A geometric approach to energy shaping[1]

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Abstract

In this thesis is initiated a more systematic geometric exploration of energy shaping. Most of the previous results have been dealt with particular cases and neither the existence nor the space of solutions has been discussed with any degree of generality. The geometric theory of partial differential equations originated by Goldschmidt and Spencer in late 1960's is utilized to analyze the partial differential equations in energy shaping. The energy shaping partial differential equations are described as a fibered submanifold of a k-jet bundle of a fibered manifold. By revealing the nature of kinetic energy shaping, similarities are noticed between the problem of kinetic energy shaping and some well-known problems in Riemannian geometry. In particular, there is a strong similarity between kinetic energy shaping and the problem of finding a metric connection initiated by Eisenhart and Veblen. We notice that the necessary conditions for the set of so-called λ-equation restricted to the control distribution are related to the Ricci identity, similarly to the Eisenhart and Veblen metric connection problem. Finally, the set of λ-equations for kinetic energy shaping are coupled with the integrability results of potential energy shaping. This gives new insights for answering some key questions in energy shaping that have not been addressed to this point. The procedure shows how a poor design of closed-loop metric can make it impossible to achieve any flexibility in the character of the possible closed-loop potential function. The integrability results of this thesis have been used to answer some interesting questions about the energy shaping method. In particular, a geometric proof is provided which shows that linear controllability is sufficient for energy shaping of linear simple mechanical systems. Furthermore, it is shown that all linearly controllable simple mechanical control systems with one degree of underactuation can be stabilized using energy shaping feedback. The result is geometric and completely characterizes the energy shaping problem for these systems. Using the geometric approach of this thesis, some new open problems in energy shaping are formulated. In particular, we give ideas for relating the kinetic energy shaping problem to a problem on holonomy groups. Moreover, we suggest that the so-called Fakras lemma might be used for investigating the stabilization condition of energy shaping.

Contents

Chapter 1

Introduction

Consider the following control problem: given a mechanical system with an unstable equilibrium at q_0, stabilize the system using feedback. One of the recent developments in the stabilization of equilibria is the *energy shaping method*. The key idea concerns the construction of a feedback for which the closed-loop system possesses the structure of a mechanical system. A feedback so obtained is called an *energy shaping feedback* and the procedure by which it is obtained is called *energy shaping*. In the classical notion of energy shaping, the *assumed* method consists of two stages: shaping the kinetic energy of the system—so-called *kinetic energy shaping*—and changing the potential energy of the system—so-called *potential energy shaping*. If such an energy shaping feedback exists, then for stability one has to ensure that the Hessian of the closed-loop potential energy is positive-definite.

The cart-pendulum, as a mechanical system with one degree of underactuation, is one of the systems that has been stabilized using the energy shaping method [16, 33]. Potential energy shaping alone can be shown to be not enough to stabilize the system; therefore kinetic energy shaping is necessary. More complicated mechanical systems with more degrees of freedom, like the spherical pendulum, have been stabilized using the energy shaping method [11]. Linear controllability is a necessary condition for stabilization using energy shaping. For linear systems, linear controllability is also a sufficient condition for the existence of a stabilizing feedback [51, 36]. Such sharp conditions for nonlinear systems do not exist in the literature. Thus the question of which mechanical systems are stabilizable using energy shaping is still unresolved. Moreover, almost all the existing results on energy shaping are based on a specific parametrization of the assumed solutions to the energy shaping problem. While the parameterizations used are sufficient for particular problems, it is not clear whether (1) a better controller would result if a richer class of feedbacks were available or (2) there are systems that are not presently amenable to stabilization by energy shaping using existing parameterizations, but which could be stabilized using energy shaping were the complete set of energy shaping feedbacks known.

Recently there have been notable attempts to investigate various features of the energy shaping problem. The first classical appearance of the notion of potential energy shaping problem is in [45]. Van der Schaft [47] made a significant geometric contribution to the problem from the Hamiltonian point of view. It turns out that this method has an extension in the Lagrangian setting called the method of Controlled Lagrangians; this has been investigated in [11, 10]. In recent work, Chang, Woolsey and others have realized that the space of possible kinetic energy feedbacks can be enlarged by considering the addition of appropriate gyroscopic forcing [16, 50]. In the Hamiltonian framework, the idea of kinetic energy shaping has been related by van der Schaft [17] to the notion of interconnection and modified into the IDA-PBC method [33]. The equivalence of the Controlled Lagrangian method and the IDA-PBC method has been addressed in [16, 9]. Both methods result in a set of partial differential equations whose solutions determine the energy shaping feedbacks. In other recent work, the possibility of finding a coordinate change for simplifying the kinetic energy shaping partial differential equations in the IDA-PBC method has been investigated [48].

A differential geometric approach to the kinetic energy shaping problem—the so-called λ-method—has been presented in [7]. In this paper, the authors propose a system of linear partial differential equations for the kinetic energy shaping problem in terms of a new variable, $\lambda = \mathbb{G}_{\mathrm{cl}}^{\sharp}\mathbb{G}_{\mathrm{ol}}^{\flat}$, where $\mathbb{G}_{\mathrm{ol}}$ and $\mathbb{G}_{\mathrm{cl}}$ are the open-loop and closed-loop metrics, respectively. The main idea of the λ-method is that it transforms the set of quasi-linear equations for kinetic energy shaping into a set of overdetermined linear partial differential equations [5]. In [6] an equivalent system of linear partial differential equations is given for the assumed procedure of kinetic energy shaping problem. Moreover, the authors investigate the compatibility conditions for the set of λ-equation in local coordinates. However, the analysis of the compatibility conditions is not complete, and many structural questions remain unanswered, even after one accounts for the results in [5, 6]. The λ-method has been modified by adding the possibility of using gyroscopic forces for enlarging the space of solutions [16]. The resulting partial differential equations remain poorly understood.

Lewis [31] has introduced an affine differential geometric approach to energy shaping in order to have a better geometric understanding of the problem and to state some of the questions that had not been addressed before. The main idea of the approach involves first understanding the existence of such an energy shaping feedback and then what such a feedback might look like. In recent work, sufficient conditions for the existence of potential energy shaping are derived assuming that kinetic energy shaping has been performed [32]. The results are based on the integrability theory for linear partial differential equations developed by Goldschmidt [21] and Spencer [44]. Although the results offer some insight, they are limited by the fact that kinetic energy shaping has been assumed to precede potential energy shaping.

In the next section a formal statement of the energy shaping problem is given. Before that, some of the basic notation used in this manuscript is presented.

Notation. The basic differential geometric notation that is used in this thesis is that of [2] and [14]. The identity map for a set S is denoted by id_S and the image of a map $f : S \to W$ by $\mathrm{Im}(f)$. For a vector space V, the set of (r, s)-tensors on V is denoted by $\mathrm{T}^r_s(\mathsf{V})$. By $\mathsf{S}_k\mathsf{V}$ and $\Lambda_k\mathsf{V}$ we denote, respectively, the set of symmetric and skew-symmetric $(0, k)$-tensors on V. The dual space of V is denoted by V^*. Let V and W be $\mathbb{R}$-vector spaces; by $\mathrm{L}(\mathsf{V}, \mathsf{W})$ we denote the set of linear maps from V to W. We shall also require symmetrizing and skew-symmetrizing maps. Thus, for $\mathsf{A} \in \mathrm{T}^0_k(\mathsf{V})$, we define the following projection maps:

$$\mathrm{Alt}(\mathsf{A})(v_1,\dots,v_k) = \frac{1}{k!}\sum_{\sigma\in\mathfrak{S}_k}(-1)^{\mathrm{sgn}(\sigma)}\mathsf{A}(v_{\sigma(1)},\dots,v_{\sigma(k)});$$
$$\mathrm{Sym}(\mathsf{A})(v_1,\dots,v_k) = \frac{1}{k!}\sum_{\sigma\in\mathfrak{S}_k}\mathsf{A}(v_{\sigma(1)},\dots,v_{\sigma(k)}),$$

where $\mathfrak{S}_k$ is the permutation group on k symbols and $\mathrm{sgn}(\sigma)$ is the parity of the permutation σ. Let A be a $(0,2)$-tensor on V. We define the flat map $\mathsf{A}^\flat : \mathsf{V}\to\mathsf{V}^*$ by $\langle\mathsf{A}^\flat(u); v\rangle = \mathsf{A}(u,v)$, $u,v\in\mathsf{V}$. The inverse of the flat map is denoted by $\mathsf{A}^\sharp : \mathsf{V}^*\to\mathsf{V}$ in case $\mathsf{A}^\flat$ is invertible. We also define a similar notation for a $(0,3)$-tensor A on V by

$$\langle\mathsf{A}^\flat(u), w\rangle = \mathsf{A}(w,u,u), \qquad u,w\in\mathsf{V}.$$

For $\mathsf{S}\subset\mathsf{V}$ and $\mathsf{W}\subset\mathsf{V}^*$ we denote

$$\mathrm{ann}(\mathsf{S}) = \{\alpha\in\mathsf{V}^* \mid \alpha(v)=0, \quad \forall\ v\in\mathsf{S}\},$$
$$\mathrm{coann}(\mathsf{W}) = \{v\in\mathsf{V} \mid \alpha(v)=0, \quad \forall\ \alpha\in\mathsf{W}\}.$$

For the purpose of using a version of the Cartan–Kähler theorem, all manifolds and maps will be assumed to be *analytic* unless otherwise stated. Many of the theorems and lemmata are still true in the smooth case. Let Q be an analytic manifold. If $\pi:\mathsf{E}\to\mathsf{Q}$ is an analytic vector bundle, $\Gamma^\omega(\mathsf{E})$ denotes the set of analytic sections of E. We often denote a bundle by a triple $(\mathsf{E},\pi,\mathsf{Q})$. We denote the tangent bundle of Q by $\pi_{\mathsf{Q}}:\mathsf{TQ}\to\mathsf{Q}$. Let $(\mathsf{E}_1,\pi_1,\mathsf{Q})$ and $(\mathsf{E}_2,\pi_2,\mathsf{Q})$ be two vector bundles over the same base manifold Q. Then the fibered product bundle is the triple $(\mathsf{E}_1\times_{\mathsf{Q}}\mathsf{E}_2, \pi_1\times_{\mathsf{Q}}\pi_2,\mathsf{Q})$, where the $\mathsf{E}_1\times_{\mathsf{Q}}\mathsf{E}_2$ is defined by

$$\big\{(u_1,u_2)\in\mathsf{E}_1\times_{\mathsf{Q}}\mathsf{E}_2 \mid \pi_1(u_1)=\pi_2(u_2)\big\}$$

and the projection map is defined through $\pi_1\times_{\mathsf{Q}}\pi_2(u_1,u_2)=\pi_1(u_1)=\pi_2(u_2)$. Furthermore, the tensor product of π_1 and π_2 with fibers $(\mathsf{E}_1)_q$ and $(\mathsf{E}_2)_q$, $q\in\mathsf{Q}$, is the vector bundle on Q with fibers $(\mathsf{E}_1)_q\otimes(\mathsf{E}_2)_q$; we denote this vector bundle by $(\mathsf{E}_1\otimes_{\mathsf{Q}}\mathsf{E}_2,\pi_1\otimes_{\mathsf{Q}}\pi_2,\mathsf{Q})$. Consider a vector bundle $(\mathsf{E},\pi,\mathsf{Q})$ and a map $\xi:\mathsf{N}\to\mathsf{Q}$. Then the pull-back bundle of π is the bundle $(\xi^*(\mathsf{E}),\xi^*(\pi),\mathsf{N})$, where $\xi^*(\mathsf{E})$ is defined to equal

$$\big\{(u,y)\in\mathsf{E}\times_{\mathsf{Q}}\mathsf{N} \mid \pi(u)=\xi(y)\big\}$$

and the projection map is defined through $\xi^*(\pi)(u,y)=y$.

The set of analytic functions on Q is denoted by $C^\omega(\mathsf{Q})$. The exterior derivative of a k-form α on Q is denoted by $\mathsf{d}\alpha$. For a $(0,k)$-tensor field A and a Riemannian metric $\mathbb{G}$ on Q, we define the $(1,k-1)$-tensor field $\mathbb{G}^\sharp\mathsf{A}$ by

$$\mathbb{G}^\sharp\mathsf{A}(\alpha,X_1,\dots,X_{k-1}) = \mathsf{A}(\mathbb{G}^\sharp(\alpha),X_1,\dots,X_{k-1}), \tag{1.1}$$

where $\alpha\in\Gamma^\omega(\mathsf{T}^*\mathsf{Q})$, $X_1,\dots,X_k\in\Gamma^\omega(\mathsf{TQ})$. Finally, we give a decomposition of the $(0,3)$-tensor fields. We call a $(0,3)$-tensor field A on Q:

a) **gyroscopic** if $\mathsf{A}(X_1,X_2,X_3) = -\mathsf{A}(X_2,X_1,X_3)$ for all $X_1,X_2,X_3\in\Gamma^\omega(\mathsf{TQ})$;

b) **torsional** if $\mathsf{A}(X_1, X_2, X_3) = -\mathsf{A}(X_1, X_3, X_2)$ for all $X_1, X_2, X_3 \in \Gamma^\omega(\mathsf{TQ})$;

c) **geodesic** if $\mathsf{A}(X_1, X_2, X_3) = \mathsf{A}(X_1, X_3, X_2)$ for all $X_1, X_2, X_3 \in \Gamma^\omega(\mathsf{TQ})$;

d) **skew** if $\mathsf{A} \in \Gamma^\omega(\Lambda_3(\mathsf{TQ}))$.

We denote the set of gyroscopic and torsional tensor fields on Q, respectively, by $\mathsf{Gyr}(\mathsf{TQ})$ and $\mathsf{Tor}(\mathsf{TQ})$. We can record the decomposition of $\mathrm{T}^0_3(\mathsf{TQ})$ as follows [31, 19]:

$$\mathrm{T}^0_3(\mathsf{TQ}) = \mathsf{S}_3(\mathsf{TQ}) \oplus (\mathsf{Gyr}(\mathsf{TQ}) \cap \ker \mathrm{Alt}) \oplus (\mathsf{Tor}(\mathsf{TQ}) \cap \ker \mathrm{Alt}) \oplus \Lambda_3(\mathsf{TQ}).$$

1.1. Statement of the problem

A *forced simple mechanical system* is a quadruple $\Sigma = (\mathsf{Q}, \mathbb{G}, V, \mathcal{F}_\mathrm{e})$, where Q is an n-dimensional manifold called the *configuration manifold*, $\mathbb{G}$ is a Riemannian metric on Q, V is a function on the configuration manifold called the *potential function*, and $\mathcal{F}_\mathrm{e} : \mathsf{TQ} \to \mathsf{T}^*\mathsf{Q}$ is a bundle map over id_Q called the *external force*. We denote by $\nabla^\mathbb{G}$ the covariant derivative with respect to the associated Levi-Civita connection. The governing equations for a forced simple mechanical system are

$$\nabla^\mathbb{G}_{\gamma'(t)}\gamma'(t) = -\mathbb{G}^\sharp \circ \mathsf{d}V(\gamma(t)) + \mathbb{G}^\sharp \mathcal{F}_\mathrm{e}(\gamma'(t)),$$

where $\gamma : I \to \mathsf{Q}$ is an analytic curve on Q.

Similarly, a *simple mechanical control system* is a quintuple $\Sigma = (\mathsf{Q}, \mathbb{G}, V, \mathcal{F}_\mathrm{e}, \mathcal{W})$, where Q is an n-dimensional manifold called the *configuration manifold*, $\mathbb{G}$ is a Riemannian metric on Q, V is a function on the configuration manifold called the *potential function*, $\mathcal{F}_\mathrm{e} : \mathsf{TQ} \to \mathsf{T}^*\mathsf{Q}$ is a bundle map over id_Q called the *external force*, and $\mathcal{W}$ is a subbundle of $\mathsf{T}^*\mathsf{Q}$ called the control subbundle [14]. The governing equations for a simple mechanical control system are

$$\nabla^\mathbb{G}_{\gamma'(t)}\gamma'(t) = -\mathbb{G}^\sharp \circ \mathsf{d}V(\gamma(t)) + \mathbb{G}^\sharp \mathcal{F}_\mathrm{e}(\gamma'(t)) + \mathbb{G}^\sharp u(\gamma'(t)),$$

where $\gamma : I \to \mathsf{Q}$ is a curve on Q and $u : \mathsf{TQ} \to \mathcal{W}$ is the assumed state feedback. A class of external forces in which we are interested is gyroscopic forces.

1.1 Definition: Let $\Sigma = (\mathsf{Q}, \mathbb{G}, V, \mathcal{F}_\mathrm{e})$ be a forced simple mechanical system. We call an external force $\mathcal{F}_\mathrm{G} : \mathsf{TQ} \to \mathsf{T}^*\mathsf{Q}$ a *gyroscopic force* if, for all $X \in \Gamma^\omega(\mathsf{TQ})$,

$$\langle X, \mathcal{F}_\mathrm{G}(X)\rangle = 0.$$

A *linear gyroscopic force* is a gyroscopic force $\mathcal{F}_{\mathrm{G},1}$ of the following form:

$$\mathcal{F}_{\mathrm{G},1}(X) = \mathbb{B}^\flat_{\mathrm{G},1}(X), \qquad X \in \Gamma^\omega(\mathsf{T}_q\mathsf{Q}),$$

where $\mathbb{B}_{G,1}$ is a skew-symmetric $(0, 2)$-tensor. A *quadratic gyroscopic force* is a gyroscopic force $\mathcal{F}_{G,2}$ with the following form:

$$\mathcal{F}_{\mathrm{G},2}(X) = \mathbb{B}^\flat_{\mathrm{G},2}(X), \qquad X \in \Gamma^\omega(\mathsf{T}_q\mathsf{Q}),$$

where $\mathbb{B}_{\mathrm{G},2}$ is a $(0, 3)$-tensor which is skew-symmetric in the first two arguments, i.e., $\mathbb{B}_{\mathrm{G},2}(X, Y, Z) = -\mathbb{B}_{\mathrm{G},2}(Y, X, Z)$, $X, Y, Z \in \Gamma^\omega(\mathsf{T}_q\mathsf{Q})$. By definition of the flat map, a quadratic gyroscopic force is defined by

$$\langle \mathcal{F}_{\mathrm{G},2}(X); Z\rangle = \mathbb{B}_{\mathrm{G},2}(Z, X, X), \qquad X, Z \in \Gamma^\omega(\mathsf{T}_q\mathsf{Q}).$$

Given an open-loop simple mechanical control system $\Sigma_{\text{ol}} = (\mathsf{Q}, \mathbb{G}_{\text{ol}}, V_{\text{ol}}, \mathcal{F}_{\text{ol}}, \mathcal{W}_{\text{ol}})$, we seek a control force such that the closed-loop system is a forced simple mechanical system $\Sigma_{\text{cl}} = (\mathsf{Q}, \mathbb{G}_{\text{cl}}, V_{\text{cl}}, \mathcal{F}_{\text{cl}})$, possibly with some external force. The reason for seeking this as the closed-loop system is that the stability analysis of the equilibria for mechanical systems is well understood [14, Chapter 6]. The class of gyroscopic forces does not change the total energy of the closed-loop system, while the addition of gyroscopic forces improves the possibility of finding a stable closed-loop system [16]. Here it is assumed that the open-loop external force $\mathcal{F}_{\text{ol}}$ is zero. Moreover, it seems that only the *quadratic* gyroscopic forces are useful in extending the space of possible closed-loop metrics [31]. The objective, therefore, can be phrased with the following definition.

1.2 Definition: Let $\Sigma_{\text{ol}} = (\mathsf{Q}, \mathbb{G}_{\text{ol}}, V_{\text{ol}}, \mathcal{F}_{\text{ol}}, \mathcal{W}_{\text{ol}})$ be an open-loop simple mechanical control system with $\mathcal{F}_{\text{ol}} = 0$. If there exists a bundle map $u_{\text{shp}} : \mathsf{TQ} \to \mathcal{W}_{\text{ol}}$ (called the control) with $u_{\text{shp}} = -u_{\text{kin}} - u_{\text{pot}}$ such that the closed-loop system is a forced simple mechanical system $\Sigma_{\text{cl}} = (\mathsf{Q}, \mathbb{G}_{\text{cl}}, V_{\text{cl}}, \mathcal{F}_{\text{cl}})$, where $\mathcal{F}_{\text{cl}}$ is a quadratic gyroscopic force with associated $(0, 3)$-tensor $\mathbb{B}$ and

1. $\mathbb{G}^{\sharp}_{\text{ol}} \circ u_{\text{kin}}(\gamma'(t)) = \nabla^{\mathbb{G}_{\text{cl}}}_{\gamma'(t)}\gamma'(t) - \nabla^{\mathbb{G}_{\text{ol}}}_{\gamma'(t)}\gamma'(t) - \mathbb{G}^{\sharp}_{\text{cl}} \circ (\mathbb{B}^{\flat}(\gamma'(t)))$,
2. $u_{\text{pot}}(\gamma(t)) = \mathbb{G}^{\flat}_{ol} \circ \mathbb{G}^{\sharp}_{cl}\mathsf{d}V_{\text{cl}}(\gamma(t)) - \mathsf{d}V_{\text{ol}}(\gamma(t))$,

then the control u_{shp} is called an *energy shaping feedback*.

1.3 Remark: Throughout this work, it is assumed that the equilibrium point $q_0 \in \mathsf{Q}$ is a regular point for $\mathcal{W}_{\text{ol}}$. Moreover, it is assumed that the control codistribution $\mathcal{W}_{\text{ol}}$ is *integrable*. These assumptions are common, even implicit, in the literature and many examples fall into this case. Nevertheless, these assumptions, especially the integrability assumptions are stringent, and relaxing them is an interesting challenge.

The conditions of Definition 1.2 contain as unknowns the closed-loop metric $\mathbb{G}_{\text{cl}}$, the closed-loop potential energy V_{cl}, and the gyroscopic $(0, 3)$-tensor field $\mathbb{B}$. One can observe that these equations involve the *first jet* of the unknowns. One can construct a set of first-order partial differential equations which completely characterize the existence of an energy shaping feedback. Let $\mathcal{W}_{\text{ol}} \subset \mathsf{T}^*\mathsf{Q}$ be a given subbundle and define the associated $\mathbb{G}_{\text{ol}}$-orthogonal projection map $P \in \Gamma^{\omega}(\mathsf{T}^*\mathsf{Q} \otimes \mathsf{TQ})$ by

$$\ker(P) = \mathbb{G}^{\sharp}_{\text{ol}}\mathcal{W}_{\text{ol}}.$$

Note that P completely prescribes $\mathcal{W}_{\text{ol}}$. We apply P to the equation from part 1 of Definition 1.2 to arrive at the following equation:

$$P(\nabla^{\mathbb{G}_{\text{cl}}}_{\gamma'(t)}\gamma'(t) - \nabla^{\mathbb{G}_{\text{ol}}}_{\gamma'(t)}\gamma'(t) - \mathbb{G}^{\sharp}_{\text{cl}} \circ \mathbb{B}^{\flat}(\gamma'(t))) = 0.$$

Assume Q is an n-dimensional manifold and $\mathcal{W}_{\text{ol}}$ is an integrable codistribution of dimension $n-m$. In adapted local coordinates, the kinetic energy shaping partial differential equation is given by

$$P^a_r(\mathbb{G}^{rl}_{\text{cl}}(\mathbb{G}_{\text{cl},lj,k} + \mathbb{G}_{\text{cl},lk,j} - \mathbb{G}_{\text{cl},kj,l}) - \mathbb{G}^{rl}_{\text{ol}}(\mathbb{G}_{\text{ol},lj,k} + \mathbb{G}_{\text{ol},lk,j} - \mathbb{G}_{\text{ol},kj,l}) - \mathbb{G}^{rl}_{\text{cl}}\mathbb{B}_{lkj}) = 0,$$

where $i,j,k,l,r \in \{1,\dots,n\}$, $a \in \{1,\dots,m\}$, and where we denote the first derivative of $\mathbb{G}_{\mathrm{cl},lj}$ with respect to q^k by $\mathbb{G}_{\mathrm{cl},lj,k}$. Similarly, let $\hat{P} : \mathsf{T}^*\mathsf{Q} \to \mathsf{T}^*\mathsf{Q}/\mathcal{W}_{\mathrm{ol}}$ be the canonical projection on to the quotient vector bundle. We have

$$\hat{P}(\mathbb{G}_{\mathrm{ol}}^{\flat} \circ \mathbb{G}_{\mathrm{cl}}^{\sharp} \mathsf{d}V_{\mathrm{cl}}(\gamma(t)) - \mathsf{d}V_{\mathrm{ol}}(\gamma(t))) = 0.$$

In local coordinates we have

$$\hat{P}_a^i(\mathbb{G}_{\mathrm{ol},ij}\mathbb{G}_{\mathrm{cl}}{}^{jk}V_{\mathrm{cl},k} - V_{\mathrm{ol},k}) = 0,$$

where $i,j,k \in \{1,\dots,n\}$, $a \in \{1,\dots,m\}$, and where we denote the first derivative of V_{cl} with respect to q^k by $V_{\mathrm{cl},k}$. For more details on the affine differential geometric setup of the energy shaping problem, see [31].

1.2. Some open problems in energy shaping

Now that the energy shaping partial differential equations have been specified, a summary of some of the fundamental questions one can now ask are provided.

P1. *Describe the set of achievable closed-loop metrics.* Until the recent paper [20], there has not been much consideration of this problem in the literature, apart from giving a geometric description of the problem [7] and some initial and incomplete integrability results [6].

P2. *Assume that one has found a closed-loop metric which solves the kinetic energy shaping problem. What are the conditions under which there exists a closed-loop potential function which satisfies the potential energy shaping problem?*

P3. *Describe the set of achievable closed-loop potential functions by allowing the closed-loop metric to vary over its achievable set.*

P4. *Give a complete description of the set of stabilizing potential energy shaping functions.* In order to have a stabilizing energy shaping feedback, the Hessian of the closed-loop potential functions should be positive-definite. The type of obstruction this condition puts on the set of achieved energy shaping feedbacks has not yet been characterized.

P5. *Describe the effect of including gyroscopic forces in the procedure of energy shaping.* An algebraic presentation of this problem has been given in [31]. Although one can extend the results of Chapter 4 to the case with gyroscopic forces, many geometric and algebraic constructions remain to be performed to clarify how the results should be interpreted in terms of stabilization.

P6. *Reconstruct some of the existing results using the sufficient conditions of Chapter 4; namely, answer the following questions:*

   (a) *Why is it always the case that one can construct an explicit solution to the set of partial differential equations for systems with one degree of underactuation?*

   (b) *Why is linear controllability a sufficient condition for existence of a stabilizing energy shaping feedback for linear systems?*

P7. *Find some interesting counterexamples.* It would be revealing to have an example for which there exists no stabilizing energy shaping feedback, even in the absence of gyroscopic forces. This might help to understand the key primary question in energy shaping: *when is it possible to stabilize a system by the energy shaping method?*

P8. *Is it possible to give a complete solution to the stabilization problem, at least in special cases? In particular, can we give a complete description of stabilization of simple mechanical systems with one degree of underactuation?* Since, for these systems, for any solution to the kinetic energy shaping partial differential equations there exists a potential function that satisfies the potential energy shaping system of partial differential equations (see [20]), it is an interesting question to see whether any/some/all of these solutions are stabilizing solutions.

1.3. Contribution of thesis

In this thesis a geometric framework is developed for stabilization of simple mechanical systems using energy shaping. In particular, the geometric theory of partial differential equations has been used to discuss the integrability of the energy shaping partial differential equations. As a summary, in this document the following answers to some of the problems of Section 1.2 are provided.

Note. Most of the integrability results of this document have been published in a recent paper [20], coauthored with Drs. Andrew D. Lewis and Abdol-Reza Mansouri.

A1. In Section 4.1 we partially answer Problem 1. Assuming that $\mathcal{W}_{\mathrm{ol}}$ is integrable, we describe a set of sufficient conditions under which one can construct a formal solution to the set of kinetic energy shaping partial differential equations in the analytic case and in the absence of gyroscopic forces. Moreover, we show that any analytic solution to the kinetic energy shaping problem satisfies these conditions. (See Theorems 4.7 and 4.15.) In Section 7.2.1 some observations are made that suggest a possible relationship between the kinetic energy shaping problem and holonomy groups. This might lead to some new insight into the sufficient conditions for kinetic energy shaping problem.

A2. Lewis [32] presented a set of sufficient conditions for Problem 2 using a geometric analysis of the potential energy shaping partial differential equations. In Section 4.3 we couple this sufficient condition with the kinetic energy shaping results. In other words, we give conditions on the closed-loop metric so that there exists a solution to the set of potential energy shaping partial differential equations. (See Theorem 4.24.)

A3. Problem 3 remains open, and even a clear geometric formulation of this problem is far from being achieved. In this document we provide one possible approach by placing the problem in the setting of geometric partial differential equations [21, 22]. In particular, we give conditions on the set of closed-loop metrics under which there exists a closed-loop potential function that satisfies the set of potential energy shaping partial differential equations.

A4. In Section 7.2.3 a promising approach is introduced for answering the question of whether a solution to the energy shaping partial differential equations is stabilizing or not. The approach relies on the so-called Farkas lemma [29]. This might be the subject of future work.

A5. The integrability results in this thesis do not include gyroscopic forces. Moreover, an algebraic description of gyroscopic forces that reveals the interaction of the gyroscopic forces and the kinetic energy shaping partial differential equations is far from being achieved. In Section 7.2.2 we suggest some future directions for this problem.

A6. Problem 6(a) has been discussed in [6] and [3]. But the results do not reveal how the geometric obstructions given by the kinetic and potential energy shaping conditions are intertwined. We give a result in Theorem 6.1 which essentially solves the problem. The second question has been posed and solved in [51, 36]. In Chapter 5, a geometric proof for this problem is presented that specializes the integrability results of Chapter 4 to linear simple mechanical control systems.

A7. In Section 4.5 an example of a simple mechanical system is given that is not stabilizable using the energy shaping method in the absence of gyroscopic forces. The integrability results of Chapter 4 are essential in devising such an example.

A8. In Chapter 6 the energy shaping problem is fully characterized for systems with one degree of underactuation and these systems are proved to be stabilizable using an energy shaping feedback. Furthermore, Example 6.6 demonstrates how it might be the case that the energy shaping for a system with one degree of underactuation is not possible via a positive-definite closed-loop metric.

This document is organized as follows. In Chapter 2, a brief review is given of the fundamental mathematical background required in this thesis. In particular, Section 2.1 gives an introduction to the geometric methods for analyzing formal integrability of partial differential equations [21, 22]. The character of the theorems in this section is mainly algebraic and may seem unmotivated to a reader unfamiliar with the formal theory of partial differential equations. A reader new to these techniques is advised that some effort will be required to become comfortable with them. A list of useful references is [21, 22, 23, 44, 34, 24, 25, 41]. In Section 2.2, we motivate the definition of a connection as a section of a jet bundle [38] in order to give a precise definition for the space of torsion free affine connections on a manifold. A geometric formulation for the partial differential equations of kinetic energy shaping is presented in Chapter 3, and the existing results for potential energy shaping [32] are recalled. The main results of the so-called λ-method [7] for kinetic energy shaping are reviewed and reproved in Section 3.2. Chapter 4 contains the formal integrability results for the partial differential equations in the energy shaping problem. Section 4.1 involves one of the main contribution of this thesis. The set of λ-equations has been proved to have an involutive symbol and to be formally integrable under a certain surjectivity condition. In other words, sufficient conditions for the existence of a formal solution to the λ-equations is given. Section 4.3 deals with the potential energy shaping problem. The set of conditions in [32] is analyzed to characterize the set of acceptable closed-loop metrics. Finally, in Section 4.4 a set of sufficient conditions is given for the total energy shaping problem. Chapter 5 is devoted to linear simple mechanical control systems. In particular, we give an algebraic proof for the result of [51, 36] which shows that linear controllability is sufficient for linear energy shaping (it is always necessary for asymptotic stabilization). In Chapter 6 systems with one degree of underactuation are considered and the result of Chapter 4 is been used to obtain Theorem 6.4, which essentially solves the stabilization problem for systems with one degree of underactuation. Chapter 7 contains the conclusions of the thesis. Moreover, some of the future directions of this thesis have been presented in this chapter: Section 7.2.1 gives some ideas for relating the kinetic energy shaping problem to problems in holonomy groups. Work in this direction might reveal the essential character of the kinetic energy shaping partial differential equations. Section 7.2.2 raises the issue of gyroscopic forces, concentrating on the limitations of gyroscopic forces as a tool for extending the solutions of the kinetic energy shaping partial differential equations. Finally, in Section 7.2.3 a new approach is suggested for dealing with the stabilization condition after solving the energy shaping partial differential equations. This approach suggests using a version of Farkas lemma [29, 8] for this stabilization condition.

Chapter 2

Mathematical preliminaries

The mathematical preliminaries that are used in this thesis are reviewed in this chapter. In particular, a summary of some basic background for modeling the system of partial differential equations in the energy shaping problem is given. The differential geometric notions used in modeling of simple mechanical systems are assumed here, and the unfamiliar reader is referred to [14, 2, 1, 30, 26] for more details. This section consists of three main parts. The first part deals with the geometric modeling of partial differential equations and the second part gives a useful definition of a connection on a vector bundle which characterizes the structure of the set of all affine connections. Finally, we present the so-called Ricci identity [15] which plays a significant role in answering some questions about the kinetic energy shaping problem.

2.1. Formal integrability of partial differential equations

In this section, we describe the main technique that we use for studying the energy shaping partial differential equations. The discussion centers around an analogue of the Cauchy–Kowalevski theorem [13] and formal integrability. We review the contributions made by Goldschmidt and Spencer in the late 1960's [21, 22, 44].

Although understanding the proofs of the main theorems in Chapter 4 depends on techniques from the formal theory of partial differential equations, we emphasize that the statement of the main results of the thesis are accessible without understanding formal methods in detail. The main integrability results in this document involve applications of the important Theorem 2.20 stated below. However, the verification of the hypothesis of this theorem typically takes some effort. In this section we describe the tools used to verify the hypothesis of Theorem 2.20.

2.1.1. Representation of a partial differential equation as a fibered submanifold of a jet bundle. We denote by $(\mathsf{E}, \pi, \mathsf{Q})$ a fibered manifold $\pi : \mathsf{E} \to \mathsf{Q}$. The *vertical bundle* of the fibered manifold π is the subbundle of $\mathsf{T}\pi : \mathsf{TE} \to \mathsf{TQ}$ given by $\mathsf{V}\pi = \ker(\mathsf{T}\pi)$. We denote by $\mathsf{J}_k\pi$ the *bundle of k-jets* [39]. A local section of π is a pair (U, ξ), where U is an open submanifold of Q and ξ is a map $\xi : U \to \mathsf{E}$ such that $\pi \circ \xi = \mathrm{id}_U$. If (ξ, U) is an analytic local section of π, we denote its k-jet by $j_k\xi$. We denote an element of $\mathsf{J}_k\pi$ by $j_k\xi(x)$, where $x \in U$. If we represent the sheaf of germs of sections of π by $\mathscr{S}_{\mathsf{Q}}(\pi)$, then j_k induces a morphism of sheaves $\mathscr{S}_{\mathsf{Q}}(\pi) \to \mathscr{S}_{\mathsf{Q}}(\mathsf{J}_k\pi)$. We let $\pi_k : \mathsf{J}_k\pi \to \mathsf{Q}$ and $\pi_l^k : \mathsf{J}_k\pi \to \mathsf{J}_l\pi$, $l \leq k$, be the canonical projections. One can show that π_k and π_l^k are surjective submersions. Moreover, $\pi_l^k : \mathsf{J}_k\pi \to \mathsf{J}_l\pi$ is an epimorphism of fibered manifolds and $(\mathsf{J}_k\pi, \pi_{k-1}^k, \mathsf{J}_{k-1}\pi)$ is an affine bundle modeled on a vector bundle over $\mathsf{J}_{k-1}\pi$ whose total space is the tensor product $\pi_{k-1}^*(\mathsf{S}_k\mathsf{T}^*\mathsf{Q}) \otimes (\pi_0^{k-1})^*\mathsf{V}\pi$; see [39]. The following definition establishes the relationship between jet bundles and systems of partial differential equations.

