{\tau, t}^{X_{\text{ref}}^T} \iota(\tau)R^\sharp(\tau)\iota^*(\tau)\Phi_{t, \tau}^{X_{\text{ref}}^{T*}}$  also satisfies the differential equation (3.7.2). Applying the **backward differentiation lemma** [Abraham, Marsden and Ratiu 1988],

$$\frac{d}{dt} \Phi_{t, \tau}^{X_{\text{ref}}^T}(v_x) = -T\Phi_{t, \tau}^{X_{\text{ref}}^T} \circ X_{\text{ref}}^T(v_x)$$

yields

$$\begin{aligned} \frac{d}{dt} \left( \Phi_{\tau, t}^{X_{\text{ref}}^T} \iota(\tau)R^\sharp(\tau)\iota^*(\tau)\Phi_{t, \tau}^{X_{\text{ref}}^{T*}} \right) &= X_{\text{ref}}^T(\Phi_{\tau, t}^{X_{\text{ref}}^T} \iota(\tau)R^\sharp(\tau)\iota^*(\tau)\Phi_{t, \tau}^{X_{\text{ref}}^{T*}}) - \\ & T_{\gamma_{\text{ref}}(\tau)} \Phi_{\tau, t}^{X_{\text{ref}}^T} \iota(\tau)R^\sharp(\tau)\iota^*(\tau)X_{\text{ref}}^{T*} \Phi_{t, \tau}^{X_{\text{ref}}^{T*}}, \end{aligned}$$

which in coordinates becomes

$$\begin{aligned} \frac{d}{dt} \left( \Phi_{\tau, t}^{X_{\text{ref}}^T} \iota(\tau)R^\sharp(\tau)\iota^*(\tau)\Phi_{t, \tau}^{X_{\text{ref}}^{T*}} \right)^{sj} &= \frac{\partial X_{\text{ref}}^s}{\partial x^\ell} (\Phi_{\tau, t_0}^{X_{\text{ref}}^T})_k^\ell R^{ki}(\tau) (\Phi_{t_0, \tau}^{X_{\text{ref}}^{T*}})_i^j + \\ & (\Phi_{\tau, t_0}^{X_{\text{ref}}^T})_k^s R^{ki}(\tau) (\Phi_{t_0, \tau}^{X_{\text{ref}}^{T*}})_\ell^j \frac{\partial X_{\text{ref}}^\ell}{\partial x^i}, \end{aligned}$$

providing the desired result. ■

The next lemma plays a central role in the rest of the proof.

**3.22 Lemma:** *If  $A_{\text{ref}}^T$  is controllable from  $v_0 \in T_{\gamma_{\text{ref}}(t_0)}M$ , then there exists a section  $X_1$  of  $L(\mathcal{A})$ , linear in  $v_0$ , such that, for the resulting trajectory  $\Upsilon_1 \in \text{Traj}(A_{\text{ref}}^T)$  and time  $t_2(v_0, t_0) \leq t_1$ , the following hold:*

- (i)  $\Upsilon_1(t_0) = v_0$  and
- (ii)  $\Upsilon_1(t) = 0_{\gamma_{\text{ref}}(t)}$  for all  $t \geq t_2(v_0, t_0)$ .

**Proof:** Inline with the standard minimum energy control in Section 1.2.2, define a vector field along  $\gamma_{\text{ref}}$  by

$$\Upsilon(\tau) = \Phi_{t_0, \tau}^{X_{\text{ref}}^T} (\Upsilon(t_0) - W^b(t_0, \tau)W^\sharp(t_0, t_1)\Upsilon(t_0)), \quad (3.7.3)$$

where  $\Upsilon(t_0) = v_0$ . The lemma will follow if (3.7.3) is a trajectory of  $A_{\text{ref}}^T$ . We claim, this is the trajectory prescribed by the following section of  $L(\mathcal{A})$ :

$$X(t) = -R(t)^\sharp \iota^*(t) \Phi_{t_0, t}^{X^{T^*}} \eta, \quad (3.7.4)$$

where  $\eta \in T_{\gamma_{\text{ref}}(t_0)}M$  satisfies  $W(t_0, t_1)\eta = \Upsilon(t_0)$ . If  $\Upsilon(\tau)$ , as defined in (3.7.3), is a trajectory, then  $\Upsilon(\tau)$  must satisfy

$$\mathcal{L}^{\gamma_{\text{ref}}, X_{\text{ref}}}(\Upsilon(\tau)) = -\iota(\tau)R(\tau)^\sharp \iota^*(\tau) \Phi_{t_0, \tau}^{X^{T^*}} \eta.$$

Consider the following coordinate computations:

$$\begin{aligned} \mathcal{L}^{\gamma_{\text{ref}}, X_{\text{ref}}}((\Phi_{t_0, \tau}^{X_{\text{ref}}^T} W^b(t_0, \tau)\eta))^i &= \\ &= \frac{d}{d\tau} \left( \Phi_{t_0, \tau}^{X_{\text{ref}}^T} W^b(t_0, \tau)\eta \right)^i - \frac{\partial X_{\text{ref}}^i}{\partial x^k}(\gamma_{\text{ref}}(\tau)) \left( \Phi_{t_0, \tau}^{X_{\text{ref}}^T} W^b(t_0, \tau)\eta \right)^k \\ &= \frac{d}{d\tau} \left( (\Phi_{t_0, \tau}^{X_{\text{ref}}^T})^i_k W^{kj}(t_0, \tau)\eta_j \right) - \\ &= \frac{\partial X_{\text{ref}}^i}{\partial x^k}(\Phi_{t_0, \tau}^{X_{\text{ref}}^T})^k_s W^{sj}(t_0, \tau)\eta_j \\ &= \frac{\partial X_{\text{ref}}^i}{\partial x^k}(\gamma_{\text{ref}}(\tau)) \left( (\Phi_{t_0, \tau}^{X_{\text{ref}}^T})^k_s W^{sj}(t_0, \tau)\eta_j \right) + \\ &= (\Phi_{t_0, \tau}^{X_{\text{ref}}^T})^i_k \frac{d}{d\tau} \left( W^{kj}(t_0, \tau) \right) \eta_j - \\ &= \frac{\partial X_{\text{ref}}^i}{\partial x^k}(\gamma_{\text{ref}}(\tau)) \left( \Phi_{t_0, \tau}^{X_{\text{ref}}^T} \right)^k_s W^{sj}(t_0, \tau)\eta_j \\ &= (\Phi_{t_0, \tau}^{X_{\text{ref}}^T})^i_k \frac{d}{d\tau} \left( W^{kj}(t_0, \tau) \right) \eta_j \\ &= (\Phi_{t_0, \tau}^{X_{\text{ref}}^T})^i_k (\Phi_{\tau, t_0}^{X_{\text{ref}}^T})^k_\ell R^{\ell s}(\tau) (\Phi_{t_0, \tau}^{X_{\text{ref}}^T})^j_s \eta_j \\ &= R^{is}(\tau) (\Phi_{t_0, \tau}^{X_{\text{ref}}^T})^j_s \eta_j = \iota(\tau)R(\tau)^\sharp \iota^*(\tau) \Phi_{t_0, \tau}^{X^{T^*}} \eta^i \end{aligned}$$

and

$$\begin{aligned} \mathcal{L}^{\gamma_{\text{ref}}, X_{\text{ref}}}(\Phi_{t_0, \tau}^{X_{\text{ref}}^T} \Upsilon(t_0)) &= \frac{d}{d\tau} \left( \Phi_{t_0, \tau}^{X_{\text{ref}}^T} \Upsilon(t_0) \right)^i - \frac{\partial X_{\text{ref}}^i}{\partial x^k} \left( \Phi_{t_0, \tau}^{X_{\text{ref}}^T} \Upsilon(t_0) \right)^k \\ &= \frac{\partial X_{\text{ref}}^i}{\partial x^k} (\Phi_{t_0, \tau}^{X_{\text{ref}}^T})^k_j \Upsilon^j(t_0) - \frac{\partial X_{\text{ref}}^i}{\partial x^k} \left( (\Phi_{t_0, \tau}^{X_{\text{ref}}^T})^k_j \Upsilon^j(t_0) \right) = 0. \end{aligned}$$

Combining the above coordinate calculations gives

$$\begin{aligned} \mathcal{L}^{\gamma_{\text{ref}}, X_{\text{ref}}}(\Upsilon(\tau)) &= \mathcal{L}^{\gamma_{\text{ref}}, X_{\text{ref}}}(\Phi_{t_0, \tau}^{X_{\text{ref}}^T} (\Upsilon(t_0) - W^b(t_0, \tau)W^\sharp(t_0, t_1)\Upsilon(t_0))) \\ &= \mathcal{L}^{\gamma_{\text{ref}}, X_{\text{ref}}}(\Phi_{t_0, \tau}^{X_{\text{ref}}^T} \Upsilon(t_0)) - \mathcal{L}^{\gamma_{\text{ref}}, X_{\text{ref}}}((\Phi_{t_0, \tau}^{X_{\text{ref}}^T} W^b(t_0, \tau)\eta)) \\ &= -\iota(\tau)R(\tau)^\sharp \iota^*(\tau) \Phi_{t_0, \tau}^{X^{T^*}} \eta, \end{aligned}$$

as desired. ■

Now the Lemmata 3.21 and 3.22 are used to prove that solutions to the Riccati equation exist as the final time in the LQR problem tends to infinity.

**3.23 Proposition:** For fixed  $t_1$ , let  $K_{t_1}(t)$  be a solution to

$$\mathcal{L}^{X_{\text{ref}}, \gamma_{\text{ref}}} K(t) - K^b(t) \iota(t) R^\sharp(t) \iota^*(t) K^b(t) + Q(t) = 0, \quad K(t_1) = 0, \quad (3.7.5)$$

for each  $t \in [t_0, t_1]$ . If  $A_{\text{ref}}^T$  is controllable, then

- (i)  $\lim_{t_1 \rightarrow \infty} K_{t_1}(t) = \bar{K}(t)$  and
- (ii)  $\bar{K}(t)$  satisfies (3.7.5).

