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Fundamental problems in geometric control theory

Andrew D. Lewis*

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Abstract

Open problems in controllability and stabilisability of analytic systems are discussed. In particular, questions like, "Can controllability and/or stabilisability be tested by solving algebraic equations?" and, "What is the relationship between controllability *from* a state and stabilisability *to* the same state?" are discussed. A main idea is a rethinking of how one might examine stabilisability by connecting it to controllability.

For analytic systems, local obstructions to controllability and stabilisability should be determined by the germ of the system at the prescribed state. A means of characterising these germs in a systematic manner is presented.

Keywords. Controllability, reachability, stabilisability

AMS Subject Classifications (2020). 93B05, 93B27, 93D15

1. Introduction

In order to put the reader in the right frame of mind at the outset, I will quote Professor Walter Noll:

> The researcher's focus is on the discovery of new results. He is the creator of new knowledge. His nightmare is to get stuck in his search or to learn that what he has found has already been discovered shortly before by somebody else. Priority is very important to him and will sometimes induce him to rush into print prematurely.
>
> The professor's focus, on the other hand, is on understanding, gaining insight into, judging the significance of, and organizing old knowledge. He is disturbed by the pile-up of undigested and ill-understood new results. He is not happy until he has been able to fit these results into a larger context. He is happy if he can find a new conceptual framework with which to unify and simplify the results that have been found by the researcher.—Noll [2008]

I unapologetically declare that I am striving to write this paper in my capacity as a professor, according to Noll's dichotomy.

For the period of the past several years I have, with my dedicated and hard-working graduate students César Aguilar and Pantelis Isaiah, been engaged in an attempt to understand the local geometric structure of analytic control systems. My particular interest has

*Professor, Department of Mathematics and Statistics, Queen's University, Kingston, ON K7L 3N6, Canada
Email: andrew.lewis@queensu.ca, URL: http://www.mast.queensu.ca/~andrew/
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been to understand controllability (which I will here call "reachability") and stabilisability (which I will annoyingly here essentially refer to as "controllability"[1]). Of course, these are venerable topics about which much has been written. Let me clearly state that I am not claiming to present "solutions" to these problems in this paper. My aim is to formulate some overarching questions related to these concepts, and to describe what properties a solution might have. I will also describe a device that I will argue captures exactly the sort of information one needs to address the local reachability and controllability problems in which I am interested.

2. A truncated critical literature review

A very good way to describe what I have been trying to do with my students is to discuss what is known about reachability and controllability (remember, I am using these terms instead of "controllability" and "stabilisability"). My objective in this literature review is not to be comprehensive and present all important contributions. Rather, I will focus on describing those parts of the research literature whose description will contribute to the objective of the paper. I hope this will not cause offence to anyone who has made substantial contributions to these areas and whose work is not mentioned.

In this section we shall assume the reader is familiar enough with the subject of the paper that we can use notions that we have not yet precisely defined. Precise definitions for the concepts needed in the paper are provided in Section 3.

2.1. A review of reachability. The subject of reachability (again, this refers to what is commonly called "controllability") is a classical one in geometric control theory. Indeed, it is probably fair to say that the career of any researcher in geometric control theory has touched upon the subject of reachability in a substantial way at some point. As is well-known, the subject of reachability of a control system from a state x_0 starts with the notion of accessibility, by which it is meant that the set of states reachable from x_0 has a nonempty interior. Accessibility is well understood, and it can be precisely tested for analytic systems using the so-called "Lie algebra rank condition" [e.g., Krener 1974, Sussmann and Jurdjevic 1972]. Other forms of reachability require that x_0 be itself contained in the interior of the set of reachable states. In terms of design of controllers, this is a desirable property, as it means that by control you can go "in all directions" from x_0. The testing for these other forms of reachability appears to be extremely challenging. First of all, as is borne out by the results of Kawski [1990] and Sontag [1988], one should give up on an easily computable test for these forms of reachability; precisely, reachability is an NP-hard decision problem. However, "not easily computable" should not be confused with "uninsightful," as we hope the following example illustrates.

2.1 Example: On $\mathbb{R}^m \times \mathbb{R}^{n-m}$ with states denoted by $(\boldsymbol{x}_1, \boldsymbol{x}_2)$ and control denoted by $\boldsymbol{u} \in \mathbb{R}^m$, we consider the system

$$\dot{\boldsymbol{x}}_1(t) = \boldsymbol{u}(t),$$
$$\dot{\boldsymbol{x}}_2(t) = \boldsymbol{F}(\boldsymbol{x}_1(t)),$$

[1] I believe that the words "reachability" and "controllability" as I use them here better capture the essence of what I am trying to do than do the standard terms "controllability" and "stabilisability."

where $\boldsymbol{F}\colon \mathbb{R}^m \to \mathbb{R}^{n-m}$ is a homogeneous polynomial function of degree at least 2. One can easily check that this system is locally reachable from $(\mathbf{0},\mathbf{0})$ if and only if $\operatorname{conv}(\operatorname{image}(\boldsymbol{F})) = \mathbb{R}^{n-m}$, cf. [Aguilar and Lewis 2012, Example 5.3]. This is a satisfying geometric condition for reachability; if one had to "guess" when this system is reachable, this would be the conclusion one would reach. However, if $\boldsymbol{F}$ has degree 4 or more, then the verifiability of this condition is known to be NP-hard [Ramana 1993]. •

The point is that we should not give up on the reachability problem, even if we know that any "solution" to the problem will be intractable. However, the NP-hard characterisations of reachability *do* give one cause to wonder just what is meant by a solution to the problem. A few components of this sort of question are pointed out by Agrachev [1999]. The most germane to us of Agrachev's questions is whether, given a reachable system, is it true that there is some finite-order of differentiation so that, any system whose derivatives up to that order agree with the reachable system, must itself be reachable. Slightly less precisely, is it possible to recognise reachable systems using only finite differentiations? This sort of question is fleshed out in the paper of Kawski [2006], who gives some very interesting examples that show just how delicate the nature of reachability can be.

In most any general treatment of reachability, a key rôle is played by the notions of "variation" and "obstruction."

A variation is to be thought of as a tangent vector representing a direction tangent to the reachable set. The exact way in which one determines a variation is crucial, and various schemes exist for describing these [e.g., Bianchini and Stefani 1993, Kawski 1988a, Kawski 1988b]. In [Aguilar 2010, Aguilar and Lewis 2008] is presented a methodology for generating control variations that has some interesting algebraic structure associated with it. We review the principal ingredient in this construction in Section 5.2.

