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Geometric interpretations of the symmetric product in affine differential geometry

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Abstract

The symmetric product of vector fields on a manifold arises when one studies the controllability of certain classes of mechanical control systems. A geometric description of the symmetric product is provided using parallel transport, along the lines of the flow interpretation of the Lie bracket. This geometric interpretation of the symmetric product is used to provide an intrinsic proof of the fact that the distributions closed under the symmetric product are exactly those distributions invariant under the geodesic flow.

Keywords. affine differential geometry, symmetric product, geodesic invariance

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1. Introduction

Given an affine connection ∇ on a manifold M, the corresponding **symmetric product** is simply given by

$$\langle X : Y\rangle = \nabla_X Y + \nabla_Y X.$$

The symmetric product for Levi-Civita connections appeared for first time in [Crouch 1981] in the study of gradient systems. This product appeared again in [Lewis and Murray 1997] where it was used to characterize the controllability of a large class of mechanical control systems. Since then, the symmetric product has been widely used to solve control theoretic problems for mechanical systems, such as motion planning [Bullo, Leonard, and Lewis 2000, Kobilarov and Marsden 2011], trackability [Barbero-Liñán and Sigalotti 2010, Bullo and Lewis 2004], and so on. We refer to [Bullo and Lewis 2004] as a general reference for control theory for mechanical systems.

The symmetric product has an interesting interpretation similar to that for the Lie bracket as it relates to integrable distributions. Let us recall the result from [Lewis 1998]. We say that a distribution $\mathcal{D}$ on M is **geodesically invariant** under an affine connection

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∇ on M if, as a submanifold of TM, $\mathcal{D}$ is invariant under the geodesic spray associated with ∇. One can then show that a distribution is geodesically invariant if and only if the symmetric product of any $\mathcal{D}$-valued vector fields is again a $\mathcal{D}$-valued vector field. We provide an intrinsic proof of this result in Section 4.

Now, for the Lie bracket, one has the well-known formula

$$[X, Y](x) = \frac{1}{2} \frac{\mathrm{d}^2}{\mathrm{d}t^2}\bigg|_{t=0} \Phi^Y_{-t} \circ \Phi^X_{-t} \circ \Phi^Y_t \circ \Phi^X_t(x), \tag{1.1}$$

where Φ^X_t denotes the flow of X [Abraham, Marsden, and Ratiu 1988, Proposition 4.2.34]. In this paper we provide for the first time a similar formula for the symmetric product, using parallel transport. This is a novel interpretation. Moreover, we use our interpretation of the symmetric product to provide a coordinate-free proof of the theorem on geodesic invariance mentioned in the preceding paragraph. The original proof in [Lewis 1998] uses coordinates, and we refer to [Bhand 2007, Bhand 2010] for an intrinsic proof using the bundle of linear frames.

Let us provide an outline of the paper. In Section 2 we provide our differential geometric notation and recall some facts that we shall use in the paper. One of the features of the paper is that it makes essential and novel use of the Baker–Campbell–Hausdorff formula and we review this in Section 2.1. In Section 3 we give various infinitesimal descriptions of the symmetric product, see Theorem 3.2. In Section 4 we use our infinitesimal descriptions of the symmetric product to prove the geodesic invariance theorem [Lewis 1998] mentioned above. One of the contributions of the paper is to give only intrinsic, coordinate-free characterizations and proofs, and as a result there are many calculations in the paper that may be of independent interest. In particular, as mentioned above, we make use of the Baker–Campbell–Hausdorff formula in a novel way in a few places.

2. Notation, background, and preliminary constructions

In this section we recall the basic facts about affine connections and tangent bundles that will be important for us. Some of our constructions are presented in detail since we give—for the first time as far as we are aware—some intrinsic definitions and proofs that are well-known using coordinates.

Here is the notation we shall use in the paper. By Id_S we denote the identity map of a set S. By $\mathbb{Z}_{\geq 0}$ and $\mathbb{R}$ we denote the set of nonnegative integers and real numbers, respectively. For the most part, we shall adopt the differential geometric conventions of [Abraham, Marsden, and Ratiu 1988]. We shall assume all manifolds are paracompact, Hausdorff, and of class $\mathcal{C}^\infty$. All maps and geometric objects will be assumed to be of class $\mathcal{C}^\infty$, and we shall frequently use the word "smooth" to mean of class $\mathcal{C}^\infty$. The set of smooth functions on a manifold M is denoted by $\mathcal{C}^\infty(M)$. For a manifold M, its tangent bundle will be denoted by $\tau_M \colon TM \to M$. If $f \colon M \to N$ is a map, its derivative is denoted by $Tf \colon TM \to TN$, and $T_x f$ denotes the restriction of f to the tangent space $T_x M$. The flow of a vector field X is denoted by Φ^X_t, i.e., the integral curve of X through x is $t \mapsto \Phi^X_t(x)$. We shall suppose that all vector fields are complete, and leave to the reader the task of modifying proofs to account for the case where flows are defined on subintervals of $\mathbb{R}$. If $\pi \colon E \to M$ is a vector bundle over M, we denote by $\Gamma^\infty(E)$ the set of smooth sections of E. Sometimes it will be convenient to denote the zero vector in the fiber E_x as 0_x. If $X \in \Gamma^\infty(TM)$ is a vector field and if $\Phi\colon M \to M$ is a diffeomorphism, the pull-back of X by Φ is given by

$$\Phi^* X = T\Phi^{-1} \circ X \circ \Phi.$$

For a vector field $X \in \Gamma^\infty(TM)$ and for a function $f \in \mathcal{C}^\infty(M)$, we denote by $\mathcal{L}_X f$ the Lie derivative of f with respect to f.

2.1. The Baker–Campbell–Hausdorff formula. One of the features of our presentation is that we use the Baker–Campbell–Hausdorff (BCH) formula, as enunciated in [Strichartz 1987], to evaluate compositions of flows in a crucial way in a few places. In this section we quickly review this formula.

The BCH formula provides a formula for the "product of exponentials" in a Lie algebra in terms of brackets of the quantities being exponentiated. First we recall the formal version of the formula, following [Serre 1992]. Consider a finite set $\boldsymbol{\xi} = \{\xi_1, \dots, \xi_p\}$ of indeterminates and let $\hat{A}(\boldsymbol{\xi})$ be the $\mathbb{R}$-algebra of formal power series in these indeterminates. To be clear about this, let $V(\boldsymbol{\xi})$ be the free $\mathbb{R}$-vector space generated by $\boldsymbol{\xi}$. Thus an element $\zeta \in V(\boldsymbol{\xi})$ is a map $\zeta\colon \boldsymbol{\xi} \to \mathbb{R}$, and the set of such maps is equipped with the pointwise operations of addition and scalar multiplication. For $k \in \mathbb{Z}_{\geq 0}$, let $T^k(V(\boldsymbol{\xi}))$ be the kth tensor power of $V(\boldsymbol{\xi})$. Then $\hat{A}(\boldsymbol{\xi}) = \prod_{k\in\mathbb{Z}_{\geq 0}} T^k(V(\boldsymbol{\xi}))$ is the direct product. Thus an element of $\hat{A}(\boldsymbol{\xi})$ is a map

$$\alpha\colon \mathbb{Z}_{\geq 0} \to \cup_{k\in\mathbb{Z}_{\geq 0}} T^k(V(\boldsymbol{\xi}))$$

such that $\alpha(k) \in T^k(V(\boldsymbol{\xi}))$. The $\mathbb{R}$-vector space $\hat{A}(\boldsymbol{\xi})$ is an algebra with the tensor product as the product. This algebra then has the natural Lie algebra structure given by commutation: $[\alpha, \beta] = \alpha\beta - \beta\alpha$. By $\hat{L}(\boldsymbol{\xi})$ we denote the Lie subalgebra of $\hat{A}(\boldsymbol{\xi})$ generated by the indeterminates $\{\xi_1, \dots, \xi_k\}$. Thus, formally, elements of $\hat{L}(\boldsymbol{\xi})$ are $\mathbb{R}$-linear combinations of Lie brackets of the indeterminates. Let $L(\boldsymbol{\xi})$ be the Lie subalgebra of $\hat{L}(\boldsymbol{\xi})$ having components in only finitely many $T^k(V(\boldsymbol{\xi}))$, i.e., the free Lie algebra generated by the indeterminates $\boldsymbol{\xi}$. One can then define a map $\exp\colon \hat{L}(\boldsymbol{\xi}) \to \hat{A}(\boldsymbol{\xi})$ by the usual formal series expression:

$$\exp(\alpha) = \sum_{k=0}^{\infty} \frac{\alpha^k}{k!}.$$

The formal Baker–Campbell–Hausdorff formula is then the unique map

$$\mathrm{BCH}\colon \underbrace{\hat{L}(\boldsymbol{\xi}) \times \cdots \times \hat{L}(\boldsymbol{\xi})}_{k \text{ copies}} \to \hat{L}(\boldsymbol{\xi})$$

satisfying

$$\exp(\alpha_1) \cdots \exp(\alpha_k) = \exp(\mathrm{BCH}(\alpha_1, \dots, \alpha_k)).$$

The component of $\mathrm{BCH}(\alpha_1, \dots, \alpha_k)$ in $T^m(V(\boldsymbol{\xi}))$ we denote by $\mathrm{BCH}_m(\alpha_1, \dots, \alpha_k)$, and we note that

$$\begin{aligned}
\mathrm{BCH}_1(\alpha_1, \dots, \alpha_k) &= \alpha_1 + \cdots + \alpha_k, \\
\mathrm{BCH}_2(\alpha_1, \dots, \alpha_k) &= \frac{1}{2} \sum_{\substack{a,b\in\{1,\dots,k\} \\ a<b}} [\alpha_a, \alpha_b].
\end{aligned} \tag{2.1}$$

Now let us recall what can be said about the BCH formula where the indeterminates are vector fields $X_1, \dots, X_k$ on a manifold M. The vector fields $X_1, \dots, X_k$ define a map $\phi\colon \{\xi_1, \dots, \xi_k\} \to \Gamma^\infty(TM)$ by $\phi(\xi_j) = X_j$, $j \in \{1, \dots, k\}$. Since $\Gamma^\infty(TM)$ is a Lie algebra, there exists a unique extension, which we also denote by ϕ, from $L(\boldsymbol{\xi})$ to $\Gamma^\infty(TM)$. It is not generally the case that the infinite formal series defining BCH converges (it does in the real analytic case) in any reasonable topology on $\Gamma^\infty(TM)$, but in [Strichartz 1987] there are useful asymptotic formulae. For our purposes, these amount to the following. For each $m \in \mathbb{Z}_{\geq 0}$

$$\Phi^{X_k}_{t_k} \cdots \Phi^{X_1}_{t_1}(x) = \sum_{j=1}^{m} \Phi_1^{\phi(\mathrm{BCH}_j(t_1 X_1, \dots, t_k X_k))}(x) + O((|t_1| + \dots + |t_k|)^{m+1}) \tag{2.2}$$

(here and subsequently, for brevity we denote composition of flows with juxtaposition). It is this formula that we shall use below.