2.1 Definition: Let $(\mathsf{E}, \pi, \mathsf{Q})$ be a fibered manifold and let $\mathsf{J}_k\pi$ be its bundle of k-jets. A *partial differential equation* is a fibered submanifold $\mathsf{R}_k \subset \mathsf{J}_k\pi$.

We denote by $\hat{\pi}_k$ the restriction of π_k to R_k. As one can see, the "equation" representation of the partial differential equation is obscure here. The following local characterization of a partial differential equation as a kernel of a fibered manifold morphism is helpful for identifying the "equation."

2.2 Proposition: *Let $(\mathsf{E}, \pi, \mathsf{Q})$ be a fibered manifold. Given a partial differential equation $\mathsf{R}_k \subset \mathsf{J}_k\pi$ and a point $p \in \mathsf{Q}$, there exists neighborhood U of p, a fibered manifold (E', π', U), an analytic section η of π', and a morphism of fibered manifolds $\Phi : \pi_k^{-1}(U) \to \mathsf{E}'$ such that*

$$\pi_k^{-1}(U) \cap \mathsf{R}_k \doteq \ker_\eta \Phi = \{u_k \in \pi_k^{-1}(U) \mid \Phi(u_k) = \eta(\pi_k(u_k))\}.$$

Proof: Because R_k is a fibered submanifold, there exists an adapted chart $(\mathcal{U}_k, \phi_k)$ for $\mathsf{J}_k\pi$ with the induced chart (U, ϕ) on Q such that

$$\phi_k(\mathcal{U}_k) \subset \phi(U) \times V \times W \subset \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^{m'}, \qquad n, m, m' \in \mathbb{Z}_{\geq 0},$$

and such that

$$\pi_k^{-1}(U) \cap \mathsf{R}_k = \{(x, v, 0) \mid x \in \phi(U),\ v \in V\}.$$

Take $\mathsf{E}' = U \times V$ and $\pi'(x, v) = x$. Taking $\Phi(\eta) = (x, v)$, where $(\phi(x), v, w) = \phi_k(\eta)$ and $\eta(x) = (x, 0)$, the result follows. ■

A morphism $\Phi : \mathsf{J}_k\pi \to \pi'$ of fibered manifolds induces a differential operator $\mathfrak{D}$ of order k which is a sheaf morphism of the form $\Phi \circ j_k : \mathscr{S}_{\mathsf{Q}}(\pi) \to \mathscr{S}_{\mathsf{Q}}(\pi')$.

2.1.2. Prolongations and symbols. The notion of *involutivity* for partial differential equations was defined by Cartan, where he used his exterior differential calculus to prove the existence of formal solutions for involutive partial differential equations of first order using Cauchy–Kowalevski theorem. This notion, which is an essential ingredient of the Cartan–Kähler theorem, relies on the important notions of *prolongation* and *symbol*. In particular, Kuranishi [28] proves that, by prolonging a partial differential equation a sufficient number of times, one obtains an involutive system. These two notions are the subject of our study in this section.

Prolongation. The process of differentiating a partial differential equation in order to arrive at a higher order partial differential equation is called prolongation. One can formalize this statement as in the following definition.

2.3 Definition: Let $(\mathsf{E},\pi,\mathsf{Q})$ be a fibered manifold and let $\mathsf{R}_k \subset \mathsf{J}_k\pi$ be a partial differential equation. The rth-*prolongation* of R_k is the subset

$$\rho_r(\mathsf{R}_k) = \mathsf{J}_r\hat{\pi}_k \cap \mathsf{J}_{k+r}\pi.$$

A partial differential equation R_k is *regular* if $\rho_r(\mathsf{R}_k)$ is a fibered submanifold of $\mathsf{J}_{k+r}\pi$ for each $r \in \mathbb{Z}_{\geq 0}$. One can represent the rth-prolongation of a partial differential equation using the associated morphism. The rth-prolongation of Φ is defined to be the unique morphism of fibered manifolds over Q, $\rho_r(\Phi) : \mathsf{J}_{r+k}\pi \to \mathsf{J}_r\pi'$, that makes the following diagram commute:

$$\begin{array}{ccc}
\mathscr{S}_{\mathsf{Q}}(\mathsf{J}_{k+r}\pi) & \xrightarrow{\rho_r(\Phi)} & \mathscr{S}_{\mathsf{Q}}(\mathsf{J}_r\pi') \\
{\scriptstyle j_{k+r}}\uparrow & & \uparrow{\scriptstyle j_r} \\
\mathscr{S}_{\mathsf{Q}}(\pi) & \xrightarrow{\mathfrak{D}} & \mathscr{S}_{\mathsf{Q}}(\pi')
\end{array}$$

It is fairly clear that, for $r, l \in \mathbb{Z}_{\geq 0}$ and $r \geq l$, we have $\pi^{k+r}_{k+l}(\rho_r(\mathsf{R}_k)) \subset \rho_l(\mathsf{R}_k)$. We adopt the notation $\hat{\pi}^{k+r}_{k+l} : \rho_r(\mathsf{R}_k) \to \rho_l(\mathsf{R}_k)$ and $\hat{\pi}_{k+r} : \rho_r(\mathsf{R}_k) \to \mathsf{Q}$ as the canonical projections. There is no guarantee that the first map is a surjective submersion; surjectivity of this map leads to the concept of *formal integrability* that will be discussed later. The following remark will be used in Section 2.1.3; for details of the proof we refer to [23, 22].

2.4 Remark: Let π be a fibered manifold as before and let $\mathsf{R}_k \subset \mathsf{J}_k\pi$ be a partial differential equation. If $\rho_r(\mathsf{R}_k)$ is a fibered submanifold of $\mathsf{J}_{k+r}\pi$, then $\rho_l(\rho_r(\mathsf{R}_k)) = \rho_{l+r}(\mathsf{R}_k)$. Since one can define $\rho_r(\mathsf{R}_k)$ as the kernel of a morphism of fibered manifolds, this follows immediately from studying the following exact commutative diagram and showing that the map γ is an isomorphism of fibred manifolds.

$$\begin{array}{ccccccc}
 & & & & 0 & & 0 \\
 & & & & \downarrow & & \downarrow \\
0 & \longrightarrow & \rho_{r+l}(\mathsf{R}_k) & \longrightarrow & \mathsf{J}_{k+r+l}\pi & \xrightarrow{\rho_{l+r}(\Phi)} & \mathsf{J}_{l+r}\pi' \\
 & & \downarrow{\scriptstyle\gamma} & & \Vert & & \downarrow \\
0 & \longrightarrow & \rho_l(\rho_r(\mathsf{R}_k)) & \longrightarrow & \mathsf{J}_{k+r+l}\pi & \xrightarrow{\rho_l(\rho_r(\Phi))} & \mathsf{J}_r\pi'_l \\
 & & & & \downarrow & & \\
 & & & & 0 & &
\end{array}$$

Whenever R_k is regular, for sake of convenience, we use R_{k+r} for the rth-prolongation.

Symbols. The highest order terms in the linearization of a partial differential equation carry valuable information about formal integrability of the equation [22]. Similarly to our approach for defining a partial differential equation, we give two equivalent formal definitions to capture these higher order terms, one as a vector bundle morphism and one as a family of subspaces.

Given $p_{k-1} \in \mathsf{J}_{k-1}\pi$, recall that $(\pi^k_{k-1})^{-1}(p_{k-1})$ has the structure of an affine space modeled on $\mathsf{S}_k\mathsf{T}^*_{\pi_{k-1}(p_{k-1})}\mathsf{Q} \otimes \mathsf{V}_{\pi_0^{k-1}(p_{k-1})}\pi$. For each $p_k \in \mathsf{J}_k\pi$ we have

$$\mathsf{V}_{p_k}\pi^k_{k-1} \simeq \mathsf{S}_k\mathsf{T}^*_{\pi_{k-1}(p_{k-1})}\mathsf{Q} \otimes \mathsf{V}_{\pi_0^{k-1}(p_{k-1})}\pi,$$

as well as a vector bundle isomorphism $\pi_k^*\mathsf{S}_k\mathsf{T}^*\mathsf{Q}\otimes(\pi_0^k)^*\mathsf{V}\pi \cong \mathsf{V}\pi_{k-1}^k$. The identification of these bundles is made implicitly in most of the literature [39] and we follow this convention.

2.5 Definition: Let $(\mathsf{E},\pi,\mathsf{Q})$ be a fibered manifold and let $\mathsf{R}_k\subset\mathsf{J}_k\pi$ be a partial differential equation. The *symbol* of R_k is the family G_k of vector spaces given by

$$\mathsf{G}_k|_{p_k} = \mathsf{V}_{p_k}\hat{\pi}_k \cap \mathsf{V}_{p_k}\pi_{k-1}^k, \quad p_k\in\mathsf{J}_k\pi.$$

Let ξ be a section of E on an open neighborhood $U\subset\mathsf{Q}$ and let $p\in U$. Let $\{f_1,\dots,f_k\}$ be $\mathbb{R}$-valued functions defined on a neighborhood U of $p\in\mathsf{Q}$ which vanish at p. As in [21], define $\epsilon_k:\mathsf{S}_k\mathsf{T}^*\mathsf{Q}\otimes\mathsf{V}\pi\to\mathsf{V}\pi_k$ by

$$\epsilon_k : (df_1\cdots df_k\otimes\xi)(p)\to j_k((\Pi_{i=1}^k f_i).\xi)(p).$$

The map ϵ_k is well-defined since the derivatives of $(\Pi_{i=1}^k f_i)$ vanish up to order $k-1$ at p. We have the following lemma.

2.6 Lemma: *Let $(\mathsf{E},\pi,\mathsf{Q})$ be a fibered manifold. We have the following short exact sequence of vector bundles over $\mathsf{J}_k\pi$:*

$$0 \longrightarrow \mathsf{S}_k\mathsf{T}^*\mathsf{Q}\otimes\mathsf{V}\pi \xrightarrow{\epsilon_k} \mathsf{V}\pi_k \xrightarrow{\mathsf{V}\pi_{k-1}^k} (\pi_{k-1}^k)^*(\mathsf{V}\pi_{k-1}) \longrightarrow 0\ .$$

The following definition introduces *the symbol map* as a morphism of vector bundles. It is crucial to understand the distinction between the definition of the symbol map as a bundle map and the definition of the symbol map at a point as a map of vector spaces. This explicit distinction is usually dropped in the literature.

2.7 Definition: Let $(\mathsf{E},\pi,\mathsf{Q})$ and $(\mathsf{E}',\pi',\mathsf{Q})$ be fibered manifolds and let $\Phi:\mathsf{J}_k\pi\to\mathsf{E}'$ be a morphism over id_{Q}. The symbol of Φ is defined to be

$$\sigma(\Phi) = \mathsf{V}\Phi\circ\epsilon_k : \pi_k^*\mathsf{S}_k\mathsf{T}^*\mathsf{Q}\otimes(\pi_0^k)^*\mathsf{V}\pi\to\mathsf{V}\pi'.$$

The following proposition relates the definition of the symbol as a family of vector spaces with that as a map.

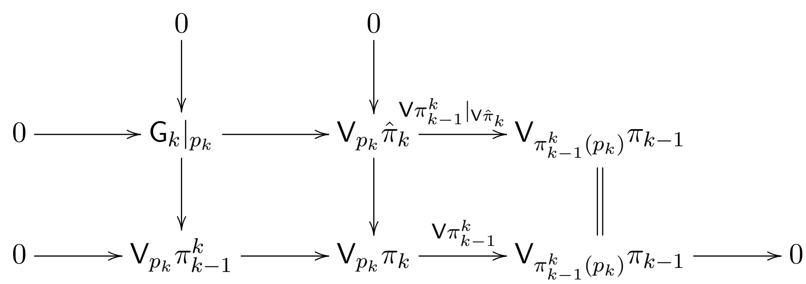
2.8 Proposition: *Let π be a fibered manifold as above and let $p_k\in\mathsf{R}_k\subset\mathsf{J}_k\pi$. Then the following sequences are exact:*

1. $0 \longrightarrow \mathsf{G}_k|_{p_k} \longrightarrow \mathsf{S}_k\mathsf{T}^*_{\pi_k(p_k)}\mathsf{Q}\otimes\mathsf{V}_{\pi_o^k(p_k)}\pi \xrightarrow{\sigma(\Phi)|_{p_k}} \mathsf{V}_{\Phi(p_k)}\pi'$;
2. $0 \longrightarrow \mathsf{G}_k|_{p_k} \longrightarrow \mathsf{V}_{p_k}\hat{\pi}_k \xrightarrow{\mathsf{V}\pi_{k-1}^k|_{\mathsf{V}\hat{\pi}_k}} \mathsf{V}_{\pi_{k-1}^k(p_k)}\pi_{k-1}$.

Sketch of the proof: The proof of exactness of the first sequence follows from the following commutative diagram:

$$\begin{array}{ccccccc}
 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathsf{G}_k|_{p_k} & \longrightarrow & \mathsf{S}_k\mathsf{T}^*_{\pi_k(p_k)}\mathsf{Q}\otimes\mathsf{V}_{\pi_0^k(p_k)}(\pi) & \xrightarrow{\sigma(\Phi)} & \mathsf{V}_{\Phi(p_k)}\pi' \\
 & & \downarrow & & \downarrow & & \| \\
0 & \longrightarrow & \mathsf{V}_{\pi_0^k(p_k)}(\hat{\pi}_k) & \longrightarrow & \mathsf{V}_{\pi_0^k(p_k)}(\pi_k) & \xrightarrow{\mathsf{V}\Phi} & \mathsf{V}_{\Phi(p_k)}\pi'
\end{array}$$

where the bottom row is exact. Similarly, for the second sequence, one should consider the following exact commutative diagram:



The second row is exact since $\mathsf{V}\pi^k_{k-1}$ is an epimorphism of vector spaces. ■

Note that G_k is not always a vector bundle over $\mathsf{V}\hat{\pi}_k$. The first statement of Proposition 2.8 reveals the relationship between Definition 2.5 and Definition 2.7, by basically identifying the kernel of the symbol map at p_k with $\mathsf{G}_k|_{p_k}$. The second statement of this proposition, along with the affine structure of jet bundles, shows that the symbol of a partial differential equation can be identified as the highest order component in the linearization of the equation.

Prolongation of symbols. We establish a process for prolonging the symbol of a partial differential equation. This process can be obtained in a purely algebraic manner [24]. Let $(\mathsf{E}, \pi, \mathsf{Q})$ be a fibered manifold and $\mathsf{R}_k \subset \mathsf{J}_k\pi$ a partial differential equation. We fix a point $p_k \in \mathsf{R}_k$, we let $x = \pi_k(p_k)$, and we let $\{e^1, \ldots, e^n\}$ be a basis for $\mathsf{T}^*_x\mathsf{Q}$. We denote an induced basis element for $\mathsf{S}_k\mathsf{T}^*_x\mathsf{Q}$ by $e^{i_1}e^{i_2}\cdots e^{i_k}$, where $i_1, \ldots, i_k \in \{1, \ldots, n\}$ satisfy $i_1 \leq i_2 \leq \cdots \leq i_n$. For $k, r \in \mathbb{Z}_{\geq 0}$, we define the natural inclusion $\Delta_{k,r} : \mathsf{S}_{k+r}\mathsf{T}^*_x\mathsf{Q} \to \mathsf{S}_r\mathsf{T}^*_x\mathsf{Q} \otimes \mathsf{S}_k\mathsf{T}^*_x\mathsf{Q}$ by

$$\Delta_{k,r} : A_{i_1\cdots i_{k+r}} e^{i_1}e^{i_2}\cdots e^{i_{k+r}} \mapsto A_{i_1\cdots i_r i_{r+1}\cdots i_{k+r}} e^{i_1}e^{i_2}\cdots e^{i_r} \otimes e^{i_{r+1}}\cdots e^{i_{k+r}}.$$

The map $\Delta_{k,r}$ can be extended naturally to

$$\Delta_{k,r} \otimes \mathrm{id}_{\mathsf{V}\pi} : \mathsf{S}_{k+r}\mathsf{T}^*_x\mathsf{Q} \otimes \mathsf{V}_{\pi^k_0(p_k)}\pi \to \mathsf{S}_r\mathsf{T}^*_x\mathsf{Q} \otimes \mathsf{S}_k\mathsf{T}^*_x\mathsf{Q} \otimes \mathsf{V}_{\pi^k_0(p_k)}\pi.$$

Let $\Phi : \mathsf{J}_k\pi \to \pi'$ be the local morphism associated to R_k and let $\sigma(\Phi)$ be the associated symbol map. With $\mathsf{G}_k|_{p_k} = \ker \sigma(\Phi)|_{p_k}$, we establish the rth-prolongation of the symbol using the following definition.

2.9 Definition: Let $\mathsf{R}_k \subset \mathsf{J}_k\pi$ be a partial differential equation. For each $p_k \in \mathsf{R}_k$ with $x = \pi_k(p_k)$, the map

$$\rho_r(\sigma(\Phi)|_{p_k}) : \mathsf{S}_{k+r}\mathsf{T}^*_x\mathsf{Q} \otimes \mathsf{V}_{\pi^k_0(p_k)}\pi \to \mathsf{S}_r\mathsf{T}^*_x\mathsf{Q} \otimes \mathsf{V}_{\Phi(p_k)}\pi',$$

defined by $(\mathrm{id}_{\mathsf{S}_r\mathsf{T}^*_x\mathsf{Q}} \otimes \sigma(\Phi)|_{p_k}) \circ (\Delta_{k,r} \otimes \mathrm{id}_{\mathsf{V}\pi})$ is called the *rth-prolongation* of $\sigma(\Phi)|_{p_k}$. Its kernel is denoted by $\rho_r(\mathsf{G}_k|_{p_k})$ and is called the rth-prolongation of the symbol.

2.10 Remark: Even if G_k is a vector bundle over $\mathsf{V}\hat{\pi}_k$, $\rho_r(\mathsf{G}_k)$ might not be a vector bundle over $\mathsf{V}\hat{\pi}_k$. In case it is, we sometimes use the notation G_{k+r} instead of $\rho_r(\mathsf{G}_k)$.

2.1.3. Formal integrability. Given a partial differential equation, we would like to study the existence of solutions. Specifically, we would like to construct the solutions of a given partial differential equation by constructing their Taylor series order by order. Since the theory we use rests on the Cauchy–Kowalevski theorem, we assume analyticity of all the data. We start by giving a formal definition for *solutions*.

2.11 Definition: Let $(\mathsf{E}, \pi, \mathsf{Q})$ be a fibered manifold and let $\mathsf{R}_k \subset \mathsf{J}_k\pi$ be a kth-order partial differential equation. A *local formal solution* of order k is a pair (ξ_k, U), where U is an open subset of Q and ξ_k is a section of R_k over U. If R_k is regular, one can define a *formal solution* of order $(k+r)$ as a pair (ξ_{k+r}, U), where ξ_{k+r} is a section of R_{k+r}.

One can come up with examples which are not "formally integrable" in the sense that one cannot iteratively construct a solution as a Taylor series; see Example 2.22.

2.12 Definition: Let $(\mathsf{E}, \pi, \mathsf{Q})$ be a fibered manifold and let $\mathsf{R}_k \subset \mathsf{J}_k\pi$ be a regular partial differential equation. Then R_k is *formally integrable* if the maps $\pi^{k+r+1}_{k+r} : \rho_{r+1}(\mathsf{R}_k) \to \rho_r(\mathsf{R}_k)$ are epimorphisms of fibered manifolds for each $r \in \mathbb{Z}_{\geq 0}$.

Note that in our definition of formally integrable we assume that R_k is regular.

2.13 Proposition: *If R_k is formally integrable then $\rho_r(\mathsf{G}_k)$ is a vector bundle over R_k for each $r \in \mathbb{Z}_{\geq 0}$.*

Proof: As R_k is formally integrable, π^{k+r+1}_{k+r} is an epimorphism and so locally of constant rank. Then the following short exact sequence

$$0 \longrightarrow \mathsf{G}_{k+r+1} \longrightarrow \mathsf{V}\hat{\pi}_{k+r+1} \longrightarrow \mathsf{V}\hat{\pi}_{k+r} \longrightarrow 0$$

yields that G_{k+r} is of constant rank. ■

The δ-sequence. Another purely algebraic construction which is used extensively for the formal theory is the δ-sequence. The δ-sequence has been utilized by Spencer [43] in the theory of deformation of structures. We describe this construction in the partial differential equation framework, omitting some details, and we construct the δ-sequence for $\mathsf{T}^*\mathsf{Q}$ which provides a characterization of the δ-operator with the fiberwise exterior derivative on the set of differential r-forms on $\mathsf{T}^*\mathsf{Q}$. Generally, there is no necessity for a manifold structure and one can give the construction of the δ-sequence in a purely algebraic fashion [24].

We start by characterizing $\Lambda_r\mathsf{T}^*\mathsf{Q} \otimes \mathsf{S}_k\mathsf{T}^*\mathsf{Q}$ as a subset of differential r-forms on $\mathsf{T}^*\mathsf{Q}$. First we give the following lemma.

2.14 Lemma: *Let V be a $\mathbb{R}$-vector space and denote by*

$$P_k(\mathsf{V}) = \{f : \mathsf{V} \to \mathbb{R} \mid f(x) = \mathsf{A}(x, \ldots, x), \quad \mathsf{A} \in \mathrm{T}^0_k\mathsf{V}\}$$

the homogenous polynomial functions of degree k. Then, for $f \in P_k(\mathsf{V})$, there exists a unique $\mathsf{A} \in \mathsf{S}_k\mathsf{V}$ such that $\mathsf{A}(x, \ldots, x) = f(x)$ for each $x \in \mathsf{V}$.

Proof: We first prove that A exists. Take $\mathsf{B} \in \mathrm{T}^0_k\mathsf{V}$. If $f(v) = \mathsf{B}(v, \dots, v)$, for $v \in \mathsf{V}$, we define A by

$$\mathsf{A}(v_1, \dots, v_k) = \mathrm{Sym}(\mathsf{B})(v_1, \dots, v_k),$$

which is symmetric by definition and $f(v) = \mathsf{A}(v, \dots, v)$. In order to prove the uniqueness, we show that if $\hat{\mathsf{A}} \in \mathsf{S}_k\mathsf{V}$ satisfies $f(v) = \hat{\mathsf{A}}(v, \dots, v)$ for all $v \in \mathsf{V}$, then $\hat{\mathsf{A}} = \mathsf{A}$. Note that if $\hat{\mathsf{A}} \in \mathrm{T}^0_2\mathsf{V}$ then

$$\sum_{\sigma \in \mathfrak{S}_2} \hat{\mathsf{A}}(v_{\sigma(1)}, v_{\sigma(2)}) = \hat{\mathsf{A}}(\sum_{j_1=1}^{2} v_{j_1}, \sum_{j_2=1}^{2} v_{j_2}) - \sum_{j_1=1}^{2} \hat{\mathsf{A}}(v_{j_1}, v_{j_1}).$$

An inductive argument shows that for $\hat{\mathsf{A}} \in \mathrm{T}^0_k\mathsf{V}$

$$\begin{aligned}\sum_{\sigma \in \mathfrak{S}_k} \hat{\mathsf{A}}(v_{\sigma(1)}, \dots, v_{\sigma(k)}) = {} & \hat{\mathsf{A}}(\sum_{j_1=1}^{k} v_{j_1}, \dots, \sum_{j_k=1}^{k} v_{j_k}) \\ & - \sum_{l=1}^{k-1} \sum_{j_1, \dots, j_l} \hat{\mathsf{A}}(v_{j_1} + \dots + v_{j_l}, \dots, v_{j_1} + \dots + v_{j_l}),\end{aligned}$$

where $\{j_1, \dots, j_l\} \subset \{1, \dots, k\}$ whose elements are distinct. Thus if $\hat{\mathsf{A}} \in \mathsf{S}_k\mathsf{V}$ satisfies $f(v) = \hat{\mathsf{A}}(v, \dots, v)$ for all $v \in \mathsf{V}$,

$$\begin{aligned}\hat{\mathsf{A}}(v_1, \dots, v_k) &= \frac{1}{k!} \sum_{\sigma \in \mathfrak{S}_k} \hat{\mathsf{A}}(v_{\sigma(1)}, \dots, v_{\sigma(k)}) \\ &= \frac{1}{k!}\big(f(v_1, \dots, v_k) - \sum_{l=1}^{k-1} \sum_{j_1, \dots, j_l} f(v_{j_1} + \dots + v_{j_l})\big) \\ &= \mathsf{A}(v_1, \dots, v_k).\end{aligned}$$

Thus A is unique. ■

2.15 Lemma: *The following map from $\Lambda_r \mathsf{T}^*_x\mathsf{Q} \otimes \mathsf{S}_k \mathsf{T}^*_x\mathsf{Q}$ to the set of differential r-forms on $\mathsf{T}^*_x\mathsf{Q}$ is a monomorphism of $\mathbb{R}$-vector spaces,*

$$\phi_{k,r}(\alpha \otimes \mathsf{A})(u)(v_1, \dots, v_r) = \mathsf{A}(u, \dots, u)\alpha(v_1, \dots, v_r), \qquad v_1, \dots, v_r \in \mathsf{T}_u\mathsf{T}^*_x\mathsf{Q} \cong \mathsf{T}^*_x\mathsf{Q},$$

where $x \in \mathsf{Q}$ and $u \in \mathsf{T}^*_x\mathsf{Q}$.

Proof: The map $\phi_{k,r}$ is linear by construction. We need to show that it is injective. Suppose that $\phi_{k,r}(\mathsf{A}_1 \otimes \alpha_1 + \dots + \mathsf{A}_i \otimes \alpha_i) = 0$. Then

$$\sum_{a=1}^{i} \mathsf{A}_a(u, \dots, u)\alpha_a(v_1, \dots, v_r) = 0$$

for all $v_1, \dots, v_r \in \mathsf{T}_x^*\mathsf{Q}$. If $u_1, \dots, u_k \in \mathsf{T}_x^*\mathsf{Q}$, then, by the previous lemma, for each $a \in \{1, \dots, i\}$ we have

$$\begin{aligned}\mathsf{A}_a(u_1, \dots, u_k) = \mathsf{A}_a(&\sum_{j_1=1}^{k} u_{j_1}, \dots, \sum_{j_k=1}^{k} u_{j_k}) \\ &- \sum_{l=1}^{k-1} \sum_{j_1,\dots,j_l} \mathsf{A}_a(u_{j_1} + \dots + u_{j_l}, \dots, u_{j_1} + \dots + u_{j_l}).\end{aligned}$$

As a result, we have

$$\begin{aligned}\sum_{a=1}^{i} A_a(u_1, \dots, u_r)\alpha_a(v_1, \dots, v_r) &= \sum_{a=1}^{i} \alpha_a(v_1, \dots, v_r)\mathsf{A}_a(\sum_{j_1=1}^{k} u_{j_1}, \dots, \sum_{j_k=1}^{k} u_{j_k}) \\ &- \sum_{a=1}^{i}\sum_{l=1}^{k-1} \sum_{j_1,\dots,j_l} \mathsf{A}_a(u_{j_1} + \dots + u_{j_l}, \dots, u_{j_1} + \dots + u_{j_l}) = 0,\end{aligned}$$

where the last equality holds by assumption; thus $\sum_{a=1}^{i} \mathsf{A}_a \otimes \alpha_a = 0$ and the map $\phi_{k,r}$ is injective. ■

The preceding characterization basically identifies the symmetric tensor part of $\Lambda_r\mathsf{T}_x^*\mathsf{Q} \otimes \mathsf{S}_k\mathsf{T}_x^*\mathsf{Q}$ with a homogeneous polynomial function of order k. Let d_r be the exterior derivative on $\mathsf{T}_x^*\mathsf{Q}$ restricted to differential r-forms. One can define a linear map $\delta_{r,k} : \Lambda_r\mathsf{T}_x^*\mathsf{Q} \otimes \mathsf{S}_k\mathsf{T}_x^*\mathsf{Q} \to \Lambda_{r+1}\mathsf{T}_x^*\mathsf{Q} \otimes \mathsf{S}_{k-1}\mathsf{T}_x^*\mathsf{Q}$ by asking that the following diagram be commutative:

$$\begin{array}{ccc} \Lambda_r\mathsf{T}_x^*\mathsf{Q} \otimes \mathsf{S}_k\mathsf{T}_x^*\mathsf{Q} & \xrightarrow{\delta_{r,k}} & \Lambda_{r+1}\mathsf{T}_x^*\mathsf{Q} \otimes \mathsf{S}_{k-1}\mathsf{T}_x^*\mathsf{Q} \\ \downarrow \phi_{k,r} & & \downarrow \phi_{k-1,r+1} \\ \Gamma^\omega(\Lambda_r\mathsf{T}_x^*\mathsf{Q}) & \xrightarrow{\mathsf{d}_r} & \Gamma^\omega(\Lambda_{r+1}\mathsf{T}_x^*\mathsf{Q}) \end{array}$$

Explicitly, for $\alpha \in \Lambda_r\mathsf{T}_x^*\mathsf{Q}$ and $\mathsf{A} \in \mathsf{S}_k\mathsf{T}_x^*\mathsf{Q}$,

$$\begin{aligned}&\delta_{r,k}(\alpha \otimes \mathsf{A})(v_1, \dots, v_{r+1}, u_1, \dots, u_{k-1}) \\ &\qquad = \sum_{j=1}^{r+1}(-1)^{j+1}k\alpha(v_1, \dots, \hat{v}_j, \dots, v_{r+1})\mathsf{A}(v_j, u_1, \dots, u_{k-1}).\end{aligned}$$

In other words, the $\delta_{r,k}$ operator imitates the exterior derivative on the space of differential forms on $\mathsf{T}_x^*\mathsf{Q}$ with polynomial coefficients when we identify the symmetric homogenous polynomials of degree k with a symmetric k-tensor.

It turns out that the following sequence, the so-called kth δ-*sequence*, is exact (here we simply denote $\delta_{r,k}$ by δ)

$$\begin{aligned}0 \longrightarrow \mathsf{S}_k\mathsf{T}_x^*\mathsf{Q} \xrightarrow{\delta} \mathsf{T}_x^*\mathsf{Q} \otimes \mathsf{S}_{k-1}\mathsf{T}_x^*\mathsf{Q} \xrightarrow{\delta} \cdots \\ \cdots \xrightarrow{\delta} \Lambda_n\mathsf{T}_x^*\mathsf{Q} \otimes \mathsf{S}_{k-n}\mathsf{T}_x^*\mathsf{Q} \longrightarrow 0\end{aligned}$$

The exactness is the Poincaré lemma for the class of differential forms. Let $\mathsf{R}_k \subset \mathsf{J}_k\pi$ be a partial differential equation. Consider the following exact and commutative diagram:

$$\begin{array}{ccccccc}
0 \longrightarrow & \Lambda_s\mathsf{T}^*\mathsf{Q} \otimes \mathsf{G}_{k+r+1} & \longrightarrow & \Lambda_s\mathsf{T}^*\mathsf{Q} \otimes \mathsf{S}_{k+r+1}\mathsf{T}^*\mathsf{Q} \otimes \mathsf{V}\pi & \xrightarrow{\sigma_{r+1}(\Phi)} & \Lambda_s\mathsf{T}^*\mathsf{Q} \otimes \mathsf{S}_{r+1}\mathsf{T}^*\mathsf{Q} \otimes \mathsf{V}\pi' \\
 & \downarrow \delta & & \downarrow \delta & & \downarrow \delta \\
0 \longrightarrow & \Lambda_{s+1}\mathsf{T}^*\mathsf{Q} \otimes \mathsf{G}_{k+r} & \longrightarrow & \Lambda_{s+1}\mathsf{T}^*\mathsf{Q} \otimes \mathsf{S}_{k+r}\mathsf{T}^*\mathsf{Q} \otimes \mathsf{V}\pi & \xrightarrow{\sigma_{r}(\Phi)} & \Lambda_{s+1}\mathsf{T}^*\mathsf{Q} \otimes \mathsf{S}_{r}\mathsf{T}^*\mathsf{Q} \otimes \mathsf{V}\pi'
\end{array}$$

The map δ induces a new δ-sequence for the symbol *at each point*. Note that sequences involving the symbol shall really be specified at each point and for the sake of simplicity we omit the point. What is more, there is no guarantee that this sequence is exact in general. Summarizing, we have the following graded differential complex

$$\begin{aligned}
0 \longrightarrow \mathsf{G}_{k+r} \xrightarrow{\ \delta\ } \mathsf{T}^*\mathsf{Q} \otimes \mathsf{G}_{k+r-1} \xrightarrow{\ \delta\ } \cdots \\
\cdots \xrightarrow{\ \delta\ } \Lambda_n\mathsf{T}^*\mathsf{Q} \otimes \mathsf{G}_{k+r-n} \longrightarrow 0\,. \qquad (2.1)
\end{aligned}$$

We denote by $\mathsf{H}^s_{k+r-s}(\mathsf{G}_k)$ the cohomology at $\Lambda_s\mathsf{T}^*\mathsf{Q} \otimes \mathsf{G}_{k+r-s}$ of this complex and we call it the *Spencer cohomology group* of degree $k+r-s$:

$$\mathsf{H}^s_{k+r-s}(\mathsf{G}_k) = \ker(\delta_{s,k+r-s})/\mathrm{Im}(\delta_{s-1,k+r+1-s}).$$

G_k is said to be *m-acyclic* if $\mathsf{H}^s_{k+r} = 0$ for all $0 \le s \le m$ and $r \ge 0$.

2.16 Definition: Let Q be an n-dimensional manifold and let $\mathsf{R}_k \subset \mathsf{J}_k\pi$ be a partial differential equation as above. If the symbol is n-acyclic it is called *involutive*.

By definition, a symbol is involutive if and only if its corresponding δ-sequences are exact. In particular, the symbol of the trivial system of partial differential equations is involutive.

2.17 Remark: The concept of involutivity is critical in the formal theory of partial differential equations, and is not easy to grasp at first glance. It is simply not possible to provide a complete review of the concept in this document. Guillemin and Sternberg relate the different interpretations of an involutive symbol and actually propose a practical method for verifying involutivity [25]. J. P. Serre's complementary note in the appendix of this paper completes the picture by relating the sequence given in Equation (2.1) to the Koszul complex.