Also important in any study of reachability are obstructions. Typically, these are variations that are intrinsically "unidirectional." It is difficult to make this notion precise in any sort of general way, so let us rather illustrate the idea with an example.

2.2 Example: We consider the system on $\mathbb{R}^3$ with states denoted by (x_1, x_2, x_3) and controls denoted by (u_1, u_2):

$$\begin{aligned}\dot{x}_1(t) &= u_1(t),\\ \dot{x}_2(t) &= u_2(t),\\ \dot{x}_3(t) &= x_1(t)^2 + \alpha x_2(t)^2.\end{aligned}$$

This is a special case of the class of systems from Example 2.1. From that example we know that the system is reachable from $(0,0,0)$ if and only if $\alpha < 0$. For $\alpha \geq 0$, the system is not reachable from $(0,0,0)$ because of "obstructions" possessed by the system. In the development of the theory, these obstructions manifest themselves via certain Lie brackets of the drift vector field $f_0 = (x_1^2 + \alpha x_2^2)\frac{\partial}{\partial x_3}$ and the control vector fields $f_a = \frac{\partial}{\partial x_a}$, $a \in \{1,2\}$. The offending Lie brackets in this example are

$$[f_1,[f_0,f_1]] = 2\frac{\partial}{\partial x_3}, \quad [f_2,[f_0,f_1]] = 2\alpha\frac{\partial}{\partial x_3}.$$

Loosely speaking, the reason that these brackets are obstructions is that the control vector fields, which are those whose sign can be controlled, appear an even number of times, and so their direction cannot be changed by changing the sign of the control. However, if $\alpha < 0$ then one can "neutralise" these two obstructions as they have the opposite sign. •

This example is generalised in [Basto-Gonçalves 1998]. Also, this sort of structure plays a key rôle in the design of motion control algorithms for a class of mechanical systems [Bullo and Lewis 2005].

Having painted a (hopelessly) general picture of how reachability has been studied in the literature, one could now ask, "What is known?" We will not try to address this here. The fact is that what is known is not in a very good state. One has a morass of separately sufficient and necessary conditions which overlap awkwardly and which all depend on the particular approach of the authors. Suffice it to say that Lie brackets of vector fields feature prominently in almost any geometric treatment of reachability theory.

We shall, however, point out three phenomena that are not consistently well handled by existing approaches to reachability.

1. *The effects of feedback transformations:* One might like to think that reachability should be feedback invariant. This is true, provided one is sufficiently careful to understand what one is saying. However, even the simplest test for local reachability, the test of reachability of the linearisation, is not robust under feedback transformation. Indeed, consider the two feedback equivalent systems

$$\begin{aligned} \dot{x}_1(t) &= x_2(t), & \dot{x}_1(t) &= x_2(t), \\ \dot{x}_2(t) &= x_3(t)u_1(t), & \dot{x}_2(t) &= x_3(t) + x_3(t)u_1(t), \\ \dot{x}_3(t) &= u_2(t), & \dot{x}_3(t) &= u_2(t). \end{aligned}$$

   The system on the left does not have a reachable linearisation, while the system on the right has a reachable linearisation. Many of the sophisticated tests for reachability in the literature share this lack of invariance under feedback transformations.

2. *Invariance under changes of coordinate:* In some sense, most of the Lie algebraic approaches are coordinate-invariant in that the Lie bracket is coordinate-invariant. However, other tools (nilpotent and homogeneous approximation spring to mind) are decidedly *not* coordinate-invariant. Also, most of the examples studied as counterexamples are polynomial systems, and this is a class of systems that is also decidedly not coordinate-invariant. This makes it difficult to understand what these counterexamples are counterexamples for. Often, they are coordinate-dependent counterexamples to coordinate-dependent or feedback-dependent statements. For this reason, these examples may not always be as useful as they seem. Or they may be; the point is that one cannot really say for sure.

3. *The effects of changing the size of the control set:* Consider the system

$$\begin{aligned} \dot{x}_1(t) &= u_1(t), \\ \dot{x}_2(t) &= u_2(t), \\ \dot{x}_3(t) &= x_1(t)^2(1 + \tfrac{1}{2}u_2(t)). \end{aligned}$$

   Suppose that the controls take values in $[-\alpha, \alpha]^2$. It is shown by Aguilar [2010, Example 5.8] that this system is reachable from $(0, 0, 0)$ if $\alpha > 2$ and is not reachable from $(0, 0, 0)$ if $\alpha < 2$. This fact will necessarily be lost in any reachability test involving brackets of the drift and control vector fields.

In all three cases, the problem with standard approaches using Lie brackets is that the usual tests involve brackets of the drift and control vector fields, and specific choices of these can affect both the actual reachability of the system and/or its satisfying any given test for reachability. It is clear that a good theory needs to account for this.

2.2. A review of controllability, née stabilisability. Remember that "controllability" here refers to what is commonly called "stabilisability," i.e., controlling states to a point. To be more clear, perhaps the most relevant concept that we are thinking about is what is commonly called "asymptotic controllability." In any case, the precise definitions shall be given in Section 3.

Issues of language aside, for some time now it has been known that controllability, i.e., that which is typically known as "asymptotic controllability," is equivalent to the existence of a stabilising feedback [Ancona and Bressan 1999, Clarke, Ledyaev, Sontag, and Subotin 1997]. That is to say, in the terminology we are using and with appropriate assumptions, controllability and stabilisability are equivalent. Therefore, the question of the existence of a stabilising feedback for a state x_0 is reduced to the problem of determining whether all states can be controlled open-loop to x_0, possibly in infinite time. The problem of whether or not a system is asymptotically controllable has not been well studied. Instead the focus of this branch of research has been on the existence and properties of the stabilising feedback laws. This is neither surprising nor inappropriate. After all, if one knows that a system can be stabilised by feedback, the obvious question is how this can be done and done well, particularly in view of the importance in applications of finding stabilising controllers.

However, the development of the theory along these lines, with the attendant focus on Lyapunov theory, has led to a substantial gulf between the study of reachability and stabilisability, the former being decidedly geometric and the latter being decidedly not. This seems like something that ought to be rectified. After all, the transference of the decidability of whether a system is controllable to a state x_0 to the existence of a control Lyapunov function is not significant gain: both are difficult problems, perhaps equally so. Moreover, some things are decidedly lost in the coarseness of the Lyapunov approach. A few examples of these are as follows.