2.2. Tangent bundle geometry. In this section we review some well-known constructions concerning tangent bundles.

We recall the definition of the vertical lift, which we regard as a vector bundle map $\mathrm{vlft}\colon TM \oplus_M TM \to TTM$ as follows. Let $x \in M$ and let $v_x, w_x \in T_x M$. The **vertical lift** of u_x to v_x is given by

$$\mathrm{vlft}(v_x, u_x) = \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} (v_x + t u_x). \tag{2.3}$$

One easily verifies that the following diagram commutes:

$$\begin{tikzcd} TM \oplus_M TM \arrow[rr, "\mathrm{vlft}"] \arrow[dr, "\tau_M \oplus \tau_M"'] & & TTM \arrow[dl, "T\tau_M \circ \tau_M"] \\ & M & \end{tikzcd}$$

The image of vlft is the **vertical subbundle** $VTM = \ker(T\tau_M)$. For a vector field X on M, the **vertical lift** of X is the vector field X^V on TM given by $X^V(v_x) = \mathrm{vlft}(v_x, X(x))$. It is evident that

$$\Phi_t^{X^V}(v_x) = v_x + tX(x). \tag{2.4}$$

In Section 2.3 we shall see that the double tangent bundle TTM has two natural vector bundle structures, one for the vector bundle $\tau_{TM}\colon TTM \to TM$ (called the primary vector bundle with the vector bundle operations denoted with a subscript "1") and one for $T\pi\colon TTM \to TM$ (called the secondary vector bundle with the vector bundle operations denoted with a subscript "2"). The vertical lift interacts with these two vector bundle structures differently in each component. Indeed, the following diagrams commute

$$\begin{tikzcd} TM \oplus_M TM \arrow[rr, "\mathrm{vlft}"] \arrow[dr, "\mathrm{pr}_1"'] & & TTM \arrow[dl, "\tau_{TM}"] \\ & TM & \end{tikzcd} \qquad \begin{tikzcd} TM \oplus_M TM \arrow[rr, "\mathrm{vlft}"] \arrow[dr, "\mathrm{pr}_2"'] & & TTM \arrow[dl, "T\tau_M"] \\ & TM & \end{tikzcd}$$

This means that

$$\begin{aligned} \mathrm{vlft}(v_1+v_2,u) &= \mathrm{vlft}(v_1,u) +_2 \mathrm{vlft}(v_2,u), \quad \mathrm{vlft}(av,u) = a\cdot_2 \mathrm{vlft}(v,u)\\ \mathrm{vlft}(v,u_1+u_2) &= \mathrm{vlft}(v,u_1) +_1 \mathrm{vlft}(v,u_2), \quad \mathrm{vlft}(v,au) = a\cdot_1 \mathrm{vlft}(v,u). \end{aligned} \tag{2.5}$$

We recall that, given $X \in \Gamma^\infty(TM)$, the **complete lift** of X is the vector field $X^C \in \Gamma^\infty(TTM)$ defined by

$$X^C(v_x) = \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} T_x\Phi_t^X(v_x).$$

Evidently,

$$\Phi_t^{X^C}(v_x) = T_x\Phi_t^X(v_x). \tag{2.6}$$

Let us determine another useful characterisation of the flow of the complete lift.

2.1 Lemma: *Let $X \in \Gamma^\infty(TM)$. Let $x_0 \in M$ and $v_0 \in T_{x_0}M$. Let $J \subseteq \mathbb{R}$ be an interval for which $0 \in \mathrm{int}(J)$ and let $\gamma\colon J \to M$ be a differentiable curve such that $\gamma'(0) = v_0$. Let $I \subseteq \mathbb{R}$ be an interval. Define $\sigma(s,t) = \Phi_t^X(\gamma(s))$ for $(s,t) \in J \times I$ and define a vector field $V_\sigma(t) = \frac{\mathrm{d}}{\mathrm{d}s}\big|_{s=0}\sigma(s,t)$ along the integral curve of X through x_0. Then the integral curve of X^C through v_0 is $t \mapsto V_\sigma(t)$.*

Proof: This is a simple computation:

$$V_\sigma(t) = \left.\frac{\mathrm{d}}{\mathrm{d}s}\right|_{s=0} \Phi_t^X(\gamma(s)) = T_{\gamma(0)}\Phi_t^X(\gamma'(0)) = T_{x_0}\Phi_t^X(v_0) = \Phi_t^{X^C}(v_0)$$

using (2.6). ■

A consequence of the lemma is that the flow of X^C is that of a linear vector field, and so, by definition of a linear vector field [Kolář, Michor, and Slovák 1993, §47.9], X^C is a vector bundle morphism according to the following diagram:

$$\begin{tikzcd} TM \arrow[r, "X^C"] \arrow[d, "\tau_M"'] & TTM \arrow[d, "T\pi"] \\ M \arrow[r, "X"'] & TM \end{tikzcd} \tag{2.7}$$

2.3. The double tangent bundle. In this section we review some of the structure of the double tangent bundle of a manifold. We shall make great use of some of the constructions in this section in our intrinsic constructions to follow. Parts of the intrinsic treatment we give of the canonical tangent bundle involution are, as far as we know, new.

We begin by recalling the two vector bundle structures for TTM, as we shall use both. The double tangent bundle is represented naturally as a vector bundle over $\tau_M\colon TM \to M$ in the following two ways:

$$\begin{tikzcd} TTM \arrow[r, "\tau_{TM}"] \arrow[d, "T\tau_M"'] & TM \arrow[d, "\tau_M"] \\ TM \arrow[r, "\tau_M"'] & M \end{tikzcd} \qquad \begin{tikzcd} TTM \arrow[r, "T\tau_M"] \arrow[d, "\tau_{TM}"'] & TM \arrow[d, "\tau_M"] \\ TM \arrow[r, "\tau_M"'] & M \end{tikzcd} \tag{2.8}$$

The vector bundle on the left we call the **primary vector bundle** and that on the right we call the **secondary vector bundle**. We shall need to introduce notation for the different vector bundle operations. If $u, v \in TTM$ satisfy $\tau_{TM}(u) = \tau_{TM}(v)$, then the sum of u and v and the scalar multiple of u by $a \in \mathbb{R}$ in the primary vector bundle are denoted by $u +_1 v$ and $a \cdot_1 u$, respectively. If $u, v \in TTM$ satisfy $T\tau_M(u) = T\tau_M(v)$, then the sum of u and v and the scalar multiple of u by $a \in \mathbb{R}$ in the secondary vector bundle are denoted by $u +_2 v$ and $a \cdot_2 u$, respectively. For the vector bundle $\tau_{TM} \colon TTM \to TM$, the vector bundle structure is the usual tangent bundle structure. We describe the vector bundle structure for $T\tau_M \colon TTM \to TM$ as follows. First note that the diagram

$$\begin{tikzcd} TM \arrow[r, "TX"] \arrow[d, "\tau_M"'] & TTM \arrow[d, "\tau_{TM}"] \\ M \arrow[r, "X"'] & TM \end{tikzcd}$$

commutes for a vector field X, giving TX as a vector bundle mapping over X. Thus the map $X \mapsto TX$ is a morphism of the secondary vector bundle structure. Now let $u, v \in TTM$ be such that $w \doteq T\tau_M(u) = T\tau_M(v)$. We consider two cases.

1. $w \neq 0$: Let $U, V \in \Gamma^\infty(TM)$ be such that $TU(w) = u$ and $TV(w) = v$. We then have
$$u +_2 v = T(U + V)(w), \quad a \cdot_2 u = T(aU)(w). \tag{2.9}$$
2. $w = 0$: In this case u and v are vertical. So we let $U, V \in \Gamma^\infty(TM)$ be such that $u = U^V \circ \tau_{TM}(u)$ and $v = V^V \circ \tau_{TM}(v)$. We then have
$$u +_2 v = (U + V)^V(\tau_{TM}(u) + \tau_{TM}(v)), \quad a \cdot_2 u = (aU)^V(a\tau_{TM}(u)). \tag{2.10}$$

We can say, motivated by (2.9), that the secondary vector bundle structure is the derivative of the vector bundle structure for $\tau_M \colon TM \to M$.

The diagrams (2.8) give a **double vector bundle** as introduced in [Pradines 1974], and studied subsequently by many authors; see [Mackenzie 2005, Chapter 9] for a general reference. A consequence of this structure is the following result that captures how the two vector bundle structures are related.

2.2 Lemma: *Let $u, v, w, z \in TTM$ satisfy*

$$T\tau_M(u) = T\tau_M(v), \quad T\tau_M(w) = T\tau_M(z), \quad \tau_{TM}(u) = \tau_{TM}(w), \quad \tau_{TM}(v) = \tau_{TM}(z)$$

and let $a, b \in \mathbb{R}$. Then the following statements hold:

1. $(u +_2 v) +_1 (w +_2 z) = (u +_1 w) +_2 (v +_1 z)$;
2. $a \cdot_1 (u +_2 v) = (a \cdot_1 u) +_2 (a \cdot_1 v)$;
3. $a \cdot_2 (u +_1 w) = (a \cdot_2 u) +_1 (a \cdot_2 w)$;
4. $a \cdot_1 (b \cdot_2 w) = b \cdot_2 (a \cdot_1 w)$.

To understand how the two vector bundle structures for TTM are related, we shall use a particular representation of points in TTM. Let ρ be a smooth map from a neighbourhood of $(0,0) \in \mathbb{R}^2$ to M. We shall use coordinates (s,t) for $\mathbb{R}^2$. For fixed s and t define $\rho_s(t) = \rho^t(s) = \rho(s,t)$, We then denote

$$\frac{\partial}{\partial t}\rho(s,t) = \frac{\mathrm{d}}{\mathrm{d}t}\rho_s(t) \in T_{\rho(s,t)}M, \quad \frac{\partial}{\partial s}\rho(s,t) = \frac{\mathrm{d}}{\mathrm{d}s}\rho^t(s) \in T_{\rho(s,t)}M.$$

Note that

$$s \mapsto \frac{\partial}{\partial t}\rho(s,t)$$

is a curve in TM for fixed t. The tangent vector field to this curve we denote by

$$s \mapsto \frac{\partial}{\partial s}\frac{\partial}{\partial t}\rho(s,t) \in T_{\frac{\partial}{\partial t}\rho(s,t)}TM.$$

We belabour the development of the notation somewhat since these partial derivatives are not the usual partial derivatives from calculus, although the notation might make one think they are. For example, we do not generally have equality of mixed partials, i.e., generally we have

$$\frac{\partial}{\partial s}\frac{\partial}{\partial t}\rho(s,t) \neq \frac{\partial}{\partial t}\frac{\partial}{\partial s}\rho(s,t).$$

Now let ρ_1 and ρ_2 be smooth maps from a neighbourhood of $(0,0) \in \mathbb{R}^2$ to M. We say two such maps are **equivalent** if

$$\frac{\partial}{\partial s}\frac{\partial}{\partial t}\rho_1(0,0) = \frac{\partial}{\partial s}\frac{\partial}{\partial t}\rho_2(0,0).$$

To the equivalence classes of this equivalence relation, we associate points in TTM by

$$[\rho] \mapsto \frac{\partial}{\partial s}\frac{\partial}{\partial t}\rho(0,0).$$

We easily verify that

$$\tau_{TM}([\rho]) = \frac{\partial}{\partial t}\rho(0,0), \quad T\tau_M([\rho]) = \frac{\partial}{\partial s}\rho(0,0). \tag{2.11}$$

Next, using the preceding representation of points in TTM, we relate the two vector bundle structures for TTM by defining a canonical involution of TTM. This is a well-known object, of course. Our development and use of this involution differs a little from what one usually sees in that it is entirely free from local coordinates. If ρ is a smooth map from a neighbourhood of $(0,0) \in \mathbb{R}^2$ into M, define another such map by $\bar{\rho}(s,t) = \rho(t,s)$. We then define the **canonical tangent bundle involution** as the map $I_M\colon TTM \to TTM$ defined by $I_M([\rho]) = [\bar{\rho}]$. Clearly $I_M \circ I_M = \mathrm{Id}_{TTM}$.