We next address the concept of a *quasi-regular* basis and a practical method for verifying involutivity [42]. Let $(\mathsf{E}, \pi, \mathsf{Q})$ be a bundle with Q an n-dimensional manifold and $x \in \mathsf{Q}$. Let $\mathsf{R}_k \subset \mathsf{J}_k\pi$ be a partial differential equation with associated symbol G_k and let $p_k \in \mathsf{R}_k$ be such that $\pi_k(p_k) = x$. Let $\{\alpha^1, \dots, \alpha^n\}$ be a basis for $\mathsf{T}^*_x\mathsf{Q}$. Let $j \in \{1, \dots, n\}$, we define

$$\mathsf{G}_{k,j}|_{(x,p_k)} = \mathsf{G}_k|_{p_k} \cap \mathsf{S}_k\Sigma_j|_x,$$

where Σ_j is the subspace of $\mathsf{T}^*_x\mathsf{Q}$ generated by $\{\alpha^{j+1}, \dots, \alpha^n\}$.

2.18 Definition: (Quasi-regular basis) Let $(\mathsf{E}, \pi, \mathsf{Q})$ be a bundle with Q an n-dimensional manifold and $x \in \mathsf{Q}$. Let $\mathsf{R}_k \subset \mathsf{J}_k\pi$ be a partial differential equation with associated symbol G_k and let $p_k \in \mathsf{R}_k$ be such that $\pi_k(p_k) = x$. A basis $\{\alpha^1, \ldots, \alpha^n\}$ for $\mathsf{T}^*_x\mathsf{Q}$ is called *quasi-regular* if

$$\dim(\mathsf{G}_{k+1}|_{p_{k+1}}) = \dim(\mathsf{G}_k|_{p_k}) + \sum_{j=1}^{n-1} \dim(\mathsf{G}_{k,j}|_{(x,p_k)}). \tag{2.2}$$

The following theorem relates the concept of involutivity to the existence of a quasi-regular basis; the proof of the theorem can be found in [42].

2.19 Theorem: (Criterion of involutivity) *Let $\mathsf{R}_k \in \mathsf{J}_k\pi$ be a partial differential equation. If there exists a quasi-regular basis for $\mathsf{T}^*_{\pi_k(p_k)}\mathsf{Q}$, the symbol G_k is involutive at $p_k \in \mathsf{R}_k$.*

We now have the required machinery for the following central theorem for formal integrability [22]. This theorem is an analogue of the Cartan–Kähler theorem and is essentially a local result.

2.20 Theorem: (Goldschmidt) *Let $(\mathsf{E}, \pi, \mathsf{Q})$ be a fibered manifold and $\mathsf{R}_k \subset \mathsf{J}_k\pi$ a partial differential equation. Assume the following hypotheses:*

1. *$\rho_1(\mathsf{R}_k)$ is a fibered submanifold of $\mathsf{J}_{k+1}\pi$;*
2. *$\hat{\pi}^{k+1}_k : \rho_1(\mathsf{R}_k) \to \mathsf{R}_k$ is an epimorphism of fibered manifolds;*
3. *G_k is 2-acyclic.*

Then R_k is formally integrable and thus, given a point $p_{k+l} \in \mathsf{R}_{k+l}$ with $\pi_{k+l}(p_{k+l}) = x \in \mathsf{Q}$, there exists an analytic solution ξ_k of the equation R_k on a neighborhood of x such that $j_l(\xi_k)(x) = p_{k+l}$, where $l \geq 1$.

Sketch of the proof: Let Φ be the local morphism associated to R_k and recall the affine structure of $\rho_1(\mathsf{R}_k)$ over R_k. Since $\rho_1(\mathsf{R}_k)$ is a fibered submanifold of $\mathsf{J}_{k+1}\pi$, we have $\mathsf{G}_{k+1} = \rho_1(\mathsf{G}_k)$ as a vector bundle over R_k and so one can define a vector bundle $\mathsf{C} = \operatorname{coker}(\rho_1(\sigma(\Phi)))$ such that the following sequence is exact:

$$0 \longrightarrow \mathsf{G}_{k+1} \longrightarrow \mathsf{S}_{k+1}\mathsf{T}^*\mathsf{Q} \otimes \mathsf{V}\pi \xrightarrow{\rho_1(\sigma(\Phi))} \mathsf{T}^*\mathsf{Q} \otimes \mathsf{V}\pi' \xrightarrow{\tau} \mathsf{C} \longrightarrow 0$$

where τ is the canonical projection onto C. The essence of the proof is the construction of a map $\kappa : \mathsf{R}_k \to \mathsf{C}$ as follows. Consider the following exact and commutative diagram where the upper row is a sequence of vector bundles on which the second row of affine bundles are modeled:

$$\begin{array}{ccccccccc}
0 & \dashrightarrow & \mathsf{G}_{k+1} & \dashrightarrow & \mathsf{S}_{k+1}\mathsf{T}^*\mathsf{Q} \otimes \mathsf{V}\pi & \overset{\rho_1(\sigma(\Phi))}{\dashrightarrow} & \mathsf{T}^*\mathsf{Q} \otimes \mathsf{V}\pi' & \overset{\tau}{\dashrightarrow} & \mathsf{C} \dashrightarrow 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \rho_1(\mathsf{R}_1) & \longrightarrow & \mathsf{J}_{k+1}\pi & \xrightarrow{\rho_1(\Phi)} & \mathsf{J}_1\pi' & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathsf{R}_k & \longrightarrow & \mathsf{J}_k\pi & \xrightarrow{\Phi} & \pi' & &
\end{array}$$

Let $p \in \mathsf{R}_k$ with $\pi_k(p) = q \in \mathsf{Q}$ and let $p' \in \mathsf{J}_{k+1}\pi$ projecting to p. By commutativity of the diagram, $\rho_1(\Phi)(p')$ projects to $\Phi(p)$. As a result,

$$\rho_1(\Phi)(p') - j_1\Phi(p) \in \mathsf{T}^*\mathsf{Q} \otimes \mathsf{V}\pi'.$$

Let

$$\kappa(p) = \tau(\rho_1(\Phi)(p') - j_1\Phi(p)). \tag{2.3}$$

One can show that this definition is independent of the choice of p' [21]. This map is called the *curvature map*. A diagram chase shows that the map κ is zero with respect to the zero section of the vector bundle C if and only if the map π_k^{k+1} is an epimorphism of affine bundles. The 2-acyclicity of the symbol implies that $\hat{\pi}_{k+r}^{k+r+1}$ is also an epimorphism of affine bundles for $r \in \mathbb{Z}_{\geq 0}$. ■

If the symbol is involutive, condition 3 is automatically satisfied. We have the following definition.

2.21 Definition: A partial differential equation R_k is called *involutive* at a point p if

1. its associated morphism is of constant rank,
2. there exists a quasi-regular basis at p, and
3. the map $\hat{\pi}_k^{k+1}$ is surjective and is of constant rank in a neighborhood of p.

2.22 Example: Let $\mathsf{Q} = \mathbb{R}^3$ and $\mathsf{E} = \mathbb{R}^3 \times \mathbb{R}$ with $\pi(x, y, z, f) = (x, y, z)$. Let X be a vector field on Q. Consider the partial differential equation

$$\mathsf{R}_{\text{grad}} = \{(x, y, z, f, f_x, f_y, f_z) \in \mathsf{J}_1\pi \mid \text{grad}(f) = X\}.$$

We show that the solutions to this partial differential equation exists if $\text{curl}(X) = 0$. The symbol map of this partial differential equation is clearly the identity map and hence involutive. Following the proof of Theorem 2.20, we have the following diagram:

$$\begin{array}{ccccccccc}
0 \dashrightarrow & \mathsf{S}_2\mathsf{T}^*\mathbb{R}^3 \otimes \mathbb{R}^3 & \dashrightarrow & \mathsf{S}_2\mathsf{T}^*\mathbb{R}^3 \otimes \mathbb{R}^3 & \dashrightarrow & \mathsf{T}^*\mathbb{R}^3 \otimes \mathsf{T}\mathbb{R}^3 & \overset{\tau}{\dashrightarrow} & \Lambda_2\mathsf{T}^*\mathbb{R}^3 \otimes \mathbb{R}^3 & \dashrightarrow 0 \\
& \downarrow & & \downarrow & & \downarrow & & & \\
0 \longrightarrow & \rho_1(\mathsf{R}_{\text{grad}}) & \longrightarrow & \mathsf{J}_2\mathbb{R}^3 & \xrightarrow{\rho_1(\text{grad})} & \mathsf{J}_1\mathbb{R}^3 & & & \\
& \downarrow & & \downarrow & & \downarrow & & & \\
0 \longrightarrow & \mathsf{R}_{\text{grad}} & \longrightarrow & \mathsf{J}_1\mathbb{R}^3 & \xrightarrow{\text{grad}} & \mathsf{T}\mathbb{R}^3 \cong \mathbb{R}^3 & & &
\end{array}$$

The map τ is defined by the composition of the alternation map and an isomorphism of $\mathsf{T}^*\mathbb{R}^3 \otimes \mathsf{T}\mathbb{R}^3$ to $\mathsf{T}^*\mathbb{R}^3 \otimes \mathsf{T}^*\mathbb{R}^3$ which maps e^i to e_i for $i \in \{1, 2, 3\}$. This alternation map in coordinates yields

$$\frac{\partial X^i}{\partial x^j} - \frac{\partial X^j}{\partial x^i} = 0,$$

which is exactly $\text{curl}(X) = 0$ as claimed.

2.2. The space of connections

In this section we fix a fibered manifold $(\mathsf{E}, \pi, \mathsf{Q})$. As before, we denote by $\mathsf{V}\pi$ and $\mathsf{J}_k(\pi)$, the vertical bundle and the bundle of k-jets of the fibered manifold π, respectively [39]. We start by defining what we mean by a connection. It is not hard to show that the definition we give is equivalent to the usual construction of a connection as a splitting of the total space of a bundle on which the connection is defined [38, 13].

2.23 Definition: A *connection* on a fibered manifold $(\mathsf{E}, \pi, \mathsf{Q})$ is a section $\mathcal{S} : \mathsf{E} \to \mathsf{J}_1\pi$ of the bundle $\pi_0^1 : \mathsf{J}_1\pi \to \mathsf{E}$.

In natural coordinates (q^i, u^a, u_k^a) for $\mathsf{J}_1\pi$, a connection has the form $(q^i, u^a) \mapsto (q^i, u^a, \mathcal{S}_k^a)$, which defines the *connection coefficients* $\mathcal{S}_k^a$, where $a \in \{1, \ldots, m\}$ and $i, k \in \{1, \ldots, n\}$. One can define the covariant derivative associated to a connection as follows.

2.24 Definition: Let $\mathcal{S} : \mathsf{E} \to \mathsf{J}_1\pi$ be a connection on a fibered manifold $(\mathsf{E}, \pi, \mathsf{Q})$. If ξ is a smooth local section of E, then the $\mathcal{S}$-*covariant differential* of ξ is the smooth local section $\nabla^{\mathcal{S}}\xi$ of $\mathsf{T}^*\mathsf{Q} \otimes \xi^*\mathsf{V}\pi$ defined by

$$\nabla^{\mathcal{S}}\xi(q) = j_1\xi(q) - \mathcal{S}(\xi(q)). \tag{2.4}$$

In natural coordinates we have

$$\nabla^{\mathcal{S}}\xi = \left(\frac{\partial \xi^a}{\partial q^i} - \mathcal{S}_i^a\right) dx^i \otimes \frac{\partial}{\partial u^a}.$$

If X is a vector field on Q, then the $\mathcal{S}$-*covariant derivative* of ξ with respect to X is the section of $\xi^*\mathsf{V}\pi$ defined by $\nabla_X^{\mathcal{S}}\xi = \nabla^{\mathcal{S}}\xi(X)$. A *linear connection* on a vector bundle $(\mathsf{E}, \pi, \mathsf{Q})$ is a connection $\mathcal{S} : \mathsf{E} \to \mathsf{J}_1\pi$ that is also a vector bundle morphism over id_{E}. In adapted coordinates (x^i, u^a) for E and (x^i, u^a, u_k^a) on $\mathsf{J}_1\pi$, a linear connection has the form $(x^i, u^a) \mapsto (x^i, u^a, \mathcal{S}_{kb}^a u^b)$ which defines the connection coefficients $\mathcal{S}_{ib}^a$, for $a, b \in \{1, \ldots, m\}$ and $i \in \{1, \ldots, n\}$.

A linear connection $\mathcal{S}$ on the vector bundle $(\mathsf{TQ}, \pi_{\mathsf{Q}}, \mathsf{Q})$ is sometimes called an *affine connection* on Q. We have the following proposition which generalizes to vector bundles.

2.25 Proposition: *The set of affine connections on a manifold* Q *is the set of sections of an affine subbundle of the vector bundle* $\mathsf{T}^*\mathsf{Q} \otimes \mathsf{J}_1\pi_{\mathsf{Q}}$ *over* Q *modeled on the vector bundle* $\mathsf{T}^*\mathsf{Q} \otimes \mathsf{T}^*\mathsf{Q} \otimes \mathsf{TQ}$.

The following proposition clarifies the structure of the space of torsion-free affine connections.

2.26 Proposition: *The set of torsion-free affine connections on a manifold* Q *is an affine subbundle of the vector bundle* $\mathsf{T}^*\mathsf{Q} \otimes \mathsf{J}_1\pi_{\mathsf{Q}}$ *over* Q *modeled on the vector bundle* $\mathsf{S}_2\mathsf{T}^*\mathsf{Q} \otimes \mathsf{TQ}$ *given by*

$$\mathsf{Aff}_0(\mathsf{Q}) = \{\Xi \in \mathsf{T}^*\mathsf{Q} \otimes \mathsf{J}_1\pi_{\mathsf{Q}} \mid \pi_0^1 \circ \Xi = \mathrm{id}_{\mathsf{TQ}}, (j_1Y - \Xi(Y))(X) - (j_1X - \Xi(X))(Y) = [X, Y]\}, \tag{2.5}$$

where $X, Y \in \Gamma^\omega(\mathsf{TQ})$ *and we think of* Ξ *as a vector bundle map from* TQ *to* $\mathsf{J}_1\pi_{\mathsf{Q}}$.

2.3. The Ricci identity

Let $(\mathsf{E}, \pi, \mathsf{Q})$ be a vector bundle. There is a bijective correspondence between the set of linear connections $\mathcal{S} : \mathsf{E} \to \mathsf{J}_1\pi$ and the type $(1,1)$-tensor fields $\mathcal{P}^H_{\mathcal{S}}$ on E, where $\mathcal{P}^H_{\mathcal{S}}$ is a projection operator of constant rank, $\mathcal{P}^H_{\mathcal{S}}(X) = 0$ for every $X \in \Gamma^\omega(\mathsf{V}\pi)$ and $\mathrm{Im}(\mathcal{P}^H_{\mathcal{S}}) \oplus \mathsf{V}\pi = \mathsf{TE}$. Such a projection is called the horizontal projection associated to the connection $\mathcal{S}$ [39]. An *integral section* of a connection $\mathcal{S}$ is an analytic local section ξ of π satisfying $j_1\xi = \mathcal{S}(\xi)$. There is no guarantee that such a section exists, even locally. The existence of such an integral section is equivalent to the vanishing of the Nijenhuis tensor of the $(1,1)$-tensor field $\mathcal{P}^H_{\mathcal{S}}$ (see [39, 27]). In other words, the Nijenhuis tensor measures the involutivity of the associated horizontal subbundle and, as a result, the Nijenhuis tensor of $\mathcal{S}$ is directly related to the curvature tensor $\mathsf{R}[\mathcal{S}]$ associated to $\mathcal{S}$. Let (q^i, u^a) be adapted local coordinates on a neighborhood U of E with $i \in \{1, \dots, n\}$ and $a \in \{1, \dots, m\}$. Also let $\{e_1, \dots, e_m\}$ be a basis for the local sections of E. The curvature tensor, $\mathsf{R}[\mathcal{S}] \in \Gamma^\omega(\mathsf{E}^* \otimes \mathsf{E} \otimes \Lambda_2\mathsf{T}^*\mathsf{Q})$, can be written as

$$\mathsf{R}[\mathcal{S}]^a_{ijb} = \frac{\partial \mathcal{S}^a_{ib}}{\partial x^j} + \mathcal{S}^a_{ic}\mathcal{S}^c_{jb} - \Big(\frac{\partial \mathcal{S}^a_{jb}}{\partial x^i} + \mathcal{S}^a_{jc}\mathcal{S}^c_{ib}\Big), \tag{2.6}$$

where $i, j \in \{1, \dots, n\}$ and $a, b, c \in \{1, \dots, m\}$. One can naturally define an induced connection on the tensor product of two vector bundles as follows. Let $(\mathsf{E}_1, \pi_1, \mathsf{Q})$ and $(\mathsf{E}_2, \pi_2, \mathsf{Q})$ be two vector bundles equipped with two linear connections $\mathcal{S}_1$ and $\mathcal{S}_2$, respectively. There is a unique connection $\mathcal{S}_1 \otimes_{\mathsf{Q}} \mathcal{S}_2$ that makes the following diagram commute:

$$\begin{array}{ccc} \mathsf{E}_1 \times_{\mathsf{Q}} \mathsf{E}_2 & \xrightarrow{\ \otimes\ } & \pi_1 \otimes_{\mathsf{Q}} \pi_2 \\ {\scriptstyle \mathcal{S}_1 \times_{\mathsf{Q}} \mathcal{S}_2}\downarrow & & \downarrow{\scriptstyle \mathcal{S}_1 \otimes_{\mathsf{Q}} \mathcal{S}_2} \\ \mathsf{J}_1\pi_2 \times_{\mathsf{Q}} \mathsf{J}_1\pi_2 & \longrightarrow & \mathsf{J}_1(\pi_1 \otimes_{\mathsf{Q}} \pi_2) \end{array}$$

For more information about the induced connection on a fibered product bundle, see [39]. One can use the same procedure to induce a connection $\mathcal{S}^*$ on the dual bundle π^*. For our purposes, we consider the tensor bundle $(\mathsf{E} \otimes_{\mathsf{Q}} \mathsf{E}, \pi \otimes_{\mathsf{Q}} \pi, \mathsf{Q})$, where $\pi : \mathsf{E} \to \mathsf{Q}$ is a vector bundle. Let (x^i) be local coordinates for Q and let (x^i, u^a) be adapted coordinates for E, where $i \in \{1, \dots, n\}$ and $a \in \{1, \dots, m\}$. Denote an analytic local section of $\mathsf{E} \otimes \mathsf{E}$ by $\xi = \xi^{ab} e_a \otimes e_b$, where $\{e_1, \dots, e_m\}$ is a basis for local sections of E and $a, b \in \{1, \dots, m\}$. Then the covariant derivative with respect to $\mathcal{S} \otimes \mathcal{S}$ can be represented by

$$\nabla^{\mathcal{S}\otimes\mathcal{S}}\xi = j_1\xi - (\mathcal{S} \otimes \mathcal{S})(\xi).$$

We have the following representation of the covariant derivative with respect to the induced connection $\mathcal{S} \otimes \mathcal{S}$ in local coordinates:

$$(\nabla^{\mathcal{S}\otimes\mathcal{S}}\xi)^{ab}_i = \frac{\partial \xi^{ab}}{\partial x^i} - \mathcal{S}^a_{ic}\xi^{cb} - \mathcal{S}^b_{ic}\xi^{ac}, \tag{2.7}$$

where $i \in \{1, \dots, n\}$ and $a, b, c \in \{1, \dots, m\}$. Using Equation (2.6), one can show that the associated curvature tensor for $\mathcal{S} \otimes \mathcal{S}$ is

$$\mathsf{R}[\mathcal{S} \otimes \mathcal{S}]^{ab}_{cdij}\xi^{cd} = \mathsf{R}[\mathcal{S}]^a_{cij}\xi^{cb} + \mathsf{R}[\mathcal{S}]^b_{cij}\xi^{ac}. \tag{2.8}$$

The vanishing of the curvature tensor is an obstruction for the involutivity of the horizontal subspace of $\mathsf{T}(\mathsf{E}\otimes\mathsf{E})$ associated with the induced connection $\mathcal{S}\otimes\mathcal{S}$. The relationship between the curvature tensor of a tensor product bundle and the curvature of the underlying bundles leads to the Ricci identity [15]. In the literature this identity is typically introduced through the following lemma.

2.27 Lemma: *Let $(\mathsf{Q}, \mathbb{G})$ be a Riemannian manifold equipped with a symmetric affine connection $\mathcal{S}$. Then the following identity holds and is called the Ricci identity:*

$$(\nabla_X \nabla_Y \mathbb{G} - \nabla_Y \nabla_X \mathbb{G} - \nabla_{[X,Y]}\mathbb{G})(Z, W) = \mathbb{G}(R(X, Y)Z, W) + \mathbb{G}(Z, R(X, Y)W), \quad (2.9)$$

where $X, Y, Z, W \in \Gamma^\omega(\mathsf{TQ})$.

Proof: The proof follows from a direct computation using Equation (2.7) to compute the covariant derivative of $\mathbb{G}$ with respect to a vector field. ■

2.28 Remark: We state the following remarks for future use.

1. Lemma 2.27 can be extended to any $(0, 2)$-tensor on Q, but for our purposes we state the lemma for a metric $\mathbb{G}$ on Q.

2. The Ricci identity appears when one tries to find a set of necessary conditions for a metric to be associated to a given symmetric affine connection; see [18, 46].

Chapter 3

Geometric formulation of partial differential equations in energy shaping

We give a formulation of the partial differential equations of the energy shaping problem using the theory of partial differential equations presented in the previous section. This formulation is an integral part of our approach since it places the energy shaping problem into the realm of the formal theory of partial differential equations.

3.1. Kinetic energy shaping

We provide a jet bundle structure associated to the kinetic energy shaping system of partial differential equations. This system of partial differential equations involves the affine subbundle description for the set of torsion free connection; see Section 2.2.

Let $(\mathsf{S}_2^+\mathsf{T}^*\mathsf{Q}, \pi_{\mathcal{G}}, \mathsf{Q})$ be the bundle of Riemannian metrics on the configuration manifold Q. One can generalize the definitions in this section by allowing metrics with other signatures; see Remark 3.7. Let $(\mathsf{B}, \pi_{\mathsf{B}}, \mathsf{Q})$ be the bundle of gyroscopic tensor fields on Q; thus $\mathsf{B} \doteq \mathsf{Gyr}(\mathsf{TQ}) \cap \ker(\mathrm{Alt})$. We have the following definition.

3.1 Definition: The *kinetic energy shaping bundle* is the fibered product bundle $(\mathsf{KS}, \pi \doteq \pi_{\mathcal{G}} \times_{\mathsf{Q}} \pi_{\mathsf{B}}, \mathsf{Q})$, where $\mathsf{KS} \doteq \mathsf{S}_2^+\mathsf{T}^*\mathsf{Q} \times_{\mathsf{Q}} \mathsf{B}$. We denote by π_1 and π_2 the projections onto the first and second factors.

In local coordinates, a typical fiber over $q \in \mathsf{Q}$ is a pair $(\mathbb{G}(q), \mathbb{B}(q))$ and coordinates for $\mathsf{J}_1\pi$ are denoted by $(q^i, \mathbb{G}_{mn}, \mathbb{B}_{lpq}, \mathbb{G}_{jk,a}, \mathbb{B}_{lpq,b})$, where we denote the derivatives of $\mathbb{G}_{jk}$ and $\mathbb{B}_{lpq}$, respectively, by $\mathbb{G}_{jk,a}$ and $\mathbb{B}_{lpq,b}$.

Define the "Levi-Civita" map $\phi_{\mathrm{LC}} : \mathsf{J}_1\pi_{\mathcal{G}} \to \mathsf{Aff}_0(\mathsf{Q})$ by $\phi_{\mathrm{LC}}(j_1\mathbb{G}) = \nabla^{\mathbb{G}}$. Let $\Sigma_{\mathrm{ol}} = (\mathsf{Q}, \mathbb{G}_{\mathrm{ol}}, V_{\mathrm{ol}}, 0, \mathcal{W}_{\mathrm{ol}})$ be a given open-loop simple mechanical control system and let $(\mathsf{KS}, \pi, \mathsf{Q})$ be the *kinetic energy shaping bundle*. We define the following projection:

$$\pi_{\mathcal{W}} : \mathbb{G}_{\mathrm{ol}}^{\sharp}(\mathsf{T}^*\mathsf{Q} \otimes \mathsf{T}^*\mathsf{Q} \otimes \mathsf{T}^*\mathsf{Q}) \to \mathbb{G}_{\mathrm{ol}}^{\sharp}(\mathsf{T}^*\mathsf{Q} \otimes \mathsf{T}^*\mathsf{Q} \otimes \mathsf{T}^*\mathsf{Q})/\mathbb{G}_{\mathrm{ol}}^{\sharp}(\mathcal{W}_{\mathrm{ol}} \otimes \mathsf{T}^*\mathsf{Q} \otimes \mathsf{T}^*\mathsf{Q}) \doteq \mathcal{K},$$

where we use the extended definition of sharp map; see Equation (1.1). We now have the required tools for defining the kinetic energy shaping partial differential equation as a submanifold of $\mathsf{J}_1\pi$.

3.2 Definition: Let $(\mathsf{KS}, \pi, \mathsf{Q})$ be the kinetic energy shaping bundle. If $\pi_{\mathcal{W}}$ and ϕ_{LC} are, respectively, the projection and the affine connection map defined above, the *kinetic energy shaping submanifold* $\mathsf{R}_{\mathrm{kin}} \subset \mathsf{J}_1\pi$ is defined by

$$\mathsf{R}_{\mathrm{kin}} = \{p \in \mathsf{J}_1\pi \mid \Phi_{\mathrm{kin}}(p) = 0\},$$

where Φ_{kin} is the *kinetic energy shaping map* given by

$$\Phi_{\mathrm{kin}}(p) = \pi_{\mathcal{W}}\left(\phi_{\mathrm{LC}}(j_1\pi_1(p)) - \phi_{\mathrm{LC}}(j_1\pi_1(p_0))\right) - \pi_{\mathcal{W}}(\pi_1(p)^\sharp \pi_2(p)),$$

where the last term involves gyroscopic forces.

One can represent the governing system of partial differential equations for the kinetic energy shaping problem by the following exact sequence:

$$0 \longrightarrow \mathsf{R}_{\mathrm{kin}} \longrightarrow \mathsf{J}_1\pi \xrightarrow{\Phi_{\mathrm{kin}}} \mathcal{K},$$

where $\mathsf{R}_{\mathrm{kin}}$ is the kernel of Φ_{kin} with respect to the zero section of $\mathcal{K}$.

3.2. The λ-method

In this section, we recall a differential geometric approach to the kinetic energy shaping problem from [7, 5]. The main idea is to transform the set of quasi-linear partial differential equations from the previous section into a set of linear partial differential equations in terms of a new variable. In the following definition we introduce a set of partial differential equations which is the main component of this equivalent system.

The following theorem gives the desired transformation.

3.3 Theorem: *Let $\Sigma_{\mathrm{ol}} = (\mathsf{Q}, \mathbb{G}_{\mathrm{ol}}, V_{\mathrm{ol}}, \mathcal{F}_{\mathrm{ol}}, \mathcal{W}_{\mathrm{ol}})$ be an open-loop simple mechanical control system. Let $P \in \Gamma^\omega(\mathsf{T}^*\mathsf{Q} \otimes \mathsf{TQ})$ be the $\mathbb{G}_{\mathrm{ol}}$-orthogonal projection as in Section 1.1. Let $\mathbb{G}_{\mathrm{cl}} \in \Gamma^\omega(\mathsf{S}_2^+\mathsf{T}^*\mathsf{Q})$ and let $\mathbb{B}$ be a quadratic gyroscopic tensor. If $\mathbb{G}_{\mathrm{ol}}^\flat = \mathbb{G}_{\mathrm{cl}}^\flat \circ \lambda$ for $\lambda \in \Gamma^\omega(\mathsf{T}^*\mathsf{Q} \otimes \mathsf{TQ})$, then the following two conditions are equivalent:*

1. $P(\nabla_X^{\mathbb{G}_{\mathrm{cl}}} X - \nabla_X^{\mathbb{G}_{\mathrm{ol}}} X - \mathbb{G}_{\mathrm{cl}}^\sharp \circ \mathbb{B}^\flat(X)) = 0$, *where* $X \in \Gamma^\omega(\mathsf{TQ})$;
2. (a) $\nabla_Z^{\mathbb{G}_{\mathrm{ol}}}(\mathbb{G}_{\mathrm{ol}}\lambda)(PX, PY) + \frac{1}{2}\langle \mathbb{B}(\lambda PX, \lambda PY) + \mathbb{B}(\lambda PY, \lambda PX), Z\rangle = 0$ *and*

   (b) $\nabla_{\lambda PX}^{\mathbb{G}_{\mathrm{ol}}} \mathbb{G}_{\mathrm{cl}}(Z, Z) + 2\mathbb{G}_{\mathrm{cl}}(\nabla_Z^{\mathbb{G}_{\mathrm{ol}}} \lambda PX, Z) = 2\mathbb{G}_{\mathrm{ol}}(\nabla_Z^{\mathbb{G}_{\mathrm{ol}}} PX, Z) - 2\langle \lambda PX, \mathbb{B}^\flat(Z)\rangle$,

   where $X, Y, Z \in \Gamma^\omega(\mathsf{TQ})$.

In order to prove this theorem we need the following lemma.

3.4 Lemma: *Let $(\mathsf{Q}, \mathbb{G})$ be a Riemannian manifold and let $\mathcal{W}$ be a codistribution on Q. Let $P \in \Gamma^\omega(\mathsf{T}^*\mathsf{Q} \otimes \mathsf{TQ})$ be the $\mathbb{G}$-orthogonal projection and let $\mathbb{G}_{\mathrm{cl}} \in \Gamma^\omega(\mathsf{S}_2^+\mathsf{T}^*\mathsf{Q})$. If*

$$P(\nabla_X^{\mathbb{G}_{\mathrm{cl}}} X - \nabla_X^{\mathbb{G}} X - \mathbb{G}_{\mathrm{cl}}^\sharp \circ \mathbb{B}^\flat(X)) = 0, \qquad \forall X \in \Gamma^\omega(\mathsf{TQ}),$$

then

1. *for* $X, Y \in \Gamma^\omega(\mathsf{TQ})$ *we have*

$$P(\nabla^{\mathbb{G}}_X Y - \nabla^{\mathbb{G}_{\rm cl}}_X Y) = -\tfrac{1}{2} P\mathbb{G}^\sharp_{\rm cl} \circ (\mathbb{B}^\flat(X+Y) - \mathbb{B}^\flat(X) - \mathbb{B}^\flat(Y)), \tag{3.1}$$

and

2. *for* $\mathbb{G} = \mathbb{G}_{\rm cl} \circ \lambda$ *for* $\lambda \in \Gamma^\omega(\mathsf{T}^*\mathsf{Q} \otimes \mathsf{TQ})$ *we have*

$$2\mathbb{G}(P(\nabla^{\mathbb{G}}_{\lambda PX} Z - \nabla^{\mathbb{G}_{\rm cl}}_{\lambda PX} Z), X) = \nabla^{\mathbb{G}}_Z(\mathbb{G}\lambda)(PX, PX), \tag{3.2}$$

where $X, Z \in \Gamma^\omega(\mathsf{TQ})$.

Proof: We begin with the first statement. Note that since the connections are torsion free,

$$\nabla^{\mathbb{G}}_X Y - \nabla^{\mathbb{G}_{\rm cl}}_X Y = \nabla^{\mathbb{G}}_Y X - \nabla^{\mathbb{G}_{\rm cl}}_Y X.$$

We have

$$\begin{aligned}
P(\nabla^{\mathbb{G}}_X Y - \nabla^{\mathbb{G}_{\rm cl}}_X Y) &= \tfrac{1}{2} P(\nabla^{\mathbb{G}}_X Y - \nabla^{\mathbb{G}_{\rm cl}}_X Y + \nabla^{\mathbb{G}}_Y X - \nabla^{\mathbb{G}_{\rm cl}}_Y X) \\
&= \tfrac{1}{2} P(\nabla^{\mathbb{G}}_{X+Y}(X+Y) - \nabla^{\mathbb{G}_{\rm cl}}_{X+Y}(X+Y)) - \tfrac{1}{2} P(\nabla^{\mathbb{G}}_X(X) - \nabla^{\mathbb{G}_{\rm cl}}_X(X)) \\
&\quad - \tfrac{1}{2} P(\nabla^{\mathbb{G}}_Y(Y) - \nabla^{\mathbb{G}_{\rm cl}}_Y(Y)) \\
&= -\tfrac{1}{2} P\mathbb{G}^\sharp_{\rm cl} \circ (\mathbb{B}^\flat(X+Y) - \mathbb{B}^\flat(X) - \mathbb{B}^\flat(Y)),
\end{aligned}$$

where $X, Y \in \Gamma^\omega(\mathsf{TQ})$.