1. *The effects of system structure on controllability:* Even in our brief review of reachability in the preceding section, we saw that there were many opportunities for the structure of the system to manifest itself in the study of its reachability. The *in toto* imposition of a Lyapunov function on a system hurdles over all structural properties of the system. This is reflected by the fact that in the literature on reachability one sees many pathological deviously constructed examples that serve to illustrate the complex nature of reachability. Such examples are not in abundance in the literature on stabilisation. Indeed, the literature on stabilisation is more dedicated to classes of systems that *are* stabilisable. This marked distinction between the two bodies of research is peculiar, given that the subjects are so closely linked, as we shall emphasise in Section 4.2. A good characterisation of the available techniques for determining the existence of Lyapunov functions is provided by Sontag.

> The search for such functions is more of an art than a science, and good physical insight into a given system plus a good amount of trial and error is

typically the only way to proceed.—Sontag [1998]

2. *Regularity of stabilising feedback:* A problem of some importance in the study of stabilising feedbacks is that of the regularity of the feedback. In particular, we refer to [Artstein 1983] for the existence of feedbacks that are smooth on a punctured neighbourhood of the equilibrium (see also the important constructive result of Sontag [1989]) and to [Brockett 1983, Coron 1990, Orsi, Praly, and Mareels 2003, Zabczyk 1989] for topological obstructions to the existence of continuous stabilisers. The Lyapunov characterisation for smooth stabilisation suffers from the same existential problems as described above. On the other hand, the topological characterisations for the existence of continuous stabilisers are testable but extremely coarse.

Given the preceding observations, our approach to the problem of stabilisability is to instead study controllability, as this can be related to reachability. However, in order to do this, we need to begin to be more precise.

3. Definitions

In this section we present the definitions we shall use in this paper regarding systems, reachability, and controllability.

3.1. System definitions.

There are many possible system definitions, and the kind of results one can prove varies with the definition one uses. But a choice has to be made, and here is ours.

3.1 Definition: A ***control system*** is a triple $\Sigma = (\mathsf{M}, \mathcal{C}, F)$ where

(i) M is a paracompact Hausdorff real analytic manifold,

(ii) $\mathcal{C}$ is a metric space, and

(iii) $F\colon \mathsf{M} \times \mathcal{C} \to \mathsf{TM}$ satisfies

  (a) $F(x, u) \in \mathsf{T}_x\mathsf{M}$,

  (b) for each $u \in \mathcal{C}$, F_u is a real analytic vector field, where $F_u(x) = F(x, u)$, and

  (c) F^T is continuous, where $F^T\colon \mathsf{TM} \times \mathcal{C} \to \mathsf{TTM}$ is defined by asking that $F^T(v_x, u) = F_u^T$, where $\cdot^T$ denotes the tangent lift.

If $\mathsf{S} \subseteq \mathsf{M}$ is a submanifold such that $F_u(x) \in \mathsf{T}_x\mathsf{S}$ for each $(x, u) \in \mathsf{S} \times \mathcal{C}$, then by $\Sigma|\mathsf{S}$ we denote the control system $\Sigma|\mathsf{S} = (\mathsf{S}, \mathcal{C}, F|\mathsf{S} \times \mathcal{C})$. •

Let us also define and provide notation for control and trajectories.

3.2 Definition: (i) A ***control*** for a control system $\Sigma = (\mathsf{M}, \mathcal{C}, F)$ is a measurable locally essentially bounded map $\mu\colon \mathbb{T} \to \mathcal{C}$ where $\mathbb{T} \subseteq \mathbb{R}$ is an interval. We denote by $\mathrm{L}^\infty_{\mathrm{loc}}(\mathbb{T}; \mathcal{C})$ the controls defined on $\mathbb{T}$.

(ii) A ***controlled trajectory*** for Σ is a pair (ξ, μ) where $\mu \in \mathrm{L}^\infty_{\mathrm{loc}}(\mathbb{T}; \mathcal{C})$ and $\xi\colon \mathbb{T} \to \mathsf{M}$ is a locally absolutely continuous curve satisfying

$$\xi'(t) = F(\xi(t), \mu(t)). \tag{3.1}$$

For $x_0 \in \mathsf{M}$, $\mu \in \mathrm{L}^\infty_{\mathrm{loc}}(\mathbb{T}; \mathcal{C})$, and $t_0, t \in \mathbb{T}$, let $\Phi^F(x_0, \mu, t_0, t) \in \mathsf{M}$ be the evaluation at t of the solution to (3.1) with initial condition $\xi(t_0) = x_0$.

(iii) Let $x_0 \in \mathsf{M}$, let $\mathbb{T}' \subseteq \mathbb{T}$, and let $t_0 \in \mathbb{T}'$. A control $\mu \in \mathrm{L}^\infty_{\mathrm{loc}}(\mathbb{T}', \mathcal{C})$ is ***admissible*** for (t_0, x_0) on $\mathbb{T}'$ if there exists a solution to the initial value problem

$$\xi'(t) = F(\xi(t), \mu(t)), \qquad \xi(t_0) = x_0, \tag{3.2}$$

defined for every $t \in \mathbb{T}'$. Let us denote by $\mathrm{Adm}^\infty_\Sigma(x_0, t_0, \mathbb{T}')$ the set of admissible controls for (t_0, x_0) on $\mathbb{T}'$. •

We shall make use of the time-reverse system associated to a control system.

3.3 Definition: If $\Sigma = (\mathsf{M}, \mathcal{C}, F)$ is a control system, the ***time-reversed system*** is $\Sigma_- = (\mathsf{M}, \mathcal{C}, -F)$, where $(-F)(x, u) = -(F(x, u))$. •

The following simple lemma is one we shall reference a few times.

3.4 Lemma: *Let $\Sigma = (\mathsf{M}, \mathcal{C}, F)$ be a control system and let $x_0, x_1 \in \mathsf{M}$. If $\mu \in \mathrm{L}^\infty_{\mathrm{loc}}([t_0, t_1]; \mathcal{C})$ and if $\mu_- \in \mathrm{L}^\infty_{\mathrm{loc}}([t_0, t_1]; \mathcal{C})$ is defined by*

$$\mu_-(t) = \mu((t_0 + t_1) - t),$$

then $x_1 = \Phi^F(x_0, \mu, t_0, t_1)$ if and only if $x_0 = \Phi^{-F}(x_1, \mu_-, t_0, t_1)$.