An interesting and useful formula connecting the complete lift and the canonical tangent bundle involution is the following.

2.3 Lemma: *For $X \in \Gamma^\infty(TM)$, $X^C = I_M \circ TX$.*

Proof: Let $v_x \in TM$ and let γ be a curve for which $\gamma'(0) = v_x$. As in Lemma 2.1, define $\sigma(s,t) = \Phi_t^X(\gamma(s))$ so that

$$X^C(v_x) = \frac{\partial}{\partial s}\frac{\partial}{\partial t}\sigma(0,0).$$

Then $\bar{\sigma}(s,t) = \Phi_s^X(\gamma(t))$ and so

$$I_M(X^C(v_x)) = \frac{\partial}{\partial t}\frac{\partial}{\partial s}\bar{\sigma}(0,0) = \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} X(\gamma(t)) = T_x X(v_x),$$

as desired. ■

We have seen in (2.9) above that the secondary vector bundle structure can be defined using the tangent functor. Referring to (2.7) we see that $X \mapsto TX$ is a morphism with respect to the primary vector bundle structure. By the preceding lemma, this gives us a way of representing the primary vector bundle operations in TTM. Indeed, if $u, v \in TTM$ satisfy $\tau_{TM}(u) = \tau_{TM}(v) \doteq w$, we consider the following two cases.

1. $w \neq 0$: In this case, via (2.9) and the preceding lemma, let $U, V \in \Gamma^\infty(TM)$ be such that $U^C(w) = u$ and $V^C(w) = v$. Then we have

$$u +_1 v = (U+V)^C(w), \quad a \cdot_1 u = (aU)^C(w). \tag{2.12}$$

2. $w = 0$: In this case, $u, v \in T_{0_x}TM$ for a suitable x. We note that

$$T_0TM \simeq T_xM \oplus T_xM,$$

cf. [Bullo and Lewis 2004, Lemma 6.33]. Thus there exists $U, V \in \Gamma^\infty(TM)$ such that

$$u = T\pi(u) \oplus U(x), \quad v = T\pi(v) \oplus V(x).$$

We then have

$$u +_1 v = (T\pi(u) + T\pi(v)) \oplus (U+V)(x), \quad a \cdot_1 u = (aT\pi(u)) \oplus (aU)(x). \tag{2.13}$$

The following result will be helpful, and is more or less clear given the preceding discussion.

2.4 Lemma: *The map I_M is a vector bundle isomorphism:*

$$\begin{tikzcd} TTM \arrow[rr, "I_M"] \arrow[dr, "T\tau_M"'] & & TTM \arrow[dl, "\tau_{TM}"] \\ & TM & \end{tikzcd}$$

Proof: A proof in natural coordinates is elementary. We shall give an intrinsic proof.

It is clear from (2.11) and the relations

$$\frac{\partial}{\partial t}\bar{\rho}(0,0) = \frac{\partial}{\partial s}\rho(0,0), \quad \frac{\partial}{\partial s}\bar{\rho}(0,0) = \frac{\partial}{\partial t}\rho(0,0)$$

that the diagram in the statement of the lemma commutes. Moreover, it is also clear that I_M is a bijection. It thus remains to show that it is a vector bundle map. Let $u, v \in TTM$ be such that $T\tau_M(u) = T\tau_M(v) \doteq w$. We then consider two cases.

$w \neq 0$: Let $U, V \in \Gamma^\infty(TM)$ be such that $TU(w) = u$ and $TV(w) = v$. Then, using Lemma 2.3 and equations (2.9) and (2.12),

$$\begin{aligned} I_M(u +_2 v) &= I_M \circ T(U+V)(w) = (U+V)^C(w) \\ &= U^C(w) +_1 V^C(w) = I_M \circ TU(w) +_1 I_M \circ TV(w) \\ &= I_M(u) +_1 I_M(v) \end{aligned}$$

and

$$\begin{aligned} I_M(a \cdot_2 u) &= I_M \circ T(aU)(w) = (aU)^C(w) \\ &= a \cdot_1 U^C(w) = a \cdot_1 I_M \circ TU(w) = a \cdot_1 I_M(u), \end{aligned}$$

as desired in this case.

$w = 0$: Let $x = \tau_M \circ T\tau_M(u) = \tau_M \circ T\tau_M(v)$. Choose $U, U', V, V' \in \Gamma^\infty(TM)$ such that

$$u = U^V(U'(x)), \quad v = V^V(V'(x)).$$

For $s \in \mathbb{R}$ define $U_s, V_s \in \Gamma^\infty(TM)$ by

$$U_s = U' + sU, \quad V_s = V' + sV.$$

Define

$$\rho(s,t) = \Phi_t^{U_s}(x), \quad \sigma(s,t) = \Phi_t^{V_s}(x),$$

and note that

$$\frac{\partial}{\partial t}\rho(s,0) = U_s(x), \quad \frac{\partial}{\partial t}\sigma(s,0) = V_s(x)$$

and so $[\rho] = u$ and $[\sigma] = v$. Now we use the Baker–Campbell–Hausdorff formula to get

$$\bar\rho(s,t) = \Phi_s^{U'+tU}(x) = \Phi_1^{sU'+stU}(x) = \Phi_t^{sU} \circ \Phi_s^{U'}(x) + O((|s|+|t|)^2).$$

Therefore,

$$\frac{\partial}{\partial t}\bar\rho(s,0) = sU(\Phi_s^{U'}(x))$$

and so

$$\frac{\partial}{\partial s}\frac{\partial}{\partial t}\bar\rho(0,0) = (0, U(x)) + (U'(x), 0) \in T_0TM \simeq T_xM \oplus T_xM.$$

Thus we have

$$I_M(u) = T\tau_M(u) \oplus U(x),$$

with a similar formula holding for v, of course. Therefore,

$$\begin{aligned} I_M(u +_2 v) &= I_M(U^V(U'(x)) +_2 V^V(V'(x))) \\ &= I_M((U+V)^V(U'(x) + V'(x))) \\ &= (U'(x) + V'(x), U(x) + V(x)) \\ &= I_M(u) +_1 I_M(v), \end{aligned}$$

using (2.10), (2.13), and the preceding calculations. Similarly,

$$I_M(a \cdot_2 u) = I_M((aU)^V(aU'(x))) = (aU'(x), aU(x)) = a \cdot_1 I_M(u),$$

as desired. ■

We close this section with a few technical lemmata that we will subsequently use in the paper.

2.5 Lemma: *If $w \in TTM$ satisfies $\tau_{TM}(w) = v$ and $T\tau_M(w) = u$ and if $z \in T_xM$, then*

$$w +_2 I_M \circ \mathrm{vlft}(u, z) = w +_1 \mathrm{vlft}(v, z).$$

Proof: Let $U, V, Z \in \Gamma^\infty(TM)$ be vector fields for which

$$U(x) = u, \quad V(x) = v, \quad Z(x) = z,$$

We consider two cases.

$u \neq 0$: In this case, write $w = TW(u)$ for some vector field $W \in \Gamma^\infty(TM)$. Then $W(x) = V(x)$. We compute

$$\begin{aligned} w +_2 I_M \circ \mathrm{vlft}(u, z) &= TW(u) +_2 I_M(Z^V(U(x))) \\ &= TW(u) +_2 (U(x) \oplus Z(x)) \\ &= TW(u) +_1 Z^V(V(x)), \end{aligned}$$

as desired in this case.

$u = 0$: Here we write $w = W^V(V(x))$ for an appropriate vector field W on M. Then

$$\begin{aligned} w +_2 I_M \circ \mathrm{vlft}(u, z) &= W^V(V(x)) +_2 I_M(Z^V(U(x))) \\ &= W^V(V(x)) +_2 (U(x) \oplus Z(x)) \\ &= W^V(V(x)) +_1 Z^V(V(x)), \end{aligned}$$

giving the lemma. ■

The proof of the following lemma is a specialization of the proof of Lemma 6.19 in [Kolář, Michor, and Slovák 1993].

2.6 Lemma: *For $X, Y \in \Gamma^\infty(TM)$ we have*

$$TY(X(x)) -_1 I_M \circ TX(Y(x)) = \mathrm{vlft}(Y(x), [X, Y](x)).$$

Proof: We use the formula

$$[X, Y](x) = \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} (\Phi_t^X)^* Y(x),$$

[Abraham, Marsden, and Ratiu 1988, Theorem 4.2.19]. Note that the curve

$$t \mapsto (\Phi_t^X)^* Y(x)$$

is a curve in T_xM passing through $Y(x)$ at $t = 0$, and so its derivative with respect to t at $t = 0$ is a vertical tangent vector in $T_{Y(x)}TM$. Note that $V_{Y(x)}TM \simeq T_xM$. We calculate

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\bigg|_{t=0}(\Phi_t^X)^*Y(x) = \frac{\mathrm{d}}{\mathrm{d}t}\bigg|_{t=0} T\Phi_{-t}^X \circ Y \circ \Phi_t^X(x) \\ &= -X^C \circ Y(x) +_1 TY \circ X(x) \\ &= TY \circ X(x) -_1 I_M \circ TX \circ Y(x),\end{aligned}$$

using Lemma 2.3 and (2.6). ■

2.4. Affine differential geometry. This section will be a very rapid overview of the affine differential geometry we shall use in this paper. We refer to [Kobayashi and Nomizu 1963] for details.

A $\mathcal{C}^\infty$-**affine connection** on a manifold M assigns to each pair $(X, Y) \in \Gamma^\infty(TM) \times \Gamma^\infty(TM)$ a vector field $\nabla_X Y \in \Gamma^\infty(TM)$, and the assignment satisfies

1. the map $(X, Y) \mapsto \nabla_X Y$ is $\mathbb{R}$-bilinear,
2. $\nabla_{fX} Y = f\nabla_X Y$, and
3. $\nabla_X(fY) = f\nabla_X Y + (\mathcal{L}_X f)Y$

for each $X, Y \in \Gamma^\infty(TM)$ and $f \in \mathcal{C}^\infty(M)$. The vector field $\nabla_X Y$ is called the **covariant derivative** of Y with respect to X.

As the expression $\nabla_X Y$ is tensorial in X, it only depends on the value of X at the point x. Hence, if $v_x \in T_xM$, we can define

$$\nabla_{v_x} Y(x) = \nabla_X Y(x) \in T_xM,$$

where X is any $\mathcal{C}^\infty$-vector field such that $X(x) = v_x$.