**Proof:** Proof of (i): Let  $\Upsilon_* \in \text{Traj}(A_{\text{ref}}^T)$  be a vector field along  $\gamma_{\text{ref}}$  that minimizes the cost

$$J(\Upsilon(t_0), t_0, t_1) = \frac{1}{2} \int_{t_0}^{t_1} Q(t)(\Upsilon(t), \Upsilon(t)) + R(t)(X(t), X(t)) dt$$

provided by Theorem 3.19. Since  $A_{\text{ref}}^T$  is controllable, there exists  $\Upsilon_1 \in \text{Traj}(A_{\text{ref}}^T)$  and a time  $t_2(\Upsilon_*(t_0), t_0) \leq t_1$  such that

1.  $\Upsilon_1(t_0) = \Upsilon_*(t_0)$  and
2.  $\Upsilon_1(t) = 0_{\gamma_{\text{ref}}(t)}$  for all  $t \geq t_2(\Upsilon_*(t_0), t_0)$ .

The idea is to choose, by Lemma 3.22, a control that steers the system to zero along the reference curve in time  $t_2 \leq t_1$  and which depends on both the initial condition  $\Upsilon_*(t_0)$  and  $t_0$ . Once system is at zero, the control is set to zero and because the system is linear the state remains zero along the reference curve.

The calculations in the proof of Theorem 3.19 show that the value of the minimal cost and the solution to the Riccati equation are related by

$$K_{t_1}(t_0)(\Upsilon_*(t_0), \Upsilon_*(t_0)) = J(\Upsilon_*(t_0), t_0, t_1).$$

The optimality of the trajectory  $\Upsilon_*(t)$  implies that

$$\begin{aligned} K_{t_1}(t_0)(\Upsilon_*(t_0), \Upsilon_*(t_0)) &= J(\Upsilon_*(t_0), t_0, t_1) \leq J(\Upsilon_1(t_0), t_0, t_1) \\ &= \frac{1}{2} \int_{t_0}^{t_2} Q(\tau)(\Upsilon_1(\tau), \Upsilon_1(\tau)) + R(\tau)(X_1(\tau), X_1(\tau)) d\tau + \\ &\quad \frac{1}{2} \int_{t_2}^{t_1} Q(\tau)(\Upsilon_1(\tau), \Upsilon_1(\tau)) + R(\tau)(X_1(\tau), X_1(\tau)) d\tau \\ &= \frac{1}{2} \int_{t_0}^{t_2} Q(\tau)(\Upsilon_1(\tau), \Upsilon_1(\tau)) + R(\tau)(X_1(\tau), X_1(\tau)) d\tau. \end{aligned}$$

The trajectory

$$\Upsilon_1(\tau) = \Phi_{t_0, \tau}^{X_{\text{ref}}^T}(\Upsilon(t_0) - W^b(t_0, \tau)W^\sharp(t_0, t_2)\Upsilon(t_0)),$$

and the control

$$X_1(\tau) = -R(\tau)^\sharp \iota^*(\tau) \Phi_{t_0, \tau}^{X_{\text{ref}}^T} W^\sharp(t_0, t_2)\Upsilon(t_0),$$

are linear in the initial condition, which implies that

$$\begin{aligned} K_{t_1}(t_0)(\Upsilon_*(t_0), \Upsilon_*(t_0)) &\leq \frac{1}{2} \int_{t_0}^{t_2} Q(\tau)(\Upsilon_1(\tau), \Upsilon_1(\tau)) + R(\tau)(X_1(\tau), X_1(\tau)) d\tau \\ &\leq f(t_0, t_2)(\Upsilon_*(t_0), \Upsilon_*(t_0)), \end{aligned}$$

where  $f(t_0, t_2)$  is a non-negative  $(0,2)$ -tensor on  $T_{\gamma_{\text{ref}}(t_0)}M$ . This ensures that, for each  $t_0$ , the bilinear map  $K_{t_1}(t_0): T_{\gamma_{\text{ref}}(t_0)}M \times T_{\gamma_{\text{ref}}(t_0)}M \rightarrow \mathbb{R}$  satisfies,

$$\sup_{t_1 \in \mathbb{R}} |K_{t_1}(t_0)(\Upsilon_*(t_0), \Upsilon_*(t_0))| \leq \sup_{t_1 \in \mathbb{R}} f(t_0, t_2)(\Upsilon_*(t_0), \Upsilon_*(t_0)) < \infty$$

Since  $K_{t_1}(t_0)(\Upsilon(t_0), \Upsilon(t_0)) = J(\Upsilon(t_0), t_0, t_1)$  and  $J(\Upsilon(t_0), t_0, t_1) \leq J(\Upsilon(t_0), t_0, t_2)$  for all  $t_1 \leq t_2$ , the map  $t_1 \mapsto K_{t_1}(t_0)(\Upsilon(t_0), \Upsilon(t_0))$  is non-decreasing as  $t_1 \rightarrow \infty$ . Using the above facts, the limit  $\bar{K}(t_0)$  is constructed as follows. Define a quadratic form  $K(t_0)$  by

$$K(t_0)(v, v) = \lim_{t_1 \rightarrow \infty} \langle K_{t_1}^\flat(t_0)v, v \rangle, \quad v \in T_{\gamma_{\text{ref}}(t_0)}M. \quad (3.7.6)$$

The limit in (3.7.6) exist because it is a bounded non-decreasing function of  $t_1$ . The polarization identity,

$$4\bar{K}(t_0)(v, w) = K(t_0)(v + w, v + w) - K(t_0)(v - w, v - w),$$

then defines a symmetric  $(0, 2)$ -tensor  $\bar{K}(t_0)$  for all  $v, w \in T_{\gamma_{\text{ref}}(t_0)}M$ .

Proof of (ii): Let  $t \in [t_0, t_1]$  and let  $K_{t_1}(t)$  be a solution to the Riccati equation with final condition  $K_{t_1}(t_1) = 0$ . By part (i) of the proof

$$\bar{K}(t) = \lim_{t_1 \rightarrow \infty} K_{t_1}(t),$$

exists. Now suppose, for  $t \in [t_0, \tau_1]$ ,  $\tau_1 \leq t_1$ , that  $P_{\tau_1}(t)$  is a solution to the Riccati equation with final condition  $P_{\tau_1}(\tau_1) = K_{t_1}(\tau_1)$ . Then, the continuity of solutions to differential equations implies that  $P_{\tau_1}(t) = K_{t_1}(t)$  for  $t \leq \tau_1 \leq t_1$ . Thus, it follows that

$$\bar{K}(t) = \lim_{t_1 \rightarrow \infty} K_{t_1}(t) = \lim_{t_1 \rightarrow \infty} P_{\tau_1}(t) = P_{\tau_1}(t),$$

with final condition  $\bar{K}(\tau_1) = P_{\tau_1}(\tau_1)$ . ■

**3.24 Theorem:** *The trajectory corresponding to the section of  $L(\mathcal{A})$  defined by*

$$X(t) = -R^\sharp(t)\iota^*(t)\bar{K}^\flat(t)\Upsilon(t) \quad (3.7.7)$$

*is optimal in the sense of Problem 3.1(ii).*

Proof: Let  $\bar{J}(\Upsilon(t_0), t_0, t_1)$  be the cost associated with (3.7.7) on the interval  $[t_0, t_1]$ .

**1 Lemma:** *The trajectory  $\Upsilon(t)$  corresponding to (3.7.7) has the associated cost  $\bar{J}(\Upsilon(t_0), t_0, \infty) := \lim_{t_1 \rightarrow \infty} \bar{J}(\Upsilon(t_0), t_0, t_1) = \bar{K}(t_0)(\Upsilon(t_0), \Upsilon(t_0))$ .*

Proof: Let  $J(\Upsilon(t_0), t_0, t_1)$  be the optimal cost. By Proposition 3.23, for  $\epsilon > 0$  there exists  $T > t_0$  such that

$$\begin{aligned} \bar{J}(\Upsilon(t_0), t_0, t_1) &\geq J(\Upsilon(t_0), t_0, t_1) \\ &= K_{t_1}(t_0)(\Upsilon(t_0), \Upsilon(t_0)) \\ &\geq \bar{K}(t_0)(\Upsilon(t_0), \Upsilon(t_0)) - \epsilon, \end{aligned}$$

for all  $t_1 \geq T$ . On the other hand,  $\bar{K}(t)$  is a solution to the Riccati equation and thus, by Theorem 3.19,

$$\bar{J}(\Upsilon(t_0), t_0, t_1) = \bar{K}(t_0)(\Upsilon(t_0), \Upsilon(t_0)) - \bar{K}(t_1)(\Upsilon(t_1), \Upsilon(t_1)) \leq \bar{K}(t_0)(\Upsilon(t_0), \Upsilon(t_0)). \quad \blacktriangledown$$



Now, by contradiction,  $\bar{J}(\Upsilon(t_0), t_0, \infty)$  is proved to be the optimal cost. By assumption  $J(\Upsilon(t_0), t_0, \infty) \leq \bar{J}(\Upsilon(t_0), t_0, \infty)$  and, if the inequality is strict, then there exist a positive constant  $C$  such that

$$0 < C \leq \bar{J}(\Upsilon(t_0), t_0, \infty) - J(\Upsilon(t_0), t_0, \infty).$$

To obtain a contradiction, choose a section of  $L(\mathcal{A})$  whose cost  $J_1(\Upsilon(t_0), t_0, \infty)$  has the property that

$$0 < \frac{C}{2} \leq \bar{J}(\Upsilon(t_0), t_0, \infty) - J_1(t_0, \infty, \Upsilon(t_0)). \quad (3.7.8)$$

For  $\epsilon = \frac{C}{4} > 0$  there exists  $T > t_0$  such that

$$\begin{aligned} J(\Upsilon(t_0), t_0, t_1) + \frac{C}{4} &= K_{t_1}(t_0)(\Upsilon(t_0), \Upsilon(t_0)) + \frac{C}{4} \\ &> \bar{K}(t_0)(\Upsilon(t_0), \Upsilon(t_0)) \\ &= \bar{J}(\Upsilon(t_0), t_0, t_1) \geq J_1(\Upsilon(t_0), t_0, t_1) + \frac{C}{2}, \quad \forall t_1 \geq T, \end{aligned}$$

which contradicts the fact that  $J(\Upsilon(t_0), t_0, t_1)$  is the optimal cost. ■

# Chapter 4

## Stability and Stabilization

In this chapter, stability and stabilization by LQR methods of the linearization are addressed to complete the geometric picture of Kalman's theory of optimal control. In Section 4.1, the stability definitions for a fixed reference vector field  $X_{\text{ref}}$  and for linear vector fields over  $X_{\text{ref}}$  are defined by using a metric on  $M$  and a metric on the fibres of  $TM$  over  $\text{image}(\gamma_{\text{ref}})$ , respectively. Such metrics are naturally induced by choosing a Riemannian metric,  $\mathbf{G}$ , on  $M$ . It is noted, that unless the state manifold is compact, these stability definitions depend on the choice of metric. In Section 4.2, Lyapunov's direct method for linear vector fields on tangent bundles is introduced. In Section 4.3, after making geometric sense of the terms "linear state-feedback" and "closed-loop system," it is proved that the linearization of an affine system is uniformly asymptotically stabilized using the linear state-feedback provided by the infinite time LQR problem.