Proof: Let $\xi(t) = \Phi^F(x_0, \mu, t_0, t)$ and take

$$\xi_-(t) = \xi((t_0 + t_1) - t).$$

We then calculate

$$\xi_-'(t) = -\xi'((t_0 + t_1) - t) = -F(\xi((t_0 + t_1) - t), \mu((t_0 + t_1) - t)) = -F(\xi_-(t), \mu_-(t)).$$

Now we note that $\xi_-(t_0) = \xi(t_1)$ so that $\xi_-(t) = \Phi^{-F}(x_1, \mu_-, t_0, t)$. Taking $t = t_1$ gives $x_0 = \Phi^{-F}(x_1, \mu_-, t_0, t_1)$. That the equality $x_0 = \Phi^{-F}(x_1, \mu_-, t_0, t_1)$ follows from $x_1 = \Phi^F(x_0, \mu, t_0, t_1)$ is obtained by replacing F with $-F$ in the preceding argument. ■

3.2. Reachability definitions. Let us first define reachable sets.

3.5 Definition: Let $\Sigma = (\mathsf{M}, \mathcal{C}, F)$ be a control system.

(i) Let $t \in \mathbb{R}_{\geq 0}$. The ***reachable set from x_0 in time t*** is

$$\mathcal{R}_\Sigma(x_0, t) = \{\Phi^F(x_0, \mu, 0, t) \mid \mu \in \mathrm{Adm}^\infty_\Sigma(x_0, 0, t, \mathbb{T}'),\ [0, t] \subseteq \mathbb{T}'\}.$$

(ii) Let $t \in \mathbb{R}_{\geq 0}$. The ***reachable set from x_0 in time at most t*** is

$$\mathcal{R}_\Sigma(x_0, \leq t) = \bigcup_{\tau \in [0, t]} \mathcal{R}_\Sigma(x_0, \tau).$$

(iii) The ***reachable set from x_0*** is

$$\mathcal{R}_\Sigma(x_0) = \bigcup_{t \in \mathbb{R}_{\geq 0}} \mathcal{R}_\Sigma(x_0, t).$$ •

With these definitions, one can easily say what is meant by reachability in its various forms.

3.6 Definition: Let $\Sigma = (\mathsf{M}, \mathcal{C}, F)$ be a control system and let $x_0 \in \mathsf{M}$.

(i) The system Σ is ***accessible*** from x_0 if $\operatorname{int}(\mathcal{R}_\Sigma(x_0)) \neq \varnothing$.

(ii) The system Σ is ***small-time accessible*** from x_0 if there exists $T \in \mathbb{R}_{>0}$ such that $\operatorname{int}(\mathcal{R}_\Sigma(x_0, \leq t)) \neq \varnothing$ for every $t \in]0, T]$.

(iii) The system Σ is ***small-time locally accessible*** from x_0 if $\Sigma|\mathcal{U}$ is small-time accessible from x_0 for every neighbourhood $\mathcal{U}$ of x_0.

(iv) The system Σ is ***reachable*** from x_0 if $x_0 \in \operatorname{int}(\mathcal{R}_\Sigma(x_0))$.

(v) The system Σ is ***small-time reachable*** from x_0 if there exists $T \in \mathbb{R}_{>0}$ such that $x_0 \in \operatorname{int}(\mathcal{R}_\Sigma(x_0, \leq t))$ for every $t \in]0, T]$.

(vi) The system Σ is ***small-time locally reachable*** from x_0 if $\Sigma|\mathcal{U}$ is small-time reachable from x_0 for every neighbourhood $\mathcal{U}$ of x_0.

(vii) The system Σ is ***totally reachable*** if $\mathcal{R}_\Sigma(x) = \mathsf{M}$ for every $x \in \mathsf{M}$.

(viii) A state $x \in \mathsf{M}$ is ***asymptotically reachable*** from x_0 if there exists $\mu \in \mathrm{L}^\infty_{\mathrm{loc}}(\mathbb{R}_{\geq 0}; \mathcal{C})$ such that $x = \lim_{t\to\infty} \Phi^F(x_0, \mu, 0, t)$. •

The terminology we use is not standard, although the concepts represented are.

The standard theory of accessibility for analytic systems [Sussmann and Jurdjevic 1972] implies that "accessible," "small-time accessible," and "small-time locally accessible" are all equivalent concepts, but we make the unnecessary trichotomy for the sake of completeness and consistency. The main result of the theory developed by Sussmann and Jurdjevic is that the three notions of accessibility are themselves equivalent to the Lie algebra rank condition which we now recall for the sake of completeness and later reference. As above, for $u \in \mathcal{C}$ we denote by F_u the real analytic vector field $F_u(x) = F(x, u)$. By $\mathscr{F}_\Sigma$ let us denote the family of vector fields $(F_u)_{u\in\mathcal{C}}$ and by $\mathscr{L}^{(\infty)}(\mathscr{F}_\Sigma)$ we denote the Lie algebra of vector fields generated by $\mathscr{F}_\Sigma$. We then define a distribution $\mathsf{L}^{(\infty)}(\mathscr{F}_\Sigma)$ by

$$\mathsf{L}^{(\infty)}(\mathscr{F}_\Sigma)_x = \operatorname{span}_{\mathbb{R}}(X(x)|\ X \in \mathscr{L}^{(\infty)}(\mathscr{F}_\Sigma)).$$

One then has that Σ is accessible from x_0 if and only if $\mathsf{L}^{(\infty)}(\mathscr{F}_\Sigma)_{x_0} = \mathsf{T}_{x_0}\mathsf{M}$.

We shall also make use of the notion of the orbit of the family of vector fields $\mathscr{F}_\Sigma$.

3.7 Definition: Let $\Sigma = (\mathsf{M}, \mathcal{C}, F)$ be a control system with $\mathscr{F}_\Sigma = (F_u)_{u\in\mathcal{C}}$ the family of vector fields defined above. The ***orbit*** of Σ through $x_0 \in \mathsf{M}$ is

$$\operatorname{Orb}(x_0; \mathscr{F}_\Sigma) = \{\Phi^{F_{u_1}}_{t_1} \circ \cdots \circ \Phi^{F_{u_k}}_{t_k}(x_0) \mid\ u_1, \dots, u_k \in \mathcal{C},\ t_1, \dots, t_k \in \mathbb{R},\ k \in \mathbb{Z}_{\geq 0}\},$$

assuming all vector fields are complete. •

A theorem of Nagano [1966] shows that, since the vector fields are real analytic, $\operatorname{Orb}(x_0; \mathscr{F}_\Sigma)$ is an immersed submanifold and that $\mathsf{T}_x\operatorname{Orb}(x_0; \mathscr{F}_\Sigma) = \mathsf{L}^{(\infty)}(\mathscr{F}_\Sigma)_x$.