Given an affine connection ∇, there exists a complementary subbundle HTM of the vertical subbundle $VTM = \ker(T\tau_M)$, i.e., $TTM = HTM \oplus VTM$. This complementary subbundle is called the **horizontal subbundle** and is constructed as follows [Kolář, Michor, and Slovák 1993]. We shall first define a map hlft: $TM \oplus_M TM \to TTM$. Let $x \in M$ and $u, v \in T_xM$. Let $X \in \Gamma^\infty(TM)$ be such that $X(x) = v$ and define

$$\mathrm{hlft}(v, u) = TX(u) -_1 \mathrm{vlft}(v, \nabla_u X), \tag{2.14}$$

where vlft is the vertical lift map from (2.3). One can easily check that hlft is indeed a vector bundle map according to both of the following commuting diagrams:

$$\begin{tikzcd} TM \oplus_M TM \arrow[rr, "\mathrm{hlft}"] \arrow[dr, "\mathrm{pr}_1"'] & & TTM \arrow[dl, "\tau_{TM}"] \\ & TM & \end{tikzcd} \qquad \begin{tikzcd} TM \oplus_M TM \arrow[rr, "\mathrm{hlft}"] \arrow[dr, "\mathrm{pr}_2"'] & & TTM \arrow[dl, "T\tau_M"] \\ & TM & \end{tikzcd}$$

Thus

$$\begin{aligned}\mathrm{hlft}(v_1 + v_2, u) &= \mathrm{hlft}(v_1, u) +_2 \mathrm{hlft}(v_2, u), \quad & \mathrm{hlft}(av, u) &= a \cdot_2 \mathrm{hlft}(v, u) \\ \mathrm{hlft}(v, u_1 + u_2) &= \mathrm{hlft}(v, u_1) +_1 \mathrm{hlft}(v, u_2), \quad & \mathrm{hlft}(v, au) &= a \cdot_1 \mathrm{hlft}(v, u).\end{aligned} \tag{2.15}$$

The horizontal subbundle is defined by

$$H_{v_x}TM = \{\mathrm{hlft}(v_x, u_x) |\ u_x \in T_xM\}.$$

At each $v_x \in TM$, the linear map $T_{v_x}\tau_M : T_{v_x}TM \to T_xM$, restricted to the horizontal subspace $H_{v_x}TM$, is an isomorphism. The inverse of this isomorphism, applied to $u_x \in T_xM$, is the **horizontal lift** of u_x to $v_x \in T_xM$:

$$\mathrm{hlft}(v_x, u_x) = (T_{v_x}\tau_M | H_{v_x}TM)^{-1}(u_x).$$

The **horizontal lift** of the vector field $X \in \Gamma^\infty(TM)$ is the vector field $X^H \in \Gamma^\infty(TTM)$ defined by $X^H(v_x) = \mathrm{hlft}(v_x, X(x))$.

The **torsion tensor** is denoted by T:

$$T(X, Y) = \nabla_X Y - \nabla_Y X - [X, Y].$$

The canonical tangent bundle involution also provides an interesting and useful way of characterising torsion-free affine connections. The following result appears in [Fisher and Laquer 1999], but with a coordinate proof. We provide an intrinsic proof that is quite a lot simpler than the proof in [Fisher and Laquer 1999].

2.7 Lemma: *With the notation preceding,*

$$\mathrm{hlft}(v_x, u_x) -_1 I_M \circ \mathrm{hlft}(u_x, v_x) = \mathrm{vlft}(v_x, T(v_x, u_x))$$

for all $u_x, v_x \in T_xM$ and all $x \in M$. As a consequence, the following statements are equivalent:

1. *∇ is torsion-free;*
2. *$\mathrm{hlft}(v_x, u_x) = I_M \circ \mathrm{hlft}(u_x, v_x)$ for all $u_x, v_x \in T_xM$ and $x \in M$;*
3. *I_M leaves the horizontal subbundle $HTM \subseteq TTM$ invariant.*

Proof: The first assertion of the lemma follows from Lemmata 2.5 and 2.6 as follows:

$$\begin{aligned}
&\mathrm{hlft}(X(x), Y(x)) -_1 I_M \circ \mathrm{hlft}(Y(x), X(x))\\
&\quad = (TX(Y(x)) -_1 \mathrm{vlft}(X(x), \nabla_Y X(x))) -_1 I_M(TY(X(x)) -_1 \mathrm{vlft}(Y(x), \nabla_X Y(x)))\\
&\quad = TX(Y(x)) -_1 \mathrm{vlft}(X(x), \nabla_Y X(x)) -_1 (I_M \circ TY(X(x)) -_2 I_M \circ \mathrm{vlft}(Y(x), \nabla_X Y(x)))\\
&\quad = TX(Y(x)) -_1 \mathrm{vlft}(X(x), \nabla_Y X(x)) -_1 (I_M \circ TY(X(x)) -_1 \mathrm{vlft}(X(x), \nabla_X Y(x)))\\
&\quad = TX(Y(x)) -_1 \mathrm{vlft}(X(x), \nabla_Y X(x))\\
&\qquad -_1 (TX(Y(x)) +_1 \mathrm{vlft}(X(x), [X, Y](x)) -_1 \mathrm{vlft}(X(x), \nabla_X Y(x)))\\
&\quad = \mathrm{vlft}(X(x), -[X, Y](x) + \nabla_X Y(x) - \nabla_Y X(x))\\
&\quad = \mathrm{vlft}(X(x), T(X(x), Y(x))),
\end{aligned}$$

for vector fields X and Y.

(1) $\Longrightarrow$ (2) This follows immediately from the first assertion of the lemma.

(2) $\Longrightarrow$ (3) This is obvious.

(3) $\Longrightarrow$ (1) Let $w \in HTM$ so that $I_M(w) \in HTM$. Write $w = \mathrm{hlft}(v_x, u_x)$ for some $x \in M$ and $u_x, v_x \in T_xM$. Since

$$\tau_{TM}(w) = v_x, \quad T\tau_M(w) = u_x,$$

we have

$$\tau_{TM}(I_M(w)) = u_x, \quad T\tau_M(I_M(w)) = v_x.$$

Since $I_M(w)$ is horizontal, we must have $I_M(w) = \mathrm{hlft}(u_x, v_x)$. It then immediately follows that $T = 0$ from the first part of the proof. ■

Given an interval $I \subseteq \mathbb{R}$ and a curve $\gamma\colon I \to M$, a **vector field along** γ is a smooth map that assigns to every $t \in I$ an element of $T_{\gamma(t)}M$. If $Y : I \to TM$ is a vector field along γ, it makes sense to define a $\mathcal{C}^\infty$-vector field along γ by

$$I \ni t \mapsto \nabla_{\gamma'(t)}\overline{Y}(\gamma(t)) \in T_{\gamma(t)}M,$$

where $\overline{Y}$ is a vector field for which $Y(t) = \overline{Y}(\gamma(t))$. This construction can be shown to be independent of the extension of Y to $\overline{Y}$. A vector field Y along γ is **parallel** if $\nabla_{\gamma'(t)}Y(t) = 0$ for each $t \in I$.

The equation $\nabla_{\gamma'(t)}Y(t) = 0$ can be regarded as a differential equation for the vector field Y along γ. If the initial value v of the vector field at $t_0 \in I$ is given, the differential equation has a unique solution $Y(t)$ for t sufficiently close to t_0. The map $\tau_\gamma^{(t,t_0)} : T_{\gamma(t_0)}M \to T_{\gamma(t)}M$ that sends $v \in T_{\gamma(t_0)}M$ to the unique vector $Y(t) \in T_{\gamma(t)}M$ defined by the solution to the initial value problem

$$\nabla_{\gamma'(t)}Y(t) = 0, \quad Y(t_0) = v,$$

is called the **parallel transport** along γ. Note that $\tau_\gamma^{(t,t_0)}$ is an isomorphism. We recall from the discussion in [Kobayashi and Nomizu 1963, page 114] the following formula:

$$\Phi_t^{X^H}(v_x) = \tau_\gamma^{(t,0)}(v_x), \tag{2.16}$$

where γ is the integral curve of the vector field X for which $\gamma(0) = x$. The covariant derivative of Y along X can also be described as follows:

$$\nabla_X Y(x) = \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} \tau_\gamma^{(0,t)}(Y(\gamma(t))), \tag{2.17}$$

where γ is the integral curve of X satisfying $\gamma(0) = x$.

A **geodesic** of an affine connection ∇ on M is a curve $\gamma\colon I \to M$ satisfying $\nabla_{\gamma'(t)}\gamma'(t) = 0$. A geodesic can also be described as a curve whose tangent vector field is parallel along itself. The geodesic equations give rise to a second-order vector field $Z \in \Gamma^\infty(TTM)$ having the property that the integral curves of Z projected to M by the natural tangent bundle projection τ_M are geodesics of ∇. This vector field Z is called **geodesic spray** for ∇. The geodesic spray can be defined using horizontal lifts as follows:

$$Z(v_x) = \operatorname{hlft}(v_x, v_x). \tag{2.18}$$

Note that while parallel transport uses "all" of the information about an affine connection, the geodesics do not, as they depend only on the symmetric part of the Christoffel symbols. This observation is made precise as follows. If ∇ is an affine connection on M, then there exists a unique torsion-free affine connection, denoted by $\overline{\nabla}$, whose geodesics are exactly those of ∇. Explicitly,

$$\overline{\nabla}_X Y = \nabla_X Y - \frac{1}{2}T(X,Y), \tag{2.19}$$

cf. Propositions 7.9 and 7.10 in Chapter III in [Kobayashi and Nomizu 1963]. Here T is the torsion of ∇. It is possible to relate the parallel transport of a connection and its torsion-free connection.

2.8 Lemma: *Let ∇ be an affine connection on M with torsion T and let $\overline{\nabla}$ be the corresponding zero-torsion affine connection. Let γ be a geodesic for both ∇ and $\overline{\nabla}$ with the same initial condition. If $V \in T_{\gamma(0)}M$ then*

$$\tau_\gamma^{(t,0)}(V) - \overline{\tau}_\gamma^{(t,0)}(V) = \overline{\tau}_\gamma^{(t,0)}\left(-\frac{1}{2}\int_0^t \overline{\tau}_\gamma^{(0,s)}\left(T(\gamma'(s), \tau_\gamma^{(s,0)}(V))\right)\,\mathrm{d}s\right), \tag{2.20}$$

where $\tau_\gamma^{t,0}$ (resp. $\overline{\tau}_\gamma^{t,0}$) is the ∇ (resp. $\overline{\nabla}$) parallel transport along γ from $T_{\gamma(0)}M$ to $T_{\gamma(t)}M$.

Proof: Let us abbreviate

$$A_V(t) = -\frac{1}{2}\int_0^t \overline{\tau}_\gamma^{(0,s)}\left(T(\gamma'(s), \tau_\gamma^{(s,0)}(V))\right)\,\mathrm{d}s$$

so that

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}A_V(t) &= \overline{\tau}_\gamma^{(0,t)}\left(-\frac{1}{2}T(\gamma'(t), \tau_\gamma^{(t,0)}(V))\right)\\ &= \overline{\tau}_\gamma^{(0,t)}\left(\overline{\nabla}_{\gamma'(t)}\tau_\gamma^{(t,0)}(V) - \nabla_{\gamma'(t)}\tau_\gamma^{(t,0)}(V)\right)\\ &= \overline{\tau}_\gamma^{(0,t)}\left(\overline{\nabla}_{\gamma'(t)}\tau_\gamma^{(t,0)}(V)\right),\end{aligned}$$

using (2.19). We also compute

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\left(\overline{\tau}_\gamma^{(0,t)}\circ\tau_\gamma^{(t,0)}(V) - V\right) &= \overline{\nabla}_{\gamma'(t)}\left(\overline{\tau}_\gamma^{(0,t)}\circ\tau_\gamma^{(t,0)}(V)\right)\\ &= \overline{\tau}_\gamma^{(0,t)}\overline{\nabla}_{\gamma'(t)}\tau_\gamma^{(t,0)}(V).\end{aligned}$$

Thus we have

$$\frac{\mathrm{d}}{\mathrm{d}t}A_V(t) = \frac{\mathrm{d}}{\mathrm{d}t}\left(\overline{\tau}_\gamma^{(0,t)}\circ\tau_\gamma^{(t,0)}(V) - V\right).$$

Since

$$A_V(t)|_{t=0} = \left(\overline{\tau}_\gamma^{(0,t)}\circ\tau_\gamma^{(t,0)}(V) - V\right)\Big|_{t=0},$$

it follows that

$$A_V(t) = \overline{\tau}_\gamma^{(0,t)}\circ\tau_\gamma^{(t,0)}(V) - V.$$

Rearranging gives the result. ■

3. Infinitesimal descriptions of the symmetric product

Now we are ready to geometrically describe the symmetric product for vector fields. We shall provide four equivalent infinitesimal descriptions of the symmetric product (some of which are related in elementary ways). To do this, we make use of the BCH formula.