For the second part, recall that for the Levi-Civita connection $\nabla^{\mathbb{G}}$ associated to $\mathbb{G}$ one can write

$$\begin{aligned}
2\mathbb{G}(\nabla^{\mathbb{G}}_X Y, Z) = L_X\mathbb{G}(Y, Z) + L_Y\mathbb{G}(Z, X) - L_Z\mathbb{G}(X, Y) \\
+ \mathbb{G}([X, Y], Z) + \mathbb{G}([Z, X], Y) - \mathbb{G}([Y, Z], X).
\end{aligned} \tag{3.3}$$

Moreover, we have

$$L_X(\mathbb{G}(Y, Z)) = \mathbb{G}(\nabla^{\mathbb{G}}_X Y, Z) + \mathbb{G}(Y, \nabla^{\mathbb{G}}_X Z). \tag{3.4}$$

Thus

$$\begin{aligned}
2\mathbb{G}(P(\nabla^{\mathbb{G}}_{\lambda PX} Z - \nabla^{\mathbb{G}_{\rm cl}}_{\lambda PX} Z), X) &= 2\mathbb{G}(\nabla^{\mathbb{G}}_{\lambda PX} Z, PX) - 2\mathbb{G}(\nabla^{\mathbb{G}_{\rm cl}}_{\lambda PX} Z, PX) \\
&= 2\mathbb{G}(\nabla^{\mathbb{G}}_{\lambda PX} Z, PX) - 2\mathbb{G}_{\rm cl}(\nabla^{\mathbb{G}_{\rm cl}}_{\lambda PX} Z, \lambda PX).
\end{aligned}$$

We use Equation (3.3) to get

$$\begin{aligned}
2\mathbb{G}(P(\nabla^{\mathbb{G}}_{\lambda PX} Z - \nabla^{\mathbb{G}_{\rm cl}}_{\lambda PX} Z), X) = \lambda PX\mathbb{G}(Z, PX) - PX\mathbb{G}(\lambda PX, Z) \\
+ \mathbb{G}([PX, \lambda PX], Z) - \mathbb{G}([Z, PX], \lambda PX) + \mathbb{G}([Z, \lambda PX], PX).
\end{aligned} \tag{3.5}$$

Expanding the terms using Equation (3.4), after some simplification, we have

$$2\mathbb{G}(P(\nabla^{\mathbb{G}}_{\lambda PX} Z - \nabla^{\mathbb{G}_{\rm cl}}_{\lambda PX} Z), X) = \mathbb{G}(\nabla^{\mathbb{G}}_Z(\lambda PX), PX) - \mathbb{G}(\nabla^{\mathbb{G}}_Z PX, \lambda PX).$$

Note that

$$L_Z(\lambda(X)) = \nabla^{\mathbb{G}}_Z(\lambda)(X) + \lambda(\nabla^{\mathbb{G}}_Z X).$$

As a result,

$$\begin{aligned}2\mathbb{G}(P(\nabla^{\mathbb{G}}_{\lambda PX}Z-\nabla^{\mathbb{G}_{\rm cl}}_{\lambda PX}Z),X)&=\mathbb{G}((\nabla^{\mathbb{G}}_{Z}\lambda)(PX)+\lambda(\nabla^{\mathbb{G}}_{Z}PX),PX)-\mathbb{G}(\nabla^{\mathbb{G}}_{Z}PX,\lambda PX)\\&=\mathbb{G}((\nabla^{\mathbb{G}}_{Z}\lambda)(PX),PX)+\mathbb{G}(\lambda(\nabla^{\mathbb{G}}_{Z}PX),PX)\\&\quad-\mathbb{G}(\nabla^{\mathbb{G}}_{Z}PX,\lambda PX)\\&=\mathbb{G}((\nabla^{\mathbb{G}}_{Z}\lambda)PX,PX).\end{aligned}$$

Since $\nabla^{\mathbb{G}}\mathbb{G}=0$, one can write

$$2\mathbb{G}(P(\nabla^{\mathbb{G}}_{\lambda PX}Z-\nabla^{\mathbb{G}_{\rm cl}}_{\lambda PX}Z),X)=\nabla^{\mathbb{G}}_{Z}(\mathbb{G}\lambda)(PX,PX),\tag{3.6}$$

which is the desired result. ■

Proof of Theorem 3.3: (1)⇒(2)] We first assume that $\mathbb{G}_{\rm cl}$ and $\mathbb{B}$ satisfy

$$P(\nabla^{\mathbb{G}_{\rm cl}}_{X}X-\nabla^{\mathbb{G}_{\rm ol}}_{X}X-\mathbb{G}^{\sharp}_{\rm cl}\circ\mathbb{B}^{\flat}(X))=0,\qquad\forall X\in\Gamma^{\omega}(\mathsf{TQ}).$$

Using the second part of Lemma 3.4 we have

$$2\mathbb{G}_{\rm ol}(P(\nabla^{\mathbb{G}_{\rm ol}}_{\lambda PX}Z-\nabla^{\mathbb{G}_{\rm cl}}_{\lambda PX}Z),X)=\nabla_{Z}(\mathbb{G}_{\rm ol}\lambda)(PX,PX).\tag{3.7}$$

By the first part of Lemma 3.4 we have

$$P(\nabla^{\mathbb{G}_{\rm ol}}_{\lambda PX}Z-\nabla^{\mathbb{G}_{\rm cl}}_{\lambda PX}Z)=-\tfrac{1}{2}P\mathbb{G}^{\sharp}_{\rm cl}\circ(\mathbb{B}^{\flat}(\lambda PX+Z)-\mathbb{B}^{\flat}(\lambda PX)-\mathbb{B}^{\flat}(Z)).$$

Using Equation (3.7) we get

$$2\mathbb{G}_{\rm ol}(-\tfrac{1}{2}P\mathbb{G}^{\sharp}_{\rm cl}\circ(\mathbb{B}^{\flat}(\lambda PX+Z)-\mathbb{B}^{\flat}(\lambda PX)-\mathbb{B}^{\flat}(Z)),X)=\nabla_{Z}(\mathbb{G}_{\rm ol}\lambda)(PX,PX),$$

which can be written as

$$\nabla_{Z}(\mathbb{G}_{\rm ol}\lambda)(PX,PX)+\langle\mathbb{B}^{\flat}(\lambda PX+Z)-\mathbb{B}^{\flat}(\lambda PX)-\mathbb{B}^{\flat}(Z),\lambda PX\rangle=0.$$

From the definition of the flat operation we have $\langle\mathbb{B}^{\flat}(Y),X\rangle=\mathbb{B}(X,Y,Y)$. Also recall that the gyroscopic tensor is antisymmetric in the first two arguments. We can then expand the right hand side of the previous equation to get

$$\nabla_{Z}(\mathbb{G}_{\rm ol}\lambda)(PX,PX)=\mathbb{B}(\lambda PX,Z,\lambda PX)=-\langle\mathbb{B}(\lambda PX,\lambda PX),Z\rangle.$$

Notice that $\mathbb{G}_{\rm ol}\lambda$ and $\nabla_{Z}(\mathbb{G}_{\rm ol}\lambda)$ are both symmetric $(0,2)$-tensors. Thus we have

$$\nabla_{Z}(\mathbb{G}_{\rm ol}\lambda)(PX,PY)+\tfrac{1}{2}\langle\mathbb{B}(\lambda PX,\lambda PY)+\mathbb{B}(\lambda PY,\lambda PX),Z\rangle=0$$

for all $X,Y\in\Gamma^{\omega}(\mathsf{TQ})$ which is (2a). In order to prove (2b) we have

$$\begin{aligned}L_{\lambda PX}\mathbb{G}_{\rm cl}(Z,Z)&=\mathbb{G}_{\rm cl}(\nabla^{\mathbb{G}_{\rm cl}}_{\lambda PX}Z,Z)+\mathbb{G}_{\rm cl}(Z,\nabla^{\mathbb{G}_{\rm cl}}_{\lambda PX}Z)\\&=2\mathbb{G}_{\rm cl}(\nabla^{\mathbb{G}_{\rm cl}}_{\lambda PX}Z,Z)\\&=2\mathbb{G}_{\rm cl}(\nabla^{\mathbb{G}_{\rm cl}}_{Z}\lambda PX-[Z,\lambda PX],Z)\\&=2\mathbb{G}_{\rm cl}(\nabla^{\mathbb{G}_{\rm cl}}_{Z}\lambda PX,Z)-2\mathbb{G}_{\rm cl}([Z,\lambda PX],Z).\end{aligned}$$

As a result,

$$\begin{aligned} L_{\lambda PX}\mathbb{G}_{\mathrm{cl}}(Z,Z) + 2\mathbb{G}_{\mathrm{cl}}([Z,\lambda PX],Z) &= 2\mathbb{G}_{\mathrm{cl}}(\nabla^{\mathbb{G}_{\mathrm{cl}}}_Z \lambda PX, Z) \\ &= 2L_Z\mathbb{G}_{\mathrm{ol}}(PX,Z) - 2\mathbb{G}_{\mathrm{ol}}(PX, \nabla^{\mathbb{G}_{\mathrm{cl}}}_Z Z) \\ &= 2L_Z\mathbb{G}_{\mathrm{ol}}(PX,Z) - 2\mathbb{G}_{\mathrm{ol}}(X, P\nabla^{\mathbb{G}_{\mathrm{cl}}}_Z Z). \end{aligned}$$

Now, from the kinetic energy shaping system of partial differential equations, we have

$$P\nabla^{\mathbb{G}_{\mathrm{cl}}}_Z Z = P\nabla^{\mathbb{G}_{\mathrm{ol}}}_Z Z + P\mathbb{G}^{\sharp}_{\mathrm{cl}}\mathbb{B}^{\flat}(Z).$$

Therefore,

$$\begin{aligned} &L_{\lambda PX}\mathbb{G}_{\mathrm{cl}}(Z,Z) + 2\mathbb{G}_{\mathrm{cl}}([Z,\lambda PX],Z) \\ &= 2L_Z\mathbb{G}_{\mathrm{ol}}(PX,Z) - 2\mathbb{G}_{\mathrm{ol}}(X, P\nabla^{\mathbb{G}_{\mathrm{ol}}}_Z Z + P\mathbb{G}^{\sharp}_{\mathrm{cl}}\mathbb{B}^{\flat}(Z)) \\ &= 2\mathbb{G}_{\mathrm{ol}}(\nabla^{\mathbb{G}_{\mathrm{ol}}}_Z PX, Z) + 2\mathbb{G}_{\mathrm{ol}}(PX, \nabla^{\mathbb{G}_{\mathrm{ol}}}_Z Z) - 2\mathbb{G}_{\mathrm{ol}}(X, P\nabla^{\mathbb{G}_{\mathrm{ol}}}_Z Z + P\mathbb{G}^{\sharp}_{\mathrm{cl}}\mathbb{B}^{\flat}(Z)). \end{aligned}$$

This gives

$$L_{\lambda PX}\mathbb{G}_{\mathrm{cl}}(Z,Z) + 2\mathbb{G}_{\mathrm{cl}}([Z,\lambda PX],Z) = 2\mathbb{G}_{\mathrm{ol}}(\nabla^{\mathbb{G}_{\mathrm{ol}}}_Z PX, Z) - 2\langle \lambda PX, \mathbb{B}^{\flat}(Z)\rangle,$$

which is (2b).

(2)⇒(1): We have to prove that if $\lambda = \mathbb{G}^{\sharp}_{\mathrm{cl}} \circ \mathbb{G}^{\flat}_{\mathrm{ol}}$ and if $\mathbb{G}_{\mathrm{cl}}$ and $\mathbb{B}$ satisfy the set of extended λ equations and the closed-loop metric equation given in part (2) of the theorem, then $(\mathbb{G}_{\mathrm{cl}}, \mathbb{B})$ is a solution to the kinetic energy shaping problem. We compute

$$\begin{aligned} &\mathbb{G}_{\mathrm{ol}}(P(\nabla^{\mathbb{G}_{\mathrm{ol}}}_X X - \nabla^{\mathbb{G}_{\mathrm{cl}}}_X X + \mathbb{G}^{\sharp}_{\mathrm{cl}}\mathbb{B}^{\flat}(X)), Z) \\ &= \mathbb{G}_{\mathrm{ol}}(\nabla^{\mathbb{G}_{\mathrm{ol}}}_X X - \nabla^{\mathbb{G}_{\mathrm{cl}}}_X X + \mathbb{G}^{\sharp}_{\mathrm{cl}}\mathbb{B}^{\flat}(X), PZ) \\ &= \mathbb{G}_{\mathrm{ol}}(\nabla^{\mathbb{G}_{\mathrm{ol}}}_X X, PZ) - \mathbb{G}_{\mathrm{cl}}(\nabla^{\mathbb{G}_{\mathrm{cl}}}_X X, \lambda PZ) + \langle \mathbb{B}^{\flat}(X), \lambda PZ\rangle \\ &= L_X\mathbb{G}_{\mathrm{ol}}(X, PZ) - \mathbb{G}_{\mathrm{ol}}(X, \nabla^{\mathbb{G}_{\mathrm{ol}}}_X PZ) \\ &\quad - L_X\mathbb{G}_{\mathrm{cl}}(X, \lambda PZ) + \mathbb{G}_{\mathrm{cl}}(X, \nabla^{\mathbb{G}_{\mathrm{cl}}}_X \lambda PZ) + \langle \mathbb{B}^{\flat}(X), \lambda PZ\rangle \\ &= -\mathbb{G}_{\mathrm{ol}}(X, \nabla^{\mathbb{G}_{\mathrm{ol}}}_X PZ) + \mathbb{G}_{\mathrm{cl}}(X, \nabla^{\mathbb{G}_{\mathrm{cl}}}_X \lambda PZ) + \langle \mathbb{B}^{\flat}(X), \lambda PZ\rangle \\ &= -\mathbb{G}_{\mathrm{ol}}(X, \nabla^{\mathbb{G}_{\mathrm{ol}}}_X PZ) + \mathbb{G}_{\mathrm{cl}}(X, \nabla^{\mathbb{G}_{\mathrm{cl}}}_{\lambda PZ} X) + \mathbb{G}_{\mathrm{cl}}(X, [X, \lambda PZ]) + \langle \mathbb{B}^{\flat}(X), \lambda PZ\rangle \\ &= -\mathbb{G}_{\mathrm{ol}}(X, \nabla^{\mathbb{G}_{\mathrm{ol}}}_X PZ) + \tfrac{1}{2}L_{\lambda PZ}\mathbb{G}_{\mathrm{cl}}(X,X) + \mathbb{G}_{\mathrm{cl}}(X, [X, \lambda PZ]) + \langle \mathbb{B}^{\flat}(X), \lambda PZ\rangle \\ &= -\mathbb{G}_{\mathrm{ol}}(X, \nabla^{\mathbb{G}_{\mathrm{ol}}}_X PZ) + \mathbb{G}_{\mathrm{ol}}(\nabla^{\mathbb{G}_{\mathrm{ol}}}_X PZ, X) - \langle \lambda PZ, \mathbb{B}^{\flat}(X)\rangle + \langle \mathbb{B}^{\flat}(X), \lambda PZ\rangle \\ &= 0. \end{aligned}$$

As a result,

$$P(\nabla^{\mathbb{G}_{\mathrm{ol}}}_X X - \nabla^{\mathbb{G}_{\mathrm{cl}}}_X X + \mathbb{G}^{\sharp}_{\mathrm{cl}}\mathbb{B}^{\flat}(X)) = 0,$$

as desired. ■

From part (2) of the previous theorem we see that the kinetic energy shaping partial differential equation is equivalent to two partial differential equations, one for λ and one for obtaining $\mathbb{G}_{\mathrm{cl}}$ from λ. We will study these partial differential equations in detail in Chapter 4, but for now let us define them.

3.5 Definition: Let Q be an n-dimensional manifold and let $\mathbb{G} \in \Gamma^\omega(\mathsf{S}_2^+\mathsf{T}^*\mathsf{Q})$ be a Riemannian metric on Q. Let $\mathcal{W} \subset \mathsf{T}^*\mathsf{Q}$ be a subbundle and let P be the associated $\mathbb{G}$-orthogonal projection map as in Section 1.1. The following set of partial differential equations with $\lambda \in \Gamma^\omega(\mathsf{T}^*\mathsf{Q} \otimes \mathsf{T}\mathsf{Q})$ and $\mathbb{B}$ a gyroscopic $(0,3)$-tensor field as dependent variables is called the *extended λ-equation*:

$$\nabla_Z^{\mathbb{G}}(\mathbb{G}\lambda)(PX, PY) + \tfrac{1}{2}\langle \mathbb{B}(\lambda PX, \lambda PY) + \mathbb{B}(\lambda PY, \lambda PX), Z\rangle = 0, \tag{3.8}$$

where $X, Y \in \Gamma^\omega(\mathsf{T}\mathsf{Q})$.

3.6 Definition: Let Q be an n-dimensional manifold and let $\mathbb{G} \in \Gamma^\omega(\mathsf{S}_2^+\mathsf{T}^*\mathsf{Q})$ be a metric on Q. Let $\mathcal{W} \subset \mathsf{T}^*\mathsf{Q}$ be a subbundle and $P \in \Gamma^\omega(\mathsf{T}^*\mathsf{Q} \otimes \mathsf{T}\mathsf{Q})$ be the associated projection map as above. Also let $\lambda \in \Gamma^\omega(\mathsf{T}^*\mathsf{Q} \otimes \mathsf{T}\mathsf{Q})$ and let $\mathbb{B}$ be a gyroscopic $(0,3)$-tensor field on Q. The following set of partial differential equations with $\mathbb{G}_{\mathrm{cl}} \in \Gamma^\omega(\mathsf{S}_2\mathsf{T}^*\mathsf{Q})$ as unknown is called the *extended closed-loop metric equation*:

$$\nabla_{\lambda PX}^{\mathbb{G}}\mathbb{G}_{\mathrm{cl}}(Z, Z) + 2\mathbb{G}_{\mathrm{cl}}(\nabla_Z^{\mathbb{G}}\lambda PX, Z) = 2\mathbb{G}(\nabla_Z^{\mathbb{G}}PX, Z) - 2\langle \lambda PX, \mathbb{B}^\flat(Z)\rangle, \tag{3.9}$$

for $X, Y, Z \in \Gamma^\omega(\mathsf{T}\mathsf{Q})$.

It is clear that the λ-equation is a first-order linear system of partial differential equations. The word *extended* is due to the presence of gyroscopic forces. Theorem 3.3 is an intrinsic version of what has been presented in [7]. Different versions of the proof have been given in [5, 6] and modified with presence of gyroscopic forces in [16]. One should note that, by solving the λ-equations, we only obtain the restriction of λ to $\Gamma^\omega(\mathcal{W}^\perp \otimes \mathsf{T}^*\mathsf{Q})$. As in [6], we assume that one solves the kinetic energy shaping problem in the following steps.

1. Find the set of pairs $(\lambda, \mathbb{B})$ which satisfy the extended λ-equations.

2. Use the set of solutions to the λ-equations to find a closed-loop metric $\mathbb{G}_{\mathrm{cl}}$ as a solution to the extended closed-loop metric system of equations. This closed-loop metric will be a solution to the kinetic energy shaping problem by the statement of Theorem 3.3. Moreover, all the solutions to the kinetic energy shaping problem can be produced by this procedure.

3.7 Remark: Note that there is no assumption on the positive-definiteness of the closed-loop metric. In other words, one may very well achieve a closed-loop metric which is not positive-definite, but which could possibly lead to a stabilizing energy shaping feedback.

3.3. The λ-method partial differential equations

In this section, we formulate the two partial differential equations for the λ-method in the language of jet bundles. We make the simplifying assumption in this section that $\mathbb{B} = 0$. Assume that $\mathcal{W}$ is an $(n-m)$-dimensional integrable subbundle of $\mathsf{T}^*\mathsf{Q}$, where m is the number of unactuated directions. In this section, for simplification, we fix the notation $\mathbb{G} = \mathbb{G}_{\mathrm{ol}}$. We use the same notation in Sections 4.1 and 4.2.

3.3.1. The equation R_L. With the assumptions above, the set of λ-equations we consider in this section is

$$\nabla^{\mathbb{G}}_Z(\mathbb{G}\lambda)(PX, PY) = 0, \qquad X, Y, Z \in \Gamma^\omega(\mathsf{TQ}),$$

where $\lambda \in \Gamma^\omega(\mathsf{T^*Q} \otimes \mathsf{TQ})$, $\mathbb{G}$ is a metric on Q and $P \in \Gamma^\omega(\mathsf{T^*Q} \otimes \mathsf{TQ})$ is the $\mathbb{G}$-orthogonal projection as before. We denote by $\mathcal{W}^\perp$ the $\mathbb{G}$-orthogonal complement of $\mathcal{W}$ and by $\lambda|_{\mathcal{W}^\perp\otimes\mathsf{TQ}}$ the restriction of λ to $\mathcal{W}^\perp \otimes \mathsf{TQ}$. Consider the bundle $\pi : \mathcal{W}^\perp \otimes \mathsf{TQ} \to \mathsf{Q}$ and let $\lambda|_{\mathcal{W}^\perp\otimes\mathsf{TQ}}(q)$ be a typical fiber element over $q \in \mathsf{Q}$. We define the bundle map

$$\begin{aligned}\Phi :&\mathsf{J}_1\pi \to \mathsf{T^*Q} \otimes \mathcal{W}^\perp \otimes \mathcal{W}^\perp\\ &j_1(\lambda|_{\mathcal{W}^\perp\otimes\mathsf{TQ}}) \mapsto \nabla^{\mathbb{G}}(\mathbb{G}\lambda)|_{\mathrm{Im}P\otimes\mathrm{Im}P},\end{aligned} \tag{3.10}$$

In an adapted coordinate system (q^i, λ^i_a) on $\mathcal{W}^\perp \otimes \mathsf{TQ}$ and $(q^i, \lambda^i_a, \lambda^i_{a,k})$ on $\mathsf{J}_1\pi$,

$$\Phi(q^i, \lambda^i_a, \lambda^i_{a,k}) = (q^i, \mathbb{G}_{ai}\lambda^i_{b,k} + \mathbb{G}_{ai,k}\lambda^i_b - \mathcal{S}^s_{ka}\mathbb{G}_{si}\lambda^i_b - \mathcal{S}^s_{kb}\mathbb{G}_{si}\lambda^i_a), \tag{3.11}$$

where $\mathcal{S}^i_{jk}$, $i, j, k \in \{1, \ldots, n\}$, are the coefficients of the Levi-Civita connection associated to $\mathbb{G}$, and $k \in \{1, \ldots, n\}$, $a, b \in \{1, \ldots, m\}$. Thus we define

$$\mathsf{R}_\mathrm{L} \doteq \{p \in \mathsf{J}_1\pi \mid \Phi(p) = 0\}$$

to be the submanifold of $\mathsf{J}_1\pi$ corresponding to the λ-equation.

3.3.2. The equation R_E. With the assumptions above, the closed-loop metric equation is

$$\nabla^{\mathbb{G}}_{\lambda PX}\mathbb{G}_{\mathrm{cl}}(Z, Z) + 2\mathbb{G}_{\mathrm{cl}}(\nabla^{\mathbb{G}}_Z\lambda PX, Z) = 2\mathbb{G}(\nabla^{\mathbb{G}}_Z PX, Z),$$

for $X, Y, Z \in \Gamma^\omega(\mathsf{TQ})$. Consider the bundle $(\mathsf{S_2T^*Q}, \pi, \mathsf{Q})$ and let $\mathcal{W}$ and P be, respectively, the integrable control codistribution and the $\mathbb{G}$-orthogonal projection, respectively, as in Section 1.1. Let λ be an automorphism of TQ and denote a section of $\mathcal{W}^\perp \otimes \mathsf{TQ}$ by $\lambda|_{\mathcal{W}^\perp\otimes\mathsf{TQ}}$. Define the bundle map $\Upsilon_1 : \mathsf{J}_1\pi \to \mathsf{T^*Q} \otimes \mathsf{S_2T^*Q}$ by $\Upsilon_1(j_1\mathbb{G}_{\mathrm{cl}}) = \nabla^{\mathbb{G}}\mathbb{G}_{\mathrm{cl}}$. Also define a bundle map

$$\Upsilon_0 : \mathsf{J}_1\pi \to \mathcal{W}^\perp \otimes \mathsf{S_2T^*Q}$$

by

$$\Upsilon_0(X, Z, Z) = 2\mathbb{G}_{\mathrm{cl}}(\nabla^{\mathbb{G}}_Z\lambda PX, Z) - 2\mathbb{G}(\nabla^{\mathbb{G}}_Z PX, Z).$$

Let $\Psi_\mathrm{E} : \mathsf{T^*Q} \otimes \mathsf{S_2T^*Q} \to \mathcal{W}^\perp \otimes \mathsf{S_2T^*Q}$ be the bundle map given by

$$\Psi_\mathrm{E}(\beta \otimes \mathsf{A})(PX) = \beta(\lambda PX) \otimes \mathsf{A},$$

where $\beta \in \Gamma^\omega(\mathsf{T^*Q})$ and $\mathsf{A} \in \Gamma^\omega(\mathsf{S_2T^*Q})$. Observe that the map Ψ_E is surjective. Finally, define $\Phi_\mathrm{E} = \Psi_\mathrm{E} \circ \Upsilon_1 + \Upsilon_0$. Thus $\mathsf{R}_\mathrm{E} = \ker \Phi_\mathrm{E}$ gives the extended closed-loop metric system of partial differential equations.

3.4. Potential energy shaping

In this section, we explore aspects of potential energy shaping. First, we recall the result of Lewis [32] regarding potential shaping after kinetic shaping has been done. Then we couple the sufficient conditions of Lewis [32] with the λ-equations from the previous section. In this way, we can understand how kinetic energy shaping can influence potential energy shaping.

3.4.1. Sufficient conditions for potential energy shaping. We recall the results for potential energy shaping after kinetic energy shaping from [32]. Denote the bundle automorphism $\mathbb{G}_{\mathrm{ol}}^{\flat} \circ \mathbb{G}_{\mathrm{cl}}^{\sharp}$ by Λ_{cl}. Define a codistribution $\mathcal{W}_{\mathrm{cl}} = \Lambda_{\mathrm{cl}}^{-1}(\mathcal{W}_{\mathrm{ol}})$ and assume that this codistribution is integrable. Let $(\mathsf{PS} \doteq \mathsf{Q} \times \mathbb{R}, \pi, \mathsf{Q})$ be the trivial vector bundle over Q, so that a section of π corresponds to a potential function via the formula $q \mapsto (q, V(q))$. We define a $\mathsf{T}^*\mathsf{Q}$-valued differential operator $\mathfrak{D}_{\mathsf{d}}(V) = \mathsf{d}V$ which induces a vector bundle map $\Phi_{\mathrm{pot}} : \mathsf{J}_1\pi \to \mathsf{T}^*\mathsf{Q}$ such that $\mathfrak{D}_{\mathsf{d}}(V)(q) = \Phi_{\mathrm{pot}}(j_1 V(q))$. Similarly to what we did for kinetic energy shaping, we denote by

$$\pi_{\mathcal{W}_{\mathrm{cl}}} : \mathsf{T}^*\mathsf{Q} \to \mathsf{T}^*\mathsf{Q}/\mathcal{W}_{cl}$$

the canonical projection.

3.8 Definition: Let $\Sigma_{\mathrm{ol}} = (\mathsf{Q}, \mathbb{G}_{\mathrm{ol}}, V_{\mathrm{ol}}, \mathcal{F}_{\mathrm{ol}}, \mathcal{W}_{\mathrm{ol}})$ be an open-loop simple mechanical control system. The submanifold $\mathsf{R}_{\mathrm{pot}} \subset \mathsf{J}_1\pi$ defined by

$$\mathsf{R}_{\mathrm{pot}} = \{p \in \mathsf{J}_1\pi \mid \pi_{\mathcal{W}_{\mathrm{cl}}} \circ \Phi_{\mathrm{pot}}(p) = \pi_{\mathcal{W}_{\mathrm{cl}}} \circ \Lambda_{\mathrm{cl}}^{-1}\mathsf{d}V_{\mathrm{ol}}\}$$

is called the *potential energy shaping submanifold*.

Let $\pi_1 : \mathsf{J}_1\pi \to \mathsf{Q}$ be the canonical projection. Lewis [32] gives a set of sufficient conditions under which the potential shaping problem has a solution. The proof follows from the integrability theory of partial differential equations; in particular, the potential energy shaping partial differential equation has an involutive symbol. We recall the definition of $(\mathbb{G}_{\mathrm{ol}}$-$\mathbb{G}_{\mathrm{cl}})$-potential energy shaping feedback from [32].

3.9 Definition: A section $\mathcal{F}$ of $\mathcal{W}$ is called a $(\mathbb{G}_{\mathrm{ol}}$-$\mathbb{G}_{\mathrm{cl}})$*-potential energy shaping feedback* if there exists a function V_{cl} on Q such that

$$\mathcal{F}(q) = \Lambda_{\mathrm{cl}}\mathsf{d}V_{\mathrm{cl}} - \mathsf{d}V_{\mathrm{ol}}, \qquad q \in \mathsf{Q}.$$

The following theorem implies when one can construct a Taylor series solution to the potential energy shaping partial differential equation order-by-order.

3.10 Theorem: *Let $\Sigma_{\mathrm{ol}} = (\mathsf{Q}, \mathbb{G}_{\mathrm{ol}}, V_{\mathrm{ol}}, \mathcal{F}_{\mathrm{ol}}, \mathcal{W}_{\mathrm{ol}})$ be an analytic open-loop simple mechanical control system. Let $\mathbb{G}_{\mathrm{cl}}$ be a closed-loop analytic metric. Let $p_0 \in \mathsf{R}_{\mathrm{pot}}$ and let $q_0 = \pi_1(p_0)$. Assume that q_0 is a regular point for $\mathcal{W}_{\mathrm{ol}}$ and that $\mathcal{W}_{\mathrm{cl}} = \Lambda_{\mathrm{cl}}^{-1}\mathcal{W}_{\mathrm{ol}}$ is integrable in a neighborhood of q_0. Then the following statements are equivalent:*

1. *there exists a neighborhood U of q_0 and an analytic $(\mathbb{G}_{\mathrm{ol}}$-$\mathbb{G}_{\mathrm{cl}})$-potential energy shaping feedback $\mathcal{F} \in \Gamma^{\omega}(\mathcal{W})$ defined on U which satisfies*

$$\Phi_{\mathrm{pot}}(p_0) = \Lambda_{\mathrm{cl}}\mathsf{d}V(q_0) - \mathsf{d}V_{\mathrm{ol}}(q_0) + \Lambda_{\mathrm{cl}}^{-1}\mathsf{d}V_{\mathrm{ol}}(q_0),$$

   for a solution V to $\mathsf{R}_{\mathrm{pot}}$;

2. *there exists a neighborhood U of q_0 such that $d(\Lambda_{\mathrm{cl}}^{-1}\mathsf{d}V_{\mathrm{ol}})(q) \in \mathsf{I}_2(\mathcal{W}_{\mathrm{cl}}|_q)$, where we denote $\mathsf{I}_2(\mathcal{W}_{\mathrm{cl}}|_q) = \mathsf{I}(\mathcal{W}_{\mathrm{cl}}|_q) \cap \Lambda_2(\mathsf{T}_q^*\mathsf{Q})$ and the algebraic ideal $\mathsf{I}(\mathcal{W}_{\mathrm{cl}}|_q)$ of $\Lambda(\mathsf{T}_q^*\mathsf{Q})$ is generated by elements of the form $\gamma \wedge \omega$ with $\gamma \in \mathcal{W}_{\mathrm{cl}}|_q$.*

The theorem gives a set of compatibility conditions for the existence of a $(\mathbb{G}_{\text{ol}}\text{-}\mathbb{G}_{\text{cl}})$-potential energy shaping feedback. Moreover, one can give a full description of the set of achievable potential energy shaping feedbacks. Let $\alpha_{\text{cl}} = \Lambda_{\text{cl}}^{-1}\mathsf{d}V_{\text{ol}}$. Let us use a coordinate system $(q^1, \dots, q^n)$ on U a neighborhood of q_0 such that

$$\mathcal{W}_{\text{cl}}|_{q_0} = \mathsf{span}(dq^{m+1}, \dots, dq^n).$$

In these local coordinates we write the one form α_{cl} as $\alpha_{\text{cl}} = \alpha_j dq^j$ and compatibility conditions become:

$$\frac{\partial \alpha_j}{\partial q^i} - \frac{\partial \alpha_i}{\partial q^j} = 0, \qquad i, j \in \{1, \dots, m\}. \tag{3.12}$$

3.11 Remark: One can make the following observations about the potential energy shaping problem.

1. The choice of $\mathbb{G}_{\text{cl}}$ affects the set of solutions that one might get for potential energy shaping. *A bad choice of $\mathbb{G}_{\text{cl}}$ might make it impossible to find any potential energy shaping feedback*. As a result, if one is able to have an understanding of the set of closed-loop energy shaping metrics, then the condition given by Equation (3.12) is an obstruction that detects the closed-loop energy shaping metrics for which their exists a potential energy shaping feedback. We give a complete description of this problem in Section 4.3.

2. Following [32], if we denote the set of all solutions for the potential shaping problem by

$$\mathsf{PS}_{q_0} = \{V_{\text{cl}} \oplus \mathcal{F}_{\text{cl}} \in C^\omega(\mathsf{Q}) \oplus \Gamma^\omega(\mathcal{W}_{\text{cl}}) \mid \mathsf{d}V_{\text{cl}} = \mathcal{F}_{\text{cl}} + \Lambda_{\text{cl}}^{-1}\mathsf{d}V_{\text{ol}},\ V_{\text{cl}}(q_0) = 0\},$$

one can describe the whole set of solutions as an affine subspace of $C^\omega(\mathsf{Q}) \oplus \Gamma^\omega(\mathcal{W}_{\text{cl}})$ modeled on the subspace

$$L(\mathsf{PS}_{q_0}) = \{f \oplus \beta \in C^\omega(\mathsf{Q}) \oplus \Gamma^\omega(\mathcal{W}_{\text{cl}}) \mid \mathsf{d}f = \beta\}.$$

3.4.2. The equation R_T. The sufficient condition for integrability of the partial differential equation in potential energy shaping, given in Equation (3.12), can be seen as a nonlinear partial differential equation with the dependent variable $\Lambda = \mathbb{G}_{\text{ol}}^\flat \circ \mathbb{G}_{\text{cl}}^\sharp$. The following commutative diagram shows the relationship between λ and Λ:

$$\begin{array}{ccc} \mathsf{T}^*\mathsf{Q} & \xrightarrow{\Lambda} & \mathsf{T}^*\mathsf{Q} \\ \big\downarrow {\scriptstyle \mathbb{G}_{\text{ol}}^\sharp} & & \big\downarrow {\scriptstyle \mathbb{G}_{\text{ol}}^\sharp} \\ \mathsf{T}\mathsf{Q} & \xrightarrow{\lambda} & \mathsf{T}\mathsf{Q} \end{array}$$

Any condition on Λ imposes conditions on λ and vice versa. Through these conditions, we can find the obstruction that the potential energy shaping integrability condition imposes on the set of solutions to the λ-equations. The solutions of the λ-equation which give rise to potential energy shaping are the ones that satisfy the partial differential equation obtained from Equation (3.12), with the dependent variable Λ. Thus we wish to analyze the integrability of the partial differential equation obtained from Equation (3.12). We make some algebraic constructions.

An algebraic construction

Let V be a finite-dimensional $\mathbb{R}$-vector space. Let D be a subspace of V^*. The algebraic ideal $\mathsf{I}(\mathsf{D})$ of $\Lambda(\mathsf{V}^*)$ is generated by elements of the form $\gamma \wedge \omega$, where $\gamma \in \mathsf{D}$. For $k \in \mathbb{Z}_{\geq 0}$ we denote $\mathsf{I}_k(\mathsf{D}) = \mathsf{I}(\mathsf{D}) \cap \Lambda_k(\mathsf{V}^*)$. For $\Theta \in \mathsf{Aut}(\mathsf{V}^*)$, we wish to understand $\mathsf{I}_2(\Theta(\mathsf{D}))$.