3.3. Controllability definitions.

Let us first define controllable sets.

3.8 Definition: Let $\Sigma = (\mathsf{M}, \mathcal{C}, F)$ be a control system.

(i) Let $t \in \mathbb{R}_{\geq 0}$. The ***controllable set to x_0 in time t*** is

$$\mathcal{C}_\Sigma(x_0, t) = \{x \in \mathsf{M} \mid x_0 = \Phi^F(x, \mu, 0, t),\ \mu \in \mathrm{Adm}^\infty_\Sigma(x, 0, t, \mathbb{T}'),\ [0, t] \subseteq \mathbb{T}'\}.$$

(ii) Let $t \in \mathbb{R}_{\geq 0}$. The ***controllable set to x_0 in time at most t*** is

$$\mathcal{C}_\Sigma(x_0, \leq t) = \bigcup_{\tau \in [0,t]} \mathcal{C}_\Sigma(x_0, \tau).$$

(iii) The ***controllable set to x_0*** is

$$\mathcal{C}_\Sigma(x_0) = \bigcup_{t \in \mathbb{R}_{\geq 0}} \mathcal{C}_\Sigma(x_0, t).$$ •

We can now give our definitions for controllability.

3.9 Definition: Let $\Sigma = (\mathsf{M}, \mathcal{C}, F)$ be a control system and let $x_0 \in \mathsf{M}$.

(i) The system Σ is ***approachable*** to x_0 if $\mathrm{int}(\mathcal{C}_\Sigma(x_0)) \neq \varnothing$.

(ii) The system Σ is ***small-time approachable*** to x_0 if there exists $T \in \mathbb{R}_{>0}$ such that $\mathrm{int}(\mathcal{C}_\Sigma(x_0, \leq t)) \neq \varnothing$ for every $t \in]0, T]$.

(iii) The system Σ is ***small-time locally approachable*** to x_0 if $\Sigma|\mathcal{U}$ is small-time approachable to x_0 for every neighbourhood $\mathcal{U}$ of x_0.

(iv) The system Σ is ***controllable*** to x_0 if $x_0 \in \mathrm{int}(\mathcal{C}_\Sigma(x_0))$.

(v) The system Σ is ***small-time controllable*** to x_0 if there exists $T \in \mathbb{R}_{>0}$ such that $x_0 \in \mathrm{int}(\mathcal{C}_\Sigma(x_0, \leq t))$ for every $t \in]0, T]$.

(vi) The system Σ is ***small-time locally controllable*** to x_0 if $\Sigma|\mathcal{U}$ is small-time controllable to x_0 for every neighbourhood $\mathcal{U}$ of x_0.

(vii) A state $x \in \mathsf{M}$ is ***asymptotically controllable*** to x_0 if there exists $\mu \in \mathrm{L}^\infty_{\mathrm{loc}}(\mathbb{R}_{\geq 0}; \mathcal{C})$ such that $x_0 = \lim_{t \to \infty} \Phi^F(x, \mu, t_0, t)$. •

The definitions here perfectly mirror those for reachability. One distinction that does arise for controllability is the importance of asymptotic controllability, as it is this notion that relates to stabilisability. Thus we flesh this out with the following definitions.

3.10 Definition: Let $\Sigma = (\mathsf{M}, \mathcal{C}, F)$ be a control system and let $x_0 \in \mathsf{M}$.

(i) The system Σ is ***asymptotically controllable*** to x_0 if x is asymptotically controllable to x_0 for every $x \in \mathsf{M}$.

(ii) The system Σ is ***locally asymptotically controllable*** to x_0 if, for every neighbourhood $\mathcal{U}$ of x_0, there exists a neighbourhood $\mathcal{V} \subseteq \mathcal{U}$ of x_0 such that $\Sigma|\mathcal{V}$ is asymptotically controllable to x_0. •

The notions of asymptotic controllability used by Ancona and Bressan [1999] and Clarke, Ledyaev, Sontag, and Subotin [1997] to prove its equivalence to the existence of a stabilising feedback include some additional conditions, mainly the condition that the open-loop controls that control states to x_0 have some Lyapunov-type stability conditions, i.e., the system with the open-loop controls should stabilise x_0 in the Lyapunov sense. We do not include such conditions here since we do not really know at this point what conditions will most naturally arise from our approach.

4. Some problems and lines for future research

In this section, with the definitions from the preceding section as backdrop, I state a few simple problems that immediately arise. Then I formulate a few less crisp problems that I believe will be fruitful for future research.

4.1. Trivial consequences of the definitions. For the sake of bookkeeping, let us record some elementary results.

4.1 Lemma: *If $\Sigma = (\mathsf{M}, \mathcal{C}, F)$ is a control system, if $x_0, x_1 \in \mathsf{M}$, and if $T \in \mathbb{R}_{>0}$, then the following statements hold:*

(i) $\mathcal{R}_\Sigma(x_0, T) = \mathcal{C}_{\Sigma_-}(x_0, T)$;

(ii) $\mathcal{R}_\Sigma(x_0, \leq T) = \mathcal{C}_{\Sigma_-}(x_0, \leq T)$;

(iii) $\mathcal{R}_\Sigma(x_0) = \mathcal{C}_{\Sigma_-}(x_0)$.

Proof: These statements all follow from Lemma 3.4. ■

A consequence of these elementary observations is the following.

4.2 Lemma: *If $\Sigma = (\mathsf{M}; \mathcal{C}; F)$ is a control system and if $x_0 \in \mathsf{M}$, then Σ is accessible from x_0 if and only if it is approachable to x_0.*

Proof: This follows from Lemma 4.1, the fact that accessibility from x_0 is equivalent to the Lie algebra rank condition holding at x_0, and the fact that the Lie algebra rank condition holds at x_0 for Σ if and only if it holds at x_0 for Σ_-. ■

4.2. Problems arising from the definitions. Since this is supposed to be a paper about some problems in geometric control theory, let us precisely state some more elementary of these problems. In each case, we first give a vague version of the question, and then a more clear statement after some discussion. In all cases we consider a control system $\Sigma = (\mathsf{M}, \mathcal{C}, F)$.

4.3 Problem: (Can asymptotic reachability occur?) The question here is whether states exist that are asymptotically reachable from a given state. Without some conditions, the answer is obviously, "Yes, it is possible." For example, for the system

$$\begin{aligned} \dot{x}_1(t) &= u(t), \\ \dot{x}_2(t) &= -x_2(t), \end{aligned} \tag{4.1}$$

the state $(0, 0)$ is asymptotically reachable from $(0, 1)$. The real problem here is the following.