Let ∇ be an affine connection. The **symmetric product** for ∇ of two vector fields X and Y on M is defined as follows:

$$\langle X : Y\rangle = \nabla_X Y + \nabla_Y X.$$

Our infinitesimal descriptions of the symmetric product, like that of (1.1) for the Lie bracket, involve concatenations of flows of vector fields. Before we state the results, let us give the various constructions we use. We let ∇ be an affine connection on M with $\overline{\nabla}$ the associated zero-torsion connection. We let $X_1, X_2 \in \Gamma^\infty(TM)$ and let $v_x \in TM$. By X_1^H and X_2^H we denote the horizontal lifts with respect to ∇ and by $\overline{X}_1^H$ and $\overline{X}_2^H$ we denote the horizontal lifts with respect to $\overline{\nabla}$. By η_1 and η_2 we denote the integral curves of X_1 and X_2, respectively, through x. We let $\tau_\gamma^{(t,0)}$ and $\overline{\tau}_\gamma^{(t,0)}$ denote the parallel transport with respect to ∇ and $\overline{\nabla}$, respectively, along a curve γ. Now define four curves Υ_1, Υ_2, Υ_3, and Υ_4 in TM as follows:

$$\begin{aligned}
\Upsilon_1(t) &= \Phi_{-t}^{X_2^V}\,\Phi_{-t}^{X_1^H}\,\Phi_t^{X_2^V}\,\Phi_t^{X_1^H}\,\Phi_{-t}^{X_1^V}\,\Phi_{-t}^{X_2^H}\,\Phi_t^{X_1^V}\,\Phi_t^{X_2^H}(v_x),\\
\Upsilon_2(t) &= \Phi_{-t}^{X_2^V}\,\Phi_{-t}^{\overline{X}_1^H}\,\Phi_t^{X_2^V}\,\Phi_t^{\overline{X}_1^H}\,\Phi_{-t}^{X_1^V}\,\Phi_{-t}^{\overline{X}_2^H}\,\Phi_t^{X_1^V}\,\Phi_t^{\overline{X}_2^H}(v_x),\\
\Upsilon_3(t) &= \Phi_{-t}^{X_2^V}\,\tau_{\eta_1}^{(0,t)}\,\Phi_t^{X_2^V}\,\tau_{\eta_1}^{(t,0)}\,\Phi_{-t}^{X_1^V}\,\tau_{\eta_2}^{(0,t)}\,\Phi_t^{X_1^V}\,\tau_{\eta_2}^{(t,0)}(v_x),\\
\Upsilon_4(t) &= \Phi_{-t}^{X_2^V}\,\overline{\tau}_{\eta_1}^{(0,t)}\,\Phi_t^{X_2^V}\,\overline{\tau}_{\eta_1}^{(t,0)}\,\Phi_{-t}^{X_1^V}\,\overline{\tau}_{\eta_2}^{(0,t)}\,\Phi_t^{X_1^V}\,\overline{\tau}_{\eta_2}^{(t,0)}(v_x).
\end{aligned}$$

Before we state the main result in this section, we have the following lemma that we shall use in its proof. This formula appears, for example, in [Crampin 2000].

3.1 Lemma: *For vector fields $X, Y \in \Gamma^\infty(TM)$ and for an affine connection ∇ on M, we have $(\nabla_X Y)^V = [X^H, Y^V]$.*

Proof: We use (1.1):

$$[X^H, Y^V](v_x) = -\frac{1}{2}\left.\frac{\mathrm{d}^2}{\mathrm{d}t^2}\right|_{t=0} \Phi_{-t}^{X^H}\,\Phi_{-t}^{Y^V}\,\Phi_t^{X^H}\,\Phi_t^{Y^V}(v_x).$$

By equations (2.4) and (2.16) and by linearity of parallel transport we compute

$$\Phi_{-t}^{X^H}\,\Phi_{-t}^{Y^V}\,\Phi_t^{X^H}\,\Phi_t^{Y^V}(v_x) = v_x - t(\tau_\eta^{(0,t)}(Y(\eta(t))) - Y(x)),$$

where η is the integral curve of X through x. Note that this is a curve in T_xM and so its derivatives will be vertical tangent vectors. We then have

$$\left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} \Phi_{-t}^{X^H}\,\Phi_{-t}^{Y^V}\,\Phi_t^{X^H}\,\Phi_t^{Y^V}(v_x) = 0$$

and by (2.17)

$$\left.\frac{\mathrm{d}^2}{\mathrm{d}t^2}\right|_{t=0} \Phi_{-t}^{X^H}\,\Phi_{-t}^{Y^V}\,\Phi_t^{X^H}\,\Phi_t^{Y^V}(v_x) = -2\operatorname{vlft}(v_x, \nabla_X Y(x)),$$

from which the lemma immediately follows. ■

With the preceding notation, we state the following theorem.

3.2 Theorem: *With the notation of the preceding paragraph, if* $\Upsilon \in \{\Upsilon_1, \Upsilon_2, \Upsilon_3, \Upsilon_4\}$*, then* $\Upsilon'(0) = 0$ *and*

$$\frac{1}{2}\left.\frac{\mathrm{d}^2}{\mathrm{d}t^2}\right|_{t=0} \Upsilon(t) = \langle X_1 : X_2\rangle^V(v_x).$$

Proof: Let us apply the BCH formulae (2.1) to the concatenation of flows defining Υ_1. It is immediately clear that

$$\phi(\mathrm{BCH}_1(tX_2^H, tX_1^V, -tX_2^H, -tX_1^V, tX_1^H, tX_2^V, -tX_1^H, -tX_2^V) = 0.$$

Some bookkeeping and the fact the flows of vertically lifted vector fields obviously commute gives

$$\begin{aligned}\phi(\mathrm{BCH}_2(tX_2^H, tX_1^V, -tX_2^H, -tX_1^V, tX_1^H, tX_2^V, -tX_1^H, -tX_2^V)\\ = t^2([X_1^H, X_2^V] + [X_2^H, X_1^V]).\end{aligned}$$

An application of (2.2) and Lemma 3.1 now gives

$$\begin{aligned}\frac{1}{2}\left.\frac{\mathrm{d}^2}{\mathrm{d}t^2}\right|_{t=0} \Upsilon_1(t) &= \frac{1}{2}\left.\frac{\mathrm{d}^2}{\mathrm{d}t^2}\right|_{t=0} \Phi_1^{t^2\langle X_1:X_2\rangle^V}(v_x)\\ &= \frac{1}{2}\left.\frac{\mathrm{d}^2}{\mathrm{d}t^2}\right|_{t=0} \Phi_{t^2}^{\langle X_1:X_2\rangle^V}(v_x) = \langle X_1 : X_2\rangle^V(v_x).\end{aligned}$$

The same argument as above gives

$$\frac{1}{2}\left.\frac{\mathrm{d}^2}{\mathrm{d}t^2}\right|_{t=0} \Upsilon_2(t) = [\overline{X}_1^H, X_2^V](v_x) + [\overline{X}_2^H, X_1^V](v_x).$$

By (2.19) we have

$$\overline{X}^H(v_x) = X^H(v_x) + \frac{1}{2}\mathrm{vlft}(v_x, T(X(x), v_x)).$$

As a result, one directly computes

$$[\overline{X}^H, Y^V] = [X^H, Y^V] - \frac{1}{2}T(X,Y)^V.$$

Skew-symmetry of the torsion then gives

$$[\overline{X}_1^H, X_2^V](v_x) + [\overline{X}_2^H, X_1^V](v_x) = [X_1^H, X_2^V](v_x) + [X_2^H, X_1^V](v_x),$$

and from this we arrive at

$$\frac{1}{2}\left.\frac{\mathrm{d}^2}{\mathrm{d}t^2}\right|_{t=0} \Upsilon_2(t) = \langle X_1 : X_2\rangle^V(v_x).$$

Given this formula and the results from the first part of the proof, we immediately have from (2.16)

$$\frac{1}{2}\left.\frac{\mathrm{d}^2}{\mathrm{d}t^2}\right|_{t=0} \Upsilon_3(t) = \frac{1}{2}\left.\frac{\mathrm{d}^2}{\mathrm{d}t^2}\right|_{t=0} \Upsilon_4(t) = \langle X_1 : X_2\rangle^V(v_x).$$

as desired. ■

The following corollary gives a geometric interpretation of what is going on with the composition of flows in the preceding theorem.

3.3 Corollary: *Let $X_1, X_2 \in \Gamma^\infty(TM)$, let ∇ be an affine connection on M, and let $x \in M$. If $\Upsilon_1 = \Upsilon_3$ are defined as preceding Theorem 3.2 while taking $v_x = 0$, we have*

$$\Upsilon_1(t) = \Upsilon_3(t) = v_x + t\left(\tau_{\eta_2}^{(0,t)}(X_1(\eta_2(t))) - X_1(x) + \tau_{\eta_1}^{(0,t)}(X_2(\eta_1(t))) - X_2(x)\right). \quad (3.1)$$

Proof: The idea is the same, but only a little longer to carry out, as the proof of Lemma 3.1. ■

The upshot of the corollary is that the conclusion of Theorem 3.2 can be rendered a little more transparent since it is more or less obvious that the first derivative of the right-hand side of (3.1) is zero and that the second derivative is twice the symmetric product.

There exists another infinitesimal description of the symmetric product along the same lines as that of Theorem 3.2. We let γ_1 and γ_2 denote geodesics with initial conditions $X_1(x)$ and $X_2(x)$, respectively. Now define two new curves Υ_3^Z and Υ_4^Z in TM as follows:

$$\begin{aligned}
\Upsilon_3^Z(t) &= \Phi_{-t}^{X_2^V}\tau_{\gamma_1}^{(0,t)}\Phi_t^{X_2^V}\tau_{\gamma_1}^{(t,0)}\Phi_{-t}^{X_1^V}\tau_{\gamma_2}^{(0,t)}\Phi_t^{X_1^V}\tau_{\gamma_2}^{(t,0)}(v_x),\\
\Upsilon_4^Z(t) &= \Phi_{-t}^{X_2^V}\overline{\tau}_{\gamma_1}^{(0,t)}\Phi_t^{X_2^V}\overline{\tau}_{\gamma_1}^{(t,0)}\Phi_{-t}^{X_1^V}\overline{\tau}_{\gamma_2}^{(0,t)}\Phi_t^{X_1^V}\overline{\tau}_{\gamma_2}^{(t,0)}(v_x).
\end{aligned}$$

With these constructions, we have the following result.