### 4.1. Stability definitions

Let  $X_{\text{ref}}$  be an  $\text{LIC}^\infty$  reference vector field for  $\mathcal{A}_{\text{ref}}$  with a reference trajectory  $\gamma_{\text{ref}}: I \rightarrow M$ . Let  $\mathbf{G}$  be a Riemannian metric on  $M$ . The Riemannian metric induces a metric on  $M$  denoted by  $d_{\mathbf{G}}: M \times M \rightarrow \mathbb{R}_+$ . If  $\text{LAC}(x_1, x_2, [0, 1])$  denotes the set of absolutely continuous curves  $\gamma$  defined on  $[0, 1]$  and satisfying  $\gamma(0) = x_0$  and  $\gamma(1) = x_1$ , then

$$d_{\mathbf{G}}(x_0, x_1) = \inf_{\gamma \in \text{LAC}(x_0, x_1, [0, 1])} \int_0^1 \sqrt{\mathbf{G}(t)(\gamma'(t), \gamma'(t))} dt.$$

The stability definitions along  $\gamma_{\text{ref}}$  are as follows.

**4.1 Definition:** Let  $X_{\text{ref}}: I \times M \rightarrow TM$  be an  $\text{LIC}^\infty$  reference vector field with  $\sup I = \infty$  and let  $\gamma_{\text{ref}}: I \rightarrow M$  be an integral curve for  $X_{\text{ref}}$  with the property that  $\gamma_{\text{ref}}: I \rightarrow M$  is defined on all of  $I$ . Let  $\mathbf{G}$  be a Riemannian metric on  $M$ . With respect to  $\gamma_{\text{ref}}$ ,  $X_{\text{ref}}$  is

- (i) **locally  $d_{\mathbf{G}}$ -stable** if, for each  $t_0 \in I$  and for each  $\epsilon > 0$ , there exists  $\delta > 0$  such that

$$d_{\mathbf{G}}(x_0, \gamma_{\text{ref}}(t_0)) < \delta \implies d_{\mathbf{G}}(\Phi_{t_0, t}^{X_{\text{ref}}}(x_0), \gamma_{\text{ref}}(t)) < \epsilon, \quad t \geq t_0;$$

- (ii) **locally uniformly  $d_{\mathbf{G}}$ -stable** if, for each  $\epsilon > 0$ , there exists  $\delta > 0$  such that

$$d_{\mathbf{G}}(x_0, \gamma_{\text{ref}}(t_0)) < \delta \implies d_{\mathbf{G}}(\Phi_{t_0, t}^{X_{\text{ref}}}(x_0), \gamma_{\text{ref}}(t)) < \epsilon, \quad t_0 \in I, \quad t \geq t_0;$$

- (iii) **locally asymptotically  $d_G$ -stable** if it is locally  $d_G$ -stable and if, for each  $t_0 \in I$  and  $\epsilon > 0$ , there exists  $\delta, T > 0$  such that

$$d_G(x_0, \gamma_{\text{ref}}(t_0)) < \delta \implies d_G(\Phi_{t_0,t}^{X_{\text{ref}}}(x_0), \gamma_{\text{ref}}(t)) < \epsilon, \quad t \geq t_0 + T;$$

- (iv) **locally uniformly asymptotically  $d_G$ -stable** if it is locally uniformly  $d_G$ -stable and if, for each  $\epsilon > 0$ , there exists  $\delta, T > 0$  such that

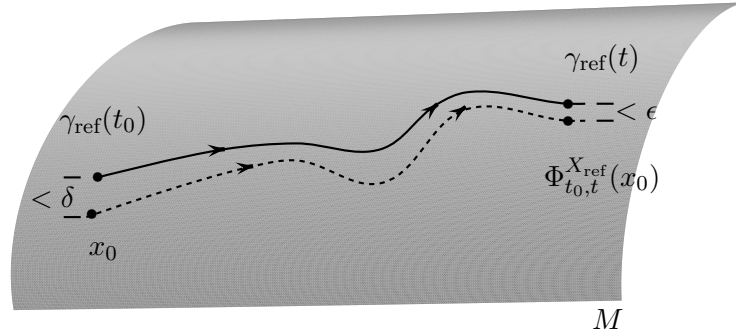
$$d_G(x_0, \gamma_{\text{ref}}(t_0)) < \delta \implies d_G(\Phi_{t_0,t}^{X_{\text{ref}}}(x_0), \gamma_{\text{ref}}(t)) < \epsilon, \quad t_0 \in I, t \geq T + t_0;$$

item **locally exponentially  $d_G$ -stable** if, for each  $t_0 \in I$ , there exists  $\delta, c_1, c_2 > 0$  such that

$$d_G(x_0, \gamma_{\text{ref}}(t_0)) < \delta \implies d_G(\Phi_{t_0,t}^{X_{\text{ref}}}(x_0), \gamma_{\text{ref}}(t)) \leq c_1 \exp(-c_2(t - t_0)), \quad t \geq t_0;$$

- (v) **locally uniformly exponentially  $d_G$ -stable** if there exists  $\delta, c_1, c_2 > 0$  such that

$$d_G(x_0, \gamma_{\text{ref}}(t_0)) < \delta \implies d_G(\Phi_{t_0,t}^{X_{\text{ref}}}(x_0), \gamma_{\text{ref}}(t)) \leq c_1 \exp(-c_2(t - t_0)), \quad t_0 \in I, t \geq t_0.$$



**Figure 4.1.** A pictorial representation of  $d_G$ -stability

**4.2 Definition:** Let  $X_{\text{ref}}$  be an  $\text{LIC}^\infty$  reference vector field with a reference trajectory  $\gamma_{\text{ref}}: I \rightarrow M$  where  $\sup I = \infty$ . Let  $G$  be a Riemannian metric on  $M$ . The linear vector field  $Y: I \times TM \rightarrow TTM$  over  $X_{\text{ref}}$  with respect to  $\gamma_{\text{ref}}$  is

- (i)  **$G$ -stable** if, for each  $t_0 \in I$ , there exists  $c(t_0) > 0$  such that

$$\|\Phi_{t_0,t}^Y(v_{\gamma_{\text{ref}}(t_0)})\|_{G(\gamma_{\text{ref}}(t))} < c(t_0) \|v_{\gamma_{\text{ref}}(t_0)}\|_{G(\gamma_{\text{ref}}(t_0))}$$

for  $t \geq t_0$ ,  $v_{\gamma_{\text{ref}}(t_0)} \in T_{\gamma_{\text{ref}}(t_0)}M$ ;

(ii) **uniformly  $\mathbf{G}$ -stable** if there exists  $c > 0$  such that

$$\|\Phi_{t_0,t}^Y(v_{\gamma_{\text{ref}}(t_0)})\|_{\mathbf{G}(\gamma_{\text{ref}}(t))} < c \|v_{\gamma_{\text{ref}}(t_0)}\|_{\mathbf{G}(\gamma_{\text{ref}}(t_0))}$$

for  $t_0 \in I$ ,  $t \geq t_0$ ,  $v_{\gamma_{\text{ref}}(t_0)} \in T_{\gamma_{\text{ref}}(t_0)}M$ ;

(iii) **asymptotically  $\mathbf{G}$ -stable** if it is  $\mathbf{G}$ -stable and if, for each  $t_0 \in I$  and for each  $\epsilon > 0$ , there exists and  $T(\epsilon, t_0) > 0$  such that

$$\|\Phi_{t_0,t}^Y(v_{\gamma_{\text{ref}}(t_0)})\|_{\mathbf{G}(\gamma_{\text{ref}}(t))} < \epsilon(t_0) \|v_{\gamma_{\text{ref}}(t_0)}\|_{\mathbf{G}(\gamma_{\text{ref}}(t_0))}$$

for  $t \geq t_0 + T(\epsilon, t_0)$ ,  $v_{\gamma_{\text{ref}}(t_0)} \in T_{\gamma_{\text{ref}}(t_0)}M$ ;

(iv) **uniformly asymptotically  $\mathbf{G}$ -stable** if it is uniformly  $\mathbf{G}$ -stable and if, for each  $\epsilon > 0$ , there exists  $T(\epsilon) > 0$  such that

$$\|\Phi_{t_0,t}^Y(v_{\gamma_{\text{ref}}(t_0)})\|_{\mathbf{G}(\gamma_{\text{ref}}(t))} < \epsilon \|v_{\gamma_{\text{ref}}(t_0)}\|_{\mathbf{G}(\gamma_{\text{ref}}(t_0))}$$

for  $t_0 \in I$ ,  $t \geq t_0 + T(\epsilon)$ ,  $v_{\gamma_{\text{ref}}(t_0)} \in T_{\gamma_{\text{ref}}(t_0)}M$ ;

(v) **exponentially  $\mathbf{G}$ -stable** if, for each  $t_0 \in I$ , there exists  $c_1(t_0), c_2(t_0) > 0$  such that

$$\|\Phi_{t_0,t}^Y(v_{\gamma_{\text{ref}}(t_0)})\|_{\mathbf{G}(\gamma_{\text{ref}}(t))} \leq c_1(t_0) \|v_{\gamma_{\text{ref}}(t_0)}\|_{\mathbf{G}(\gamma_{\text{ref}}(t_0))} \exp(-c_2(t_0)(t - t_0))$$

for  $t \geq t_0$ ,  $v_{\gamma_{\text{ref}}(t_0)} \in T_{\gamma_{\text{ref}}(t_0)}M$ ;

(vi) **uniformly exponentially  $\mathbf{G}$ -stable** if there exists  $c_1, c_2 > 0$  such that

$$\|\Phi_{t_0,t}^Y(v_{\gamma_{\text{ref}}(t_0)})\|_{\mathbf{G}(\gamma_{\text{ref}}(t))} \leq c_1 \|v_{\gamma_{\text{ref}}(t_0)}\|_{\mathbf{G}(\gamma_{\text{ref}}(t_0))} \exp(-c_2(t - t_0))$$

for  $t_0 \in I$ ,  $t \geq t_0$ ,  $v_{\gamma_{\text{ref}}(t_0)} \in T_{\gamma_{\text{ref}}(t_0)}M$ .