> *Given $x_0 \in \mathsf{M}$ and $x_1 \in \mathrm{Orb}(x_0, \mathscr{F}_\Sigma)$, is it possible that x_1 is only asymptotically reachable from x_0?* •

4.4 Problem: (Does reachability imply controllability?) If Σ is reachable (in some sense) from x_0, does this imply that Σ is also controllable to x_0 (in some similar sense)? One variation of this question is answered in the early paper of Sussmann [1979]. Here it is shown that if Σ is totally reachable then it is controllable to every point. Actually (since the previous sentence is obviously true), what is shown is that there exists a piecewise analytic feedback for the system that stabilises a given point in finite-time. However, we are interested in a local version of this result.

> *If Σ is small-time locally reachable from x_0, does it follow that Σ is small-time locally controllable to x_0?*

This question was actually answered in the affirmative by my student Pantelis Isaiah [2011]. Furthermore, he shows, following Sussmann, that if Σ[2] is small-time locally controllable from x_0, then, for any $T \in \mathbb{R}_{>0}$, there exists a piecewise analytic feedback on $\mathcal{R}_\Sigma(x_0, < T)$ that asymptotically stabilises (in the sense of Lyapunov) x_0. A key ingredient in this construction is the observation by Grasse [1992, Proposition 5.2] that, for analytic systems, small-time local reachability of Σ from x_0 is equivalent to small-time local reachability of Σ_- from x_0. •

4.5 Problem: (Does controllability imply reachability?) The real question of interest here is, "Does stabilisability imply reachability," keeping in mind that our notion of controllability is intended to stand-in for stabilisability. But for this variation to make sense, one must keep in mind that stabilisability is connected to *asymptotic* controllability [Ancona and Bressan 1999, Clarke, Ledyaev, Sontag, and Subotin 1997]. In this formulation, it is certainly not the case that asymptotic controllability implies reachability of any kind; the example (4.1) suffices to demonstrate this. Therefore, the pertinent formulation here is the following.

> *If $\Sigma|\mathrm{Orb}(x_0, \mathscr{F}_\Sigma)$ is locally asymptotically controllable to x_0, does it follow that Σ is small-time locally reachable from x_0?*

Note that on orbits the notion of asymptotic controllability to x_0 reduces to that of controllability to x_0 provided that we have an affirmative answer to Problem 4.3. In such event, we get an affirmative answer to the problem here by virtue of Lemma 3.4 and the result of Grasse [1992] on equivalence of small-time local reachability of Σ and Σ_-. •

4.6 Remark: In all three questions posed above, the reader will observe that the orbit plays a rôle. This is, in my opinion, not too surprising since the restriction to an orbit is a natural thing to do using the principle that "control theory take place on orbits of a system." However, there are many places where this principle does not apply, and interesting questions arise from considering the stabilisation of a system from states that are not in the orbit of the state to be stabilised. A perfect illustration of this is provided by bilinear systems, i.e., systems of the form

$$\dot{\boldsymbol{x}}(t) = \Big(\boldsymbol{A} + \sum_{j=1}^{m} u_j(t)\boldsymbol{B}_j\Big)\boldsymbol{x}(t),$$

[2] It is required that the control set $\mathcal{C}$ be compact and that, for each $T \in \mathbb{R}_{>0}$, there is a compact set containing all trajectories defined on $[0, T]$

for $\boldsymbol{x} \in \mathbb{R}^n$, $\boldsymbol{u} \in \mathbb{R}^m$, and $\boldsymbol{A}, \boldsymbol{B}_1, \dots, \boldsymbol{B}_m \in \mathbb{R}^{n\times n}$. While it is true that these systems arise in a natural way [see Elliott 2009], the fact is that, for a bilinear system, $\mathrm{Orb}(\boldsymbol{0}, \mathscr{F}_\Sigma) = \{\boldsymbol{0}\}$. Therefore, reachability from and controllability to $\boldsymbol{0}$ for bilinear systems is "singular" in some sense. A little more precisely, notice that, if we regard bilinear systems as control-affine systems, $\boldsymbol{0}$ is a singular point for the distribution generated by the control vector fields. Indeed, it is a particularly bad singular point because all control vector fields vanish at $\boldsymbol{0}$!

Thus we see that, while the restriction to $\mathrm{Orb}(x_0, \mathscr{F}_\Sigma)$ is quite natural from a control theoretic point of view, there are some important omissions made by this restriction. And we have no suggestions on how to handle this here, except to say that algebraic geometry will be involved. •

4.3. General lines of research and larger problems. The more or less elementary problems of the preceding section suggest that the subjects of reachability and stabilisability should not form disjoint areas of activity. However, as I attempted to emphasise in Section 2, in the present literature there is almost no "crosstalk" between the reachability and the stabilisation communities. This is regrettable, and here I formulate a few problems that I hope can serve to build a bridge between these bodies of work and the researchers contributing to them.

4.7 Problem: (Are Lie algebra constructions related to Lyapunov functions?) Existing tools for testing reachability are often based on the structure of the Lie algebra $\mathscr{L}^{(\infty)}(\mathscr{F}_\Sigma)$. It seems natural that, given the connections between reachability and controllability seen in the preceding section, the Lie algebra structure should be reflected in any Lyapunov function possessed by the system, especially in a neighbourhood of the state x_0 being stabilised. A specific sort of problem that comes to mind from thinking along these lines pertains to the regularity of Lyapunov functions. The general constructions for Lyapunov functions for asymptotically controllable systems yield Lyapunov functions that are *a priori* highly irregular [Sontag and Sussmann 1995]. In practice, however, Lyapunov functions, even when nonsmooth, are not *that* nonsmooth. One imagines that for analytic systems such as we are considering here, the resulting Lyapunov functions are somehow nicer than the merely nonsmooth Lyapunov functions of Sontag and Sussmann. •

4.8 Problem: (Do Lie algebra constructions shed light on the regularity of stabilising feedbacks?) The problem here is obviously related to the considerations put forward in Problem 4.7. Obstructions to the existence of smooth or continuous stabilising feedbacks have normally come from linearisation [Brockett 1983] or have a topological character [Brockett 1983, Coron 1990, Orsi, Praly, and Mareels 2003, Zabczyk 1989]. These linearisation and/or topological characterisations are well-known to be far from sharp. One can easily imagine, I think, that the structure of the Lie algebra $\mathscr{L}^{(\infty)}(\mathscr{F}_\Sigma)$ should be connected to the smoothness of stabilising feedback. As far as I am aware, there are no results along these lines whatsoever in the literature. •