3.4 Theorem: *With the notation of the preceding paragraph, if $\Upsilon \in \{\Upsilon_3^Z, \Upsilon_4^Z\}$, then $\Upsilon'(0) = 0$ and*

$$\frac{1}{2}\left.\frac{\mathrm{d}^2}{\mathrm{d}t^2}\right|_{t=0}\Upsilon(t) = \langle X_1 : X_2\rangle^V(v_x).$$

Proof: Corollary 3.3 also applies to $\Upsilon \in \{\Upsilon_3^Z, \Upsilon_4^Z\}$ by replacing η_1 and η_2 by geodesics γ_1 and γ_2, respectively. From Corollary 3.3 it is easy to see that the first derivative of Υ_3^Z at $t = 0$ is zero. The second derivative at $t = 0$ is

$$\begin{aligned}
\left.\frac{\mathrm{d}^2}{\mathrm{d}t^2}\right|_{t=0}\Upsilon_3^Z(t) &= 2\mathrm{vlft}\left(v_x, \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0}(\tau_{\gamma_1}^{(0,t)}(X_2(\gamma_1(t)))) + \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0}(\tau_{\gamma_2}^{(0,t)}(X_1(\gamma_2(t))))\right)\\
&= 2\mathrm{vlft}\left(v_x, \nabla_{\gamma_1'(0)}X_2(x) + \nabla_{\gamma_2'(0)}X_1(x)\right),
\end{aligned}$$

using (2.17). Note that the last expression only depends on the values and the first derivatives of the geodesics at zero. Since $\gamma_1'(0) = X_1(x)$ and $\gamma_2'(0) = X_2(x)$, the theorem follows for Υ_3^Z.

The result can be proved for Υ_4^Z using Lemma 2.8 since the parallel transport in Υ_4^Z is defined along geodesics. If γ is a geodesic and $V \in T_{\gamma(t)}M$, then (2.20) can be rewritten as follows:

$$\overline{\tau}_\gamma^{(0,t)}(V) = \tau_\gamma^{(0,t)}(V) + \frac{1}{2}\overline{\tau}_\gamma^{(0,t)}\left(\int_t^0 \overline{\tau}_\gamma^{(t,s)}\left(T(\gamma'(s), \tau_\gamma^{(s,t)}(V))\right)\mathrm{d}s\right), \quad (3.2)$$

By Corollary 3.3 and (3.2) we have

$$\begin{aligned}
\Upsilon_4^Z(t) = \Upsilon_3^Z(t) + t\Bigg(&\frac{1}{2}\overline{\tau}_{\gamma_2}^{(0,t)}\left(\int_t^0 \overline{\tau}_{\gamma_2}^{(t,s)}\left(T(\gamma_2'(s), \tau_{\gamma_2}^{(s,t)}(X_1(\gamma_2(t))))\right)\mathrm{d}s\right)\\
&+\frac{1}{2}\overline{\tau}_{\gamma_1}^{(0,t)}\left(\int_t^0 \overline{\tau}_{\gamma_1}^{(t,s)}\left(T(\gamma_1'(s), \tau_{\gamma_1}^{(s,t)}(X_2(\gamma_1(t))))\right)\mathrm{d}s\right)\Bigg).
\end{aligned}$$

As a result, with the abbreviation

$$A(t) = \frac{1}{2}\overline{\tau}_{\gamma_2}^{(0,t)}\left(\int_t^0 \overline{\tau}_{\gamma_2}^{(t,s)}\left(T(\gamma_2'(s), \tau_{\gamma_2}^{(s,t)}(X_1(\gamma_2(t))))\right)\,\mathrm{d}s\right) + \frac{1}{2}\overline{\tau}_{\gamma_1}^{(0,t)}\left(\int_t^0 \overline{\tau}_{\gamma_1}^{(t,s)}\left(T(\gamma_1'(s), \tau_{\gamma_1}^{(s,t)}(X_2(\gamma_1(t))))\right)\,\mathrm{d}s\right),$$

we have

$$\Upsilon_4^Z(0) = \Upsilon_3^Z(0), \qquad \frac{\mathrm{d}}{\mathrm{d}t}\Upsilon_4^Z(t) = \frac{\mathrm{d}}{\mathrm{d}t}\Upsilon_3^Z(t) + A(t) + t\frac{\mathrm{d}}{\mathrm{d}t}A(t),$$
$$\left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0}\Upsilon_4^Z(t) = \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0}\Upsilon_3^Z(t) + A(0) = \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0}\Upsilon_1(t) = 0,$$

since trivially $A(0) = 0$.

Now

$$\frac{\mathrm{d}^2}{\mathrm{d}t^2}\Upsilon_4^Z(t) = \frac{\mathrm{d}^2}{\mathrm{d}t^2}\Upsilon_3^Z(t) + 2\frac{\mathrm{d}}{\mathrm{d}t}A(t) + t\frac{\mathrm{d}^2}{\mathrm{d}t^2}A(t).$$

At $t = 0$,

$$\left.\frac{\mathrm{d}^2}{\mathrm{d}t^2}\right|_{t=0}\Upsilon_4^Z(t) = \left.\frac{\mathrm{d}^2}{\mathrm{d}t^2}\right|_{t=0}\Upsilon_3^Z(t) + 2\left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0}A(t).$$

Note that

$$\left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0}A(t) = -T(X_2(x), X_1(x)) - T(X_1(x), X_2(x)) = 0.$$

Thus the result follows for Υ_4^Z. ■

4. Characterization of geodesically invariant distributions

In the preceding section we provided an interpretation of the symmetric product that is similar to the composition of flows formula (1.1) for the Lie bracket. In this section we provide an interpretation of the symmetric product rather like that which Frobenius's Theorem provides for the Lie bracket. The theorem we prove here has already appeared in [Lewis 1998]. However, we provide a proof that is somewhat more elegant and also builds upon some independently interesting constructions using distributions.

4.1. Constructions using distributions. Let M be a n-dimensional manifold with $\mathcal{D}$ a distribution on M. Distributions in this paper will always be smooth and of locally constant rank. Let $\tau_{\mathcal{D}}\colon TM/\mathcal{D} \to M$ be the quotient vector bundle, and let $\pi_{\mathcal{D}}\colon TM \to TM/\mathcal{D}$ be the canonical projection. Note that the following diagram commutes:

$$\begin{tikzcd} TM \arrow[rr, "\pi_{\mathcal{D}}"] \arrow[dr, "\tau_M"'] & & TM/\mathcal{D} \arrow[dl, "\tau_{\mathcal{D}}"] \\ & M & \end{tikzcd}$$

At $0_x \in TM/\mathcal{D}$, there exists the following natural splitting

$$T_{0_x}TM/\mathcal{D} \simeq T_xM \oplus (T_xM/\mathcal{D}_x), \tag{4.1}$$

cf. [Bullo and Lewis 2004, Lemma 6.33]. Hence we define the projection $\mathrm{pr}_2\colon T_{0_x}TM/\mathcal{D} \to T_xM/\mathcal{D}_x$ onto the second component of the splitting in (4.1). The projection onto the first factor is simply $T_{0_x}\tau_{\mathcal{D}}$.

In the following result we give a characterization of vector fields tangent to subbundles that will be useful for us.

4.1 Proposition: *Let $\sigma\colon E \to M$ and $\tau\colon F \to M$ be vector bundles and let $f\colon E \to F$ be a surjective vector bundle morphism over the identity. A vector field Y on E is tangent to $\ker(f)$ if and only if*

$$\mathrm{pr}_2(T_{e_x}f \circ Y(e_x)) = 0_x$$

for every $e_x \in \ker(f)$.

Proof: For $x \in M$ the isomorphism of $T_{0_x}F$ with $T_xM \oplus F_x$ is given explicitly by

$$X_{0_x} \mapsto (T_{0_x}\tau(X_{0_x}), \mathrm{pr}_2(X_{0_x})).$$

Now note that, thinking of $\ker(f) = f^{-1}(Z(F))$ ($Z(F)$ is the zero section of F regarded as a submanifold of F) as a submanifold of E we have, for each $e_x \in \ker(f)$,

$$T_{e_x}\ker(f) = \{X_{e_x} \in T_{e_x}E|\ T_{e_x}f(X_{e_x}) \in T_{0_x}Z(F)\}$$

(see [Abraham, Marsden, and Ratiu 1988, Theorem 3.5.12]). Since $T_{0_x}Z(F) = \mathrm{image}(T_{0_x}\tau)$ we have that $X_{e_x} \in T_{e_x}\ker(f)$ if and only if

$$\mathrm{pr}_2(T_{e_x}f(X_{e_x})) = 0,$$

as desired. ■

The following result is a particular case of Proposition 4.1, noting that $\mathcal{D} = \ker(\pi_{\mathcal{D}})$.

4.2 Corollary: *Let $\mathcal{D} \subseteq TM$ be a distribution. A vector field Y on TM is tangent to $\mathcal{D}$ if and only if*

$$\mathrm{pr}_2((T_{v_x}\pi_{\mathcal{D}} \circ Y)(v_x)) = 0_x$$

for every $v_x \in \mathcal{D}$.

A corollary to this corollary, and one that will be useful for us, is the following.

4.3 Corollary: *A vector field $X \in \Gamma^\infty(TM)$ takes values in a distribution $\mathcal{D}$ on M if and only if X^V is tangent to $\mathcal{D}$.*

Proof: First suppose that X is $\mathcal{D}$-valued. If $v_x \in \mathcal{D}$ then, for any $t \in \mathbb{R}$,

$$v_x + tX(x) \in \mathcal{D} \implies \pi_{\mathcal{D}}(v_x + tX(x)) = 0_x \implies T_{v_x}\pi_{\mathcal{D}}(X^V(v_x)) = 0,$$

giving X^V tangent to $\mathcal{D}$ by the previous corollary.

Conversely, suppose that X^V is tangent to $\mathcal{D}$. Then, by the previous corollary,

$$\mathrm{pr}_2(T_{v_x}\pi_{\mathcal{D}}(X^V(v_x))) = 0_x$$

for every $v_x \in \mathcal{D}$. Since X^V is vertical and since $\pi_{\mathcal{D}}$ is a vector bundle mapping, $T_{v_x}\pi_{\mathcal{D}} \circ X^V$ is vertical. Thus

$$T_{0_x}\tau_{\mathcal{D}}(T_{v_x}\pi_{\mathcal{D}} \circ X^V)(v_x) = 0_x.$$

This implies that both components of $T_{v_x}\pi_{\mathcal{D}} \circ X^V(v_x)$ are zero in the decomposition $T_{0_x}TM/\mathcal{D} \simeq T_xM \oplus (T_xM/\mathcal{D}_x)$. Thus, reversing the calculations from the first part of the proof, we conclude that $X(x) \in \mathcal{D}$. ■

4.2. The geodesic invariance theorem. Let us define the objects of interest.

4.4 Definition: A distribution $\mathcal{D}$ on M is **geodesically invariant** under an affine connection ∇ on M if, for every geodesic $\gamma\colon I \to M$ for which $\gamma'(t_0) \in \mathcal{D}_{\gamma(t_0)}$ for some $t_0 \in I$, it holds that $\gamma'(t) \in \mathcal{D}_{\gamma(t)}$ for every $t \in I$.