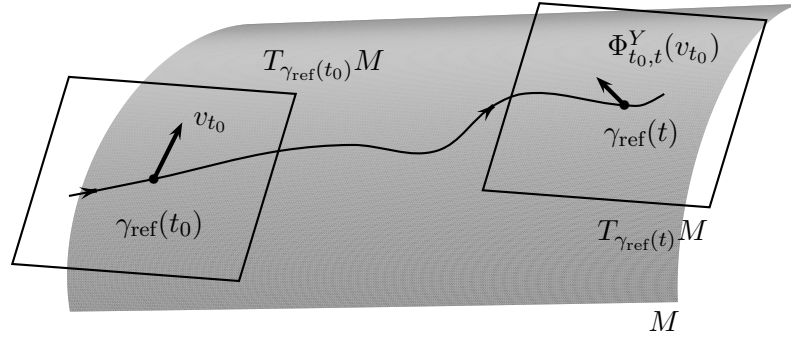
**4.3 Example:** To illustrate how stability along a non-compact reference trajectory depends on the choice metric, consider an  $\text{LIC}^\infty$  linear vector field on the tangent bundle  $\text{pr}_1: T\mathbb{R} \simeq \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$ , where  $\text{pr}_1$  is projection onto the first factor, by  $X(\tau, x) = ((\tau, x), (1, -\tau x))$ . The projected vector field on the base space is  $\pi X = \frac{\partial}{\partial \tau}$  and following diagram commutes:

$$\begin{array}{ccc} \mathbb{R} \times \mathbb{R} & \xrightarrow{X} & T(\mathbb{R} \times \mathbb{R}) \\ \pi \downarrow & & \downarrow T\pi \\ \mathbb{R} & \xrightarrow{\pi X} & \mathbb{R} \times \mathbb{R} \end{array}$$

The flow of  $X$  for each  $t_0 \in \mathbb{R}$  is the map  $\Phi_{t_0,t}^X: x_{t_0} \mapsto \exp(\frac{-t^2+t_0^2}{2})x_t$ ,  $x_{t_0} \in T_{t_0}\mathbb{R} = \{t_0\} \times \mathbb{R}$ ,  $t > t_0$ . Now, choose a Riemannian metric such that  $\mathbf{G}_{xx}(t) = \exp(\lambda(t^2 - t_0^2))$ . Then for any initial time  $t_0$ ,

$$\begin{aligned} \|\Phi_{t_0,t}^X(x_{t_0})\|_{\mathbf{G}(t)}^2 &= \exp(\lambda(t^2 - t_0^2)) \left( \exp\left(\frac{-t^2 + t_0^2}{2}\right) x_{t_0} \right)^2 \\ &= \exp((\lambda - 1)(t^2 - t_0^2)) \mathbf{G}(t_0)(x_{t_0}, x_{t_0}) \\ &= \exp((\lambda - 1)(t^2 - t_0^2)) \|x_{t_0}\|_{\mathbf{G}(t_0)}^2, \end{aligned}$$

and the vector field  $X$  is,



**Figure 4.2.** A pictorial representation of  $\mathbf{G}$ -stability

1. not  $\mathbf{G}$ -stable for  $\lambda > 1$  and
2.  $\mathbf{G}$ -stable for  $\lambda = 1$  with  $c(t_0) > 1$ . •

The above definitions, being for linear vector fields, hold both at a global and local level and are analogous with the standard stability definitions for linear time-varying ordinary differential equations. The following diagram provides the correspondence between each type of stability:

$$\begin{array}{ccccc}
 UES & \rightleftarrows & UAS & \longrightarrow & US \\
 \downarrow & & \downarrow & & \downarrow \\
 ES & \longrightarrow & AS & \longrightarrow & S
 \end{array} \tag{4.1.1}$$

The implications given by transitivity of “if-then” are left off for the sake of clarity. The only non-trivial implication is  $UAS \implies UES$ , and we show this in Proposition 4.4 below. In the statement of the result, let  $Y$  be a linear vector field over  $X_{\text{ref}}$  and let  $\|\Phi_{t_0,t}^Y\|_{\mathbf{G}}$  denote the norm of the linear map  $\Phi_{t_0,t}^Y|_{T_{\gamma_{\text{ref}}(t_0)}M}: T_{\gamma_{\text{ref}}(t_0)}M \rightarrow T_{\gamma_{\text{ref}}(t)}M$  induced by the norms on  $T_{\gamma_{\text{ref}}(t_0)}M$  and  $T_{\gamma_{\text{ref}}(t)}M$ .

**4.4 Proposition:** *The following statements are equivalent:*

- (i)  $Y$  is uniformly asymptotically  $\mathbf{G}$ -stable with respect to  $\gamma_{\text{ref}}$ ;
- (ii)  $Y$  is uniformly exponentially  $\mathbf{G}$ -stable with respect to  $\gamma_{\text{ref}}$ ;
- (iii) there exists constants  $c_1, c_2 > 0$  such that  $\|\Phi_{t_0,t}^Y\|_{\mathbf{G}} \leq c_1 \exp(-c_2(t - t_0))$ .

**Proof:** (i)  $\implies$  (ii) Since the system is uniformly  $\mathbf{G}$ -stable, there exists  $c > 0$  such that

$$\|\Phi_{t_0,t}^Y(v_{\gamma_{\text{ref}}(t_0)})\|_{\mathbf{G}(\gamma_{\text{ref}}(t))} \leq c \|v_{\gamma_{\text{ref}}(t_0)}\|_{\mathbf{G}(\gamma_{\text{ref}}(t_0))}$$

for all  $t_0 \in I$ ,  $t \geq t_0$ , and  $v_{\gamma_{\text{ref}}(t_0)} \in T_{\gamma_{\text{ref}}(t_0)}M$ . Therefore,

$$\|\Phi_{t_0,t}^Y\|_{\mathbf{G}} = \sup_{\|v_{\gamma_{\text{ref}}(t_0)}\|_{\mathbf{G}(\gamma_{\text{ref}}(t_0))}=1} \|\Phi_{t_0,t}^Y(v_{\gamma_{\text{ref}}(t_0)})\|_{\mathbf{G}(\gamma_{\text{ref}}(t))} \leq c$$

for all  $t_0 \in I$  and  $t \geq t_0$ . Now choose  $T > 0$  such that

$$\|\Phi_{t_0,t}^Y(v_{\gamma_{\text{ref}}(t_0)})\|_{\mathbf{G}(\gamma_{\text{ref}}(t))} \leq \frac{\|v_{\gamma_{\text{ref}}(t_0)}\|_{\mathbf{G}(\gamma_{\text{ref}}(t_0))}}{\mathbf{e}}, \quad t \geq t_0 + T, \quad v_{\gamma_{\text{ref}}(t_0)} \in T_{\gamma_{\text{ref}}(t_0)}M,$$

where  $\mathbf{e}$  denotes the base of the natural logarithm. Define a function  $N: [t_0, \infty) \rightarrow \mathbb{Z}_+$  by asking that  $N(t)$  is the smallest integer for which  $t \leq t_0 + N(t)T$ . Divide the interval  $[t_0, (N(t) - 1)T]$  into  $N(t) - 1$  subintervals of length  $T$ , and compute

$$\Phi_{t_0,t}^Y = \Phi_{t_0+(N(t)-1)T,t}^Y \circ \Phi_{t_0+(N(t)-2)T,t_0+(N(t)-1)T}^Y \circ \cdots \circ \Phi_{t_0,t_0+T}^Y.$$

Thus,

$$\begin{aligned} \|\Phi_{t_0,t}^Y\|_{\mathbf{G}} &\leq \|\Phi_{t_0+(N(t)-1)T,t}^Y\|_{\mathbf{G}} \prod_{j=1}^{N(t)-1} \|\Phi_{t_0+(j-1)T,t_0+jT}^Y\|_{\mathbf{G}} \\ &\leq (ce)\mathbf{e}^{-N(t)} \leq (ce)\mathbf{e}^{-(t-t_0)/T} = c_1 \exp(-c_2(t - t_0)), \end{aligned}$$

with  $c_1 = ce$  and  $c_2 = \frac{1}{T}$ . Therefore,

$$\begin{aligned} \|\Phi_{t_0,t}^Y(v_{\gamma_{\text{ref}}(t_0)})\|_{\mathbf{G}(\gamma_{\text{ref}}(t))} &\leq \|\Phi_{t_0,t}^Y\|_{\mathbf{G}} \|v_{\gamma_{\text{ref}}(t_0)}\|_{\mathbf{G}(\gamma_{\text{ref}}(t_0))} \leq \\ &c_1 \|v_{\gamma_{\text{ref}}(t_0)}\|_{\mathbf{G}(\gamma_{\text{ref}}(t_0))} \exp(-c_2(t - t_0)), \end{aligned}$$

as desired.

(ii)  $\Rightarrow$  (iii) Compute

$$\|\Phi_{t_0,t}^Y\|_{\mathbf{G}} = \sup_{\|v_{\gamma_{\text{ref}}(t_0)}\|_{\mathbf{G}(\gamma_{\text{ref}}(t_0))}=1} \|\Phi_{t_0,t}^Y(v_{\gamma_{\text{ref}}(t_0)})\|_{\mathbf{G}(\gamma_{\text{ref}}(t))} = c_1 \exp(-c_2(t - t_0)).$$

(iii)  $\Rightarrow$  (i) Let  $\epsilon > 0$ , take  $T = \max\{-\ln(\epsilon/c_1)/c_2, 1/c_2\}$  and  $t \geq t_0 + T$ , then compute

$$\begin{aligned} \|\Phi_{t_0,t}^Y(v_{\gamma_{\text{ref}}(t_0)})\|_{\mathbf{G}(\gamma_{\text{ref}}(t_0))} &\leq \|\Phi_{t_0,t}^Y\|_{\mathbf{G}} \|v_{\gamma_{\text{ref}}(t_0)}\|_{\mathbf{G}(\gamma_{\text{ref}}(t_0))} \\ &\leq c_1 \|v_{\gamma_{\text{ref}}(t_0)}\|_{\mathbf{G}(\gamma_{\text{ref}}(t_0))} \exp(-c_2(t - t_0)) \\ &\leq c_1 \|v_{\gamma_{\text{ref}}(t_0)}\|_{\mathbf{G}(\gamma_{\text{ref}}(t_0))} \exp(-c_2 T) \leq \epsilon \|v_{\gamma_{\text{ref}}(t_0)}\|_{\mathbf{G}(\gamma_{\text{ref}}(t_0))}, \end{aligned}$$

as desired ■

## 4.2. Lyapunov theory for linear vector fields on tangent bundles

In this section Lyapunov's direct method for linear vector fields on tangent bundles is introduced. The objective of the direct or "second" method is to infer the stability of a linear vector field  $Y$  with respect to a reference trajectory  $\gamma_{\text{ref}}$  without explicit knowledge of the flow of the vector field in question. As in the preceding, a reference vector field