3.12 Lemma: *We have* $\mathsf{I}_2(\Theta(\mathsf{D})) = (\Theta \otimes \Theta)(\mathsf{I}_2(\mathsf{D}))$.

Proof: Let $\{v^1, \dots, v^n\}$ be a basis for V^*, and suppose that

$$\mathsf{D} = \mathsf{span}\{v^{m+1}, v^{m+2}, \dots, v^n\}.$$

One can identify $\mathsf{I}_2(\Theta(\mathsf{D}))$ by

$$\mathsf{I}_2(\Theta(\mathsf{D})) = \mathsf{span}\{\Theta(v^j) \wedge \Theta(v^i) \mid m+1 \leq j \leq n,\ 1 \leq i \leq n\}.$$

If one extends Θ to $\Theta \otimes \Theta$ on $\mathsf{V}^* \otimes \mathsf{V}^*$ in the usual way, we have

$$\Theta \otimes \Theta(\mathsf{I}_2(\mathsf{D})) = \mathsf{span}\{\Theta(v^j) \wedge \Theta(v^i) \mid m+1 \leq j \leq n,\ 1 \leq i \leq n\},$$

as desired. ■

3.13 Lemma: *Let α be an analytic local section of* $(\mathsf{T}^*\mathsf{Q}, \pi, \mathsf{Q})$ *in a neighborhood U of* Q *and let* $\mathsf{D} \subset \mathsf{T}^*\mathsf{Q}$ *be a subbundle. Then* $\mathsf{d}\alpha \in \mathsf{I}_2(\Theta(\mathsf{D}))$ *if and only if*

$$\Theta^{-1} \otimes \Theta^{-1}(\mathsf{d}\alpha) \in \mathsf{I}_2(\mathsf{D}).$$

Proof: Note that $(\Theta \otimes \Theta)^{-1} = \Theta^{-1} \otimes \Theta^{-1}$ and so the proof follows from the previous lemma. ■

3.14 Proposition: *Let* Q *be an n-dimensional manifold and let $\beta \neq 0$ and α be analytic local sections of* $\mathsf{T}^*\mathsf{Q}$ *such that* $\alpha = \Theta(\beta)$, *where* $\Theta \in \mathsf{Aut}(\mathsf{T}^*\mathsf{Q})$. *Let U be a neighborhood of* $p \in \mathsf{Q}$. *Given* D, *an integrable subbundle of* $\mathsf{T}^*\mathsf{Q}$ *with adapted local coordinates* $\{dq^{m+1}, \dots, dq^n\}$ *as above, we have* $\mathsf{d}\alpha \in \mathsf{I}_2(\Theta(\mathsf{D}))$ *in this neighborhood if and only if*

$$\left(\Delta^k_i \Delta^p_j - \Delta^k_j \Delta^p_i\right)\left(\frac{\partial \Theta^l_k}{\partial x^p}\beta_l + \Theta^l_k \frac{\partial \beta_l}{\partial x^p}\right) = 0,$$

where $i, j \in \{1, \dots, m\}$ *and we denote* Θ^{-1} *by* Δ.

Proof: Using Lemma 3.13 we have

$$\mathsf{d}\alpha = \Delta^k_i \Delta^p_j \left(\frac{\partial \Theta^l_k}{\partial x^p}\beta_l + \Theta^l_k \frac{\partial \beta_l}{\partial x^p}\right) dq^i \wedge dq^j.$$

The proof follows since $\Lambda_2 \mathsf{T}^*\mathsf{Q} = \mathsf{I}_2(\mathsf{D}) \oplus \Lambda_2(\mathsf{D}^\perp)$. ■

Partial differential equation

We consider the system of partial differential equations of Proposition 3.14 with the automorphism Θ as unknown. One can easily observe that this system of partial differential equations is equivalent to the system of partial differential equations one would obtain by assuming Equation (3.12) as a partial differential equation with unknown Λ^{-1}. We consider *nonlinear* partial differential equations as described briefly in Section 2.1. More details on the formal integrability of nonlinear systems of partial differential equations can be found in [22].

Let $(\mathsf{Q}, \mathbb{G})$ be an n-dimensional Riemannian manifold and let $\mathcal{W} \subset \mathsf{T}^*\mathsf{Q}$ be a subbundle. Consider the vector bundle $(\pi, (\mathcal{W}^\perp \otimes_{\mathsf{Q}} \mathsf{TQ}) \oplus (\mathcal{W}^\perp \otimes_{\mathsf{Q}} \mathsf{TQ}), \mathsf{Q})$ with a typical fiber $(q, \Theta(q) \oplus \Delta(q))$ and denote its first jet bundle by $\mathsf{J}_1\pi$. We define the following system of partial differential equations in a neighborhood U of $q_0 \in \mathsf{Q}$:

$$\mathsf{R}_{\mathrm{T}} = \{j_1(\Theta \oplus \Delta) \in \mathsf{J}_1\pi \mid \Phi_{\mathrm{T}}(j_1(\Theta \oplus \Delta)) = 0\},$$

where Φ_{T} can be written in adapted local coordinates

$$\Phi_{\mathrm{T}}(j_1(\Theta \oplus \Delta)) = \left(\Delta_i^k \Delta_j^r - \Delta_j^k \Delta_i^r\right) \left(\frac{\partial \Theta_k^l}{\partial q^r} \beta_l + \Theta_k^l \frac{\partial \beta_l}{\partial q^r}\right),$$

where $i, j, k \in \{1, \dots, m\}$ and $\beta \in \Gamma^\omega(\mathsf{T}^*\mathsf{Q})$. This system of partial differential equations is quasi-linear and so we need to use Definition 2.7 to find the symbol. We look at the linearization of the partial differential equation about a given reference point. Let $\bar{\Theta}(q) \oplus \bar{\Delta}(q)$ be a typical fiber of π. If we linearize the system about this point we have

$$\begin{aligned} \frac{d}{dt}\Big|_{t=0} & \Big(\Big((\Delta_i^k t + \bar{\Delta}_i^k)(\Delta_j^p t + \bar{\Delta}_j^r) - (\Delta_j^k t + \bar{\Delta}_j^k)(\Delta_i^p t + \bar{\Delta}_i^r) \Big) \\ & \times \left(\frac{\partial(\Theta_k^l t + \bar{\Theta}_k^l)}{\partial q^r} \beta_l + (\Theta_k^l t + \bar{\Theta}_k^l) \beta_l \right) \Big) \\ & = \left(\Delta_i^k \bar{\Delta}_j^r + \Delta_j^r \bar{\Delta}_i^k - \Delta_j^k \bar{\Delta}_i^r - \Delta_i^r \bar{\Delta}_j^k \right) \frac{\partial \bar{\Theta}_k^l}{\partial q^r} \beta_l \\ & + \left(\bar{\Delta}_i^k \bar{\Delta}_j^r - \bar{\Delta}_j^k \bar{\Delta}_i^r \right) \left(\frac{\partial \Theta_k^l}{\partial q^r} \beta_l + \Theta_k^l \frac{\partial \beta_l}{\partial q^r} \right). \end{aligned}$$

The effect on the reference point of the linearization should be investigated carefully. For now, we consider the linearization of the system about a point $p \in \mathsf{J}_1\pi$ with $\pi_0^1(p) = (\mathrm{id}|_{\mathsf{TQ} \otimes_{\mathsf{Q}} \mathcal{W}^\perp} \otimes \mathrm{id}|_{\mathsf{TQ} \otimes_{\mathsf{Q}} \mathcal{W}^\perp})$. The reason for this choice is that the identity solution for Θ refers to the open-loop system which is always a solution to the energy shaping problem. Thus we study the linearization of the nonlinear system about the open-loop solution. We emphasize that the involutivity results of Section 4.3 are independent of the choice of point of linearization. We have the following linearization of Φ_{T} about p:

$$\mathsf{V}_p(\Phi_{\mathrm{T}})(j_1(\Theta \oplus \Delta)) = \left(\frac{\partial \Theta_i^l}{\partial q^j} \beta_l - \frac{\partial \Theta_j^l}{\partial q^i} \beta_l \right) + (\Theta_i^l \frac{\partial \beta_l}{\partial q^j} - \Theta_j^l \frac{\partial \beta_l}{\partial q^i}), \tag{3.13}$$

where we have utilized the fact that $\mathsf{VJ}_1\pi \cong \mathsf{J}_1\mathsf{V}\pi$ [34].

Chapter 4

Formal integrability of energy shaping partial differential equations

In this chapter we present some integrability results for the partial differential equations appearing in the previous chapter. The results rely on the formal integrability theorem of Goldshmidt, Theorem 2.20.

4.1. Formal integrability of R_L

In this section, we apply the integrability theorem of Goldschmidt to the λ-equation. The statement of the result in this section is Theorem 4.7 which gives sufficient conditions for formal integrability of the λ-equation. The proof of this theorem requires the machinery of Section 2.1. However, the statement of the result can be understood without understanding the details of the proof.

4.1.1. The symbol of R_L. Recall the definition of the partial differential equation R_L given in Section 3.3.1. We start by computing the symbol map for R_L.

4.1 Lemma: *The symbol map* $\sigma(\mathsf{R}_\mathrm{L}) : \mathsf{T}^*\mathsf{Q} \otimes \mathcal{W}^\perp \otimes \mathsf{TQ} \to \mathsf{T}^*\mathsf{Q} \otimes \mathcal{W}^\perp \otimes \mathcal{W}^\perp$ *is defined by*

$$\sigma(\mathsf{R}_\mathrm{L})(\mathsf{A})(X, PY, PZ) = \mathsf{A}(X, PY, \mathbb{G}^\flat(PZ)), \qquad \mathsf{A} \in \Gamma^\omega(\mathsf{T}^*\mathsf{Q} \otimes \mathcal{W}^\perp \otimes \mathsf{TQ}),$$

where $X, Y, Z \in \Gamma^\omega(\mathsf{TQ})$.

Proof: This can be shown using the affine structure of $\mathsf{J}_1\pi$ as follows. Take $p_1, p_2 \in \mathsf{J}_1\pi$ such that $\pi_0^1(p_1) = \pi_0^1(p_2)$. Then $\mathsf{A} = p_1 - p_2 \in \mathsf{T}^*\mathsf{Q} \otimes \mathcal{W}^\perp \otimes \mathsf{TQ}$, by the affine structure of $\mathsf{J}_1\pi$. Now one can define the symbol map to be $\Phi(p_2 - p_1)$, where Φ is defined in Section 3.3.1. Using Equation (3.11), one can observe that

$$\Phi(p_1 - p_2)(X, PY, PZ) = \mathsf{A}(X, PY, \mathbb{G}^\flat(PZ)),$$

since $\pi_0^1(p_1) = \pi_0^1(p_2)$. In local coordinates, this identifies the highest order term of the partial differential equations. ■

Let us determine the symbol $\mathsf{G}_1(\mathsf{R}_\mathrm{L})$ and its prolongation.

4.2 Lemma: *The following sequence is short exact:*

$$0 \longrightarrow \mathsf{G}_1(\mathsf{R}_\mathrm{L}) \longrightarrow \mathsf{T}^*\mathsf{Q} \otimes \mathcal{W}^\perp \otimes \mathsf{TQ} \xrightarrow{\sigma(\mathsf{R}_\mathrm{L})} \mathsf{T}^*\mathsf{Q} \otimes \mathcal{W}^\perp \otimes \mathcal{W}^\perp \longrightarrow 0 \qquad (4.1)$$

Proof: By the previous lemma one can write $\sigma(\mathsf{R}_\mathrm{L}) = \mathrm{id}_{\mathsf{T}^*\mathsf{Q}\otimes\mathcal{W}^\perp} \otimes (\mathbb{G}^\flat \circ P)$. This map is surjective since the image of TQ under $\mathbb{G}^\flat \circ P$ is isomorphic to $\mathcal{W}^\perp$. ■

Let $\{e^1, \dots, e^n\}$ be a basis for $\mathsf{T}^*_{q_0}\mathsf{Q}$ for $q_0 \in \mathsf{Q}$ and let Σ_j be the subspace of $\mathsf{T}^*_{q_0}\mathsf{Q}$ generated by $\{e^{j+1}, \dots, e^n\}$. Then we have the following lemma, similar to Lemma 4.2.

4.3 Lemma: *The following sequence is short exact:*

$$0 \longrightarrow \mathsf{G}_{1,j}(\mathsf{R}_\mathrm{L}) \longrightarrow \Sigma_j \otimes \mathcal{W}^\perp \otimes \mathsf{TQ} \xrightarrow{\sigma(\mathsf{R}_\mathrm{L})} \Sigma_j \otimes \mathcal{W}^\perp \otimes \mathcal{W}^\perp \longrightarrow 0$$

where $\mathsf{G}_{1,j}(\mathsf{R}_\mathrm{L}) = \mathsf{G}_1(\mathsf{R}_\mathrm{L}) \cap \Sigma_j$.

The following lemma characterizes the prolonged symbol $\rho_1(\mathsf{G}_1(\mathsf{R}_\mathrm{L}))$.

4.4 Lemma: *The following sequence is short exact:*

$$0 \longrightarrow \rho_1(\mathsf{G}_1(\mathsf{R}_\mathrm{L})) \longrightarrow \mathsf{S}_2\mathsf{T}^*\mathsf{Q} \otimes \mathcal{W}^\perp \otimes \mathsf{TQ} \xrightarrow{\rho_1(\sigma(\mathsf{R}_\mathrm{L}))} \mathsf{T}^*\mathsf{Q} \otimes \mathsf{T}^*\mathsf{Q} \otimes \mathcal{W}^\perp \otimes \mathcal{W}^\perp \xrightarrow{\tau} \mathsf{C} \longrightarrow 0$$

where $\mathsf{C} = \Lambda^2\mathsf{T}^*\mathsf{Q} \otimes \mathcal{W}^\perp \otimes \mathcal{W}^\perp$ *and* τ *is given by*

$$\tau(\mathsf{A})(X_1, X_2, Y_1, Y_2) = \mathsf{A}(X_1, X_2, Y_1, Y_2) - \mathsf{A}(X_2, X_1, Y_1, Y_2), \qquad (4.2)$$

where $X_1, X_2 \in \Gamma^\omega(\mathsf{T}^*\mathsf{Q})$ *and* $Y_1, Y_2 \in \Gamma^\omega(\mathcal{W}^\perp)$.

Proof: Note that τ is, up to a constant, the alternation map on the first two elements and so is surjective. Moreover, we have

$$\rho_1(\sigma(\mathsf{R}_\mathrm{L}))(\mathsf{A})(X_1, X_2, Y_1, Y_2) = \mathsf{A}(X_1, X_2, Y_1, \mathbb{G}^\flat Y_2),$$

as a consequence of the fact that $\tau \circ \rho_1(\sigma(\mathsf{R}_\mathrm{L}))$ is zero since A is symmetric in the first two arguments. Thus the image of $\rho_1(\sigma(\mathsf{R}_\mathrm{L}))$ is $\mathsf{S}_2\mathsf{T}^*\mathsf{Q}\otimes\mathcal{W}^\perp\otimes\mathcal{W}^\perp$ and so $\mathsf{C} = \Lambda^2\mathsf{T}^*\mathsf{Q}\otimes\mathcal{W}^\perp\otimes\mathcal{W}^\perp$. ■

4.5 Lemma: *The symbol* $\mathsf{G}_1(\mathsf{R}_\mathrm{L})$ *is involutive.*

Proof: We will show that any basis $\{e^1, \dots, e^n\}$ for $\mathsf{T}^*_{q_0}\mathsf{Q}$ is a quasi-regular basis. This is just a dimension count. From Lemmata 4.2 and 4.3 we have

$$\dim(\mathsf{G}_1(\mathsf{R}_\mathrm{L})) = nm(n-m),$$
$$\dim(\mathsf{G}_{1,j}(\mathsf{R}_\mathrm{L})) = (n-j)m(n-m).$$

We compute

$$\dim(\mathsf{G}_1(\mathsf{R}_\mathrm{L})) + \sum_{j=1}^{n-1} \dim(\mathsf{G}_{1,j}(\mathsf{R}_\mathrm{L})) = \tfrac{1}{2}nm(n+1)(n-m).$$

On the other side, using the exactness of Lemma 4.4, one computes

$$\begin{aligned}\dim(\rho_1(\mathsf{G}_1(\mathsf{R}_\mathrm{L}))) &= \frac{n(n+1)mn}{2} + \frac{n(m)^2(n-1)}{2} - n^2(m)^2 \\ &= \tfrac{1}{2}nm(n+1)(n-m);\end{aligned}$$

thus the $\mathsf{G}_1(\mathsf{R}_\mathrm{L})$ is involutive. ■

4.1.2. Involutivity of R_L.

4.6 Theorem: *The set of λ-equations R_L is involutive if, for $p \in \mathsf{R}_\mathrm{L}$, we have*

$$\tau(\rho_1(\Phi)(p_2) - 0) = 0,$$

where p_2 is any point in $\mathsf{J}_2(\pi)$ that projects to p.

Proof: We shall verify the conditions of Theorem 2.20. Note that $\rho_1(\mathsf{G}_1(\mathsf{R}_\mathrm{L}))$ is isomorphic to $\mathsf{S}_2\mathsf{T}^*\mathsf{Q} \otimes \mathcal{W}^\perp \otimes \mathbb{G}^\sharp\mathcal{W}$. Therefore, it is a vector bundle on the open subset of Q for which $\mathbb{G}^\sharp\mathcal{W}$ is a vector bundle. Let C be the cokernel of $\rho_1(\sigma(\Phi)))$. Then $\mathsf{G}(\mathsf{R}_\mathrm{L})$ is involutive and so the system of partial differential equations R_L is involutive if the curvature map $\kappa : \mathsf{R}_\mathrm{L} \to \mathsf{C}$, defined as in Equation (2.3), is zero. We have the following exact commutative diagram:

$$\begin{array}{ccccccccccc}
0 & \longrightarrow & \rho_1(\mathsf{G}_1(\mathsf{R}_\mathrm{L})) & \longrightarrow & \mathsf{S}_2\mathsf{T}^*\mathsf{Q} \otimes \mathcal{W}^\perp \otimes \mathsf{TQ} & \xrightarrow{\rho_1(\sigma(\Phi))} & \mathsf{T}^*\mathsf{Q} \otimes \mathsf{T}^*\mathsf{Q} \otimes \mathcal{W}^\perp \otimes \mathcal{W}^\perp & \xrightarrow{\tau} & \mathsf{C} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & & & \\
0 & \longrightarrow & \rho_1(\mathsf{R}_\mathrm{L}) & \longrightarrow & \mathsf{J}_2\pi & \xrightarrow{\rho_1(\Phi)} & \mathsf{J}_1\pi' & & & & \\
 & & \downarrow & & \downarrow & & \downarrow & & & & \\
0 & \longrightarrow & \mathsf{R}_\mathrm{L} & \longrightarrow & \mathsf{J}_1\pi & \xrightarrow{\Phi} & \mathsf{T}^*\mathsf{Q} \otimes \mathcal{W}^\perp \otimes \mathcal{W}^\perp & & & &
\end{array}$$

where we denote by π' the bundle $(\mathsf{T}^*\mathsf{Q} \otimes \mathcal{W}^\perp \otimes \mathcal{W}^\perp, \pi', \mathsf{Q})$. Let $p \in \mathsf{R}_\mathrm{L}$ so that $\pi(p) = q$ for $q \in \mathsf{Q}$. Therefore, $\Phi(p) = 0$. Take $p_2 \in \mathsf{J}_2\pi$ projecting to p and define $\xi = \rho_1(\Phi)(p_2) \in \mathsf{J}_1\pi'$. By commutativity of the diagram, ξ projects to $0 \in \mathsf{T}^*\mathsf{Q} \otimes \mathcal{W}^\perp \otimes \mathcal{W}^\perp$, so we take $\kappa(p) = \tau(\xi - 0)$. It is easy to show that the definition of κ is independent of the choice of p_2 [34, 35]. By the discussion in Section 2.1, p is in the image of the projection of $\rho_1(\mathsf{R}_\mathrm{L})$ to R_L if and only if $\kappa(p) = 0$. ■

Using Theorem 4.6 and the map τ defined by Equation (4.2), we can write the following intrinsic formula for κ:

$$\begin{aligned}
&\kappa(j_1\lambda)(Z, W, PX, PY) \\
&\quad = \nabla^{\mathbb{G}}_W\big(\nabla^{\mathbb{G}}_Z(\mathbb{G}\lambda)(PX, PY)\big) - \nabla^{\mathbb{G}}_Z\big(\nabla^{\mathbb{G}}_W(\mathbb{G}\lambda)(PX, PY)\big) - \nabla^{\mathbb{G}}_{[W,Z]}(\mathbb{G}\lambda)(PX, PY),
\end{aligned} \tag{4.3}$$

where $X, Y, Z, W \in \Gamma^\omega(\mathsf{TQ})$. This leads to the following theorem which gives an explicit expression for the compatibility conditions of the λ-equations.

4.7 Theorem: *Let $(\mathsf{Q}, \mathbb{G})$ be an analytic Riemannian manifold of dimension n and let $\mathcal{S}$ be the Levi-Civita connection on Q with the associated curvature tensor R. Let $\mathcal{W} \subset \mathsf{T}^*\mathsf{Q}$ be an analytic integrable subbundle and let $P \in \Gamma^\omega(\mathsf{T}^*\mathsf{Q} \otimes \mathsf{TQ})$ be the associated $\mathbb{G}$-orthogonal projection as above. If the partial differential equation*

$$\begin{aligned}
(\mathbb{G}\lambda)(\mathsf{R}(PX, PY)W, Z) + (\mathbb{G}\lambda)(W, \mathsf{R}(PX, PY)Z)& \\
+ \nabla^{\mathbb{G}}_Z(\mathbb{G}\lambda)(\nabla^{\mathbb{G}}_W PX, PY) + \nabla^{\mathbb{G}}_Z(\mathbb{G}\lambda)(\nabla^{\mathbb{G}}_W PY, PX)& \\
- \nabla^{\mathbb{G}}_W(\mathbb{G}\lambda)(\nabla^{\mathbb{G}}_Z PX, PY) - \nabla^{\mathbb{G}}_W(\mathbb{G}\lambda)(\nabla^{\mathbb{G}}_Z PY, PX) = 0&
\end{aligned} \tag{4.4}$$

is satisfied in a neighborhood of $\lambda_0 \in \Gamma^\omega(\mathsf{T}^*\mathsf{Q} \otimes \mathsf{TQ})$, then the set of λ-equations has a solution in a neighborhood of λ_0. Moreover, any solution to the λ-equations will satisfy Equation (4.4).

Proof: From Theorem 4.6 and involutivity of the symbol of R_{L}, a sufficient condition for the existence of solutions to the λ-equations is that the curvature map be zero. We have

$$\nabla^{\mathbb{G}}_{Z}(\mathbb{G}\lambda)(PX, PY) = \mathbb{G}((\nabla^{\mathbb{G}}_{Z}\lambda)(PX), PY)$$

for all $X, Z \in \Gamma^{\omega}(\mathsf{TQ})$. Thus

$$\begin{aligned}\nabla^{\mathbb{G}}_{W}\big(\nabla^{\mathbb{G}}_{Z}(\mathbb{G}\lambda)(PX, PY)\big) &= \nabla^{\mathbb{G}}_{W}\big(\mathbb{G}((\nabla^{\mathbb{G}}_{Z}\lambda)(PX), PY)\big)\\ &= \mathbb{G}(\nabla^{\mathbb{G}}_{W}(\nabla^{\mathbb{G}}_{Z}\lambda(PX)), PY) + \mathbb{G}(\nabla^{\mathbb{G}}_{Z}\lambda(PX), \nabla^{\mathbb{G}}_{W}PY),\end{aligned}$$

where $X, Y, Z, W \in \Gamma^{\omega}(\mathsf{TQ})$.

As a result, one can compute

$$\begin{aligned}&\nabla^{\mathbb{G}}_{W}\big(\nabla^{\mathbb{G}}_{Z}(\mathbb{G}\lambda)(PX, PY)\big)\\ &\quad= \mathbb{G}(\nabla^{\mathbb{G}}_{W}\nabla^{\mathbb{G}}_{Z}\lambda(PX), PY) + \nabla^{\mathbb{G}}_{Z}(\mathbb{G}\lambda)(\nabla^{\mathbb{G}}_{W}PX, PY) + \nabla^{\mathbb{G}}_{Z}(\mathbb{G}\lambda)(PX, \nabla^{\mathbb{G}}_{W}PY).\end{aligned}$$

We conclude that

$$\begin{aligned}&\nabla^{\mathbb{G}}_{W}\big(\nabla^{\mathbb{G}}_{Z}(\mathbb{G}\lambda)(PX, PY)\big) - \nabla^{\mathbb{G}}_{Z}\big(\nabla^{\mathbb{G}}_{W}(\mathbb{G}\lambda)(PX, PY)\big) - \nabla^{\mathbb{G}}_{[W,Z]}(\mathbb{G}\lambda)(PX, PY)\\ &\quad= \mathbb{G}(\nabla^{\mathbb{G}}_{W}\nabla^{\mathbb{G}}_{Z}\lambda(PX), PY) - \mathbb{G}(\nabla^{\mathbb{G}}_{Z}\nabla^{\mathbb{G}}_{W}\lambda(PX), PY) - \mathbb{G}(\nabla^{\mathbb{G}}_{[W,Z]}\lambda(PX), PY)\\ &\qquad+ \nabla^{\mathbb{G}}_{Z}(\mathbb{G}\lambda)(\nabla^{\mathbb{G}}_{W}PX, PY) + \nabla^{\mathbb{G}}_{Z}(\mathbb{G}\lambda)(\nabla^{\mathbb{G}}_{W}PY, PX)\\ &\qquad- \nabla^{\mathbb{G}}_{W}(\mathbb{G}\lambda)(\nabla^{\mathbb{G}}_{Z}PX, PY) - \nabla^{\mathbb{G}}_{W}(\mathbb{G}\lambda)(\nabla^{\mathbb{G}}_{Z}PY, PX) = 0.\end{aligned}$$

Equation (4.4) then follows using the Ricci identity. The necessity of this condition is clear since any formal solution of the λ-equation satisfies Equation (4.3) by definition. ■

4.2. Formal integrability of R_{E}

We prove that the system of partial differential equations for the closed-loop metric has an involutive symbol and is formally integrable under a certain surjectivity condition.

4.8 Assumption: An additional assumption is that $\lambda(\mathsf{coann}(\mathcal{W}))$ is involutive. This assumption is not necessary for proving the involutivity of the symbol, however, it helps simplifying the compatibility conditions. Recall that a similar assumption has been used in Theorem 3.10.

The main result here is Theorem 4.15. Again, this result can be understood separately from the details of its proof.

4.2.1. The symbol of R_{E}. We have the symbol map $\sigma(\mathsf{R}_{\mathrm{E}}) : \mathsf{T}^*\mathsf{Q} \otimes \mathsf{S}_2\mathsf{T}^*\mathsf{Q} \to \mathcal{W}^{\perp} \otimes \mathsf{S}_2\mathsf{T}^*\mathsf{Q}$ for the partial differential equation R_{E} given by

$$\sigma(\mathsf{R}_{\mathrm{E}})(\beta \otimes \mathsf{A})(PX) = \beta(\lambda PX) \otimes \mathsf{A},$$

where $\beta \in \Gamma^{\omega}(\mathsf{T}^*\mathsf{Q})$ and $\mathsf{A} \in \Gamma^{\omega}(\mathsf{S}_2\mathsf{T}^*\mathsf{Q})$.

4.9 Lemma: *We have* $\mathsf{G}(\mathsf{R}_{\mathrm{E}}) \doteq \ker(\sigma(\mathsf{R}_{\mathrm{E}}))$ *is isomorphic to* $\mathcal{W} \otimes \mathsf{S}_2\mathsf{T}^*\mathsf{Q}$.

Proof: If $\beta \otimes \mathsf{A} \in \ker(\sigma(\mathsf{R}_{\mathrm{E}}))$ then $\beta(\lambda PX) = 0$ and thus $\beta \in \mathsf{ann}(\lambda(\mathsf{coann}(\mathcal{W})))$. Since λ is an isomorphism, we have that $\ker(\sigma(\mathsf{R}_{\mathrm{E}}))$ is of dimension $(n-m) \times (n \times (n+1)/2)$ and so is isomorphic to $\mathcal{W} \otimes \mathsf{S}_2\mathsf{T}^*\mathsf{Q}$ as claimed. ■

Let $\{e^1, \dots, e^n\}$ be a basis for $\mathsf{T}^*_{q_0}\mathsf{Q}$ such that $\{e^1, \dots, e^{n-m}\}$ spans $\mathcal{W}$ and let Σ_j be the subspace of $\mathsf{T}^*_{q_0}\mathsf{Q}$ generated by $\{e^{j+1}, \dots, e^n\}$. Consider the restriction

$$\sigma(\mathsf{R}_{\mathrm{E}})|_{\Sigma_j \otimes \mathsf{S}_2\mathsf{T}^*\mathsf{Q}} : \Sigma_j \otimes \mathsf{S}_2\mathsf{T}^*\mathsf{Q} \to \mathcal{W}^{\perp} \otimes \mathsf{S}_2\mathsf{T}^*\mathsf{Q},$$

of the symbol map to $\Sigma_j \otimes \mathsf{S}_2\mathsf{T}^*\mathsf{Q}$. We have the following lemma.

4.10 Lemma: *We have*

$$\mathsf{G}(\mathsf{R}_{\mathrm{E}})_{1,j} \doteq \ker(\sigma(\mathsf{R}_{\mathrm{E}})|_{\Sigma_j \otimes \mathsf{S}_2\mathsf{T}^*\mathsf{Q}}) = (\Sigma_j \cap \mathcal{W}) \otimes \mathsf{S}_2\mathsf{T}^*\mathsf{Q}.$$

Proof: The proof follows by Lemma 4.9 with restricting the symbol $\sigma(\mathsf{R}_{\mathrm{E}})$ to $\Sigma_j \otimes \mathsf{S}_2\mathsf{T}^*\mathsf{Q}$. ■

Define a map

$$\rho_1(\sigma(\mathsf{R}_{\mathrm{E}})) : \mathsf{S}_2\mathsf{T}^*\mathsf{Q} \otimes \mathsf{S}_2\mathsf{T}^*\mathsf{Q} \to \mathsf{T}^*\mathsf{Q} \otimes \mathcal{W}^{\perp} \otimes \mathsf{S}_2\mathsf{T}^*\mathsf{Q},$$

by $\rho_1(\sigma(\mathsf{R}_{\mathrm{E}})) = \mathrm{id}_{\mathsf{T}^*\mathsf{Q}} \otimes \sigma(\mathsf{R}_{\mathrm{E}})$. Explicitly,

$$\rho_1(\sigma(\mathsf{R}_{\mathrm{E}}))(\Pi \otimes \mathsf{A})(X, PY, Z, W) = \Pi(\lambda(PY), X) \otimes \mathsf{A}(Z, W),$$

where $\Pi \in \Gamma^{\omega}(\mathsf{S}_2\mathsf{T}^*\mathsf{Q})$. This map then identifies the prolongation of $\sigma(\mathsf{R}_{\mathrm{E}})$. We have the following lemma.

4.11 Lemma: *The following sequence is short exact:*

$$\mathsf{S}_2\mathsf{T}^*\mathsf{Q} \otimes \mathsf{S}_2\mathsf{T}^*\mathsf{Q} \xrightarrow{\rho_1(\sigma(\mathsf{R}_{\mathrm{E}}))} \mathsf{T}^*\mathsf{Q} \otimes \mathcal{W}^{\perp} \otimes \mathsf{S}_2\mathsf{T}^*\mathsf{Q} \xrightarrow{\tau} \Lambda_2\mathcal{W}^{\perp} \otimes \mathsf{S}_2\mathsf{T}^*\mathsf{Q} \longrightarrow 0$$

where τ *is the canonical projection onto cokernel of* $\rho_1(\sigma(\mathsf{R}_{\mathrm{E}}))$.

The proof of Lemma 4.11 follows immediately from the following lemma.

4.12 Lemma: *Let* V *be an* n*-dimensional vector space and let* W *be an* m*-dimensional subspace of* V^*. *Let* γ *be an automorphism on* V. *Suppose* ϕ_γ *is the map from* $\mathsf{S}_2\mathsf{V}^*$ *to* $\mathsf{V}^* \otimes \mathsf{V}^*$ *defined by* $\phi_\gamma(\mathsf{A})(u, v) = \mathsf{A}(\gamma u, v)$, *where* $u, v \in \mathsf{V}$. *Then the following diagram is exact and commutative:*

$$\begin{array}{ccccccc}
 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \\
\mathsf{S}_2\mathsf{V}^* & \longrightarrow & \mathsf{W} \otimes \mathsf{V}^* & \longrightarrow & \Lambda_2\mathsf{W} & \longrightarrow & 0 \\
\| & & \downarrow & & \downarrow & & \\
\mathsf{S}_2\mathsf{V}^* & \xrightarrow{\phi_\gamma} & \mathsf{V}^* \otimes \mathsf{V}^* & \xrightarrow{\tau_\gamma} & \Lambda_2\mathsf{V}^* & \longrightarrow & 0
\end{array}$$

where τ_γ *is given by*

$$\tau_\gamma(\mathsf{B})(u, v) = \mathsf{B}(\gamma^{-1}u, v) - \mathsf{B}(\gamma^{-1}v, u),$$

and $\mathsf{B} \in \mathsf{V}^* \otimes \mathsf{V}^*$.

Proof: It is immediate by definition that $\ker(\phi_\gamma) = 0$. Thus the image of ϕ_γ is *isomorphic* to $\mathsf{S}_2\mathsf{V}^*$ and the cokernel is of dimension $\frac{n(n-1)}{2}$ and thus *isomorphic* to $\Lambda_2\mathsf{V}^*$. The map τ_γ precisely prescribes the projection to the cokernel since $\tau_\gamma \circ \phi_\gamma(\mathsf{A}) = 0$ for all $\mathsf{A} \in \mathsf{S}_2\mathsf{V}^*$. The first row of the diagram is the restriction of the second row by restricting γ to $\mathsf{W} \otimes \mathsf{V}$. We only need to verify that the cokernel of this restriction is isomorphic to $\Lambda_2\mathsf{W}$. This follows since

$$\tau_\gamma(\mathsf{B})(u_1 \oplus 0, v_1 \oplus v_2) = -\tau_\gamma(\mathsf{B})(v_1 \oplus 0, u_1 \oplus 0) = \tau_\gamma(\mathsf{B})(u_1 \oplus 0, v_1 \oplus 0),$$

where $\mathsf{B} \in \mathsf{W} \otimes \mathsf{V}^*$, i.e., $\tau_\gamma(\mathsf{B}) \in \Lambda_2\mathsf{W}$. ■

An immediate corollary of Lemma 4.11 is that the kernel of $\rho_1(\sigma(\mathsf{R}_\mathrm{E}))$ is isomorphic to $\mathsf{S}_2\mathcal{W} \otimes \mathsf{S}_2\mathsf{T}^*\mathsf{Q}$. One could also verify this in adapted local coordinates. By definition, $\Pi \otimes \mathsf{A} \in \ker(\rho_1(\sigma(\mathsf{R}_\mathrm{E})))$ if and only if $\Pi_{kj}\lambda^j_a = 0$, where $j, k \in \{1, \dots, n\}$ and $a \in \{1, \dots, m\}$. Since λ is an isomorphism, all of the equations that describe the kernel are independent; thus a dimension count gives that the space of all $\Pi \otimes \mathsf{A}$ which are in the kernel is isomorphic to $\mathsf{S}_2\mathcal{W} \otimes \mathsf{S}_2\mathsf{T}^*\mathsf{Q}$.

4.13 Proposition: *The symbol of* R_E *is involutive.*

Proof: Let $\{e^1, \dots, e^n\}$ be a basis for $\mathsf{T}^*_{q_0}\mathsf{Q}$ for $q_0 \in \mathsf{Q}$ and let Σ_j be the subspace of $\mathsf{T}^*_{q_0}\mathsf{Q}$ generated by $\{e^{j+1}, \dots, e^n\}$. We show that this yields a quasi-linear basis for $\mathsf{T}^*_{q_0}\mathsf{Q}$. Using Lemmata 4.9 and 4.10 we have

$$\dim(\mathsf{G}(\mathsf{R}_\mathrm{E})) = \tfrac{1}{2}(n-m)n(n+1),$$
$$\dim(\mathsf{G}(\mathsf{R}_\mathrm{E})_{1,j}) = \begin{cases} \tfrac{1}{2}(n-m-j)n(n+1), & 1 \le j < n-m, \\ 0, & n-m \le j < 0. \end{cases}$$

We compute

$$\dim(\mathsf{G}(\mathsf{R}_\mathrm{E})) + \sum_{j=1}^{n-m} \dim(\mathsf{G}(\mathsf{R}_\mathrm{E})_{1,j}) = \tfrac{1}{4}n(n-m)(n-m+1)(n+1).$$

On the other side, Lemma 4.11 implies that

$$\dim(\rho_1(\mathsf{G}(\mathsf{R}_\mathrm{E}))) = \frac{(n-m)(n-m+1)n(n+1)}{4} = \frac{1}{4}n(n-m)(n-m+1)(n+1),$$

as desired. ■

4.2.2. Involutivity of R_E.

The following theorem applies Goldschmidt's theorem to R_E.