Before we prove the main result in this section, we need to prove a few technical lemmata. The first relates the horizontal lift of a vector field to the complete lift of the same vector field. A special case of this formula is given by [Bhand 2007, Bhand 2010], with an intrinsic proof using frame bundles in [Bhand 2010].

4.5 Lemma: *If $X \in \Gamma^\infty(TM)$ then*

$$X^C(v_x) = X^H(v_x) +_1 \mathrm{vlft}(v_x, \nabla_{v_x} X(x) + T(X(x), v_x)),$$

for every $v_x \in TM$.

Proof: A direct proof in coordinates is, of course, elementary. However, we shall provide an intrinsic proof to keep in the spirit of our intrinsic proof of Theorem 4.8 below.

Let $v_x \in TM$ and let $Y \in \Gamma^\infty(TM)$ be such that $Y(x) = v_x$. Note that

$$\left.\frac{\mathrm{d}}{\mathrm{d}s}\right|_{s=0} \Phi_t^X \Phi_s^Y(x) = T_x\Phi_t^X(Y(x)).$$

Also compute

$$\begin{aligned}\left.\frac{\mathrm{d}}{\mathrm{d}s}\right|_{s=0} \Phi_t^X \Phi_s^Y(x) &= \left.\frac{\mathrm{d}}{\mathrm{d}s}\right|_{s=0} \Phi_s^Y \Phi_t^X \Phi_{-t}^X \Phi_{-s}^Y \Phi_t^X \Phi_s^Y(x)\\ &= Y(\Phi_t^X(x)) + T_x\Phi_t^X \left(\left.\frac{\mathrm{d}}{\mathrm{d}s}\right|_{s=0} \Phi_{-t}^X \Phi_{-s}^Y \Phi_t^X \Phi_s^Y (\Phi_t^X(x))\right).\end{aligned}$$

Note that

$$\begin{aligned}&\mathrm{BCH}_1(sY, tX, -sY, -tX) = 0,\\ &\mathrm{BCH}_2(sY, tX, -sY, -tX) = st[Y, X].\end{aligned}$$

Therefore, using (2.1),

$$\begin{aligned}\left.\frac{\mathrm{d}}{\mathrm{d}s}\right|_{s=0} \Phi_{-t}^X \Phi_{-s}^Y \Phi_t^X \Phi_s^Y (\Phi_t^X(x)) &= \left.\frac{\mathrm{d}}{\mathrm{d}s}\right|_{s=0} \Phi_1^{st[Y,X]}(\Phi_t^X(x))\\ &= \left.\frac{\mathrm{d}}{\mathrm{d}s}\right|_{s=0} \Phi_s^{t[Y,X]}(\Phi_t^X(x)) = t[Y, X](\Phi_t^X(x)).\end{aligned}$$

Putting the above calculations together gives

$$T_x\Phi_t^X(Y(x)) = Y(\Phi_t^X(x)) - t[X, Y](\Phi_t^X(x)).$$

Thus, recalling (2.6),

$$\Phi_t^{-X^H} \Phi_t^{X^C}(Y(x)) = \tau_{\gamma_-}^{(t,0)}(Y(\Phi_t^X(x)) - t[X, Y](\Phi_t^X(x))),$$

where γ_- is the integral curve of $-X$ through $\Phi_t^X(x)$, where we have used (2.16). If γ is the integral curve of X through x note that $\tau_{\gamma_-}^{(t,0)} = \tau_\gamma^{(0,t)}$. Now we compute

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} \Phi_t^{-X^H}\Phi_t^{X^C}(Y(x)) = & \ \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} \tau_\gamma^{(0,t)}(Y(\Phi_t^X(x)) - t[X,Y](\Phi_t^X(x))) \\ = & \ \nabla_X Y(x) - [X,Y](x) = \nabla_Y X(x) + T(X(x), Y(x)),\end{aligned}$$

using (2.17). Note that since X^C and X^H are both vector fields over X, it follows that

$$t \mapsto \tau_\gamma^{(0,t)}(Y(\Phi_t^X(x)))$$

is a curve in T_xM. Thus the derivative of this curve at $t = 0$ is in $V_{Y(x)}TM$. Thus we have shown that

$$\frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} \Phi_t^{-X^H}\Phi_t^{X^C}(v_x) = \mathrm{vlft}(v_x, \nabla_{v_x} X(x) + T(X(x), v_x)). \tag{4.2}$$

Finally, by the BCH formula, we have

$$\Phi_t^{-X^H}\Phi_t^{X^C}(v_x) = \Phi_1^{\phi(\mathrm{BCH}_1(-tX^H, tX^C))}(v_x) + O(|t|^2) = \Phi_t^{X^C - X^H}(v_x) + O(|t|^2).$$

Differentiating with respect to t and evaluating at $t = 0$, using (4.2), gives the result. ■

Another useful lemma is the following.

4.6 Lemma: *If Z is the geodesic spray for an affine connection and if $X, Y \in \Gamma^\infty(TM)$, then*

$$[X^V, [Z, Y^V]] = \langle X : Y \rangle^V.$$

Proof: Again, a proof in coordinates is easy, but we give an intrinsic proof.

We use the following formula for the Lie bracket [Abraham, Marsden, and Ratiu 1988, Theorem 4.2.19]:

$$[U, V](v_x) = \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} (\Phi_t^U)^* V(v_x),$$

for vector fields U and V on TM. Note that $\Phi_t^{Y^V} = \mathrm{Id}_{TM} + tY \circ \tau_M$, using (2.4), and so

$$Z \circ \Phi_t^{Y^V}(v_x) = \mathrm{hlft}(v_x + tY(x), v_x + tY(x)),$$

using (2.18). Now note that for $U \in \Gamma^\infty(TM)$,

$$T\tau_M = T\tau_M \circ TU \circ T\tau_M,$$

using the fact that $\tau_M \circ U = \mathrm{Id}_M$. It follows that, if $W \in TTM$, the expression $W +_2 TU \circ T\tau_M(W)$ makes sense. Thus we have

$$T\Phi_{-t}^{Y^V}(W) = W -_2 T(tY) \circ T\tau_M(W).$$

Thus

$$\begin{aligned}(\Phi_t^{Y^V})^* Z(v_x) &= T\Phi_{-t}^{Y^V} \circ Z \circ \Phi_t^{Y^V}(v_x) \\ &= \mathrm{hlft}(v_x + tY(x), v_x + tY(x)) -_2 T(tY)(v_x + tY(x)).\end{aligned}$$

We need to differentiate this expression with respect to t. To do this, let us define $\Upsilon\colon \mathbb{R}^2 \to TTM$ by

$$\Upsilon(s,t) = \mathrm{hlft}(v_x + sY(x), v_x + tY(x)) -_2 T(sY)(v_x + tY(x))$$

and $\iota\colon \mathbb{R} \to \mathbb{R}^2$ by $\iota(t) = (t,t)$. Note that

$$(\Phi_t^{Y^V})^* Z(v_x) = \Upsilon \circ \iota(t)$$

and so

$$\left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} (\Phi_t^{Y^V})^* Z(v_x) = T_1\Upsilon(1,1) +_1 T_2\Upsilon(1,1),$$

where $T_1\Upsilon$ and $T_2\Upsilon$ denote the partial derivatives of Υ, cf. [Abraham, Marsden, and Ratiu 1988, Proposition 3.3.13]. Thus we have

$$T_1\Upsilon(1,1) = \left.\frac{\mathrm{d}}{\mathrm{d}s}\right|_{s=0} \Upsilon(s,0), \quad T_2\Upsilon(1,1) = \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} \Upsilon(0,t).$$

The second of these expressions is readily calculated:

$$\begin{aligned}
\left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} \Upsilon(0,t) &= \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} \mathrm{hlft}(v_x, v_x + tY(x)) \\
&= \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} \big(\mathrm{hlft}(v_x, v_x) +_1 t \cdot_1 \mathrm{hlft}(v_x, Y(x))\big) \\
&= \mathrm{hlft}(v_x, Y(x)),
\end{aligned}$$

using (2.15). For the first, note that

$$\tau_{TM}(\mathrm{hlft}(v_x + sY(x), v_x) -_2 T(sY)(v_x)) = v_x, \quad T\tau_M(\mathrm{hlft}(v_x + sY(x), v_x) -_2 T(sY)(v_x)) = v_x.$$

Hence the tangent vector to the curve $0 \mapsto \Upsilon(s,0)$ at every time s lies in the vertical subspace $V_{\Upsilon(s,0)}(TTM) \simeq T_{v_x}TM \simeq T_xM \oplus T_xM$. In fact, to be more precise, it lies in the second copy of T_xM. In other words

$$\frac{\mathrm{d}}{\mathrm{d}s}\Upsilon(s,0) \in \{\mathrm{vlft}(\Upsilon(s,0), \mathrm{vlft}(0_x, w_x)) |\ w_x \in T_xM\} \simeq 0_x \oplus T_xM \subseteq V_{\Upsilon(s,0)}(TTM),$$

where the first vlft lifts from TTM to $VTTM$ and the second lifts from TM to VTM. Thus,

$$\left.\frac{\mathrm{d}}{\mathrm{d}s}\right|_{s=0} \big(\mathrm{hlft}(v_x + sY(x), v_x) -_2 T(sY)(v_x)\big) = \mathrm{vlft}(\mathrm{hlft}(v_x, v_x), \mathrm{hlft}(Y(x), v_x) -_2 TY(v_x)).$$

Then, using the definition of hlft from (2.14) and Lemma 2.2 we obtain

$$\begin{aligned}
\mathrm{hlft}(Y(x), v_x) -_2 TY(v_x) &= (TY(v_x) -_1 \mathrm{vlft}(Y(x), \nabla_v Y(x))) -_2 TY(v_x) \\
&= (TY(v_x) +_1 \mathrm{vlft}(Y(x), -\nabla_v Y(x))) -_2 (TY(v_x) +_1 \xi_1(Y(x))) \\
&= (TY(v_x) -_2 TY(v_x)) +_1 (\mathrm{vlft}(Y(x), -\nabla_v Y(x)) -_2 \xi_1(Y(x))) \\
&= \mathrm{vlft}(0_x, -\nabla_v Y(x)),
\end{aligned}$$

where $\xi_1 \colon TM \to TTM$ is the zero section relative to the primary vector bundle structure. According to the identification mentioned above, we have

$$\left.\frac{\mathrm{d}}{\mathrm{d}s}\right|_{s=0} \big(\mathrm{hlft}(v_x + sY(x), v_x) -_2 T(sY)(v_x)\big) \\ = \mathrm{vlft}(\mathrm{hlft}(v_x, v_x), \mathrm{vlft}(0_x, -\nabla_v Y(x))) \simeq -\mathrm{vlft}(v_x, \nabla_v Y(x)).$$

Putting the above together gives

$$[Y^V, Z](v_x) = \mathrm{hlft}(v_x, Y(x)) -_1 \mathrm{vlft}(v_x, \nabla_{v_x} Y).$$

In like manner we compute

$$(\Phi_t^{X^V})^*[Y^V, Z](v_x) = \\ \big(\mathrm{hlft}(v_x + tX(x), Y(x)) -_1 \mathrm{vlft}(v_x + tX(x), \nabla_{v_x+tX(x)} Y)\big) -_2 T(tX)(Y(x)).$$