$X_{\text{ref}} \in \text{LIC}^\infty(TM)$  with reference trajectory  $\gamma_{\text{ref}}$  is fixed. Let  $Y \in \text{LIC}^\infty(TTM)$  be a linear vector field over  $X_{\text{ref}} \in \text{LIC}^\infty(TM)$  with integral curve  $\Upsilon$ . In the classical linear systems theory, a Lyapunov candidate is defined by a quadratic form. In turn its derivative along trajectories of the linear system is also a quadratic form. The geometric definition that is provided here is analogous. A **Lyapunov candidate**  $V$  for  $Y$  is a symmetric  $(0, 2)$ -tensor field along  $\gamma_{\text{ref}}$  such that, for each  $t \geq t_0$ ,  $V(t)$  is positive-definite. The derivative of  $V$  along an integral curve  $\Upsilon$  of  $Y$  is given by

$$\frac{d}{dt}V(t)(\Upsilon(t), \Upsilon(t)) = \langle \mathcal{L}^{X_{\text{ref}}, \gamma_{\text{ref}}}(V^\flat(t)\Upsilon(t)); \Upsilon(t) \rangle + \langle V^\flat(t)\Upsilon(t); \mathcal{L}^{X_{\text{ref}}, \gamma_{\text{ref}}}(\Upsilon(t)) \rangle. \quad (4.2.1)$$

For notational convenience, define a symmetric quadratic form  $Q_V$  along  $\gamma_{\text{ref}}$  by  $-Q_V(t)(\Upsilon(t), \Upsilon(t)) = \frac{d}{dt}V(t)(\Upsilon(t), \Upsilon(t))$ . By choosing the natural coordinates for the tangent bundle, a linear vector field  $Y$  over  $X$  has the form

$$Y(t, x, v) = X^i(t, x) \frac{\partial}{\partial x^i} + Y_j^i(t, x) v^j \frac{\partial}{\partial v^i}.$$

Then (4.2.1) as a coordinate expression becomes

$$-Q_{V,ij}(t) = \dot{V}_{ij}(t) + V_{mj}(t)Y_i^m(t, x) + V_{im}(t)Y_j^m(t, x),$$

which resembles the classical differential Lyapunov equation. The stability of  $Y$  with respect to  $\gamma_{\text{ref}}$ , in the sense of Section 4.1, may be characterized in terms of  $V$  and  $Q_V$ .

**4.5 Proposition:** *Let  $X_{\text{ref}}, \gamma_{\text{ref}}, Y, \Upsilon, V$  and  $Q_V$  be as above. Then a linear vector field  $Y: I \times TM \rightarrow TTM$  with respect to  $\gamma_{\text{ref}}$  is*

(i) **G-stable** if  $Q_V(t)(\Upsilon(t), \Upsilon(t)) \geq 0$  and there exist a positive constant  $\alpha$  such that

$$\alpha \|v_{\gamma_{\text{ref}}(t)}\|_{\mathbf{G}(\gamma_{\text{ref}}(t))}^2 \leq V(t)(v_{\gamma_{\text{ref}}(t)}, v_{\gamma_{\text{ref}}(t)}),$$

for all  $v_{\gamma_{\text{ref}}(t)} \in T_{\gamma_{\text{ref}}(t)}M$ ,  $t \geq t_0$ ;

(ii) **uniformly G-stable** if  $Y$  is **G-stable** and there exists a positive constant  $\beta$  such that

$$V(t)(v_{\gamma_{\text{ref}}(t)}, v_{\gamma_{\text{ref}}(t)}) \leq \beta \|v_{\gamma_{\text{ref}}(t)}\|_{\mathbf{G}(\gamma_{\text{ref}}(t))}^2,$$

for all  $v_{\gamma_{\text{ref}}(t)} \in T_{\gamma_{\text{ref}}(t)}M$ ,  $t \geq t_0$ ;

(iii) **uniformly asymptotically G-stable** if  $Y$  is uniformly **G-stable** and there exists a positive constant  $\eta$  such that

$$Q_V(t)(\Upsilon(t), \Upsilon(t)) > \eta \|\Upsilon(t)\|_{\mathbf{G}(\gamma_{\text{ref}}(t))}^2, \quad \forall t \geq t_0.$$

**Proof:** Proof of (i): Fix  $t_0 \in I$  and let  $v_{\gamma_{\text{ref}}(t_0)} \in T_{\gamma_{\text{ref}}(t_0)}M$ . Let  $\Upsilon$  be an integral curve of  $Y$  such that  $\Upsilon(t_0) = v_{\gamma_{\text{ref}}(t_0)}$ . Since  $V$  is a Lyapunov candidate,

$$\begin{aligned} & \int_{t_0}^t \langle \mathcal{L}^{X_{\text{ref}}, \gamma_{\text{ref}}}(V^\flat(\sigma)\Upsilon(\sigma)); \Upsilon(\sigma) \rangle + \langle V^\flat(\sigma)\Upsilon(\sigma); \mathcal{L}^{X_{\text{ref}}, \gamma_{\text{ref}}}(\Upsilon(\sigma)) \rangle \, d\sigma \\ &= - \int_{t_0}^t Q_V(\sigma)(\Upsilon(\sigma), \Upsilon(\sigma)) \, d\sigma \\ &= V(t)(\Phi_{t_0, t}^Y(v_{\gamma_{\text{ref}}(t_0)}), \Phi_{t_0, t}^Y(v_{\gamma_{\text{ref}}(t_0)})) - V(t_0)(v_{\gamma_{\text{ref}}(t_0)}, v_{\gamma_{\text{ref}}(t_0)}). \end{aligned}$$

Given the hypothesis,

$$Q_V(t)(\Upsilon(t), \Upsilon(t)) \geq 0 \text{ and } \alpha \|v_{\gamma_{\text{ref}}(t)}\|_{\mathbf{G}(\gamma_{\text{ref}}(t))}^2 \leq V(t)(v_{\gamma_{\text{ref}}(t)}, v_{\gamma_{\text{ref}}(t)})$$

for all  $v_{\gamma_{\text{ref}}(t)} \in T_{\gamma_{\text{ref}}(t)}M$ ,  $t \geq t_0$ . Thus, the following inequality holds:

$$\alpha \|\Phi_{t_0,t}^Y(v_{\gamma_{\text{ref}}(t_0)})\|_{\mathbf{G}(\gamma_{\text{ref}}(t))}^2 \leq V(t)(\Phi_{t_0,t}^Y(v_{\gamma_{\text{ref}}(t_0)}), \Phi_{t_0,t}^Y(v_{\gamma_{\text{ref}}(t_0)})) \leq V(t_0)(v_{\gamma_{\text{ref}}(t_0)}, v_{\gamma_{\text{ref}}(t_0)}).$$

For fixed  $t_0$ ,  $V(t_0)(v_{\gamma_{\text{ref}}(t_0)}, v_{\gamma_{\text{ref}}(t_0)})$  is a positive-definite symmetric bilinear form on the vector space  $T_{\gamma_{\text{ref}}(t_0)}M$  which implies that there exists  $\kappa_{t_0} > 0$  such that  $V(t_0)(v_{\gamma_{\text{ref}}(t_0)}, v_{\gamma_{\text{ref}}(t_0)}) \leq \kappa_{t_0} \|v_{\gamma_{\text{ref}}(t_0)}\|_{\mathbf{G}(\gamma_{\text{ref}}(t_0))}^2$ . The result follows:

$$\|\Phi_{t_0,t}^Y(v_{\gamma_{\text{ref}}(t_0)})\|_{\mathbf{G}(\gamma_{\text{ref}}(t))} \leq \sqrt{\frac{\kappa_{t_0}}{\alpha}} \|v_{\gamma_{\text{ref}}(t_0)}\|_{\mathbf{G}(\gamma_{\text{ref}}(t_0))}.$$

Proof of (ii): Using similar arguments as in part (i), for  $t \geq t_0$  and the added hypothesis yield

$$\alpha \|\Phi_{t_0,t}^Y(v_{\gamma_{\text{ref}}(t_0)})\|_{\mathbf{G}(\gamma_{\text{ref}}(t))}^2 \leq V(t_0)(v_{\gamma_{\text{ref}}(t_0)}, v_{\gamma_{\text{ref}}(t_0)}) \leq \beta \|v_{\gamma_{\text{ref}}(t_0)}\|_{\mathbf{G}(\gamma_{\text{ref}}(t_0))}^2,$$

and result follows:

$$\|\Phi_{t_0,t}^Y(v_{\gamma_{\text{ref}}(t_0)})\|_{\mathbf{G}(\gamma_{\text{ref}}(t))} \leq \sqrt{\frac{\beta}{\alpha}} \|v_{\gamma_{\text{ref}}(t_0)}\|_{\mathbf{G}(\gamma_{\text{ref}}(t_0))}.$$