4.14 Theorem: *The partial differential equation* R_E *is involutive if, for* $p \in \mathsf{R}_\mathrm{E}$*, we have*

$$\tau(\rho_1(p_2) - 0) = 0,$$

where p_2 *is any point in* $\mathsf{J}_2\pi$ *that projects to* p*.*

Proof: Since the symbol is involutive, the proof follows along the same lines as that of Theorem 4.6. ■

One can construct the curvature map, similar to the one for R_{L}, using the map τ defined in Lemma 4.11:

$$\begin{aligned}&\kappa(j_1\mathbb{G}_{\mathrm{cl}})(\lambda PX, \lambda PY, Z, Z)\\&\quad = \nabla^{\mathbb{G}}_{\lambda PY}[\nabla^{\mathbb{G}}_{\lambda PX}(\mathbb{G}_{\mathrm{cl}})(Z,Z)] - \nabla^{\mathbb{G}}_{\lambda PX}[\nabla^{\mathbb{G}}_{\lambda PY}(\mathbb{G}_{\mathrm{cl}})(Z,Z)] - \nabla^{\mathbb{G}}_{[\lambda PY,\lambda PX]}(\mathbb{G}_{\mathrm{cl}})(Z,Z). \quad (4.5)\end{aligned}$$

Note that, by Assumption 4.8, there exists a $\zeta \in \Gamma^{\omega}(\mathsf{TQ})$ such that $[\lambda PY, \lambda PX] = \lambda P\zeta$. We state the following theorem which provides the obstruction to finding a closed-loop metric.

4.15 Theorem: *Let $(\mathsf{Q}, \mathbb{G})$ be an analytic Riemannian manifold of dimension n and let $\mathcal{S}$ be the Levi-Civita connection on Q with associated curvature tensor R. Let $\mathcal{W} \subset \mathsf{T}^*\mathsf{Q}$ be an analytic integrable subbundle and let $P \in \Gamma^{\omega}(\mathsf{T}^*\mathsf{Q} \otimes \mathsf{TQ})$ be the associated $\mathbb{G}$-orthogonal projection as above. Let λ be an automorphism on TQ which satisfies Assumption 4.8. If the first-order partial differential equation*

$$\begin{aligned}&2\nabla^{\mathbb{G}}_{\lambda PY}(\mathbb{G}_{\mathrm{cl}})([Z,\lambda PX],Z) - 2\nabla^{\mathbb{G}}_{\lambda PX}(\mathbb{G}_{\mathrm{cl}})([Z,\lambda PY],Z)\\&+ 2\mathbb{G}_{\mathrm{cl}}(\nabla^{\mathbb{G}}_{\lambda PY}(\nabla^{\mathbb{G}}_{Z}\lambda PX) - \nabla^{\mathbb{G}}_{\lambda PX}(\nabla^{\mathbb{G}}_{Z}\lambda PY), Z) + 2\mathbb{G}(\nabla^{\mathbb{G}}_{\lambda PY}(\nabla^{\mathbb{G}}_{Z}PX) - \nabla^{\mathbb{G}}_{\lambda PX}(\nabla^{\mathbb{G}}_{Z}PY), Z)\\&\quad + 2\mathbb{G}_{\mathrm{cl}}(\nabla^{\mathbb{G}}_{\lambda PY}Z, \nabla^{\mathbb{G}}_{Z}\lambda PX) - 2\mathbb{G}_{\mathrm{cl}}(\nabla^{\mathbb{G}}_{\lambda PX}Z, \nabla^{\mathbb{G}}_{Z}\lambda PY) + 2\mathbb{G}(\nabla^{\mathbb{G}}_{\lambda PX}Z, \nabla^{\mathbb{G}}_{Z}\lambda PY)\\&\quad\quad - 2\mathbb{G}_{\mathrm{cl}}(\nabla^{\mathbb{G}}_{\lambda PY}Z, \nabla^{\mathbb{G}}_{Z}\lambda PX) + 2\mathbb{G}_{\mathrm{cl}}(\nabla^{\mathbb{G}}_{Z}\lambda\zeta, Z) - 2\mathbb{G}(\nabla^{\mathbb{G}}_{Z}\zeta, Z) = 0 \quad (4.6)\end{aligned}$$

is satisfied in a neighborhood of $\mathbb{G}_{\mathrm{cl}} \in \Gamma^{\omega}(\mathsf{S}_2\mathsf{T}^*\mathsf{Q})$ for $X, Y, Z \in \Gamma^{\omega}(\mathsf{TQ})$ and $\zeta \in \Gamma^{\omega}(\mathsf{TQ})$ as above, then the set of closed-loop metric equations has a solution in a neighborhood of $\mathbb{G}_{\mathrm{cl}}$. Moreover, any solution to the closed-loop metric equations will satisfy Equation (4.6).

Proof: The proof follows from a direct computation using Equation (4.5) and the Ricci identity (similar to Theorem 4.7). One also uses the following identity

$$\begin{aligned}&(\mathbb{G}_{\mathrm{cl}})(\mathsf{R}(Z,Z)\lambda PY, \lambda PX) + (\mathbb{G}_{\mathrm{cl}})(\lambda PY, \mathsf{R}(Z,Z)\lambda PX)\\&\qquad = (\mathbb{G})(\mathsf{R}(Z,Z)\lambda PY, PX) + (\mathbb{G})(PY, \mathsf{R}(Z,Z)\lambda PX) = 0,\end{aligned}$$

which holds by the Ricci identity, since R is the curvature associated to the Levi-Civita connection $\nabla^{\mathbb{G}}$. ■

4.3. Formal integrability of R_{T}

We prove that the system of partial differential equations relating the λ-equations to the potential energy shaping equations is formally integrable under a surjectivity condition. The main result here is Theorem 4.24. As with the previous two sections, this result can be understood separately from the details of its proof.

4.3.1. The symbol of R_{T}. In this section we construct the symbol map for the quasi-linear partial differential equation Φ_{T} defined in Section 3.4.2. Note that the linearization $\mathsf{V}\Phi_{\mathrm{T}}$ about a point $p \in \mathsf{J}_1\pi$ with

$$\pi_0^1(p) = (\mathrm{id}|_{\mathsf{TQ}\otimes_{\mathsf{Q}}\mathcal{W}^{\perp}} \otimes \mathrm{id}|_{\mathsf{TQ}\otimes_{\mathsf{Q}}\mathcal{W}^{\perp}}),$$

can be reduced to a map on $\mathsf{J}_1\mathsf{V}\pi|_{(\mathcal{W}^\perp\otimes_{\mathsf{Q}}\mathsf{TQ})\oplus 0}$ since there is no Δ involved in the linearization (see Equation (3.13)). As we showed in Section 3.4.2, the linearization $\mathsf{V}\Phi_{\mathrm{T}}$ can be identified as a map from $\mathsf{J}_1\mathsf{V}\pi$ to $\mathsf{V}\pi'$, where π' denotes the bundle $(\pi', \Lambda_2\mathcal{W}^\perp, \mathsf{Q})$. We identify $\mathsf{V}_{\pi_0^k(p)}\pi$ and $\mathsf{V}_{\Phi_{\mathrm{T}}(p)}\pi'$ with $(\mathcal{W}^\perp\otimes_{\mathsf{Q}}\mathsf{TQ})|_{\pi_0^1(p)}$ and $\Lambda_2(\mathcal{W}^\perp)|_{\Phi(p)}$, respectively. In this section, we wish to define the symbol map $\sigma(\mathsf{R}_{\mathrm{T}})$ of R_{T} as a morphism of vector bundles from $\pi_1^*\mathsf{T}^*\mathsf{Q}\otimes(\pi_0^1)^*\mathsf{V}\pi$ to $\mathsf{V}\pi'$. In order to do this, we start with some algebraic constructions. We will often drop the points of evaluation for simplicity of notation.

Consider the alternation map Alt acting on the $(0,2)$-tensors and denote the restriction of $2\mathsf{Alt}$ to $(\mathcal{W}^\perp\otimes\mathsf{T}^*\mathsf{Q})|_{\pi_0^1(p)}$ by $\check{\sigma}$. Explicitly, if we have $\mathsf{b}\in\mathcal{W}^\perp_{\pi_1(p)}$ and $\mathsf{c}\in\mathsf{T}^*_{\pi_1(p)}\mathsf{Q}$, then

$$\check{\sigma}(\mathsf{b}\otimes\mathsf{c})(u_1\oplus u_2, v_1\oplus v_2) = \mathsf{b}(u_1)\mathsf{c}(v_1\oplus v_2) - \mathsf{b}(v_1)\mathsf{c}(u_1\oplus u_2).$$

4.16 Lemma: *We have* $\ker(\check{\sigma}) = \mathsf{S}_2\mathcal{W}^\perp$ *and* $\mathrm{Im}(\check{\sigma}) = \mathsf{I}_2(\mathcal{W}^\perp)$.

Proof: Since $\check{\sigma}$ is the restriction of the alternation map, one can easily observe that $\ker(\check{\sigma}) = \ker(\mathsf{Alt})\cap(\mathcal{W}^\perp\otimes\mathsf{T}^*\mathsf{Q})$. Clearly $\mathsf{S}_2\mathcal{W}^\perp\subset\mathsf{S}_2\mathsf{T}^*\mathsf{Q}\cap(\mathcal{W}^\perp\otimes\mathsf{T}^*\mathsf{Q})$. Moreover, for any $\Theta\in\Gamma^\omega(\mathsf{S}_2\mathsf{T}^*\mathsf{Q}\cap(\mathcal{W}^\perp\otimes\mathsf{T}^*\mathsf{Q}))$, we have

$$\Theta(u_1\oplus u_2, v_1\oplus v_2) = \Theta(0\oplus v_2, u_1\oplus u_2) = \Theta(0\oplus u_2, 0\oplus v_2).$$

Thus $\Theta\in\Gamma^\omega(\mathsf{S}_2\mathcal{W}^\perp)$ and as a result, we have $\mathsf{S}_2\mathsf{T}^*\mathsf{Q}\cap(\mathcal{W}^\perp\otimes\mathsf{T}^*\mathsf{Q}) = \mathsf{S}_2\mathcal{W}^\perp$. Recalling that $\Lambda_2(\mathsf{T}^*\mathsf{Q}) = \Lambda_2(\mathcal{W})\oplus\Lambda_2(\mathcal{W}^\perp)\oplus(\mathcal{W}\otimes\mathcal{W}^\perp)$ and using the definition of $\check{\sigma}$, one can observe that $\mathrm{Im}(2\mathsf{Alt}) = \mathrm{Im}(\check{\sigma})\cup\Lambda_2(\mathcal{W})$. ■

The symbol map $\sigma(\mathsf{R}_{\mathrm{T}})|_p$ can be characterized as the composition

$$\mathsf{T}^*\mathsf{Q}\otimes(\mathcal{W}^\perp\otimes_{\mathsf{Q}}\mathsf{TQ}) \xrightarrow{\tilde{\sigma}} \mathsf{I}_2(\mathcal{W}^\perp) \xrightarrow{\mathsf{p}} \Lambda_2\mathcal{W}^\perp,$$

where p is the canonical projection of $\mathsf{I}_2(\mathcal{W}^\perp)$ onto $\Lambda_2\mathcal{W}^\perp$ and

$$\tilde{\sigma}(\mathsf{b}\otimes\mathsf{c}\otimes v) = \beta(v)\check{\sigma}(\mathsf{b}\otimes\mathsf{c}),$$

with $\beta\in\mathsf{T}^*_{\pi_1(p)}\mathsf{Q}$, $\mathsf{b}\in\mathcal{W}^\perp_{\pi_1(p)}$, $\mathsf{c}\in\mathsf{T}^*_{\pi_1(p)}\mathsf{Q}$ and $v\in\mathsf{T}_{\pi_1(p)}\mathsf{Q}$. In local coordinates, $\sigma(\mathsf{R}_{\mathrm{T}})$ captures the highest order derivatives in Equation (3.13). We now identify the kernel of $\sigma(\mathsf{R}_{\mathrm{T}})$.

4.17 Lemma: *The following sequence is short exact:*

$$0 \longrightarrow \mathsf{G}(\mathsf{R}_{\mathrm{T}}) \longrightarrow \mathsf{T}^*\mathsf{Q}\otimes(\mathcal{W}^\perp\otimes_{\mathsf{Q}}\mathsf{TQ}) \xrightarrow{\sigma(\mathsf{R}_{\mathrm{T}})} \Lambda_2(\mathcal{W}^\perp),$$

where

$$\mathsf{G}(\mathsf{R}_{\mathrm{T}}) \cong \Big(\mathsf{S}_2\mathcal{W}^\perp\otimes\mathsf{TQ}\Big)\oplus\Big((\mathcal{W}\otimes\mathcal{W}^\perp)\otimes\mathsf{TQ}\Big)\oplus\Big(\Lambda_2\mathcal{W}^\perp\otimes\mathsf{coann}(\beta)\Big).$$

Proof: Clearly $\mathsf{T}^*\mathsf{Q}\otimes(\mathcal{W}^\perp\otimes\mathsf{coann}(\beta))\subset\ker(\sigma(\mathsf{R}_{\mathrm{T}}))$. Lemma 4.16 yields $\mathsf{S}_2\mathcal{W}^\perp\otimes\mathsf{TQ}\subset\ker(\sigma(\mathsf{R}_{\mathrm{T}}))$. Finally $\tilde{\sigma}((\mathcal{W}\otimes\mathcal{W}^\perp)\otimes\mathsf{TQ})\subset\ker(\mathsf{p})$. If $v\notin\mathsf{coann}(\beta)$ then, by definition, the image of $\Lambda_2\mathcal{W}^\perp\otimes v$ under $\sigma(\mathsf{R}_{\mathrm{T}})$ is $\Lambda_2\mathcal{W}^\perp$. ■

Let $\{e^1,\dots,e^n\}$ be a basis for $\mathsf{T}^*_{\pi_1(p)}\mathsf{Q}$. Let Σ_j be the subspace of $\mathsf{T}^*_{\pi_1(p)}\mathsf{Q}$ generated by $\{e^{j+1},\dots,e^n\}$ and define $\mathsf{M^*}_j = \mathcal{W}^\perp\cap\Sigma_j$ and $\mathsf{M^*}_j^\perp = \mathcal{W}^\perp\cap\Sigma_j^\perp$. Let $\mathsf{I}_2(\mathsf{M^*}_j) = \mathsf{I}(\mathsf{M^*}_j)\cap\Lambda_2(\mathcal{W}^\perp)$. The following lemma can be proved along the same lines as Lemma 4.17.

4.18 Lemma: *The following sequence is short exact:*

$$0 \longrightarrow \mathsf{G}(\mathsf{R}_{\mathrm{T}})_{1,j} \longrightarrow \Sigma_j \otimes (\mathcal{W}^{\perp} \otimes_{\mathsf{Q}} \mathsf{TQ}) \xrightarrow{\sigma(\mathsf{R}_{\mathrm{T}})} \mathsf{I}_2(\mathsf{M}^*{}_j)\ ,$$

where

$$\mathsf{G}(\mathsf{R}_{\mathrm{T}})_{1,j} \cong \big(\mathsf{S}_2\mathsf{M}^*_{\mathsf{j}} \otimes \mathsf{TQ}\big) \oplus \Big(((\mathcal{W} \cap \Sigma_j) \otimes \mathcal{W}^{\perp}) \otimes \mathsf{TQ}\Big) \oplus \Big(((\mathcal{W}^{\perp} \cap \Sigma_j) \otimes \mathcal{W}^{\perp})/\mathsf{S}_2\mathsf{M}^*_j \otimes \mathsf{coann}(\beta)\Big)\,.$$

We define the prolongation map $\rho_1(\sigma(\mathsf{R}_{\mathrm{T}}))$ as $\mathrm{id}_{\mathsf{T}^*\mathsf{Q}} \otimes \sigma(\mathsf{R}_{\mathrm{T}})$. Explicitly,

$$\rho_1(\sigma(\mathsf{R}_{\mathrm{T}})) = (\mathrm{id}_{\mathsf{T}^*\mathsf{Q}} \otimes \mathsf{p}) \circ \rho_1(\tilde{\sigma}(\mathsf{R}_{\mathrm{T}})),$$

where

$$\rho_1(\tilde{\sigma}(\mathsf{R}_{\mathrm{T}}))(\mathsf{C} \otimes \mathsf{b} \otimes v)(w_1 \oplus w_2, u_1 \oplus u_2, z_1 \oplus z_2) = \beta(v)\big(\mathsf{C}(w_1 \oplus w_2, u_1 \oplus u_2)\mathsf{b}(z_1) - \mathsf{C}(w_1 \oplus w_2, z_1 \oplus z_2)\mathsf{b}(u_1)\big),$$

and $\beta \in \mathsf{T}^*_{\pi_1(p)}\mathsf{Q}$, $\mathsf{b} \in \mathcal{W}^{\perp}_{\pi_1(p)}$, $\mathsf{C} \in \mathsf{S}_2\mathsf{T}^*_{\pi_1(p)}\mathsf{Q}$, and $v \in \mathsf{T}_{\pi_1(p)}\mathsf{Q}$.

4.19 Lemma: *The following sequence is short exact:*

$$0 \longrightarrow \rho_1(\mathsf{G}(\mathsf{R}_{\mathrm{T}})) \longrightarrow \mathsf{S}_2\mathsf{T}^*\mathsf{Q} \otimes (\mathcal{W}^{\perp} \otimes_{\mathsf{Q}} \mathsf{TQ}) \xrightarrow{\rho_1(\sigma(\mathsf{R}_{\mathrm{T}}))} \mathsf{T}^*\mathsf{Q} \otimes \Lambda_2\mathcal{W}^{\perp}$$

where

$$\rho_1(\mathsf{G}(\mathsf{R}_{\mathrm{T}})) \cong \Big(\mathsf{S}_3\mathcal{W}^{\perp} \otimes \mathsf{TQ}\Big) \oplus \Big((\mathcal{W} \otimes \mathcal{W}^{\perp} \otimes \mathcal{W}^{\perp}) \otimes \mathsf{TQ}\Big) \oplus \Big((\mathsf{S}_2\mathcal{W} \otimes \mathcal{W}^{\perp}) \otimes \mathsf{TQ}\Big) \oplus \Big(((\mathsf{S}_2\mathcal{W}^{\perp} \otimes \mathcal{W}^{\perp})/\mathsf{S}_3\mathcal{W}^{\perp}) \otimes \mathsf{coann}(\beta)\Big)\,.$$

The proof of this lemma requires the following lemma.

4.20 Lemma: *Let* V *be an* n*-dimensional* $\mathbb{R}$*-vector space and define* $\rho : \mathsf{S}_2\mathsf{V}^* \otimes \mathsf{V}^* \to \mathsf{V}^* \otimes \Lambda_2\mathsf{V}^*$ *by*

$$\rho(\mathsf{A} \otimes \alpha)(u, v, w) = \mathsf{A}(u, v)\alpha(w) - \mathsf{A}(u, w)\alpha(v).$$

Then the following sequence is short exact

$$0 \longrightarrow \mathsf{S}_3\mathsf{V}^* \longrightarrow \mathsf{S}_2\mathsf{V}^* \otimes \mathsf{V}^* \xrightarrow{\rho} \mathsf{V}^* \otimes \Lambda_2\mathsf{V}^* \longrightarrow \Lambda_3\mathsf{V}^* \longrightarrow 0\ ,$$

where τ *is defined by*

$$\tau(\mathsf{B})(u, v, w) = \mathsf{B}(u, v, w) + \mathsf{B}(v, w, u) + \mathsf{B}(w, u, v).$$

Proof: The kernel of ρ consists of elements of $\mathrm{T}^0_3(\mathsf{V}^*)$ which are symmetric in the first two entries and the last two entries. Thus $\ker(\rho) = \mathsf{S}_3\mathsf{V}^*$. The map τ is surjective, since if $\mathsf{C} \in \Lambda_3\mathsf{V}^*$ then $\mathsf{C} = \tau(\frac{1}{3}\mathsf{C})$. Furthermore, one can easily check that $\tau \circ \rho = 0$; thus the sequence is short exact. ■

We now are ready to prove Lemma 4.19.

Proof of Lemma 4.19: By definition of $\rho_1(\sigma(\mathsf{R}_{\mathrm{T}}))$,

$$\mathsf{S}_2\mathsf{T}^*\mathsf{Q} \otimes \mathcal{W}^{\perp} \otimes \mathrm{coann}(\beta) \subset \ker \rho_1(\sigma(\mathsf{R}_{\mathrm{T}})).$$

Let $v \notin \mathrm{coann}(\beta)$ and take $\mathsf{C} \in \mathsf{S}_2\mathsf{T}^*_{\pi_1(p)}\mathsf{Q}$ and $\mathsf{b} \in \mathcal{W}^{\perp}_{\pi_1(p)}$ such that $\mathsf{C}\otimes\mathsf{b} \in \mathsf{S}_3\mathcal{W}^{\perp}_{\pi_1(p)}$. Then

$$\begin{aligned} &\rho_1(\sigma(\mathsf{R}_{\mathrm{T}}))(\mathsf{C}\otimes\mathsf{b}\otimes v)(w_1\oplus w_2, u_1\oplus u_2, z_1\oplus z_2) \\ &= \beta(v)\big(\mathsf{C}(w_1\oplus 0, u_1\oplus 0)\mathsf{b}(z_1) - \mathsf{C}(w_1\oplus 0, z_1\oplus 0)\mathsf{b}(u_1)\big) \\ &= 0, \end{aligned}$$

by symmetry in the last two entries; thus

$$\mathsf{S}_3\mathcal{W}^{\perp} \otimes \mathsf{TQ} \subset \ker(\rho_1(\sigma(\mathsf{R}_{\mathrm{T}}))).$$

Furthermore, $(\mathcal{W}^{\perp}\otimes\mathcal{W}\otimes\mathcal{W}^{\perp})\otimes\mathsf{TQ}$ and $(\mathsf{S}_2\mathcal{W}\otimes\mathcal{W}^{\perp})\otimes\mathsf{TQ}$ are subsets of $\ker(\rho_1(\sigma(\mathsf{R}_{\mathrm{T}})))$, because the image of these two sets under $\rho_1(\tilde{\sigma}(\mathsf{R}_{\mathrm{T}}))$ is in the kernel of $\mathrm{id}_{\mathsf{T}^*\mathsf{Q}}\otimes\mathsf{p}$. Finally, the elements of $\mathsf{S}_2\mathsf{T}^*\mathsf{Q}\otimes(\mathcal{W}^{\perp}\otimes_{\mathsf{Q}}\mathsf{TQ})$ that look like $\mathsf{C}\otimes\mathsf{b}\otimes v$, where $\mathsf{C}\otimes\mathsf{b}\in(\mathsf{S}_2\mathcal{W}^{\perp}_{\pi_1(p)}\otimes\mathcal{W}^{\perp}_{\pi_1(p)})\backslash\mathsf{S}_3\mathcal{W}^{\perp}_{\pi_1(p)}$ and $v\notin\mathrm{coann}(\beta)$, have nonzero image under $\rho_1(\sigma(\mathsf{R}_{\mathrm{T}}))$, see Lemma 4.20. ■

4.21 Proposition: *The symbol of* R_{T} *is involutive.*

Proof: We will show that any basis $\{e^1,\ldots,e^n\}$ is a quasi-regular basis. Using Lemmata 4.17 and 4.18 we have

$$\begin{aligned} \dim(\mathsf{G}(\mathsf{R}_{\mathrm{T}})) &= \frac{m(m+1)}{2}n + mn(n-m) + \frac{m(m-1)}{2}(n-1), \\ \dim(\mathsf{G}(\mathsf{R}_{\mathrm{T}})_{1,j}) &= n\frac{(m-j+1)(m-j)}{2} + mn(n-m) \\ &\quad + [(m-j)m - \frac{(m-j)(m-j+1)}{2}](n-1), \qquad j<m, \\ \dim(\mathsf{G}(\mathsf{R}_{\mathrm{T}})_{1,j}) &= mn(n-j), \qquad j\geq m. \end{aligned}$$

As a result, we compute

$$\begin{aligned} \sum_{j=1}^{n-1}\dim(\mathsf{G}(\mathsf{R}_{\mathrm{T}})_{1,j}) + \dim(\mathsf{G}(\mathsf{R}_{\mathrm{T}})) &= \sum_{j=1}^{m-1}\Big(n\frac{(m-j+1)(m-j)}{2} + mn(n-m) \\ &\quad + [(m-j)m - \frac{(m-j)(m-j+1)}{2}](n-1)\Big) \\ &\quad + \sum_{j=m}^{n-1}(mn(n-j)) + n\frac{m(m+1)}{2} \\ &\quad + mn(n-m) + \frac{m(m-1)}{2}(n-1) \\ &= \frac{1}{6}mn(m+1)(m+2) + \tfrac{1}{2}mn(n+m+1)(n-m) \\ &\quad + \frac{1}{3}m(m-1)(m+1)(n-1). \end{aligned}$$

On the other side, by Lemma 4.19 we have

$$\begin{aligned}\dim(\rho_1(\mathsf{G}(\mathsf{R}_\mathrm{T}))) &= n\frac{m(m+1)(m+2)}{6} + nm^2(n-m) + nm\frac{(n-m)(n-m+1)}{2} \\ &\quad + \big(m\frac{m(m+1)}{2} - \frac{m(m+1)(m+2)}{6}\big)(n-1) \\ &= \frac{1}{6}mn(m+1)(m+2) + \tfrac{1}{2}mn(n+m+1)(n-m) \\ &\quad + \frac{1}{3}m(m-1)(m+1)(n-1)\end{aligned}$$

which completes the proof. ■

4.3.2. Involutivity of R_T. To compute the curvature map for R_T we use the following lemma.

4.22 Lemma: *The following sequence is exact:*

$$\mathsf{S}_2\mathsf{T}^*\mathsf{Q}\otimes(\mathcal{W}^\perp\otimes_\mathsf{Q}\mathsf{TQ}) \xrightarrow{\rho_1(\sigma(\mathsf{R}_\mathrm{T}))} \mathsf{T}^*\mathsf{Q}\otimes\Lambda_2\mathcal{W}^\perp \xrightarrow{\tau} \Lambda_3\mathcal{W}^\perp\oplus(\mathcal{W}\otimes\Lambda_2\mathcal{W}^\perp) \longrightarrow 0$$

where τ is the projection to $\operatorname{coker}(\rho_1(\sigma(\mathsf{R}_\mathrm{T})))$ *given by*

$$\begin{aligned}&\tau(\mathsf{b})(v_1\oplus v_2,u,w) = \\ &\qquad (\mathsf{b}(v_1,u,w)+\mathsf{b}(u,w,v_1)+\mathsf{b}(w,v_1,u)) + \mathsf{b}(v_2,u,w), \qquad v_1,u,w\in\mathcal{W}^\perp,\ v_2\in\mathcal{W}.\end{aligned}$$

Proof: Recall that $\mathsf{S}_2\mathsf{T}^*\mathsf{Q} = \mathsf{S}_2\mathcal{W}\oplus\mathsf{S}_2\mathcal{W}^\perp\oplus(\mathcal{W}\otimes\mathcal{W}^\perp)$. Using Lemma 4.19 and since

$$\Big(\mathsf{S}_3\mathcal{W}^\perp\otimes\mathsf{TQ}\Big)\oplus\Big((\mathcal{W}\otimes\mathcal{W}^\perp\otimes\mathcal{W}^\perp)\otimes\mathsf{TQ}\Big)\subseteq\ker(\rho_1(\sigma(\mathsf{R}_\mathrm{T}))),$$

we observe that

$$\mathcal{W}\otimes\Lambda_2\mathcal{W}^\perp\subseteq\operatorname{coker}(\rho_1(\sigma(\mathsf{R}_\mathrm{T}))).$$

Moreover, by Lemma 4.20

$$\Lambda_3\mathcal{W}^\perp\subseteq\operatorname{coker}(\rho_1(\sigma(\mathsf{R}_\mathrm{T}))).$$

Finally, a dimension count using Lemma 4.19 then shows that the sequence is exact. ■

As a consequence of the previous computations, we have the following theorem.

4.23 Theorem: *The partial differential equation R_T is involutive if, for $p\in\mathsf{R}_\mathrm{T}$, we have*

$$\tau(\rho_1(\Phi)(p_2)-0)=0,$$

where p_2 is any point in $\mathsf{J}_2\pi$ that projects to p.

Proof: The proof follows by verifying conditions of Theorem 2.20. Notice that $\rho_1(\mathsf{G}_1(\mathsf{R}_\mathrm{L}))$ is a vector bundle on the open subset on which $\mathcal{W}$ is a vector bundle. Since $\mathsf{G}(\mathsf{R}_\mathrm{T})$ is an involutive symbol, the system of partial differential equations R_T is involutive if the curvature map κ defined as follows is zero:

$$\kappa:\mathsf{R}_\mathrm{T}\to\Lambda_3\mathcal{W}^\perp\oplus(\mathcal{W}\otimes\Lambda_2\mathcal{W}^\perp), \tag{4.7}$$

with $\kappa(p)=\tau(\rho_1(\Phi)(p_2)-0)$, where p_2 is any point in $\mathsf{J}_2\pi$ that projects to p. ■

Recall the definition of R_pot from Section 3.4.1. We have the following theorem.

4.24 Theorem: *Let* $\Sigma_{\text{ol}} = (\mathsf{Q}, \mathbb{G}_{\text{ol}}, V_{\text{ol}}, \mathcal{F}_{\text{ol}}, \mathcal{W}_{\text{ol}})$ *be an analytic open-loop simple mechanical control system. Let* $p_0 \in \mathsf{R}_{\text{pot}}$ *and let* $q_0 = \pi_1(p_0)$*. Assume that* q_0 *is a regular point for* $\mathcal{W}_{\text{ol}}$ *and that there exists a bundle automorphism* Θ *on* $\mathsf{T}^*\mathsf{Q}$ *defined on a neighborhood* U *of* q_0 *such that* Θ *satisfies the following equation in the neighborhood* U*:*

$$[\kappa(\Theta)]_{rij} = \left(\frac{\partial^2\Theta^l_i}{\partial q^r \partial q^j}\frac{\partial V_{\text{ol}}}{\partial q^l} - \frac{\partial^2\Theta^l_j}{\partial q^r \partial q^i}\frac{\partial V_{\text{ol}}}{\partial q^l}\right) + \left(\frac{\partial\Theta^l_i}{\partial q^j}\frac{\partial^2 V_{\text{ol}}}{\partial q^r \partial q^l} - \frac{\partial\Theta^l_j}{\partial q^i}\frac{\partial^2 V_{\text{ol}}}{\partial q^r \partial q^l}\right) + \left(\frac{\partial\Theta^l_i}{\partial q^r}\frac{\partial^2 V_{\text{ol}}}{\partial q^j \partial q^l} - \frac{\partial\Theta^l_j}{\partial q^r}\frac{\partial^2 V_{\text{ol}}}{\partial q^i \partial q^l}\right) + \left(\Theta^l_i\frac{\partial^3 V_{\text{ol}}}{\partial q^r \partial q^j \partial q^l} - \Theta^l_j\frac{\partial^3 V_{\text{ol}}}{\partial q^r \partial q^i \partial q^l}\right) = 0,$$

where $i, j \in \{1, \dots, m\}$*,* $l \in \{1, \dots, n\}$ *and* $r \in \{m+1, \dots, n\}$*. Then there exists an analytic closed-loop energy shaping metric* $\mathbb{G}_{\text{cl}}$ *prescribed by* $\mathbb{G}^\flat_{\text{cl}} = \Theta \circ \mathbb{G}^\flat_{\text{ol}}$ *and an analytic* $(\mathbb{G}_{\text{ol}}\text{-}\mathbb{G}_{\text{cl}})$*-potential energy shaping feedback* $\mathcal{F} \in \Gamma^\omega(\mathcal{W}_{\text{ol}})$ *defined on* U *which satisfies* $\Phi_{\text{pot}}(p_0) = \Theta^{-1}\mathsf{d}V(q_0) - \mathsf{d}V_{\text{ol}}(q_0) + \Theta\mathsf{d}V_{\text{ol}}(q_0)$ *for a solution* V *to* R_{pot}*.*

Proof: Observe that the system of partial differential equations R_{T}, with $\Theta = \Lambda^{-1}_{\text{cl}}$ and $\beta = \mathsf{d}V_{\text{ol}}$, gives the sufficient conditions for existence of a $(\mathbb{G}_{\text{ol}}\text{-}\mathbb{G}_{\text{cl}})$-potential energy shaping feedback; see Equation (3.12). Using Theorem 4.23, this partial differential equation is integrable if the curvature map given by Equation (4.7) is zero. By calculating the τ map, defined in Lemma 4.22, in local coordinates and after some simplifications the curvature map can be written as

$$[\kappa(\Theta)] = \left(\frac{\partial^2\Theta^l_i}{\partial q^r \partial q^j}\frac{\partial V_{\text{ol}}}{\partial q^l} - \frac{\partial^2\Theta^l_j}{\partial q^r \partial q^i}\frac{\partial V_{\text{ol}}}{\partial q^l} + \frac{\partial\Theta^l_i}{\partial q^j}\frac{\partial^2 V_{\text{ol}}}{\partial q^r \partial q^l} - \frac{\partial\Theta^l_j}{\partial q^i}\frac{\partial^2 V_{\text{ol}}}{\partial q^r \partial q^l} + \frac{\partial\Theta^l_i}{\partial q^r}\frac{\partial^2 V_{\text{ol}}}{\partial q^j \partial q^l} - \frac{\partial\Theta^l_j}{\partial q^r}\frac{\partial^2 V_{\text{ol}}}{\partial q^i \partial q^l} + \Theta^l_i\frac{\partial^3 V_{\text{ol}}}{\partial q^r \partial q^j \partial q^l} - \Theta^l_j\frac{\partial^3 V_{\text{ol}}}{\partial q^r \partial q^i \partial q^l}\right) dq^r \otimes dq^i \wedge dq^j,$$

where $i, j \in \{1, \dots, m\}$, $l \in \{1, \dots, n\}$ and $r \in \{m+1, \dots, n\}$, as desired. ■

4.4. Summary of integrability results

In this section we give a summary of the theorems we have obtained in the previous sections. Moreover, we state a procedure that clarifies how one should perform the energy shaping method so that certain problems—such as having a closed-loop energy shaping metric for which no potential energy shaping is possible—will not arise. This procedure reveals some of the fundamental properties of energy shaping partial differential equations that have not been understood in the literature to date.