We differentiate this expression as above, in this case defining

$$\Upsilon(s,t) = \big(\mathrm{hlft}(v_x + sX(x), Y(x)) -_1 \mathrm{vlft}(v_x + sX(x), \nabla_{v_x+tX(x)} Y)\big) -_2 T(sX)(Y(x)).$$

The two expressions we need to differentiate are then

$$\Upsilon(s,0) = \big(\mathrm{hlft}(v_x + sX(x), Y(x)) -_1 \mathrm{vlft}(v_x + sX(x), \nabla_{v_x} Y)\big) -_2 T(sX)(Y(x))$$

and

$$\Upsilon(0,t) = \big(\mathrm{hlft}(v_x, Y(x)) -_1 \mathrm{vlft}(v_x, \nabla_{v_x+tX(x)} Y)\big).$$

The second of these is easily differentiated:

$$\left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} \Upsilon(0,t) = -\mathrm{vlft}(v_x, \nabla_X Y(x)),$$

using (2.5). For the first, we first note that

$$\left.\frac{\mathrm{d}}{\mathrm{d}s}\right|_{s=0} \big(\mathrm{hlft}(v_x + sX(x), Y(x)) -_1 \mathrm{vlft}(v_x + sX(x), \nabla_{v_x} Y)\big) -_2 T(sX)(Y(x)) \\ = \left.\frac{\mathrm{d}}{\mathrm{d}s}\right|_{s=0} \mathrm{hlft}(v_x + sX(x), Y(x)) -_2 T(sX)(Y(x))$$

since the vertical component of the second term is independent of s. Now we can proceed as above to compute

$$\left.\frac{\mathrm{d}}{\mathrm{d}s}\right|_{s=0} \Upsilon(s,0) = \left.\frac{\mathrm{d}}{\mathrm{d}s}\right|_{s=0} \big(\mathrm{hlft}(v_x + sX(x), Y(x)) -_2 T(sX)(Y(x))\big) \\ = \mathrm{vlft}(\mathrm{hlft}(v_x, Y(x)), \mathrm{hlft}(X(x), Y(x)) -_2 TX(Y(x))).$$

Then, as above, using the definition of hlft from (2.14) and Lemma 2.2 we obtain

$$\begin{aligned} \mathrm{hlft}(X(x), &Y(x)) -_2 TX(Y(x)) \\ &= (TX(Y(x)) -_1 \mathrm{vlft}(X(x), \nabla_Y X(x))) -_2 (TX(Y(x)) +_1 \xi_1(X(x))) \\ &= \mathrm{vlft}(X(x), -\nabla_Y X(x)) -_2 \xi_1(X(x)) \\ &= \mathrm{vlft}(0_x, -\nabla_Y X(x)). \end{aligned}$$

According to the identification mentioned above, we have

$$\left.\frac{\mathrm{d}}{\mathrm{d}s}\right|_{s=0} \big(\mathrm{hlft}(v_x + sY(x), v_x) -_2 T(sY)(v_x)\big) \\ = \mathrm{vlft}(\mathrm{hlft}(v_x, Y(x)), \mathrm{vlft}(0_x, -\nabla_Y X(x))) \simeq -\mathrm{vlft}(v_x, \nabla_X Y(x)).$$

Putting the preceding calculations together and appropriately identifying vertical tangent vectors gives

$$[X^V, [Y^V, Z]](v_x) = -\mathrm{vlft}(v_x, \nabla_X Y(x)) -_1 \mathrm{vlft}(v_x, \nabla_Y X(x)) = -\mathrm{vlft}(v_x, \langle X : Y\rangle(x)),$$

which is the result. ■

For $X \in \Gamma^\infty(TM)$ let us denote

$$\mathcal{D}_X = \{\alpha X(x)|\ x \in M,\ \alpha \in \mathbb{R}\}.$$

With this notation, the last technical lemma upon which we shall draw is the following.

4.7 Lemma: *A distribution $\mathcal{D}$ is geodesically invariant if and only if, for each $X \in \Gamma^\infty(\mathcal{D})$ and for each $v_x \in \mathcal{D}_X$, $X^H(v_x) \in T_{v_x}\mathcal{D}$.*

Proof: First suppose that $X^H(v_x) \in T_{v_x}\mathcal{D}$ for every $X \in \Gamma^\infty(\mathcal{D})$ and every $v_x \in \mathcal{D}_X$. Let $v_x \in \mathcal{D}$ and let $X \in \Gamma^\infty(\mathcal{D})$ be such that $X(x) = v_x$. (This is possible as follows. Since $\mathcal{D}$ is smooth and constant rank, there exists linearly independent smooth local generators $X_1, \dots, X_k$ for $\mathcal{D}$ about x. Write

$$v_x = \alpha_1 X_1(x) + \dots + \alpha_k X_k(x)$$

and let $f_1, \dots, f_k : M \to \mathbb{R}$ be such that $f_j(x) = \alpha_j$ and such that $f_1, \dots, f_k$ vanish outside a sufficiently small neighbourhood of x. Then take

$$X = f_1 X_1 + \dots + f_k X_k.)$$

By hypothesis, $X^H(X(x)) \in T_{v_x}\mathcal{D}$. By (2.18) and the definition of X^H it follows that $Z(v_x) \in T_{v_x}\mathcal{D}$. As $v_x \in \mathcal{D}$ is arbitrary, it follows that Z is tangent to $\mathcal{D}$, meaning that $\mathcal{D}$ is geodesically invariant.

Conversely, suppose that $Z(v_x) \in T_{v_x}\mathcal{D}$ for every $v_x \in \mathcal{D}$. Let $X \in \Gamma^\infty(\mathcal{D})$ and let $v_x \in \mathcal{D}_X$. Thus $v_x = \alpha X(x)$ for some $\alpha \in \mathbb{R}$. We then have

$$T_{v_x}\mathcal{D} \ni Z(v_x) = \mathrm{hlft}(v_x, v_x) = \alpha \mathrm{hlft}(v_x, X(x)) = \alpha X^H(v_x),$$

using (2.18). We then consider two cases. First of all, suppose that $X(x) = 0_x$. Then $v_x = 0_x$ and so $Z(v_x) = X^H(v_x) = 0_{v_x}$ and we trivially have $X^H(v_x) \in T_{v_x}\mathcal{D}$. If $X(x) \neq 0_x$ then our computation just preceding gives $X^H(v_x) = \alpha^{-1} Z(v_x) \in T_{v_x}\mathcal{D}$. ■

We can now state the main result in this section. While this result is known [Lewis 1998], we provide here a self-contained intrinsic proof using the tools developed in the paper.

4.8 Theorem: ([**Lewis 1998**]) *Let $\mathcal{D}$ be a distribution on a manifold M with an affine connection ∇. The following are equivalent:*

1. *$\mathcal{D}$ is geodesically invariant;*
2. *$\langle X : Y\rangle \in \Gamma^\infty(\mathcal{D})$ for every $X, Y \in \Gamma^\infty(\mathcal{D})$;*
3. *$\nabla_X X \in \Gamma^\infty(\mathcal{D})$ for every $X \in \Gamma^\infty(\mathcal{D})$.*

Proof: (1) $\implies$ (2) The proof of this in [Lewis 1998] makes use of the formula from Lemma 4.6 which was only derived there in coordinates. We reproduce this proof here, but now it is a self-contained intrinsic proof since we have an intrinsic proof of Lemma 4.6. We also provide a second proof using our composition formula from Theorem 3.4 for the symmetric product.

First proof: Let $X, Y \in \Gamma^\infty(\mathcal{D})$. It is clear that since $\mathcal{D}$ is geodesically invariant, Z is tangent to $\mathcal{D}$. Moreover, by Corollary 4.3, X^V and Y^V are tangent to $\mathcal{D}$. By the formula (1.1), it follows that all Lie brackets involving Z, X^V, and Y^V are also tangent to $\mathcal{D}$. In particular, $[X^V, [Z, Y^V]]$ is tangent to $\mathcal{D}$ and so, by Corollary 4.3 and Lemma 4.6, $\langle X : Y\rangle$ is tangent to $\mathcal{D}$.

Second proof: By Theorem 3.4 we know that

$$\frac{1}{2}\left.\frac{\mathrm{d}^2}{\mathrm{d}t^2}\right|_{t=0} \Upsilon_3^Z(t) = \langle X : Y\rangle^V(v_x).$$

In particular this is true for every $X, Y \in \Gamma^\infty(\mathcal{D})$. According to Corollary 4.2 we only have to prove that

$$\mathrm{pr}_2\left(T_{v_x}\pi_{\mathcal{D}} \circ \left.\frac{\mathrm{d}^2}{\mathrm{d}t^2}\right|_{t=0} \Upsilon_3^Z(t)\right) = 0_x$$

for every $v_x \in \mathcal{D}$. First assume that $X = Y$. By adapting conveniently Corollary 3.3 to Υ_3^Z we have

$$\frac{1}{2}\left.\frac{\mathrm{d}^2}{\mathrm{d}t^2}\right|_{t=0} \Upsilon_3^Z(t) = \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} \tau_{\gamma_X}^{(0,t)}(X(\gamma_X(t))),$$

where γ_X is a geodesic such that $\gamma_X(0) = x$ and $\gamma_X'(0) = X(x)$. By the Leibniz rule,

$$\left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} \tau_{\gamma_X}^{(0,t)}(X(\gamma_X(t))) = \left(\left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} \tau_{\gamma_X}^{(0,t)}\right)(X(x)) = -Z(X(x)).$$

As $\mathcal{D}$ is geodesically invariant by hypothesis, Z is tangent to $\mathcal{D}$. Using Corollary 4.2 and the polarization identity for the symmetric product, the result follows.

(2) $\implies$ (3) This follows from the definition of the symmetric product.

(3) $\implies$ (1) Let $X \in \Gamma^\infty(\mathcal{D})$ and let $v_x \in \mathcal{D}_X$. Since $X \in \Gamma^\infty(\mathcal{D})$, $\pi_{\mathcal{D}} \circ X(y) = 0_y$ for every $y \in M$. Thus

$$T_{X(x)}\pi_{\mathcal{D}} \circ T_x X(u_x) = u_x \oplus 0_x,$$

using the identification $T_{0_x}(TM/\mathcal{D}) \simeq T_x M \oplus T_x M/\mathcal{D}_x$. This gives, in particular,

$$0_x = \mathrm{pr}_2 \circ T_{X(x)}\pi_{\mathcal{D}} \circ T_x X(X(x)) = \mathrm{pr}_2 \circ T_{X(x)}\pi_{\mathcal{D}}(X^H(X(x)) + (\nabla_X X)^V(X(x)))$$

using Lemma 4.5. By hypothesis and by Corollary 4.3, $(\nabla_X X)^V$ is tangent to $\mathcal{D}$. Therefore, by Corollary 4.2,

$$\mathrm{pr}_2 \circ T_{X(x)}\pi_{\mathcal{D}}((\nabla_X X)^V(X(x))) = 0_x.$$

Another appeal to Corollary 4.2 then allows us to conclude that $X^H(X(x)) \in T_{X(x)}\mathcal{D}$. Linearity of horizontal lift implies that $X^H(v_x) \in T_{v_x}\mathcal{D}$ for all $v_x \in \mathcal{D}_X$, and the theorem follows from Lemma 4.7. ■

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