Proof of (iii): For  $\epsilon > 0$  it is shown that for  $T = \frac{\beta}{\eta\epsilon}$  we have

$$\|\Phi_{t_0,t}^Y(v_{\gamma_{\text{ref}}(t_0)})\|_{\mathbf{G}(\gamma_{\text{ref}}(t))}^2 < \epsilon \|v_{\gamma_{\text{ref}}(t_0)}\|_{\mathbf{G}(\gamma_{\text{ref}}(t_0))}^2,$$

for all  $t \geq t_0 + T$ . Let  $\epsilon > 0$ ,  $T = \frac{\beta}{\eta\epsilon}$ , and suppose

$$\|\Phi_{t_0,t}^Y(v_{\gamma_{\text{ref}}(t_0)})\|_{\mathbf{G}(\gamma_{\text{ref}}(t))}^2 \geq \epsilon \|v_{\gamma_{\text{ref}}(t_0)}\|_{\mathbf{G}(\gamma_{\text{ref}}(t_0))}^2,$$

for  $t \geq t_0 + T$ . Using the arguments, from the proof of (i),

$$\begin{aligned} 0 < \alpha \|\Phi_{t_0,t}^Y(v_{\gamma_{\text{ref}}(t_0)})\|_{\mathbf{G}(\gamma_{\text{ref}}(t))}^2 &\leq V(t)(\Phi_{t_0,t}^Y(v_{\gamma_{\text{ref}}(t_0)}), \Phi_{t_0,t}^Y(v_{\gamma_{\text{ref}}(t_0)})) \\ &= V(t_0)(v_{\gamma_{\text{ref}}(t_0)}, v_{\gamma_{\text{ref}}(t_0)}) - \\ &\quad \int_{t_0}^t Q_V(\sigma)(\Phi_{t_0,\sigma}^Y(v_{\gamma_{\text{ref}}(t_0)}), \Phi_{t_0,\sigma}^Y(v_{\gamma_{\text{ref}}(t_0)})) \, d\sigma \\ &< \beta \|v_{\gamma_{\text{ref}}(t_0)}\|_{\mathbf{G}(\gamma_{\text{ref}}(t_0))}^2 - \int_{t_0}^t \eta \|\Phi_{t_0,\sigma}^Y(v_{\gamma_{\text{ref}}(t_0)})\|_{\mathbf{G}(\gamma_{\text{ref}}(\sigma))}^2 \, d\sigma \\ &\leq \beta \|v_{\gamma_{\text{ref}}(t_0)}\|_{\mathbf{G}(\gamma_{\text{ref}}(t_0))}^2 - \\ &\quad \int_{t_0}^{t_0+T} \eta \|\Phi_{t_0,\sigma}^Y(v_{\gamma_{\text{ref}}(t_0)})\|_{\mathbf{G}(\gamma_{\text{ref}}(\sigma))}^2 \, d\sigma \\ &\leq \beta \|v_{\gamma_{\text{ref}}(t_0)}\|_{\mathbf{G}(\gamma_{\text{ref}}(t_0))}^2 - \int_{t_0}^{t_0+T} \eta\epsilon \|v_{\gamma_{\text{ref}}(t_0)}\|_{\mathbf{G}(\gamma_{\text{ref}}(t_0))}^2 \, d\sigma \\ &\leq (\beta - \eta\epsilon T) \|v_{\gamma_{\text{ref}}(t_0)}\|_{\mathbf{G}(\gamma_{\text{ref}}(t_0))}^2 \\ &= 0, \end{aligned}$$

giving a contradiction. ■



### 4.3. Stabilization of the linearization

Let  $X_{\text{ref}}$  be a reference vector field for the affine system  $\mathcal{A}$  with  $\gamma_{\text{ref}}: I \rightarrow M$  a reference trajectory. To avoid complications, suppose that  $L(\mathcal{A})$  has constant rank and so is a vector bundle over  $M$ . Let  $L(TM; L(\mathcal{A}))$  be the set of vector bundle mappings from  $TM$  to  $L(\mathcal{A})$  over  $\text{id}_M$ . A **linear state feedback** along  $\gamma_{\text{ref}}$  is then a section  $F$  of the bundle consisting of the fibers of  $L(TM; L(\mathcal{A}))$  over  $\gamma_{\text{ref}}$ . Thus  $F$  assigns to each point  $t \in I$  a linear map  $F(t): T_{\gamma_{\text{ref}}(t)}M \rightarrow L(\mathcal{A})_{(t, \gamma_{\text{ref}}(t))}$ . For such a linear state feedback, the **closed-loop system** is then the linear  $\text{LIC}^\infty$  vector field on  $TM$  defined by  $X_{\text{ref}}^F(t, v_x) = X_{\text{ref}}^T(t, v_x) + \text{vlft}(F(t)(v_x))$ . Note that the integral curves of  $X_{\text{ref}}^F$  with initial conditions projecting to  $\gamma_{\text{ref}}$  will project to  $\gamma_{\text{ref}}$ . Therefore, given a linear state feedback  $F$ , the stability of the linear vector field  $X_{\text{ref}}^F$  relative to  $\mathbf{G}$  is as defined in Section 4.1.

To obtain uniform asymptotic stability, various uniformity bounds on the problem data are required and, as in Section 1.2.3, the class of linear systems is restricted to those that are uniformly controllable.

**4.6 Definition:** Let  $\mathcal{A}$  be a time-dependent affine system on  $M$  with  $X_{\text{ref}}$  a reference vector field and  $\gamma_{\text{ref}}: I \rightarrow M$  a reference trajectory. Let  $\mathbf{G}$  be a Riemannian metric on  $M$ .

- (i) The linearisation  $\mathcal{A}_{\text{ref}}^T$  along  $\gamma_{\text{ref}}$  is  **$\mathbf{G}$ -uniformly controllable** if there exist strictly increasing functions  $\alpha, \beta: \mathbb{R}_+ \rightarrow \mathbb{R}_+$  with  $\alpha(0) = 1, \beta(0) = 1$ , such that

(a)

$$\|\Phi_{\tau, t}^{X_{\text{ref}}^T}(v_\tau)\|_{\mathbf{G}(\gamma_{\text{ref}}(t))} \leq \alpha(|t - \tau|) \|v_\tau\|_{\mathbf{G}(\gamma_{\text{ref}}(\tau))}, \quad (4.3.1)$$

$$\|\Phi_{\tau, t}^{X_{\text{ref}}^{T*}}(\eta_\tau)\|_{\mathbf{G}(\gamma_{\text{ref}}(t))} \leq \beta(|t - \tau|) \|\eta_\tau\|_{\mathbf{G}(\gamma_{\text{ref}}(\tau))}, \quad (4.3.2)$$

for all  $t, \tau \in I$ ,  $v_\tau \in T_{\gamma_{\text{ref}}(\tau)}M$ , and  $\eta_\tau \in T_{\gamma_{\text{ref}}(\tau)}^*M$ , and

- (b) there exists a constant  $\sigma$  such that, for each  $t \in I$ ,

$$0 < \alpha_3(\sigma) \|\eta_t\|_{\mathbf{G}(\gamma_{\text{ref}}(t))}^2 < W(t, t + \sigma)(\eta_t, \eta_t) < \alpha_4(\sigma) \|\eta_t\|_{\mathbf{G}(\gamma_{\text{ref}}(t))}^2,$$

where  $\eta_t \in T_{\gamma_{\text{ref}}(t)}^*M$  and  $W(t, \tau)$  is defined in (3.7.1).

- (ii) A Riemannian metric  $\mathbf{G}$  is  **$X_{\text{ref}}$ -compatible** if the linearisation  $\mathcal{A}_{\text{ref}}^T$  along  $\gamma_{\text{ref}}$  is  $\mathbf{G}$ -uniformly controllable.

Consider the following optimal closed-loop system  $X_{\text{ref}}^{\bar{K}}: I \times TM \rightarrow TTM$  defined by

$$X_{\text{ref}}^{\bar{K}}(t, v_x) = X_{\text{ref}}^T(t, v_x) - \text{vlft}((\iota(t)R^\sharp(t)\iota^*(t)\bar{K}^\flat(t))(v_x)). \quad (4.3.3)$$

Is  $X_{\text{ref}}^{\bar{K}}$  with respect to  $\gamma_{\text{ref}}$  uniformly asymptotically  $\mathbf{G}$ -stable? In the classical approach, Kalman [1960] proves this using the optimal cost associate with the infinite time LQR problem as a Lyapunov function. The approach here is analogous.

**4.7 Theorem: (LQR stability)** *Let  $X_{\text{ref}}$  be a reference vector field for an affine system  $\mathcal{A}$  along with a differentiable reference trajectory  $\gamma_{\text{ref}}$ . Let  $Q$  and  $R$  be as above with the property that there exists constants  $\alpha_1, \alpha_2, \beta_1, \beta_2 > 0$  such that*

$$\begin{aligned} \alpha_1 \|v_{\gamma_{\text{ref}}(t)}\|_{\mathbf{G}(\gamma_{\text{ref}}(t))}^2 &\leq Q(t)(v_{\gamma_{\text{ref}}(t)}, v_{\gamma_{\text{ref}}(t)}) \leq \alpha_2 \|v_{\gamma_{\text{ref}}(t)}\|_{\mathbf{G}(\gamma_{\text{ref}}(t))}^2, \\ \beta_1 \|\iota(t)w_{\gamma_{\text{ref}}(t)}\|_{\mathbf{G}(\gamma_{\text{ref}}(t))}^2 &\leq R(t)(w_{\gamma_{\text{ref}}(t)}, w_{\gamma_{\text{ref}}(t)}) \leq \beta_2 \|\iota(t)w_{\gamma_{\text{ref}}(t)}\|_{\mathbf{G}(\gamma_{\text{ref}}(t))}^2, \end{aligned}$$

where  $v_{\gamma_{\text{ref}}(t)} \in T_{\gamma_{\text{ref}}(t)}M$ ,  $w_{\gamma_{\text{ref}}(t)} \in L(\mathcal{A})_{\gamma_{\text{ref}}(t)}$ . If the Riemannian metric  $\mathbf{G}$  is  $X_{\text{ref}}$ -compatible, then the closed-loop system

$$\mathcal{A}_{\text{ref}}^{\bar{K}}(t, v_x) = \{X_{\text{ref}}^T(t, v_x) - \text{vft}((\iota(t)R^\sharp(t)\iota^*(t)\bar{K}^\flat(t))(v_x)) \mid (t, v_x) \in \mathbb{R} \times TM\}.$$

is uniformly asymptotically  $\mathbf{G}$ -stable.

**Proof:** It suffices to show the existence of a Lyapunov candidate for the linear vector field  $X_{\text{ref}}^{\bar{K}}: I \times TM \rightarrow TTM$  over  $X_{\text{ref}}$  defined by

$$X_{\text{ref}}^{\bar{K}}(t, v_x) = X_{\text{ref}}^T(t, v_x) - \text{vft}((\iota(t)R^\sharp(t)\iota^*(t)\bar{K}^\flat(t))(v_x)).$$

Recall that a Lyapunov candidate for  $X_{\text{ref}}^{\bar{K}}$  is a symmetric  $(0, 2)$ -tensor field along  $\gamma_{\text{ref}}$  that is positive-definite for each  $t \geq t_0$ . Thus the optimal feedback  $\bar{K}$  is a Lyapunov candidate for  $X_{\text{ref}}^{\bar{K}}$ . Thus, the vector field  $X_{\text{ref}}^{\bar{K}}$  with integral curve  $\Upsilon$  is uniformly asymptotically  $\mathbf{G}$ -stable if

1. there exists positive constants  $c_1$ ,  $c_2$ , and  $c$  such that

$$c_1 \|v_{\gamma_{\text{ref}}(t)}\|_{\mathbf{G}(\gamma_{\text{ref}}(t))}^2 \leq \bar{K}(v_{\gamma_{\text{ref}}(t)}, v_{\gamma_{\text{ref}}(t)}) \leq c_2 \|v_{\gamma_{\text{ref}}(t)}\|_{\mathbf{G}(\gamma_{\text{ref}}(t))}^2 \quad \text{and}$$

2.  $\langle \mathcal{L}^{X_{\text{ref}}, \gamma_{\text{ref}}}(\bar{K}^\flat(t)\Upsilon(t)); \Upsilon(t) \rangle + \langle \bar{K}^\flat(t)\Upsilon(t); \mathcal{L}^{X_{\text{ref}}, \gamma_{\text{ref}}}(\Upsilon(t)) \rangle < -c \|\Upsilon(t)\|_{\mathbf{G}(\gamma_{\text{ref}}(t))}^2$ .