1. **Kinetic energy shaping**: Find the set of bundle automorphisms λ on TQ which satisfy the sufficient conditions of Theorem 4.7 and denote it by $\hat{\mathsf{S}}_{\text{K}}$. Use the sufficient conditions of Theorem 4.15 to find the set of $\lambda \in \hat{\mathsf{S}}_{\text{K}}$ for which there exists a closed-loop metric $\mathbb{G}_{\text{cl}}$ and denote it by S_{K}.

2. **Potential energy shaping**: Find the set of bundle automorphisms Θ on $\mathsf{T}^*\mathsf{Q}$ which satisfy the sufficient conditions of Theorem 4.24 and denote it by $\hat{\mathsf{S}}_{\text{P}}$ and let

$$\mathsf{S}'_{\text{P}} = \{\Theta^{-1} \mid \Theta \in \hat{\mathsf{S}}_{\text{P}}\}.$$

The set of bundle automorphisms S'_{P} induces a set of bundle automorphisms on TQ by

$$\mathsf{S}_{\mathrm{P}} \doteq \{\mathbb{G}_{\mathrm{ol}}^{\sharp}\Lambda\mathbb{G}_{\mathrm{ol}}^{\flat} \mid \Lambda \in \mathsf{S}'_{\mathrm{P}}\}.$$

Note that, by Theorem 3.10, for each $\lambda \in \mathsf{S}_{\mathrm{P}}$ there exists a V_{cl} which satisfies the potential energy shaping partial differential equations.

3. **Total energy shaping**: The intersection $\mathsf{S}_{\mathrm{P}} \cap \mathsf{S}_{\mathrm{K}}$ yields the set of λ such that
   (a) there exists a closed-loop metric which is a solution to the kinetic energy shaping problem and
   (b) more importantly, potential energy shaping is possible and as a result, energy shaping is possible.
4. Determine the set of closed-loop potential functions V_{cl} with positive-definite Hessian at the desired point. It would be interesting to have a geometric characterization of this. Some ideas for this are addressed in Section 7.2.3.

4.5. A simple mechanical control system with no energy shaping feedback

We emphasize the importance of the sufficient conditions obtained in the previous sections by designing a class of linearly controllable systems that are not stabilizable by the energy shaping method in the absence of gyroscopic forces. We postpone the extension of this problem in the presence of gyroscopic forces for the future work; see Section 7.2.2.

4.25 Example: Let $\mathsf{Q} = \mathbb{R}^3$ and consider a simple mechanical control system $\Sigma_{\mathrm{ol}} = (\mathbb{R}^3, \mathbb{G}, V_{\mathrm{ol}}, 0, \mathcal{W}_{\mathrm{ol}})$ as follows.

1. The open-loop metric is
$$\mathbb{G} = M dq^1 \otimes dq^1 + M dq^2 \otimes dq^2 + ((q^1)^2 + (q^2)^2 + 1) dq^3 \otimes dq^3,$$
where $M \in \mathbb{R}_{>0}$.
2. The open-loop potential function V_{ol} is
$$V_{\mathrm{ol}} = (q^1)^2 + q^3 q^2 + q^3 q^1 + p(q^1, q^2),$$
with $p = O((q^1)^{k_1}(q^2)^{k_2})$, $k_1 + k_2 \geq 3$.
3. The control subbundle is $\mathcal{W} = \mathrm{span}\{dq^3\}$.

The system is linearly controllable at the origin $q_0 = \mathbf{0} \in \mathbb{R}^3$. Furthermore, the system is not stable at q_0. So we wish to proceed with the energy shaping method. We have the following proposition.

4.26 Proposition: *Σ_{ol} is stabilizable at the origin by energy shaping method if and only if $p \equiv 0$.*

In order to prove this proposition we need the following lemma.

4.27 Lemma: *For the above example, the set of analytic closed-metrics for which there exists a potential energy shaping feedback is*

$$\{\mathcal{M}dq^1 \otimes dq^1 + \mathcal{M}dq^2 \otimes dq^2 + (q^1)^2 + \mathcal{Z}(q^1, q^2, q^3)dq^3 \otimes dq^3 \mid \mathcal{M}, \mathcal{C} \in \mathbb{R},\ \mathcal{Z} \in C^\omega(\mathsf{Q})\}.$$

Proof: For this example, one can easily check that the only nonzero Christoffel symbols are $\mathcal{S}^1_{33}$, $\mathcal{S}^2_{33}$, $\mathcal{S}^3_{31}$, and $\mathcal{S}^3_{32}$. Thus the λ-equations are the following:

1. $$\begin{cases} \frac{\partial}{\partial q^1}(\mathbb{G}_{11}\lambda^1_1) = 0, \\ \frac{\partial}{\partial q^2}(\mathbb{G}_{11}\lambda^1_1) = 0, \\ \frac{\partial}{\partial q^3}(\mathbb{G}_{11}\lambda^1_1) - 2\mathcal{S}^3_{31}\mathbb{G}_{33}\lambda^3_1 = 0; \end{cases}$$

2. $$\begin{cases} \frac{\partial}{\partial q^1}(\mathbb{G}_{22}\lambda^2_2) = 0, \\ \frac{\partial}{\partial q^2}(\mathbb{G}_{22}\lambda^2_2) = 0, \\ \frac{\partial}{\partial q^3}(\mathbb{G}_{22}\lambda^2_2) - 2\mathcal{S}^3_{32}\mathbb{G}_{33}\lambda^3_2 = 0; \end{cases}$$

3. $$\begin{cases} \frac{\partial}{\partial q^1}(\mathbb{G}_{11}\lambda^1_2) = 0, \\ \frac{\partial}{\partial q^2}(\mathbb{G}_{11}\lambda^1_2) = 0, \\ \frac{\partial}{\partial q^3}(\mathbb{G}_{11}\lambda^1_2) - \mathcal{S}^3_{31}\mathbb{G}_{33}\lambda^3_2 - \mathcal{S}^3_{32}\mathbb{G}_{33}\lambda^3_1 = 0; \end{cases}$$

4. $$\begin{cases} \frac{\partial}{\partial q^1}(\mathbb{G}_{22}\lambda^2_1) = 0, \\ \frac{\partial}{\partial q^2}(\mathbb{G}_{22}\lambda^2_1) = 0, \\ \frac{\partial}{\partial q^3}(\mathbb{G}_{22}\lambda^2_1) - \mathcal{S}^3_{32}\mathbb{G}_{33}\lambda^3_1 - \mathcal{S}^3_{31}\mathbb{G}_{33}\lambda^3_2 = 0. \end{cases}$$

From the first set of equations we conclude that $q^1\lambda^3_2 = g(q^3)$, where $g \in C^\omega(\mathsf{Q})$. Similarly, we have $q^2\lambda^3_1 = h(q^3)$, where $h \in C^\omega(\mathsf{Q})$. Substituting this into the third set of equations, one can conclude that $h = g \equiv 0$. Furthermore, since $\mathbb{G}_{\mathrm{cl}}$ is required to be symmetric, $\lambda^1_3 = \lambda^2_3 = 0$. Thus the set of solutions to the λ-equation in a neighborhood of the origin is given by

$$\lambda = c_{11}dq^1 \otimes dq^1 + c_{12}(dq^1 \otimes dq^2 + dq^2 \otimes dq^1) + c_{22}dq^2 \otimes dq^2 + z(q^1, q^2, q^3)dq^3 \otimes dq^3, \quad (4.8)$$

where $c_{11}, c_{12}, c_{22} \in \mathbb{R}$ and $z \in C^\omega(\mathsf{Q})$. Recalling the definition of Θ in Proposition 3.14, we have

$$\Theta = C_{11}dq^1 \otimes dq^1 + C_{12}(dq^1 \otimes dq^2 + dq^2 \otimes dq^1) + C_{22}dq^2 \otimes dq^2 + Z(q^1, q^2, q^3)dq^3 \otimes dq^3,$$

where $C_{11}, C_{12}, C_{22} \in \mathbb{R}$ and $Z \in C^\omega(\mathsf{Q})$. We denote the set of all such Θ by S_K as in Section 4.4. Our main goal is to find the subset S_K which leads to a potential energy shaping feedback. Since $\mathcal{W}_{\mathrm{ol}}$ is invariant under Θ, the potential energy shaping compatibility condition (see Equation (3.12)) in a neighborhood of the origin is given by

$$(C_{11} - C_{22})\frac{\partial^2 V_{\mathrm{ol}}}{\partial q^1 \partial q^2} + C_{12}\frac{\partial^2 V_{\mathrm{ol}}}{\partial q^2 \partial q^2} - C_{12}\frac{\partial^2 V_{\mathrm{ol}}}{\partial q^1 \partial q^1} = 0;$$

thus we conclude that $C_{11} = C_{22}$ and $C_{12} = 0$. ■

Proof of Proposition 4.26: Since $c_{11} = c_{22}$ and $c_{12} = 0$, the potential energy shaping partial differential equations for V_{cl} reads

$$\frac{\partial V_{\mathrm{cl}}}{\partial q^i} = a\frac{\partial V_{\mathrm{cl}}}{\partial q^i},$$

where $i \in \{1, 2\}$ and $a \in \mathbb{R}\backslash\{0\}$. Thus we have

$$V_{\mathrm{cl}} = a((q^1)^2 + q^3q^2 + q^3q^1 + p(q^1, q^2)) + f(q^3),$$

where $a \in \mathbb{R}\backslash\{0\}$ and $f \in C^{\omega}(\mathsf{Q})$. The Hessian of the closed-loop potential function has the following form at the origin.

$$\mathsf{Hess}(V_{\mathrm{cl}})(q_0) = a\begin{pmatrix} 2 & 0 & 1 \\ 0 & 0 & 1 \\ 1 & 1 & b \end{pmatrix},$$

where $a \in \mathbb{R}\backslash\{0\}$ and $b \in \mathbb{R}$ are arbitrary parameters. Thus it is clear that the Hessian cannot be made positive-definite for any choice of a and b. If $p \equiv 0$, then C_{12} need not be zero and thus the Hessian can be made positive-definite using this extra arbitrary parameter. ■

Chapter 5

Energy shaping for linear simple mechanical systems

In this chapter, a geometric proof is given that, for linear simple mechanical control systems, linear controllability is sufficient for the existence of a stabilizing energy shaping feedback. Although the same result has been proved in the Lagrangian setting [51] and in the Hamiltonian setting [36], the proofs are not constructive and do not reveal the structure of the partial differential equations for energy shaping. Our proof relies on an adaptation of the integrability results of Chapter 4 to linear simple mechanical systems. Moreover, we clarify the role of kinetic energy shaping in the construction of a stabilizing energy shaping feedback for linear simple mechanical systems.

5.1. Linear simple mechanical systems

In this section we recall the algebraic formulation of linear simple mechanical systems from [14]. We use this formulation to give an algebraic description of the energy shaping problem for linear simple mechanical control systems in the next section. We start by defining what we mean by a linear simple mechanical system.

5.1 Definition: A *linear simple mechanical control system* is a quadruple $\Sigma = (\mathsf{V}, \mathsf{M}, \mathsf{K}, F)$, where

1. V is an n-dimensional $\mathbb{R}$-vector space,
2. M is an inner product on V,
3. K is a symmetric $(0,2)$-tensor on V, and
4. $F \in \mathrm{L}(\mathbb{R}^m, \mathsf{V}^*)$ corresponds to the controls.

The governing equations for a linear simple mechanical system are

$$\ddot{x}(t) + \mathsf{M}^\sharp \mathsf{K}^\flat (x(t)) = \mathsf{M}^\sharp \circ F(u(x(t), \dot{x}(t))),$$

where $t \mapsto x(t)$ is curve in V and $u : \mathsf{V} \oplus \mathsf{V} \to \mathbb{R}^m$.

If $F = 0$ we call Σ a linear simple mechanical system and we denote it by the triple $\Sigma = (\mathsf{V}, \mathsf{M}, \mathsf{K})$. The equations of motion for a linear simple mechanical system in the absence of external forces can equivalently be defined by $\dot{\mathsf{x}} = A_\Sigma(\mathsf{x})$, where $\mathsf{x} \in \mathsf{V} \oplus \mathsf{V}$ and

$$A_\Sigma = \begin{pmatrix} 0 & \mathrm{id}_{\mathsf{V}} \\ -\mathsf{M}^\sharp \mathsf{K}^\flat & 0 \end{pmatrix},$$

is a linear map on $\mathsf{V} \oplus \mathsf{V}$.

5.2 Lemma: *For a linear simple mechanical system $\Sigma = (\mathsf{V}, \mathsf{M}, \mathsf{K})$ the eigenvalues of $\mathsf{M}^\sharp \mathsf{K}^\flat$ are real.*

Proof: The proof is immediate since $\mathsf{M}^\sharp \mathsf{K}^\flat$ is symmetric with respect to the inner product M. ■

The following proposition follows immediately from the stability analysis of linear systems; see [14].

5.3 Proposition: *Let $\Sigma = (\mathsf{V}, \mathsf{M}, \mathsf{K})$ be a simple mechanical system with the equilibrium configuration $0 \in \mathsf{V}$. Then the system is stable if and only if $\mathsf{M}^\sharp \mathsf{K}^\flat$ is positive-definite.*

5.2. Energy shaping for linear simple mechanical control systems

Although the statement of the energy shaping problem for linear simple mechanical systems can be given in a purely algebraic fashion, in this section we study the energy shaping problem for these systems using the energy shaping partial differential equations. The advantage of such a treatment is (1) characterizing the space of linear stabilizing solutions to the energy shaping partial differential equations and (2) describing the space of closed-loop metrics for which there exists a closed-loop potential function which satisfies the potential energy shaping partial differential equations. It is worth reminding ourselves that the achievable closed-loop systems by performing energy shaping on a linear simple mechanical control system are not necessarily all linear simple mechanical systems. What we need in this section, however, is that the space of linear closed-loop systems is large enough to stabilize a linear simple mechanical control system using energy shaping. When we seek a linear solution to the energy shaping problem, we call the procedure the linear energy shaping. We start this section by presenting the closed-loop systems achievable using linear energy shaping.

5.2.1. The algebra of linear energy shaping. Let $\Sigma_{\mathrm{ol}} = (\mathsf{V}, \mathsf{M}_{\mathrm{ol}}, \mathsf{K}_{\mathrm{ol}}, F_{\mathrm{ol}})$ be a linear simple mechanical system. Let us define $E_{\Sigma_{\mathrm{ol}}} \subset \mathrm{L}(\mathsf{V}, \mathsf{V})$ to be the subset of linear maps A that satisfy the following conditions:

1. $A = \mathsf{M}_{\mathrm{ol}}^\sharp \mathsf{K}_{\mathrm{ol}}^\flat + \mathsf{M}_{\mathrm{ol}}^\sharp \circ F_{\mathrm{ol}} \circ L$, where $L \in \mathrm{L}(\mathsf{V}, \mathbb{R}^m)$;
2. A is diagonalizable over $\mathbb{R}$.

The following proposition shows that the set $E_{\Sigma_{\mathrm{ol}}}$ prescribes the closed-loop systems achievable by linear energy shaping.

5.4 Proposition: *Let* $\Sigma_{\rm ol} = (\mathsf{V}, \mathsf{M}_{\rm ol}, \mathsf{K}_{\rm ol}, F_{\rm ol})$ *be a linear simple mechanical control system and define* $E_{\Sigma_{\rm ol}}$ *as above. Let* $\mathsf{M}_{\rm cl}$ *be an inner product on* V *and let* $\mathsf{K}_{\rm cl} \in \mathsf{S}_2\mathsf{V}^*$*. The following statements are equivalent.*

1. *There exists a linear feedback* $u : \mathsf{V} \oplus \mathsf{V} \to \mathbb{R}^m$ *given by* $u(x, v) = -L(x)$ *for which the dynamics of the closed-loop system are those of the linear simple mechanical system* $\Sigma_{\rm cl} = (\mathsf{V}, \mathsf{M}_{\rm cl}, \mathsf{K}_{\rm cl})$*.*
2. $\mathsf{M}^\sharp_{\rm cl}\mathsf{K}^\flat_{\rm cl} \in E_{\Sigma_{\rm ol}}$*.*

Proof: The governing equation for $\Sigma_{\rm ol}$ is

$$\begin{pmatrix} \dot{x}(t) \\ \dot{v}(t) \end{pmatrix} = \begin{pmatrix} 0 & \mathrm{id}_{\mathsf{V}} \\ -\mathsf{M}^\sharp_{\rm ol}\mathsf{K}^\flat_{\rm ol} & 0 \end{pmatrix} \begin{pmatrix} x(t) \\ v(t) \end{pmatrix} + \begin{pmatrix} 0 \\ \mathsf{M}^\sharp_{\rm ol} \circ F_{\rm ol} \end{pmatrix} u(t).$$

So the closed-loop system has the following form

$$\begin{pmatrix} \dot{x}(t) \\ \dot{v}(t) \end{pmatrix} = \begin{pmatrix} 0 & \mathrm{id}_{\mathsf{V}} \\ -\mathsf{M}^\sharp_{\rm ol}\mathsf{K}^\flat_{\rm ol} - \mathsf{M}^\sharp_{\rm ol} \circ F_{\rm ol} \circ L & 0 \end{pmatrix} \begin{pmatrix} x(t) \\ v(t) \end{pmatrix},$$

for $L \in \mathrm{L}(\mathsf{V}, \mathbb{R}^m)$. As a result, the closed-loop system has the dynamics of $\Sigma_{\rm cl}$ if and only if $\mathsf{M}^\sharp_{\rm cl}\mathsf{K}^\flat_{\rm cl} = \mathsf{M}^\sharp_{\rm ol}\mathsf{K}^\flat_{\rm ol} + \mathsf{M}^\sharp_{\rm ol} \circ F_{\rm ol} \circ L$. So if the closed-loop system has the dynamics of $\Sigma_{\rm cl}$, then $\mathsf{M}^\sharp_{\rm cl}\mathsf{K}^\flat_{\rm cl} \in E_{\Sigma_{\rm ol}}$, since $\mathsf{M}^\sharp_{\rm cl}\mathsf{K}^\flat_{\rm cl}$ is diagonalizable. Conversely, if $\mathsf{M}^\sharp_{\rm cl}\mathsf{K}^\flat_{\rm cl} \in E_{\Sigma_{\rm ol}}$, then by definition there exists $L \in \mathrm{L}(\mathsf{V}, \mathbb{R}^m)$ such that $\mathsf{M}^\sharp_{\rm cl}\mathsf{K}^\flat_{\rm cl} = \mathsf{M}^\sharp_{\rm ol}\mathsf{K}^\flat_{\rm ol} + \mathsf{M}^\sharp_{\rm ol} \circ F_{\rm ol} \circ L$. ■

5.2.2. The energy shaping partial differential equations. The goal in this section is to obtain the linear energy shaping solutions from the energy shaping partial differential equations and show that, for all linear simple mechanical systems, these solutions lie in the subspace $E_{\Sigma_{\rm ol}}$. Then, if the open-loop system $\Sigma_{\rm ol}$ is linearly controllable, one can design $F_{\rm ol} \circ L$ such that $\mathsf{M}^\sharp_{\rm cl}\mathsf{K}^\flat_{\rm cl}$ is positive-definite. Thus a stabilizing energy shaping feedback is achieved.

Recall that for each $\mathsf{K}_{\rm ol} \in \mathsf{S}_2\mathsf{V}^*$ there exists a function $V_{\rm ol}$ such that $\mathsf{K}_{\rm ol}(v, v) = V_{\rm ol}(v)$, for all $v \in \mathsf{V}$. We adapt the energy shaping definition, Definition 1.2, to the linear simple mechanical systems to arrive at the following

$$\mathsf{M}^\flat_{\rm ol}\mathsf{M}^\sharp_{\rm cl}\mathsf{d}V_{\rm cl}(x(t)) - \mathsf{d}V_{\rm ol}(x(t)) = F_{\rm ol} \circ u_{\rm pot}(x(t)), \tag{5.1}$$

where $\mathsf{M}_{\rm cl}$ is an inner product on V, $V_{\rm cl}$ is the closed-loop potential function, and $u_{\rm pot} : \mathsf{V} \oplus \mathsf{V} \to \mathbb{R}^m$ is a feedback. The following proposition shows that the linear solutions of Equation (5.1) lie in $E_{\Sigma_{\rm ol}}$.

5.5 Proposition: *Let* $\Sigma_{\rm ol} = (\mathsf{V}, \mathsf{M}_{\rm ol}, \mathsf{K}_{\rm ol}, F_{\rm ol})$ *be a linear simple mechanical control system. Let* $\mathsf{M}_{\rm cl}$ *be an inner product on* V *and let* $\mathsf{K}_{\rm cl} \in \mathsf{S}_2\mathsf{V}^*$*. The following statements are equivalent:*

1. *there exists a linear feedback* $u_{\rm pot} : \mathsf{V} \oplus \mathsf{V} \to \mathbb{R}^m$ *given by* $u_{\rm pot}(x, v) = -L(x)$ *which satisfies Equation (5.1);*
2. $\mathsf{M}^\sharp_{\rm cl}\mathsf{K}^\flat_{\rm cl} \in E_{\Sigma_{\rm ol}}$*.*

Proof: We denote the projection of V^* onto $\mathsf{V}^*/\mathrm{Im}(F_{\mathrm{ol}})$ by P. As a result, if M_{cl} and V_{cl} satisfy Equation (5.1)

$$P\big(\mathsf{M}_{\mathrm{ol}}^{\flat}\mathsf{M}_{\mathrm{cl}}^{\sharp}\mathsf{d}V_{\mathrm{cl}} - \mathsf{d}V_{\mathrm{ol}}\big) = 0. \tag{5.2}$$

Note that by Theorem 3.10 this partial differential equation is involutive provided that

$$\mathsf{d}(\mathsf{M}_{\mathrm{cl}}^{\flat}\mathsf{M}_{\mathrm{ol}}^{\sharp}\mathsf{d}V_{\mathrm{ol}}) \in \mathsf{I}_2(W_{\mathrm{cl}}), \tag{5.3}$$

where $W_{\mathrm{cl}} = \mathsf{M}_{\mathrm{cl}}^{\flat}\mathsf{M}_{\mathrm{ol}}^{\sharp}(\mathrm{Im}(F_{\mathrm{ol}}))$. Note that we are only interested in the solutions to Equation (5.2) that lead to a linear closed-loop system. Thus one can write this equation as

$$P\big(\mathsf{M}_{\mathrm{ol}}^{\flat}\mathsf{M}_{\mathrm{cl}}^{\sharp}\mathsf{K}_{\mathrm{cl}}(v) - \mathsf{K}_{\mathrm{ol}}(v)\big) = 0,$$

where $v \in \mathsf{V}$. This holds if and only if

$$\mathsf{M}_{\mathrm{ol}}^{\flat}\mathsf{M}_{\mathrm{cl}}^{\sharp}\mathsf{K}_{\mathrm{cl}}(v) - \mathsf{K}_{\mathrm{ol}}(v) \in \mathrm{Im}(F_{\mathrm{ol}})$$

for all $v \in \mathsf{V}$. Equivalently,

$$\mathsf{M}_{\mathrm{cl}}^{\sharp}\mathsf{K}_{\mathrm{cl}}^{\flat} - \mathsf{M}_{\mathrm{ol}}^{\sharp}\mathsf{K}_{\mathrm{ol}}^{\flat} = \mathsf{M}_{\mathrm{ol}}^{\sharp} \circ F_{\mathrm{ol}} \circ L, \tag{5.4}$$

where $L \in \mathrm{L}(\mathsf{V}, \mathbb{R}^m)$, i.e., $\mathsf{M}_{\mathrm{cl}}^{\sharp}\mathsf{K}_{\mathrm{cl}}^{\flat} \in E_{\Sigma_{\mathrm{ol}}}$. In order to proceed with the proof, we show that, in fact, the integrability condition of Equation (5.3) holds for all solutions given by Equation (5.4). Let $P_{\mathcal{E}}$ be the projection of V^* onto $\mathsf{V}^*/W_{\mathrm{cl}}$. Then Equation (5.3) can be written as

$$\mathsf{M}_{\mathrm{cl}}(\mathsf{M}_{\mathrm{ol}}^{\sharp}\mathsf{K}_{\mathrm{ol}}^{\flat}(v_1), v_2) - \mathsf{M}_{\mathrm{cl}}(\mathsf{M}_{\mathrm{ol}}^{\sharp}\mathsf{K}_{\mathrm{ol}}^{\flat}(v_1), v_2) = 0, \tag{5.5}$$

for all $v_1, v_2 \in \mathrm{coann}(W_{\mathrm{cl}})$. If $v \in \mathrm{coann}(W_{\mathrm{cl}})$, then $\mathsf{M}_{\mathrm{cl}}^{\flat}\mathsf{M}_{\mathrm{ol}}^{\sharp}(\alpha)(v) = 0$ for any $\alpha \in \mathrm{Im}(F_{\mathrm{ol}})$. This implies that $v \in \mathrm{coann}(\mathrm{Im}(F_{\mathrm{ol}}))$, since $\mathsf{M}_{\mathrm{cl}}^{\flat}\mathsf{M}_{\mathrm{ol}}^{\sharp}$ is an isomorphism. If Equation (5.4) holds, we have $\mathsf{M}_{\mathrm{cl}}^{\sharp}\mathsf{K}_{\mathrm{cl}}^{\flat}(v) = \mathsf{M}_{\mathrm{ol}}^{\sharp}\mathsf{K}_{\mathrm{ol}}^{\flat}(v)$ for all $v \in \mathrm{coann}(\mathrm{Im}(F_{\mathrm{ol}}))$. Thus Equation (5.5), for all $v_1, v_2 \in \mathrm{coann}(\mathrm{Im}(F_{\mathrm{ol}}))$, can be written as

$$\mathsf{K}_{\mathrm{cl}}(v_1, v_2) - \mathsf{K}_{\mathrm{cl}}(v_2, v_1) = 0,$$

which holds by symmetry of K_{cl}. ∎

5.6 Theorem: *Let $\Sigma_{\mathrm{ol}} = (\mathsf{V}, \mathsf{M}_{\mathrm{ol}}, \mathsf{K}_{\mathrm{ol}}, F_{\mathrm{ol}})$ be a linear simple mechanical control system which is linearly controllable. Then there exists a linear energy shaping feedback which stabilizes the system.*

Proof: By Proposition 5.5, for any inner product M_{cl} on V and $\mathsf{K}_{\mathrm{cl}} \in \mathsf{S}_2\mathsf{V}^*$ such that $\mathsf{M}_{\mathrm{cl}}^{\sharp}\mathsf{K}_{\mathrm{cl}}^{\flat} \in E_{\Sigma_{\mathrm{ol}}}$, there exists a linear feedback $u : \mathsf{V} \oplus \mathsf{V} \to \mathbb{R}^m$ given by $u(x, v) = -L(x)$, with $L \in \mathrm{L}(\mathsf{V}, \mathbb{R}^m)$, for which the dynamics of the closed-loop system are those of the linear simple mechanical system $\Sigma_{\mathrm{cl}} = (\mathsf{V}, \mathsf{M}_{\mathrm{cl}}, \mathsf{K}_{\mathrm{cl}})$. Then, by pole placement, one can design $F_{\mathrm{ol}} \circ L$ such that $\mathsf{M}_{\mathrm{cl}}^{\sharp}\mathsf{K}_{\mathrm{cl}}^{\flat} \in E_{\Sigma_{\mathrm{ol}}}$ is positive-definite. ∎

One can choose a basis of eigenvectors for $\mathsf{M}_{\mathrm{cl}}^{\sharp}\mathsf{K}_{\mathrm{cl}}^{\flat}$ and by requiring that this basis be orthonormal, one can define M_{cl} and thus pull apart $\mathsf{M}_{\mathrm{cl}}^{\sharp}\mathsf{K}_{\mathrm{cl}}^{\flat}$ into its components. In following, we present an example of linear energy shaping.

5.7 Example: Let $\mathsf{V} = \mathbb{R}^2$ and consider the stabilization of the following system around the origin using energy shaping:

1. $\mathsf{M}_{\text{ol}} = dx^1 \otimes dx^1 + dx^2 \otimes dx^2$;
2. $V_{\text{ol}} = -(x^1)^2 + 2x^1x^2 + (x^2)^2$;
3. $F_{\text{ol}} = \mathsf{span}\{dx^2\}$.

This system is clearly linearly controllable. We shall find a closed-loop system which has the same dynamics as $\Sigma_{\text{ol}} = (\mathsf{M}_{\text{ol}}, V_{\text{ol}}, F_{\text{ol}})$. We compute

$$E_{\Sigma_{\text{ol}}} = \left\{ A \in \mathsf{D}(\mathbb{R}^{2\times 2}) \mid A = \begin{pmatrix} -2 & 2 \\ L_1 & L_2 \end{pmatrix},\ L_1, L_2 \in \mathbb{R} \right\},$$

where we denoted the space of diagonalizable two by two matrices by $\mathsf{D}(\mathbb{R}^{2\times 2})$. We are looking for an inner product M_{cl} and $\mathsf{K}_{\text{cl}} \in \mathsf{S}_2\mathbb{R}^2$ such that $\mathsf{M}^\sharp_{\text{cl}}\mathsf{K}^\flat_{\text{cl}} \in E_{\Sigma_{\text{ol}}}$; see Proposition 5.5. Suppose that the closed-loop inner product is characterized as

$$\mathsf{M}^\sharp_{\text{cl}} = \begin{pmatrix} a & b \\ b & c \end{pmatrix},$$

where $a, b, c \in \mathbb{R}$. Furthermore, suppose that $\mathsf{K}_{\text{cl}} = \mathsf{K}_{ij} dx^i \otimes dx^j$, where $\mathsf{K}_{ij} \in \mathbb{R}$ for $i, j \in \{1, 2\}$, and $\mathsf{K}_{21} = \mathsf{K}_{12}$. Then the prolongation of the energy shaping partial differential equation, Equation (5.2), reads

$$\begin{aligned} a\mathsf{K}_{11} + a\mathsf{K}_{12} &= -2, \\ a\mathsf{K}_{12} + b\mathsf{K}_{22} &= 2. \end{aligned}$$

Thus

$$\mathsf{M}^\sharp_{\text{cl}}\mathsf{K}^\flat_{\text{cl}} = \begin{pmatrix} -2 & 2 \\ \frac{b}{a}(-2 - b\mathsf{K}_{12}) + c\mathsf{K}_{12} & \frac{c}{b}(2 - a\mathsf{K}_{12}) + b\mathsf{K}_{12} \end{pmatrix},$$

which clearly belongs to $E_{\Sigma_{\text{ol}}}$. It is easy to check that $A \in E_{\Sigma_{\text{ol}}}$ is positive-definite if and only if $L_1 < 0$, $L_2 > 2$, and $-L_1 > L_2$. Moreover, since the desired closed-loop inner product is positive-definite (note that, in general, this assumption is not necessary), we have $a > 0$, $c > 0$ and $ac - b^2 > 0$. If we incorporate this assumption, we can choose the arbitrary variables a, b, c, K_{12} such that the closed-loop system is stable at the origin. For example, the closed-loop simple mechanical system is stable with choosing $a = \frac{37}{10}$, $b = \frac{42}{10}$, $c = \frac{48}{10}$, and $\mathsf{K}_{12} = -\frac{705}{10}$.

Chapter 6

Energy shaping for systems with one degree of underactuation

Numerous systems considered in the literature on energy shaping have one degree of underactuation. In this chapter we show that all *linearly controllable* simple mechanical control systems with one degree of underactuation can be stabilized using an energy shaping feedback, with closed-loop metrics which are not necessarily positive-definite. The results fully solve the problem of stabilization of systems with one degree of underactuation. First, in Theorem 6.1, we show that any solution to the kinetic energy shaping partial differential equations gives rise to a closed-loop potential function. Then we investigate, in a geometric fashion, if there exists any stabilizing solution.

6.1. Formal integrability of potential energy shaping partial differential equations

We first show that for systems with one degree of underactuation the potential energy shaping partial differential equations is always involutive. The following theorem is an immediate corollary of Theorem 4.15.

6.1 Theorem: *If Σ_{ol} is a simple mechanical control system with one degree of underactuation, for each bundle automorphism that satisfies the λ-equation, there exists a closed-loop metric and a closed-loop potential function that satisfy the energy shaping partial differential equations.*

Proof: Note that the projection map τ in Lemma 4.22 is the zero map for $m = 1$ and so the closed-loop metric equation is involutive by Theorem 4.15. Moreover, Equation (3.12) vanishes for $m = 1$. ■

6.2. Stabilization of systems with one degree of underactuation

In this section, we wish to determine the stabilizing solutions to the energy shaping partial differential equations for systems with one degree of underactuation. Throughout this section, let Q be an n-dimensional analytic manifold and $\Sigma_{\mathrm{ol}} = (\mathsf{Q}, \mathbb{G}, V_{\mathrm{ol}}, \mathcal{W}_{\mathrm{ol}})$ be an open-loop simple mechanical control system with one degree of underactuation. We denote the Hessian of a potential function V at $q_0 \in \mathsf{Q}$ by $\mathsf{Hess}(V)(q_0) \in \mathsf{S}_2\mathsf{T}^*_{q_0}\mathsf{Q}$. In particular, we denote the Hessian of the open-loop potential function and the closed-loop potential function at the equilibrium point by $\mathsf{Hess}(V_{\rm ol})(q_0)$ and $\mathsf{Hess}(V_{\rm cl})(q_0)$, respectively.

Since the compatibility conditions of Theorem 3.10 always satisfy for systems with one degree of underactuation, Theorem 6.1, one can study the prolongation of the potential energy shaping partial differential equations instead of the original partial differential equations. We shall perform this in the same fashion as in Chapter 5. Let $(q^1,\dots,q^n)$ be local coordinates in a neighborhood U of $q_0 \in \mathsf{Q}$ such that $\mathcal{W}_{\rm ol} = \operatorname{span}\{dq^2,\dots,dq^n\}$ and let P be the projection of $\mathsf{T}^*\mathsf{Q}$ onto $\mathsf{span}\{dq^1\}$.

If we prolong the potential energy shaping partial differential equation and evaluate the result at the origin, noting that $\mathsf{d}V_{\rm cl}(q_0) = 0$, we have

$$P\big(\mathbb{G}^\flat(q_0)\mathbb{G}^\sharp_{\rm cl}(q_0)\mathsf{Hess}V_{\rm cl}(v)(q_0) - \mathsf{Hess}V_{\rm ol}(v)(q_0)\big) = 0,$$

where $v \in \mathsf{T}_{q_0}\mathsf{Q}$, i.e.,

$$\mathbb{G}^\sharp_{\rm cl}(q_0)\mathsf{Hess}^\flat(V_{\rm cl})(q_0) - \mathbb{G}^\sharp(q_0)\mathsf{Hess}^\flat(V_{\rm ol})(q_0) = \mathbb{G}^\sharp(q_0)(u|_{q_0}), \tag{6.1}$$

where $u : \mathsf{TQ} \to \mathcal{W}_{\rm ol}$. If the system is linearly controllable, then one can design a control such that $\mathbb{G}^\sharp(q_0)\mathsf{Hess}^\flat(V_{\rm ol})(q_0) + \mathbb{G}^\sharp(q_0)(u|_{q_0})$ is diagonalizable and positive-definite. It is important to note that this does not necessarily imply that there exist $\mathbb{G}_{\rm cl}$ and $V_{\rm cl}$ such that $\mathbb{G}^\sharp_{\rm cl}(q_0)\mathsf{Hess}^\flat(V_{\rm cl})(q_0)$ is positive-definite, since the kinetic energy shaping partial differential equation puts restrictions on the achievable closed-loop metrics. However, we will show that, for systems with one degree of underactuation, the space of solutions of the kinetic energy shaping partial differential equations is large enough so that $\mathbb{G}^\sharp_{\rm cl}(q_0)\mathsf{Hess}^\flat(V_{\rm cl})(q_0)$ can be made positive-definite. We do this in the following steps.