4.9 Problem: (Can Lyapunov approaches to stabilisation be used for motion planning?) The strength of the Lyapunov-based theory of stabilisation is that, provided one knows a Lyapunov function, the design of a stabilising feedback is greatly facilitated [e.g., Sontag 1989]. For reachability, the attendant design problem is motion planning. It stands to reason that a Lyapunov function controlling *to* a point should shed light on controllers that reach *from* a point. This is another completely unexplored idea, as far as I can see. Ideas from the theory of finite-time stability and stabilisation are probably useful here [Dorato 2006, Moulay and Perruquetti 2005]. •

5. An approach to understanding the local structure of control systems

Now that we have described some of the problems we will address, let us say something about how one might attack these problems. The problem, clearly, is to appropriately describe the control theoretic properties of a system in a neighbourhood of some state x_0. Just what are "control theoretic properties," and how might one describe these? I will argue that studying piecewise constant controls captures enough of the information about the system. There is no theorem here to precisely capture what I want to say (although proving such a theorem would be a biproduct of this line of research), but instead I will point out two facts.

1. To study the reachability of analytic systems such as we are considering, it is sufficient to consider piecewise constant controls. This is pointed out by Grasse [1992].
2. The "sample and hold" control strategy of [Clarke, Ledyaev, Sontag, and Subotin 1997] for constructing stabilising feedbacks makes obvious use of piecewise constant controls.

Together, we hope that this provides enough evidence (although there is more) that studying the action of piecewise constant controls is sufficient to characterise the "control theoretic properties" of a system. We also mention that, based on our observations at the end of Section 2.1, one wants a theory that is (1) feedback-invariant, (2) coordinate-invariant, and (3) able to properly account for the character of the control set. In this section we present a construction that satisfies all of the above constraints.

The constructions in this section originate in the PhD thesis of my graduate student César Aguilar [2010]. These have been reported upon elsewhere [Aguilar and Lewis 2008, Aguilar and Lewis 2012], so we will be a little sketchy here. In particular, we shall assume the reader can parse the algebra and jet bundle notation we are about to introduce, and which is explained in the above references.

1. $\mathrm{S}^k(\mathsf{V})$: the symmetric tensors of degree k on a $\mathbb{R}$-vector space V
2. $\mathrm{S}^{\leq k}(\mathsf{V})$: $\oplus_{j=1}^{k}\mathrm{S}^j(\mathsf{V})$
3. $\mathrm{L}(\mathsf{U};\mathsf{V})$: the set of linear maps between $\mathbb{R}$-vector spaces U and V
4. $\mathrm{Hom}(\mathsf{A};\mathsf{B})$: the set of algebra homomorphisms of $\mathbb{R}$-algebras A and B
5. $\mathsf{J}^k\pi$: the kth jet bundle of a vector bundle $\pi\colon \mathsf{E}\to\mathsf{M}$
6. $\mathsf{T}^{*k}_x\mathsf{M}$: the $\mathbb{R}$-algebra of k-jets of functions at x taking the value 0 at x
7. $\mathsf{J}^k_{(x,y)}(\mathsf{M};\mathsf{N})$: the kth-order jets of mappings from $x\in\mathsf{M}$ to $y\in\mathsf{N}$, thought of as a homomorphism of the $\mathbb{R}$-algebras $\mathsf{J}^k_y\mathsf{N}$ and $\mathsf{T}^{*k}_x\mathsf{M}$
8. $j^k\Phi(x)$: the k jet at x of a mapping $\Phi\in C^\infty(\mathsf{M};\mathsf{N})$

5.1. Multitrajectories and variations. Our basic construction for investigating controllability is the following. By Φ_t^ξ we denote the flow of a vector field ξ. Thus $t \mapsto \Phi_t^\xi(x_0)$ is the integral curve of ξ through x_0. By $\pi_{\mathsf{TM}} \colon \mathsf{TM} \to \mathsf{M}$ we denote the tangent bundle projection.

5.1 Definition: Let M be a manifold, let $x_0 \in \mathsf{M}$, and let $\boldsymbol{\xi} = (\xi_1, \dots, \xi_p) \subseteq \Gamma^\infty(\pi_{\mathsf{TM}})$ be a family of smooth vector fields such that $\boldsymbol{\xi}_j$ is complete for every $j \in \{1, \dots, p\}$.

(i) The C^∞-map

$$\begin{aligned}\Phi^{\boldsymbol{\xi}}_{x_0} \colon \mathbb{R}^p &\to \mathsf{M}\\ (t_1, \dots, t_p) &\mapsto \Phi^{\xi_1}_{t_1} \circ \cdots \circ \Phi^{\xi_p}_{t_p}(x_0)\end{aligned}$$

is the ***$\boldsymbol{\xi}$-multitrajectory***.

(ii) A ***positive p-end-time variation*** is a C^∞-map $\boldsymbol{\tau} \colon \mathbb{R}_{\geq 0} \to \mathbb{R}^p_{\geq 0}$ with the property that $\boldsymbol{\tau}(0) = \mathbf{0}_p$. The set of positive p-end-time variations is denoted by ET_p^+.

(iii) Let $\boldsymbol{\tau}$ be a positive p-end-time variation. The ***order*** of the pair $(\boldsymbol{\xi}, \boldsymbol{\tau})$ at x_0, denoted $\mathrm{ord}_{x_0}(\boldsymbol{\xi}, \boldsymbol{\tau})$, is the smallest positive integer k such that

$$j^k(\Phi^{\boldsymbol{\xi}}_{x_0} \circ \boldsymbol{\tau})(0) \neq 0_{x_0}$$

(derivatives are assumed to be taken from the right). If no such k exists then the order is taken to be ∞.

(iv) Let $\boldsymbol{\tau}$ be a positive p-end-time variation such that $(\boldsymbol{\xi}, \boldsymbol{\tau})$ has finite order $k = \mathrm{ord}_{x_0}(\boldsymbol{\xi}, \boldsymbol{\tau})$. The ***$(\boldsymbol{\xi}, \boldsymbol{\tau})$-variation*** at x_0 is the curve $\nu_{\boldsymbol{\xi},\boldsymbol{\tau}}(x_0) \colon s \mapsto \Phi^{\boldsymbol{\xi}}_{x_0} \circ \boldsymbol{\tau}(s)$ and the ***$(\boldsymbol{\xi}, \boldsymbol{\tau})$-infinitesimal variation*** is the tangent vector $V_{\boldsymbol{\xi},\boldsymbol{\tau}}(x_0) \in \mathsf{T}_{x_0}\mathsf{M}$ defined by

$$V_{\boldsymbol{\xi},\boldsymbol{\tau}}(x_0) = j^k \nu_{\boldsymbol{\xi},\boldsymbol{\tau}}(x_0)(0) \in \mathrm{S}^k(\mathbb{R}^*) \otimes \mathsf{T}_{x_0}\mathsf{M} \simeq \mathsf{T}_{x_0}\mathsf{M}$$

(derivatives are assumed to be taken from the right). •

The idea of these constructions is that they provide a way of studying variations obtained using piecewise constant trajectories. These constructions can be related to the usual Lie algebraic characterisations of variations, and this is done by Aguilar [2010].