**Proof of (1):** The upper bound is obtained in a similar fashion as the standard case in Theorem 1.7. Using Lemma 3.22 with  $t_2 = t_0 + \sigma$ , where  $\sigma$  is the constant prescribed by  $\mathbf{G}$ -uniform controllability, and using the added hypotheses on the cost data gives

$$\bar{K}(t_0)(\Upsilon_*(t_0), \Upsilon_*(t_0)) \leq \frac{1}{2} \int_{t_0}^{t_2} \alpha_2 \|\Upsilon_1(\tau)\|_{\mathbf{G}(\gamma_{\text{ref}}(\tau))}^2 + \beta_2 \|\iota(\tau)X_1(\tau)\|_{\mathbf{G}(\gamma_{\text{ref}}(\tau))}^2 \, d\tau, \quad (4.3.4)$$

where the trajectory is given by

$$\Upsilon_1(\tau) = \Phi_{t_0, \tau}^{X_{\text{ref}}^T}(\Upsilon(t_0) - W^\flat(t_0, \tau)W^\sharp(t_0, t_2)\Upsilon(t_0)).$$

From Lemma 3.22 the corresponding section of  $L(\mathcal{A})$  is

$$X_1(\tau) = -R(\tau)^\sharp \iota^*(\tau) \Phi_{t_0, \tau}^{X_{\text{ref}}^T} \eta,$$

where  $\eta \in T_{\gamma_{\text{ref}}(t_0)}^*M$  satisfies  $W^\flat(t_0, t_2)\eta = \Upsilon(t_0)$ . First consider an upper bound on the trajectory given the following computation:

$$\begin{aligned} \|\Upsilon_1(\tau)\|^2 &= \|\Phi_{t_0, \tau}^{X_{\text{ref}}^T}(\Upsilon(t_0) - W^\flat(t_0, \tau)W^\sharp(t_0, t_2)\Upsilon(t_0))\|_{\mathbf{G}(\gamma_{\text{ref}}(\tau))}^2 \\ &\leq \|\Phi_{t_0, \tau}^{X_{\text{ref}}^T}\|_{\mathbf{G}_{\gamma_{\text{ref}}, \text{op}}}^2 \|(W^\flat(t_0, t_2) - W^\flat(t_0, \tau))W^\sharp(t_0, t_2)\|_{\mathbf{G}_{\gamma_{\text{ref}}, \text{op}}}^2 \|\Upsilon(t_0)\|_{\mathbf{G}(\gamma_{\text{ref}}(t_0))}^2 \\ &\leq \|\Phi_{t_0, \tau}^{X_{\text{ref}}^T}\|_{\mathbf{G}_{\gamma_{\text{ref}}, \text{op}}}^2 \|W^\sharp(t_0, t_2)\|_{\mathbf{G}_{\gamma_{\text{ref}}, \text{op}}}^2 \|\Upsilon(t_0)\|_{\mathbf{G}(\gamma_{\text{ref}}(t_0))}^2 \\ &\leq c_3(|\tau - t_0|)\alpha_3(\sigma) \|\Upsilon(t_0)\|_{\mathbf{G}(\gamma_{\text{ref}}(t_0))}^2, \end{aligned}$$

where

1. line 3 follows from the observations that the operator  $(W^b(t_0, t_2) - W^b(t_0, \tau))$ , is non-negative, zero at  $\tau = t_2$ , takes the form  $W^b(t_0, t_2)$  at  $\tau = t_0$ , and has a derivative with respect to  $\tau$  that is negative for all  $\tau \in [t_0, t_2]$ ;
2. line 4 follows from  $\mathbf{G}$ -uniform controllability since  $t_2 = t_0 + \sigma$ .

Similar calculations provide an upper bound on  $X_1(\tau)$ . Consider the following computation:

$$\begin{aligned} \|\iota(\tau)X_1(\tau)\|_{\mathbf{G}(\gamma_{\text{ref}}(\tau))}^2 &= \|\iota(\tau)R(\tau)\sharp\iota^*(\tau)\Phi_{t_0, \tau}^{X^{T^*}}\eta\|_{\mathbf{G}(\gamma_{\text{ref}}(\tau))}^2 \\ &\leq c_4(|t_0 - \tau|)\left\|\frac{d}{d\tau}W(t_0, \tau)\eta\right\|_{\mathbf{G}(\gamma_{\text{ref}}(t_0))}^2 \\ &\leq c_4(|t_0 - \tau|)\frac{d}{d\tau}\alpha_4(|\tau - t_0|)\|\eta\|_{\mathbf{G}(\gamma_{\text{ref}}(t_0))}^2, \end{aligned}$$

where

1. line 2 uses Lemma 3.21 and the  $\mathbf{G}$ -uniform controllability hypothesis;
2. line 3 uses the derivative of the  $\mathbf{G}$ -uniform controllability condition.

The above computations, together with (4.3.4), imply that

$$\begin{aligned} \bar{K}(t_0)(\Upsilon_*(t_0), \Upsilon_*(t_0)) &\leq \frac{1}{2} \int_{t_0}^{t_2} \alpha_2 c_3(|\tau - t_0|) \alpha_3(\sigma) \|\Upsilon(t_0)\|_{\mathbf{G}(\gamma_{\text{ref}}(t_0))}^2 + \\ &\quad \beta_2 \alpha_3(\sigma) c_4(|t_0 - \tau|) \left\| \frac{d}{d\tau} W(t_0, \tau) \right\|_{\mathbf{G}_{\gamma_{\text{ref}}, \text{op}}}^2 \|\Upsilon(t_0)\|_{\mathbf{G}(\gamma_{\text{ref}}(t_0))}^2 \, d\tau \\ &= \frac{1}{2} \int_0^{t_2 - t_0} \alpha_2 \alpha_3 c_3(s) \|\Upsilon(t_0)\|_{\mathbf{G}(\gamma_{\text{ref}}(t_0))}^2 + \\ &\quad \beta_2 \alpha_3 c_4(s) \frac{d}{ds} \alpha_4(s) \|\Upsilon(t_0)\|_{\mathbf{G}(\gamma_{\text{ref}}(t_0))}^2 \, ds \\ &\leq c_2(\sigma) \|\Upsilon(t_0)\|_{\mathbf{G}(\gamma_{\text{ref}}(t_0))}^2. \end{aligned}$$

The lower bound is constructed using the same procedure as above since the inverse of  $\bar{K}(t)$  exists for each  $t$  and satisfies the Riccati equation,

$$\mathcal{L}^{X_{\text{ref}}, \gamma_{\text{ref}}} P(t) - P(t)Q(t)P(t) + \iota(t)R^\sharp(t)\iota^*(t) = 0,$$

of the “dual system.”

Proof of 2: The derivative of  $\bar{K}$  along a trajectory of the closed looped system,

$$\begin{aligned} \frac{d\bar{K}}{dt}(\Upsilon(t), \Upsilon(t)) &= \langle \mathcal{L}^{X_{\text{ref}}, \gamma_{\text{ref}}}(\bar{K}^b(t)\Upsilon(t)); \Upsilon(t) \rangle + \langle \bar{K}^b(t)\Upsilon(t); \mathcal{L}^{X_{\text{ref}}, \gamma_{\text{ref}}}(\Upsilon(t)) \rangle \\ &= - \langle Q^b(t)\Upsilon(t); \Upsilon(t) \rangle - \langle \bar{K}^b(t)\Upsilon(t); \iota(t)R^\sharp(t)\iota^*(t)\bar{K}^b(t)\Upsilon(t) \rangle, \end{aligned}$$

is negative as required. ■

# Chapter 5

## Conclusions and future work

### 5.1. Conclusions

In this thesis a geometric understanding of Jacobian linearization along a non-trivial reference trajectory and Kalman's theory of optimal control is given for affine systems on a differentiable manifold  $M$ . The basis of this geometric formulation involved embedding the reference trajectory  $\gamma_{\text{ref}}$  for an affine system  $\mathcal{A}$  on  $M$  as an integral curve of a  $\text{LIC}^\infty$  reference vector field  $X_{\text{ref}}$  on  $M$ . Given  $X_{\text{ref}}$ , differentiation along the reference trajectory is defined by Sussmann's Lie derivative operator,  $\mathcal{L}^{X_{\text{ref}}, \gamma_{\text{ref}}}$ . This differential operator together with the tangent, cotangent, and vertical lifts, were used to clarify the geometric structure corresponding to what is commonly done in the standard approach as outlined in Section 1.1.

### 5.2. Future work

The geometry of Jacobian Linearization as presented in this thesis provides a framework in which to approach other standard concepts in control theory associated with linearization.

**5.2.1. Open questions about stability and stabilization.** Using the definitions in Section 4.1, consider the stability  $X_{\text{ref}}^T$  along  $\gamma_{\text{ref}}$  relative to the metric  $d_{\mathbb{G}}$  and consider the stability of the linearisation  $X_{\text{ref}}^T$  relative to  $\mathbb{G}$ .

**5.1 Question:** Does uniform asymptotic stability of the linearisation imply uniform exponential stability of  $X_{\text{ref}}$ ?

In the standard case, this is of course well known and follows from Lyapunov's second method. The required Lyapunov function to show uniform exponential stability, in fact, is the same function used to guarantee the uniform asymptotic stability of the linearization [Vidyasagar 2002]. For a time-varying linear system that is uniformly asymptotically stable, such a Lyapunov function is defined by

$$V(t) = \int_t^\infty \Phi^*(\sigma, t) M(\sigma) \Phi(\sigma, t) d\sigma, \quad (5.2.1)$$

where  $M: I \rightarrow L(\mathbb{R}^n; \mathbb{R}^n)$  is bounded and symmetric for each  $t \in I$  [Brockett 1970]. To answer Question 5.1 in a geometric context requires first making sense of (5.2.1). Then,

the next hurdle is to “transfer” a Lyapunov function for the linear vector field on  $TM$  to Lyapunov function for  $X_{\text{ref}}$  on  $M$ .