1. We first identify a simple class of solutions to the λ-equation using Proposition 6.2.

2. We show that this class of solutions is large enough to ensure that Equation (6.1) holds with $\mathbb{G}^\sharp_{\rm cl}(q_0)\mathsf{Hess}^\flat(V_{\rm cl})(q_0)$ diagonalizable and positive-definite.

Let U be a neighborhood of the equilibrium point $q_0 \in \mathsf{Q}$ and let $(q^1,\dots,q^n)$ be local coordinates on U. In order to find the class of solutions mentioned in 1, we need to make some observations about the kinetic energy shaping partial differential equations for systems with one degree of underactuation. For these systems, the λ-equation in adapted local coordinates is given by

$$\frac{\partial}{\partial q^k}(\mathbb{G}_{1i}\lambda^i_1) - 2\mathcal{S}^s_{k1}\mathbb{G}_{si}\lambda^i_1 = 0, \tag{6.2}$$

where $\mathcal{S}^i_{jk}$, for $i,j,k \in \{1,\dots,n\}$, are the Levi-Civita connection coefficients associated to $\mathbb{G}$ and $i,k,s \in \{1,\dots,n\}$. Suppose we are seeking solutions to the λ-equation that in local coordinates look like $\lambda(q) = \lambda^k_i dq^i \otimes \frac{\partial}{\partial q^k}$, where $\lambda^k_i \in \mathbb{R}$ and $q \in U$, i.e., λ is constant. Then one can write Equation (6.2) as follows:

$$\left(\frac{\partial\mathbb{G}_{11}}{\partial q^k} - 2\mathcal{S}^i_{k1}\mathbb{G}_{i1}\right)\lambda^1_1 + \left(\frac{\partial\mathbb{G}_{12}}{\partial q^k} - 2\mathcal{S}^i_{k1}\mathbb{G}_{i2}\right)\lambda^2_1 + \cdots + \left(\frac{\partial\mathbb{G}_{1n}}{\partial q^k} - 2\mathcal{S}^i_{k1}\mathbb{G}_{in}\right)\lambda^n_1 = 0. \tag{6.3}$$

Because $\mathcal{S}$ is the Levi-Civita connection for $\mathbb{G}$, the first term vanishes, leaving λ_1^1 arbitrary. One can rewrite Equation (6.3) in the following fashion:

$$\sum_{i=1}^{n}\sum_{j=2}^{n}(\mathcal{S}_{kj}^i\mathbb{G}_{i1} - \mathcal{S}_{k1}^i\mathbb{G}_{ij})\lambda_1^j = 0, \tag{6.4}$$

where $k \in \{1,\dots,n\}$. Thus, if $\lambda_2^j = 0$ for $j \in \{2,\dots,n\}$, $\lambda(q)$ is a solution to the λ-equation. Note that we further require that $\lambda(q) \circ \mathbb{G}^\sharp(q)$ is symmetric. In the following, we try to describe the space of such solutions of the λ-equation in an algebraic fashion.

Let V be an n-dimensional $\mathbb{R}$-vector space and let $\mathsf{G} \in \mathsf{S}_2\mathsf{V}$ be a nondegenerate symmetric tensor. Let $\Phi_\mathsf{G} : \mathsf{V}^* \otimes \mathsf{V} \to \Lambda_2\mathsf{V}$ be the map defined by

$$\Phi_\mathsf{G}(\mathsf{A})(v_1, v_2) = \mathsf{A} \circ \mathsf{G}(v_1, v_2) - \mathsf{A} \circ \mathsf{G}(v_2, v_1),$$

where $v_1, v_2 \in \mathsf{V}$. The space of all tensors, $\mathsf{A} \in \mathsf{V}^* \otimes \mathsf{V}$, such that $\mathsf{A} \circ \mathsf{G}$ is symmetric belongs to the kernel of Φ_G and thus is of dimension $\frac{n(n+1)}{2}$, we denote this subspace by S_G. Let $\{e_1,\dots,e_n\}$ be a basis for V and let $\{e^1,\dots,e^n\}$ be its dual. Let $\mathsf{W} \subset \mathsf{V}^*$ be the vector subspace generated by $\{e^2,\dots,e^n\}$ and denote its complement by E. We denote by $\tilde{\mathsf{S}}$ the space of all $\mathsf{A} \in \mathsf{V}^* \otimes \mathsf{V}$ such that, if $v \in \operatorname{coann}(\mathsf{W})$, then $\mathsf{A}(v) \in \operatorname{coann}(\mathsf{W})$, for all $v \in \mathsf{V}$. A tensor $\mathsf{A} \in \tilde{\mathsf{S}}$ can be written as

$$\mathsf{A} = \mathsf{A}_1^1 e^1 \otimes e_1 + \sum_{i=2}^{n}\sum_{j=1}^{n}\mathsf{A}_i^j e^i \otimes e_j,$$

where $\mathsf{A}_1^1 \in \mathbb{R}$ and $\mathsf{A}_i^j \in \mathbb{R}$ for $i \in \{2,\dots,n\}$ and $j \in \{1,\dots,n\}$. Thus the dimension of $\tilde{\mathsf{S}}$ is $n(n-1)+1$. If we denote the restriction of the map Φ_G to $\tilde{\mathsf{S}}$ by $\Phi_\mathsf{G}|_{\tilde{\mathsf{S}}} : \tilde{\mathsf{S}} \to \Lambda_2\mathsf{V}$, then $\ker(\Phi_\mathsf{G}|_{\tilde{\mathsf{S}}})$ is of dimension $\frac{n(n-1)}{2}+1$. If we additionally require that $\mathsf{A} \in \ker(\Phi_\mathsf{G}|_{\tilde{\mathsf{S}}})$ be nondegenerate, we obtain a $\frac{n(n-1)}{2}$-dimensional subspace of $\mathsf{V}^* \otimes \mathsf{V}$.

Let Q be an n-dimensional analytic manifold and $\Sigma_{\text{ol}} = (\mathsf{Q}, \mathbb{G}, V_{\text{ol}}, \mathcal{W}_{\text{ol}})$ be an open-loop simple mechanical control system with one degree of underactuation. Let U be a neighborhood of the equilibrium point $q_0 \in \mathsf{Q}$ and let $(q^1,\dots,q^n)$ be local coordinates on U such that $\mathcal{W}_{\text{ol}}|_q = \operatorname{span}\{dq^2,\dots,dq^n\}$, where $q \in U$. In following, we define a subspace of $\mathsf{T}_q^*\mathsf{Q} \otimes \mathsf{T}_q\mathsf{Q}$ which is large enough for stabilization of systems with one degree of underactuation. Consider the space of solutions to the λ-equation that in local coordinates look like $\lambda(q) = \lambda_i^j dq^i \otimes \frac{\partial}{\partial q^j} \in \mathsf{T}_q^*\mathsf{Q} \otimes \mathsf{T}_q\mathsf{Q}$, where $\lambda_i^j \in \mathbb{R}$ and $q \in U$, and satisfies the followings

1. $\lambda(q) \circ \mathbb{G}^\sharp(q)$ is symmetric and nondegenerate;
2. if $v \in \operatorname{coann}(\operatorname{span}\{dq^1\})$ then $\lambda(v) \in \operatorname{coann}(\operatorname{span}\{dq^1\})$ for all $v \in \mathsf{T}_q\mathsf{Q}$.

We denote this subspace by $\mathscr{S}$. The following proposition is a corollary of the algebraic discussion above.

6.2 Proposition: *$\mathscr{S}$ is an $\frac{n(n-1)}{2}$-dimensional subspace of $\mathsf{T}_q^*\mathsf{Q} \otimes \mathsf{T}_q\mathsf{Q}$.*

We wish to show that the space of solutions of the λ-equation, described in Proposition 6.2, is large enough to guarantee that $\mathbb{G}_{\text{cl}}^\sharp(q_0)\mathsf{Hess}^\flat(V_{\text{cl}})(q_0)$ can be made diagonalizable and with positive real eigenvalues. If $\lambda(q) \in \mathscr{S}$, then Equation (6.1) gives

$$\mathsf{Hess}^\flat(V_{\mathrm{cl}})(q_0)(\frac{\partial}{\partial q^1}, \frac{\partial}{\partial q^j}) = \frac{1}{\lambda_1^1}\mathsf{Hess}^\flat(V_{\mathrm{ol}})(q_0)(\frac{\partial}{\partial q^1}, \frac{\partial}{\partial q^j}), \tag{6.5}$$

$$\mathbb{G}^\sharp_{\mathrm{cl}}(q_0)(dq^1, dq^j) = \lambda_1^1\mathbb{G}^\sharp(q_0)(dq^1, dq^j), \tag{6.6}$$

where $j \in \{1, \dots, n\}$. As a result, we have the following proposition.

6.3 Proposition: *Let Q be an n-dimensional analytic manifold and $\Sigma_{\mathrm{ol}} = (\mathsf{Q}, \mathbb{G}, V_{\mathrm{ol}}, \mathcal{W}_{\mathrm{ol}})$ be an open-loop simple mechanical control system with one degree of underactuation. Let U be a neighborhood of the equilibrium point $q_0 \in \mathsf{Q}$ and let $(q^1, \dots, q^n)$ be local coordinates on U such that $\mathcal{W}_{\mathrm{ol}}|_q = \mathrm{span}\{dq^2, \dots, dq^n\}$, where $q \in U$. Suppose that*

$$A = \mathbb{G}^\sharp(q_0)\mathsf{Hess}^\flat(V_{\mathrm{ol}})(q_0) + \mathbb{G}^\sharp(q_0)(u|_{q_0})$$

is diagonalizable with real eigenvalues, where $u|_{q_0} : \mathsf{T}_{q_0}\mathsf{Q} \to \mathcal{W}_{\mathrm{ol}}|_{q_0}$. Then there exists a closed-loop metric $\mathbb{G}_{\mathrm{cl}}$ and a potential function V_{cl} such that

1. *$\mathbb{G}^\flat = \mathbb{G}^\flat_{\mathrm{cl}} \circ \lambda$, where $\lambda \in \mathscr{S}$,*
2. *$\mathbb{G}^\sharp_{\mathrm{cl}}(q_0)\mathsf{Hess}^\flat(V_{\mathrm{cl}})(q_0) = A$.*

Proof: We only need to show that if 1 holds, then $\mathbb{G}_{\mathrm{cl}}$ and V_{cl} can be selected so that 2 holds. Using Equations (6.5) and (6.6), we can write $\mathbb{G}^\sharp_{\mathrm{cl}}(q_0)$ in coordinates as

$$\begin{pmatrix} \lambda_1^1 a & \lambda_1^1 \mathsf{B} \\ \lambda_1^1 \mathsf{B}^T & \mathsf{C} \end{pmatrix},$$

where $a \in \mathbb{R}$, $\mathsf{B} \in \mathrm{L}(\mathbb{R}^{n-1}, \mathbb{R})$, and $\mathsf{C} \in \mathsf{S}_2\mathbb{R}^{n-1}$ are such that $a = \mathbb{G}^\sharp(dq^1, dq^1)$ and $\mathsf{B}(dq^1, dq^j) = \mathbb{G}^\sharp(dq^1, dq^j)$ for all $j \in \{2, \dots, n\}$. Similarly, $\mathsf{Hess}^\flat(V_{\mathrm{cl}})(q_0)$ can be written as

$$\begin{pmatrix} \frac{1}{\lambda_1^1} k & \frac{1}{\lambda_1^1}\mathcal{B} \\ \frac{1}{\lambda_1^1}\mathcal{B}^T & \mathcal{C} \end{pmatrix},$$

where $k \in \mathbb{R}$, $\mathcal{B} \in \mathrm{L}(\mathbb{R}^{n-1}, \mathbb{R})$, and $\mathcal{C} \in \mathsf{S}_2\mathbb{R}^{n-1}$ are such that $k = \mathsf{Hess}^\flat(V_{\mathrm{ol}})(q_0)(\frac{\partial}{\partial q^1}, \frac{\partial}{\partial q^1})$ and $\mathcal{B}(\frac{\partial}{\partial q^1}, \frac{\partial}{\partial q^j}) = \mathsf{Hess}^\flat(V_{\mathrm{ol}})(q_0)(\frac{\partial}{\partial q^1}, \frac{\partial}{\partial q^j})$ for all $j \in \{2, \dots, n\}$. Thus we have

$$\mathbb{G}^\sharp_{\mathrm{cl}}(q_0)\mathsf{Hess}^\flat(V_{\mathrm{cl}})(q_0) = \mathbb{G}^\sharp_{\mathrm{ol}}(q_0)\mathsf{Hess}^\flat(V_{\mathrm{01}})(q_0) + \begin{pmatrix} 0 & 0 \\ L_1 & L_2 \end{pmatrix},$$

where

1. $L_1 = k\mathsf{B}^T + \frac{1}{\lambda_1^1}\mathsf{C}\mathcal{B}^T \in \mathrm{L}(\mathbb{R}, \mathbb{R}^{n-1})$ and
2. $L_2 = \lambda_1^1\mathsf{B}\mathcal{B}^T + \mathsf{C}\mathcal{C}^T \in \mathrm{L}(\mathbb{R}^{n-1} \times \mathbb{R}^{n-1})$

can be set to any value by appropriate choice of C and $\mathcal{C}$. ■

6.4 Theorem: *Let $\Sigma_{\text{ol}} = (\mathsf{Q}, \mathbb{G}_{\text{ol}}, V_{\text{ol}}, \mathcal{W}_{\text{ol}})$ be a linearly controllable open-loop simple mechanical control system with one degree of underactuation and with $q_0 \in \mathsf{Q}$ an equilibrium point. Then the system is stabilizable at q_0 using an energy shaping feedback.*

Proof: The involutivity of the energy shaping partial differential equations ensures that formal solutions exist. If the system is linearly controllable, then one can design a control such that $\mathbb{G}^\sharp(q_0)\mathsf{Hess}^\flat(V_{\text{ol}})(q_0) + \mathbb{G}^\sharp(q_0)(u|_{q_0})$ is diagonalizable and positive-definite. Proposition 6.3 then guarantees that $\mathbb{G}_{\text{cl}}$ can be found such that it satisfies the kinetic energy shaping partial differential equations, by choosing $\lambda \in \mathscr{S}$, and taking

$$\mathbb{G}_{\text{cl}}^\sharp(q_0)\mathsf{Hess}^\flat(V_{\text{cl}})(q_0) = \mathbb{G}^\sharp(q_0)\mathsf{Hess}^\flat(V_{\text{ol}})(q_0) + \mathbb{G}^\sharp(q_0)(u|_{q_0})$$

to be diagonalizable with positive real eigenvalues. ■

Note that this proof does not require that the closed-loop metric be positive-definite and in fact, there are cases for which energy shaping is not possible with positive-definite closed-loop metrics; an example of this is presented in Example 6.6. The following proposition clarifies when it is necessary to perform kinetic energy shaping for systems with one degree of underactuation.

6.5 Proposition: *Let Q be an n-dimensional manifold and let $\Sigma_{\text{ol}} = (\mathsf{Q}, \mathbb{G}, V_{\text{ol}}, \mathcal{W}_{\text{ol}})$ be a linearly controllable simple mechanical system. Let $(q^1, \dots, q^n)$ be coordinates on a neighborhood U of $q_0 \in \mathsf{Q}$ such that $\mathcal{W}_{\text{ol}} = \text{span}\{dq^2, \dots, dq^n\}$. If $\mathsf{Hess}(V_{\text{ol}})(\frac{\partial}{\partial q^1}, \frac{\partial}{\partial q^1}) > 0$, the system can be stabilized around its equilibrium point q_0 without kinetic energy shaping.*

Proof: We shall show that Σ_{ol} is stabilizable using an energy shaping feedback with $\mathbb{G}_{\text{cl}} = \mathbb{G}$. Equation (6.1) then reads

$$\mathsf{Hess}^\flat(V_{\text{cl}})(q_0) = \mathsf{Hess}^\flat(V_{\text{ol}})(q_0) + u|_{q_0},$$

where u is a feedback. Note that since $\mathsf{Hess}(V_{\text{cl}})$ is symmetric, it is positive-definite if and only if all of its principal minors are positive. The first principal minor of $\mathsf{Hess}^\flat(V_{\text{cl}})$ is positive. Then, by linear controllability, one can choose the controls so that the system is stabilizable at the equilibrium point q_0, similar to Proposition 6.3. ■

Next, we present an example of energy shaping for simple mechanical systems with one degree of underactuation for which the energy shaping is possible *only* via a closed-loop metric that is not positive-definite.

6.6 Example: Consider the stabilization problem for a simple mechanical control system $\Sigma = (\mathbb{R}^2, \mathbb{G}, V_{\text{ol}}, 0, \mathsf{W}_{\text{ol}})$ at the origin $q_0 = \mathbf{0} \in \mathbb{R}^2$, where

1. $\mathbb{G} = ((q^2)^2 + 1)dq^1 \otimes dq^2 + ((q^1)^2 + 1)dq^2 \otimes dq^2$,
2. $V_{\text{ol}} = -(q^1)^2 + 2q^1q^2 + (q^2)^2$, and
3. $\mathcal{W}_{\text{ol}} = \{dq^2\}$.

This system is linearly controllable at the origin. We show that, for any solution of the λ-equation, the constant term in the Taylor expansion of λ_1^2 is always zero. In order to show this, we need to modify Equation (6.4) by adding an extra term, since λ, in a neighborhood of q_0, is not necessarily chosen from $\mathscr{S}$. We have

$$\sum_{i=1}^{n}(\mathbb{G}_{1i}\frac{\partial\lambda_1^i}{\partial q^k}+\sum_{j=2}^{n}(\mathcal{S}_{kj}^i\mathbb{G}_{i1}-\mathcal{S}_{k1}^i\mathbb{G}_{ij})\lambda_1^j)=0,$$

for all $k\in\{1,\dots,n\}$. For this example, by substituting the nonzero Christoffel symbols, we have

$$((q^2)^2+1)\frac{\partial\lambda_1^1}{\partial q^1}+2q^2\lambda_1^2=0, \tag{6.7}$$

$$((q^2)^2+1)\frac{\partial\lambda_1^1}{\partial q^2}-2q^1\lambda_1^2=0. \tag{6.8}$$

It is clear that $\lambda_1^1(q_0)$ can be chosen arbitrarily. Consider formal expressions for λ_1^2 and λ_1^1:

$$\begin{aligned}\lambda_1^1&=C_{00}+C_{10}q^1+C_{01}q^2+C_{20}(q^1)^2+C_{02}(q^2)^2+C_{11}q^1q^2+\dots,\\ \lambda_1^2&=D_{00}+D_{10}q^1+D_{01}q^2+D_{20}(q^1)^2+D_{02}(q^2)^2+D_{11}q^1q^2+\dots,\end{aligned}$$

where $C_{ij},D_{ij}\in\mathbb{R}$ for $i,j\in\mathbb{Z}_{\geq 0}$. If λ_1^1 and λ_1^2 satisfy Equations (6.7) and (6.8), then $C_{11}=D_{00}=0$, i.e., $\lambda_1^2(q_0)=0$. Thus the closed-loop metric at the origin has the form $\mathbb{G}_{\mathrm{cl}}(q_0)=\frac{1}{a}dq^1\otimes dq^1+\frac{1}{c}dq^2\otimes dq^2$, where $a,c\in\mathbb{R}\backslash\{0\}$ and $\lambda_1^1(q_0)=a$. Equation (6.5) implies that

$$\mathsf{Hess}^\flat(V_{\mathrm{cl}})(q_0)=\begin{pmatrix}\frac{-2}{a} & \frac{2}{a}\\ \frac{2}{a} & k\end{pmatrix},$$

where $k\in\mathbb{R}$. Thus

$$\mathbb{G}_{\mathrm{cl}}^\sharp(q_0)\mathsf{Hess}^\flat(V_{\mathrm{cl}})(q_0)=\begin{pmatrix}-2 & 2\\ \frac{2c}{a} & ck\end{pmatrix}.$$

From the discussion in Example 5.7, we have to choose $\frac{2c}{a}<0$ and $ck>2$ in order to make $\mathbb{G}_{\mathrm{cl}}^\sharp(q_0)\mathsf{Hess}^\flat(V_{\mathrm{cl}})(q_0)$ positive-definite, i.e., none of the achievable closed-loop metrics is positive-definite. However, one can choose $a,c,k\in\mathbb{R}$ so that $\mathbb{G}_{\mathrm{cl}}^\sharp(q_0)\mathsf{Hess}^\flat(V_{\mathrm{cl}})(q_0)$ is positive-definite, for example $a=-\frac{191}{100}$, $c=\frac{43}{10}$, and $k=1$.

6.7 Remark: If we take the open-loop metric given by

$$\mathbb{G}=((q^2)^2+1)dq^1\otimes dq^2+((q^1)^2+1)dq^2\otimes dq^2+2q^1q^2(dq^1\otimes dq^2+dq^2\otimes dq^1),$$

then $\lambda_1^2(q_0)$ need not be zero and the system can be shown to be stabilizable by the energy shaping method with a positive-definite closed-loop metric, similar to the linear system given in Example 5.7. This reveals that a slight change in the structure of the open-loop Levi-Civita connection has a huge impact on the achievable closed-loop metrics. In Section 7.2.1, we provide some observations that suggest a possible relationship between the holonomy group generated by the closed-loop metric and the one generated by the open-loop metric.

Chapter 7

Conclusions and future directions

7.1. Conclusions

In this thesis a geometric framework for stabilization of simple mechanical systems using the energy shaping method is developed. The geometric theory of partial differential equations has been used to show that the partial differential equations involved in the energy shaping method are integrable under a surjectivity condition. The geometric framework has been utilized to reveal the obstructions to the energy shaping method. This geometric approach has been used to obtain a geometric proof that linear controllability is sufficient for energy shaping for linear simple mechanical systems. Furthermore, the problem of stabilization of systems with one degree of underactuation is completely resolved. This approach gives some new insights for answering key questions in energy shaping that have not been addressed in the existing literature. Some of these new open problems are outlined in the next section.

7.2. Future directions

Understanding the geometry of the kinetic energy shaping problem and the interaction of gyroscopic forces in the dynamics of kinetic energy shaping should be the main focus of the future directions of this thesis. Furthermore, an algebraic formulation of the positive-definiteness of the Hessian of the closed-loop potential function is another open problem in energy shaping. In following, we describe some of these future directions in more detail.

7.2.1. Kinetic energy shaping via holonomy groups. The most interesting open problem in the energy shaping method is the characterization of closed-loop kinetic energy shaping metrics. A deep understanding of the integrability conditions of kinetic energy shaping partial differential equations is far from being achieved. Here, we propose a possible direction for investigating this problem. The main motivation for the approach is a theorem on holonomy groups by Schmidt [40]. Given a metric, the theorem provides the conditions for a connection on a connected manifold to be a metric connection. As the kinetic energy shaping partial differential equations is, in fact, the metric-connection problem restricted to a distribution, one might hope to rephrase the kinetic energy shaping problem in terms of the holonomy groups. In the following we review the result of [40].

Let Q be a connected manifold equipped with an affine connection $\mathcal{S}$. Then, for any path τ between two points of the manifold, parallel transport along τ defines a linear map between the tangent spaces at the two points which we denote by $L(\tau)$; here the tangent spaces are regarded as vector spaces and the parallel transport is a linear isomorphism of vector spaces [26]. This linear map is an isometry if the connection is a Levi-Civita connection.

7.1 Definition: Let $p \in \mathsf{Q}$, where Q is a connected manifold. Denote by $C(p)$ the loop space at pq, i.e., all the continuous closed curves starting and ending at p. We call the set of all linear transformations in the tangent space at p defined by the parallel transport along elements of $C(p)q$ the *holonomy group at* p and we denote it by $\Phi(p)$.

It can be shown easily that for connected manifolds the holonomy groups at different points are isomorphic [26]. A connection can only be the Levi-Civita connection of a metric $\mathbb{G}$ on Q if the holonomy group is a subgroup of the orthogonal group corresponding to the metric $\mathbb{G}$ (metric preserving and parallel preserving properties). The following theorem of [40] shows that the converse is true.

7.2 Theorem: *Let $\mathcal{S}$ be a torsion free connection on a connected manifold Q whose holonomy keeps a metric $\mathbb{G}$ invariant. Then $\mathcal{S}$ is the Levi-Civita connection of a metric which has the same signature as g.*

7.3 Remark: Let Q be a simply connected manifold. Then $\Phi(p)$ is a *connected* Lie subgroup of the group of linear transformations in the tangent space at p. Therefore, $\Phi(p)$ is uniquely determined by its Lie algebra $\mathfrak{g}(p)$. So the metric $\mathbb{G}_p(X, Y)$ is invariant under $\Phi(p)$ if and only if

$$\mathbb{G}(A(X), Y) + \mathbb{G}(X, A(Y)) = 0 \tag{7.1}$$

for all $A \in \mathfrak{g}(p)$. Since the elements of $\mathfrak{g}(p)$ for a simply connected analytic manifold are generated by the curvature and its covariant derivatives [26], this in fact includes the local integrability result of [18]. Although the holonomy group $\Phi(p)$ is not necessarily connected, one can obtain a similar result on a universal covering of Q and so the result can be extended to the case that the manifold is *not* simply connected; see [40].

Key question: *Is there a relationship between the holonomy group of the Levi-Civita connection associated to $\mathbb{G}_{\mathrm{ol}}$ and that of $\mathbb{G}_{\mathrm{cl}}$?*
We observe that this gives rise to the metric-connection problem restricted to a distribution. A proper answer to this key question may resolve the mystery behind the kinetic energy shaping process and, furthermore, may give a global proof for the sufficient conditions of energy shaping.

7.2.2. Gyroscopic forces. It is well-known that the presence of gyroscopic forces can enlarge the space of solutions of the kinetic energy shaping partial differential equations. Although the integrability results of this thesis have been obtained in the absence gyroscopic forces, the involutivity results hold even in the presence of gyroscopic forces. These computations are omitted from the thesis. One can ask the following key questions.

1. *Describe the integrability conditions of the kinetic energy shaping problem in the presence of gyroscopic forces in an algebro-geometric fashion.*

2. *Extend the results of Example 4.25 to a system for which there exists no energy shaping feedback even at the presence of gyroscopic forces.*

7.2.3. Stabilization condition via the Farkas lemma. After possibly solving the energy shaping partial differential equations, one has to check if the Hessian of the closed-loop potential function at the equilibrium point is positive-definite. In this section, we provide a possible algebro-geometric framework for this positive-definiteness condition using the Farkas lemma [29]. The type of obstruction the stabilization condition puts on the set of achieved energy shaping feedbacks has not yet been characterized in an algebro-geometric fashion. We show that one can possibly investigate this question using tools from convex analysis [4, 49, 12].

We denote the cone of positive-semidefinite $\mathbb{R}$-bilinear maps on V^* by $\mathsf{S}_2^{\succeq 0}\mathsf{V}$; that is

$$\mathsf{S}_2^{\succeq 0}\mathsf{V} = \{A \in \mathsf{V} \otimes \mathsf{V} \mid A(\alpha, \beta) \geq 0, \quad \forall \alpha, \beta \in \mathsf{V}^*\}.$$

We use $\succeq$ as a partial order on the positive semidefinite cone: $A \succeq B$ if $A - B$ is positive semidefinite.

Some properties of $\mathsf{S}_2^{\succeq 0}\mathsf{V}$. We start this section by showing that the set of all positive semidefinite symmetric $\mathbb{R}$-bilinear maps forms a convex cone in $\mathsf{S}_2\mathsf{V}$.

7.4 Lemma: *The set of all symmetric positive semidefinite symmetric $\mathbb{R}$-bilinear maps $\mathsf{S}_2^{\succeq 0}\mathsf{V}$ is a convex cone.*

Proof: Let $A_1, A_2 \in \mathsf{S}_2^{\succeq 0}\mathsf{V}$. This implies that $A_1(x, x), A_2(x, x) \geq 0$ for all $x \in \mathsf{V}^*$. As a result,

$$c_1 A_1(x, x) + c_2 A_2(x, x) \geq 0, \qquad c_1, c_2 \in \mathbb{R}_{\geq 0},$$

which implies $c_1 A_1 + c_2 A_2 \succeq 0$ for all $c_2, c_2 \in \mathbb{R}_{\geq 0}$ as required. ■

Note that the interior of $\mathsf{S}_2^{\succ 0}\mathsf{V}$ consists of all positive-definite symmetric $\mathbb{R}$-bilinear maps. The dual cone for the positive-semidefinite cone is defined by

$$\mathsf{S}_2^{\succeq 0}\mathsf{V}^* = \{Y \in \mathsf{S}_2\mathsf{V}^* \mid \forall A \in \mathsf{S}_2^{\succeq 0}\mathsf{V}, \quad A(Y) \geq 0\}.$$

The following theorem is a corollary of the so-called Hadamard product theorem and gives an important characterization of the positive-semidefinite symmetric $\mathbb{R}$-bilinear maps [49].

7.5 Theorem: **(Fejer)** $A \in \mathsf{S}_2^{\succeq 0}\mathsf{V}$ *if and only if* $A(Y) \geq 0$ *for all* $Y \in \mathsf{S}_2^{\succeq 0}\mathsf{V}^*$*.*

Algebraic extended Farkas lemma. It is well-known that it is not possible to generalize the classical Farkas lemma [8] to nonpolyhedral cones due to the closedness assumption in the Farkas lemma; see [29]. In this section, we reformulate the closedness sufficient condition of [4] in an algebraic fashion. Let V be an n-dimensional $\mathbb{R}$-vector space and let W be an m-dimensional $\mathbb{R}$-vector space. Suppose that we have a linear map

$$\mathcal{A} : \mathsf{S}_2\mathsf{V} \to \mathsf{W},$$

and define $\mathsf{K} \doteq \mathrm{Im}(\mathcal{A})|_{\mathsf{S}_2^{\succeq 0}\mathsf{V}}$. The following lemma provides a *Slater-type constraint qualification* that enables us to extend the Farkas lemma [8] for the positive-semidefinite cone [29], for proof see [4].

7.6 Lemma: (Algebraic version of closedness condition) *If* $\mathcal{A}(\alpha) \in \mathsf{S}_2^{\succeq 0}\mathsf{V}^*$ *for some* $\alpha \in \mathsf{W}^*$ *then* K *is closed in* W *with the topology induced from* $\mathbb{R}^m$.

Now we can state an algebraic version of the so-called extended Farkas lemma.

7.7 Lemma: (Algebraic version of extended Farkas lemma) *Let* V *and* W *be, respectively,* n *and* m*-dimensional* $\mathbb{R}$*-vector spaces. Let* $\mathcal{A} \in \mathsf{W} \otimes \mathsf{S}_2\mathsf{V}^*$ *and* $b \in \mathsf{W}$. *Assume that* $\mathcal{A}(\alpha) \in \mathsf{S}_2^{\succeq 0}\mathsf{V}^*$ *for some* $\alpha \in \mathsf{W}^*$. *Then the following statements are equivalent:*

1. *there exists* $X \in \mathsf{S}_2^{\succeq 0}\mathsf{V}$ *such that* $\mathcal{A}(X) = b$;
2. $b(\alpha) \geq 0$ *for all* α *for which* $\mathcal{A}(\alpha) \in \mathsf{S}_2^{\succeq 0}\mathsf{V}^*$.

Proof: Suppose that (1) is true. Then, by the Fejer Theorem, Theorem 7.5,

$$b(\alpha) = \mathcal{A}(X)(\alpha) = \mathcal{A}(\alpha)(X) \geq 0.$$

Conversely, suppose that there exists no $X \in \mathsf{S}_2^{\succeq 0}\mathsf{V}$ such that $\mathcal{A}(X) = b$, i.e., b is not in K. Since the cone K is closed by Lemma 7.6, there exists a hyperplane that separates b and K (see so-called separation theorem [37]), i.e., there exists some $\alpha \in W^*$ such that $b(\alpha) < 0$ and $\mathcal{A}(X)(\alpha) \geq 0$ for all $X \in \mathsf{S}_2^{\succeq 0}\mathsf{V}$ which, by the virtue of Fejer Theorem, means that $\mathcal{A}(\alpha) \in \mathsf{S}_2^{\succeq 0}\mathsf{V}^*$. ■

Stabilization condition. Suppose that we want to stabilize $\Sigma_{\rm ol} = (\mathsf{Q}, \mathbb{G}_{\rm ol}, V_{\rm ol}, 0, \mathcal{W}_{\rm ol})$ at an equilibrium point $q_0 \in \mathsf{Q}$. If there exists a $\lambda \in \Gamma^\omega(\mathsf{T}^*\mathsf{Q} \otimes \mathsf{TQ})$ that satisfies the kinetic energy shaping partial differential equations of Theorem 3.3 and yields a positive-definite closed-loop metric on Q, then one needs to check whether there exists a closed-loop potential function that satisfies the potential energy shaping partial differential equations. One can take local coordinates around q_0 in which λ and the closed-loop potential function satisfy the following partial differential equation:

$$\lambda_a^i \frac{\partial V_{\rm cl}}{\partial q^i} - \frac{\partial V_{\rm ol}}{\partial q^a} = 0, \tag{7.2}$$

where $i \in \{1, \ldots, n\}$ and $a \in \{1, \ldots, m\}$. Moreover, we require that the Hessian of $V_{\rm cl}$ at q_0 be positive-definite. Prolonging the above equation and evaluating at q_0 we have

$$\lambda_a^i(q_0) \frac{\partial^2 V_{\rm cl}}{\partial q^k \partial q^i}(q_0) - \frac{\partial^2 V_{\rm ol}}{\partial q^k \partial q^a}(q_0) = 0, \tag{7.3}$$

since $\mathsf{d}V_{\rm ol}(q_0) = \mathsf{d}V_{\rm cl}(q_0) = 0$. In the following, we use Lemma 7.7 to investigate the possibility of having a positive-definite solution to Equation (7.3).

Suppose that λ is a solution to the kinetic energy shaping partial differential equations. One can observe that positive-definiteness of the Hessian of $V_{\rm cl}$ at q_0, along with Equation (7.3), is equivalent to the intersection of the cone of positive-definite symmetric $\mathbb{R}$-bilinear maps on $\mathsf{T}_{q_0}^*\mathsf{Q}$ with the affine subspace prescribed by Equation (7.3) being nonempty. Thus a version of the extended Farkas Lemma 7.7 might be suitable for determining if the intersection is nonempty. The challenge, however, is deriving an appropriate version of Lemma 7.6 that allows us to use the extended Farkas lemma. In fact, one can extend Equation (7.3) in such a way that it satisfies the closedness condition of Lemma 7.6. Thus a necessary condition for existence of a positive-definite Hessian that satisfies Equation (7.3) can be achieved. But it is not clear whether such a result is sufficient or not.

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