5.2. A system-independent algebraic construction. In this section, motivated by our constructions above, we make two purely algebraic constructions involving jet bundles, and which do not depend on the system.

The first observation we make is that for $k \in \mathbb{Z}_{>0}$ we have

$$j^k(\Phi^{\boldsymbol{\xi}}_{x_0} \circ \boldsymbol{\tau})(0) = j^k\boldsymbol{\tau}(0) \circ j^k\Phi^{\boldsymbol{\xi}}_{x_0}(\mathbf{0}_p),$$

where we think of

$$\begin{aligned}j^k\boldsymbol{\tau}(0) &\in \mathrm{Hom}((\mathbb{R}^p)^{*k}; (\mathbb{R})^{*k}),\\ j^k\Phi^{\boldsymbol{\xi}}_{x_0}(\mathbf{0}_p) &\in \mathrm{Hom}(\mathsf{T}^{*k}_{x_0}\mathsf{M}; (\mathbb{R}^p)^{*k})\end{aligned}$$

as homomorphisms of $\mathbb{R}$-algebras, and where we use the abbreviation $(\mathbb{R}^p)^{*k} = \mathsf{J}^k_{(\mathbf{0}_p,0)}(\mathbb{R}^p; \mathbb{R})$. This shows that it is important to know the character of $j^k\Phi^{\boldsymbol{\xi}}_{x_0}(\mathbf{0}_p)$. Indeed, this object, when restricted to the case when the vector fields $\boldsymbol{\xi}$ are from $\mathscr{F}_\Sigma$, encodes fundamental information concerning the structure of the system Σ.

Let us denote by TM^p the p-fold Whitney sum of TM with itself, and denote by $\pi^p_{\mathsf{TM}}\colon \mathsf{TM}^p \to \mathsf{M}$ the canonical projection. For a family $\boldsymbol{\xi} = (\xi_1, \dots, \xi_p)$ of C^∞-vector fields on M, let us denote by $\boldsymbol{\xi}$ the corresponding section of TM^p, accepting a convenient abuse of notation. We define a map

$$\begin{aligned}\Delta_k\colon \mathsf{V} &\to \mathrm{S}^{\leq k}(\mathsf{V})\\ v &\mapsto v \oplus (v \otimes v) \oplus \cdots \oplus (v \otimes \cdots \otimes v).\end{aligned}$$

For $\mathbb{R}$-algebras A and B we recall that $\mathrm{Hom}(\mathsf{A};\mathsf{B}) \subseteq \mathrm{L}(\mathsf{A};\mathsf{B})$—i.e., homomorphisms of algebras are linear maps—but $\mathrm{Hom}(\mathsf{A};\mathsf{B})$ is not a subspace in general.

We now have the following theorem, the most complete proof of which can be found in the thesis of Aguilar [2010].

5.2 Theorem: *For each $k, p \in \mathbb{Z}_{>0}$ there exists a unique map*

$$\mathscr{T}_p^{\,k}(x_0) \in \mathrm{L}(\mathrm{S}^{\leq k}(\mathsf{J}^{k-1}_{x_0}\pi^p_{\mathsf{TM}}); \mathrm{L}(\mathsf{T}^{*k}_{x_0}\mathsf{M}; (\mathbb{R}^p)^{*k}))$$

such that

$$\mathscr{T}_p^{\,k}(x_0)(\Delta_k(j^{k-1}\boldsymbol{\xi}(x_0))) = j^k\Phi^{\boldsymbol{\xi}}_{x_0}(\mathbf{0}_p)$$

for every family $\boldsymbol{\xi} = (\xi_1, \dots, \xi_p)$ of C^∞-vector fields. Moreover, the diagram

$$\begin{tikzcd} \Delta_1(\mathsf{J}^0_{x_0}\pi^p_{\mathsf{TM}}) \arrow[d, "\mathscr{T}_p^{1}(x_0)"'] & \Delta_2(\mathsf{J}^1_{x_0}\pi^p_{\mathsf{TM}}) \arrow[l] \arrow[d, "\mathscr{T}_p^{2}(x_0)"'] & \Delta_3(\mathsf{J}^2_{x_0}\pi^p_{\mathsf{TM}}) \arrow[l] \arrow[d, "\mathscr{T}_p^{3}(x_0)"'] & \cdots \arrow[l] \\ \mathrm{Hom}(\mathsf{T}^{*1}_{x_0}\mathsf{M}; (\mathbb{R}^p)^{*1}) & \mathrm{Hom}(\mathsf{T}^{*2}_{x_0}\mathsf{M}; (\mathbb{R}^p)^{*2}) \arrow[l] & \mathrm{Hom}(\mathsf{T}^{*3}_{x_0}\mathsf{M}; (\mathbb{R}^p)^{*3}) \arrow[l] & \cdots \arrow[l] \end{tikzcd}$$

commutes, where the horizontal arrows are the canonical projections.

The diagram from the Theorem 5.2 allows us to take the limit as $k \to \infty$ in a natural (i.e., projective) way, and so one can intrinsically capture the switching behaviour of any p vector fields up to infinite order in a single algebraic object.

In [Aguilar 2010, Aguilar and Lewis 2008] a procedure is given to construct variations using the formalism above. In [Aguilar and Lewis 2012] these variations are shown to be sufficient to determine the reachability of a class of homogeneous control systems, and also to produce a bound on the order of the variation needed to prove reachability. Thus, for these systems, the question of Agrachev [1999] on the finite determinability of reachability is answered in the affirmative.

5.3. Summary. We have introduced in this section a new tool that should be useful for studying the structure of the germ of an analytic system at a point x_0. It remains, of course, to put this tool to use to prove some interesting or useful control theoretic results. Preliminary explorations show that our approach is useful for studying reachability. However, the big step of connecting the infinitesimal information provided by Theorem 5.2 to the problem of controllability to x_0 is something that is unexplored.

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