If  $X_{\text{ref}}$  is not stable with respect to some choice of metric, then perhaps it is possible to stabilize it under feedback. In Section 4.3 the linearization  $\mathcal{A}_{\text{ref}}^T$  of an affine system  $\mathcal{A}$  along a non-trivial reference trajectory  $\gamma_{\text{ref}}$  was uniformly asymptotically stabilized using a linear state-feedback obtained from solving an infinite time LQR problem. In the setting of Section 1.1, a stabilizing linear state-feedback would be then implemented to locally uniformly exponentially stabilize the non-linear system along the reference trajectory. This is easily done in the standard setup since the state space is naturally identified with each tangent space. From a geometric point of view this raises the following question.

**5.2 Question:** For a linear state-feedback  $F: I \rightarrow L(TM; L(\mathcal{A}))$  which uniformly asymptotically stabilizes the linearization  $\mathcal{A}_{\text{ref}}^T$ , how can  $F$  be implemented with the affine system  $\mathcal{A}$ ? And once a method of implementation is understood, is it ensured that the affine system  $\mathcal{A}$  is locally uniformly exponentially stabilized along the reference trajectory?

The feedback implementation problem amounts to interpreting geometrically the process of choosing coordinates on a neighbourhood of the reference trajectory. If  $\gamma_{\text{ref}}$  is assumed to be differentiable, then  $\text{image}(\gamma_{\text{ref}})$  is an immersed submanifold of  $M$  and has a well defined pullback bundle  $\gamma_{\text{ref}}^*(TM)$ . Now, consider a neighbourhood  $J \subset I$  of  $t$ , referring to [Lang 2002], the normal bundle  $N(\gamma_{\text{ref}})$  is a vector bundle over  $\gamma_{\text{ref}}(J)$  with total space  $N(\gamma_{\text{ref}}) = \gamma_{\text{ref}}^*(TM)|_J / T\gamma_{\text{ref}}|_{TJ}$ . Typically, a tabular neighbourhood around  $\gamma_{\text{ref}}(J)$  is constructed using the exponential map, provided by the Levi-Civita connection, [Lang 2002]. Furthermore, the exponential provides a diffeomorphism from a neighbourhood  $W$  of the zero section of  $N(\gamma_{\text{ref}})$  to a neighbourhood  $U$  in  $M$  and the closed-loop system can be define in  $U$  by

$$X_{\text{ref}}^{F,G}(t, x) = X_{\text{ref}}(t, x) + \exp(F(t)(\exp^{-1}(x))).$$

Another possible approach is to replace the exponential map with a diffeomorphism that acts like the exponential. An “exponential-like” map, called a near identity diffeomorphism at  $x_0$ , for  $x_0 \in M$  is provided by [Bullo and Lewis 2005, Section 10.3.3] as a means to implementing a linear-state feedback on a non-linear system.

**5.2.2. The free interval Maximum principle.** In Chapter 3 an LQR version of the Maximum Principle was proved using fixed interval multi-needle variations. In general, it is possible to allow variations where the time interval of the multi-needle variations is allowed to vary. The definition of free interval multi-needle variations is as follows.

**5.3 Definition:** Let  $t_0, t_1 \in \mathbb{R}$  satisfy  $t_0 < t_1$ . Let  $\hat{\mathcal{A}}_{\text{ref}}^T$  be an extended system with initial conditions  $\hat{\Upsilon}(t_0)$  and  $X$  a section of  $L(\mathcal{A})$ .

- (i) **Free interval multi-needle variation data** is a pair  $(\Theta, \Psi)$  where  $\Theta = \{\theta_1, \dots, \theta_k\}$  is fixed interval multi-needle variation data and where  $\Psi = (\tau, \delta\tau) \in [t_0, t_1] \times \mathbb{R}$  satisfies  $\tau > \tau_j$  where  $\tau$  and each  $\tau_j$  associated with  $\theta_j$  are Lebesgue points,  $j \in \{1, \dots, k\}$ .
- (ii) The **variation of  $X$**  associated to the free interval multi-needle variation data  $(\Theta, \Psi)$

is a map  $(s, t) \mapsto X_{(\Theta, \Psi)}(s, t)$  defined by

$$X_{(\Theta, \Psi)}(s, t) = \begin{cases} X_{\Theta}(t), & t \in [t_0, \tau], \\ X(\tau), & t \in (\tau, \tau + s\delta\tau], \\ X_{\Theta}(t - s\delta\tau), & t \in (\tau + s\delta\tau, t_1 + s\delta\tau], \end{cases} \quad (5.2.2)$$

when  $\delta\tau \geq 0$  and by

$$X_{(\Theta, \Psi)}(s, t) = \begin{cases} X_{\Theta}(t), & t \in [t_0, \tau + s\delta\tau], \\ X_{\Theta}(t - s\delta\tau), & t \in (\tau + s\delta\tau, t_1 + s\delta\tau], \end{cases} \quad (5.2.3)$$

when  $\delta\tau < 0$ .

- (iii) Let  $\Sigma(X_{(\Theta, \Psi)}(s, \cdot), \hat{\Upsilon}(t_0), t_0, \cdot)$  be the trajectory corresponding to  $X_{(\Theta, \Psi)}(s, \cdot)$  with the free interval multi-needle variation data  $(\Theta, \Psi)$ . The **free interval multi-needle variation** associated with  $X$  is defined by

$$v_{(\Theta, \Psi)} = \left. \frac{d}{ds} \right|_{s=0} \Sigma(X_{(\Theta, \Psi)}(s, \cdot), \hat{\Upsilon}(t_0), t_0, \cdot) \quad (5.2.4)$$

and projects to  $\gamma_{\text{ref}}$ .

Following the fixed interval case, the next step is to understand the form of free interval multi-needle variations at Lebesgue points. In the standard case, the dynamics of the extended system appear and naïvely following this setup it is tempting to write

$$v_{(\Theta, \Psi)} = \sum_{j=1}^k \Phi_{\tau_j, t} v_{\theta_j} + \delta\tau \Phi_{\tau, t} (X_{\text{ref}}^T(\tau, \Upsilon(\tau)) + \text{vertical}(X(\tau))(\Upsilon(\tau)), L(\Upsilon(\tau), X(\tau))),$$

where  $\Sigma(X(t), \hat{\Upsilon}(t_0), t_0, t) = (\Upsilon(t), \Upsilon^0(t))$ . However, the dynamics of the extend system  $(X_{\text{ref}}^T(\tau, \Upsilon(\tau)) + \text{vertical}(X(\tau))(\Upsilon(\tau)), L(\Upsilon(\tau), X(\tau)))$  take their values in the different space than that to which  $\Phi$  can be applied. Thus, the free interval version of the Maximum Principle needs formulating.

# Bibliography

- Abraham, R., Marsden, J. E., and Ratiu, T. S. [1988] *Manifolds, tensor analysis, and applications*, 2nd edition, number 75 in Applied Mathematical Sciences, Springer-Verlag, ISBN 0-387-96790-7.
- Athans, M. and Falb, P. L. [1966] *Optimal control. An introduction to the theory and its applications*, McGraw-Hill Book Co., New York.
- Bellman, R. [1960] *Introduction to matrix analysis*, McGraw-Hill Book Co., Inc., New York.
- Bianchini, R. M. and Stefani, G. [1993] *Controllability along a trajectory: a variational approach*, SIAM Journal on Control and Optimization, **31**(4), 900–927.
- Brockett, R. W. [1970] *Finite dimensional linear systems*, John Wiley and Sons, New York, New York, ISBN 471-10585-6.
- Bullo, F. and Lewis, A. D. [2005] *Geometric control of mechanical systems*, volume 49 of *Texts in Applied Mathematics*, Springer-Verlag, New York, ISBN 0-387-22195-6.
- Dunford, N. and Schwartz, J. T. [1963] *Linear operators. Part II: Spectral theory. Self adjoint operators in Hilbert space*, Interscience Publishers John Wiley & Sons New York-London.
- Hirschorn, R. M. and Lewis, A. D. [2002] *Geometric local controllability: Second-order conditions*, Preprint, available online at <http://penelope.mast.queensu.ca/~andrew/>.
- Ikeda, M., Maeda, H., and Kodama, S. [1972] *Stabilization of linear systems*, Journal of the Society of Industrial and Applied Mathematics, Series A Control, **10**(4), 716–729.
- Kalman, R. E. [1960] *Contributions to the theory of optimal control*, Bol. Soc. Mat. Mexicana (2), **5**, 102–119.
- Khalil, H. K. [1996] *Nonlinear systems*, 2nd edition, Prentice-Hall, Englewood Cliffs, NJ, ISBN 0-13-228024-8.
- Lang, S. [2002] *Introduction to differentiable manifolds*, 2nd edition, Universitext, Springer-Verlag, New York, ISBN 0-387-95477-5.
- Lee, E. B. and Markus, L. [1986] *Foundations of optimal control theory*, 2nd edition, Robert E. Krieger Publishing Co. Inc., Melbourne, FL, ISBN 0-89874-807-0.

- Pontryagin, L. S., Boltyanskii, V. G., Gamkrelidze, R. V., and Mishchenko, E. F. [1961] *Matematicheskaya teoriya optimalnykh protsessov*, Gosudarstv. Izdat. Fiz.-Mat. Lit., Moscow, reprint of translation: [Pontryagin, Boltyanskii, Gamkrelidze and Mishchenko 1986].
- [1986] *The Mathematical Theory of Optimal Processes*, Gordon & Breach Science Publishers, New York, ISBN 2-88124-134-4, reprint of 1962 translation from the Russian by K. N. Trirogoff.
- Sontag, E. D. [1998] *Mathematical control theory: Deterministic finite dimensional systems*, 2nd edition, number 6 in Texts in Applied Mathematics, Springer-Verlag, New York-Heidelberg-Berlin, ISBN 0-387-98489-5.
- Sussmann, H. J. [1997] *An introduction to the coordinate-free maximum principle*, in *Geometry of Feedback and Optimal Control*, B. Jakubczyk and W. Respondek, editors, pages 463–557, Dekker Marcel Dekker, New York.
- Vidyasagar, M. [2002] *Nonlinear systems analysis*, volume 42 of *Classics in Applied Mathematics*, Society for Industrial and Applied Mathematics (SIAM), Philadelphia, PA, ISBN 0-89871-526-1, reprint of the second (1993) edition.