Mathematical models for geometric control theory

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Abstract

Just as an explicit parameterisation of system dynamics by state, i.e., a choice of coordinates, can impede the identification of general structure, so it is too with an explicit parameterisation of system dynamics by control. However, such explicit and fixed parameterisation by control is commonplace in control theory, leading to definitions, methodologies, and results that depend in unexpected ways on control parameterisation. In this paper a framework is presented for modelling systems in geometric control theory in a manner that does not make any choice of parameterisation by control; the systems are called "tautological control systems." For the framework to be coherent, it relies in a fundamental way on topologies for spaces of vector fields. As such, classes of systems are considered possessing a variety of degrees of regularity: finitely differentiable; Lipschitz; smooth; real analytic. In each case, explicit geometric seminorms are provided for the topologies of spaces of vector fields that enable straightforward descriptions of time-varying vector fields and control systems. As part of the development, theorems are proved for regular (including real analytic) dependence on initial conditions of flows of vector fields depending measurably on time. Classes of "ordinary" control systems are characterised that interact with the regularity under consideration in a comprehensive way. In this framework, for example, the statement that "a smooth or real analytic control-affine system is a smooth or real analytic control system" becomes a theorem. Correspondences between ordinary control systems and tautological control systems are carefully examined, and trajectory correspondence between the two classes is proved for control-affine systems and for systems with general control dependence when the control set is compact.

Keywords. Geometric control theory, families of vector fields, topologies for spaces of vector fields, real analyticity, time-varying vector fields, linearisation

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1. Introduction

One can study nonlinear control theory from the point of view of applications, or from a more fundamental point of view, where system structure is a key element. From the practical point of view, questions that arise are often of the form, "How can we…", for example, "How can we steer a system from point A to point B?" or, "How can we stabilise this unstable equilibrium point?" or, "How can we manoeuvre this vehicle in the most efficient manner?" From a fundamental point of view, the problems are often of a more existential nature, with, "How can we" replaced with, "Can we". These existential questions are often very difficult to answer in any sort of generality.

As one thinks about these fundamental existential questions and looks into the quite extensive existing literature, one comes to understand that the question, "What is a control system?" is one whose answer must be decided upon with some care. One also begins to understand that structure coming from common physical models can be an impediment to general understanding. For example, in a real physical model, states are typically physical quantities of interest, e.g., position, current, quantity of reactant X, and so the explicit labelling of these is natural. This labelling amounts to a specific choice of coordinates, and it is now well understood that such specific choices of coordinates obfuscate structure, and so are to be avoided in any general treatment. In like manner, in a real physical model, controls are likely to have meaning that one would like to keep track of, e.g., force, voltage, flow. The maintenance of these labels in a model provides a specific parameterisation of the inputs to the system, completely akin to providing a specific coordinate parameterisation for states. However, while specific coordinate parameterisations have come (by many) to be understood as a bad idea in a general treatment, this is not the case for specific control parameterisations; models with fixed control parameterisation are commonplace in control theory. In contrast to the situation with dependence of *state* on parameterisation, the problem of eliminating dependence of *control* on parameterisation is not straightforward. In our discussion below we shall overview some of the common models for control systems, and some ways within these modelling frameworks for overcoming the problem of dependence on control parameterisation. As we shall see, the common models all have some disadvantage or other that must be confronted when using these models. In this paper we provide a means for eliminating explicit parameterisation of controls that, we believe, overcomes the problems with existing techniques. Our idea has some of its origins in the work on "chronological calculus" of Agrachev and Gamkrelidze [1978] (see also [Agrachev and Sachkov 2004]), but the approach we describe here is more general (in ways that we will describe below) and more fully developed as concerns its relationship to control theory (chronological calculus is primarily a device for understanding time-varying vector fields and flows). There are some ideas similar to ours in the approach of Sussmann [1998], but there are also some important differences, e.g., our families of vector fields are time-invariant (corresponding to vector fields with frozen control values) while Sussmann considers families of time-varying vector fields (corresponding to selecting an open-loop control). Also, the work of Sussmann does not touch on real analytic systems.

We are interested in models described by ordinary differential equations whose states are in a finite-dimensional manifold. Even within this quite narrow class of control systems, there is a lot of room to vary the models one might consider. Let us now give a brief outline of the sorts of models and methodologies of this type that are commonly present in the literature.

1.1. Models for geometric control systems: pros and cons. By this time, it is well-understood that the language of systems such as we are considering should be founded in differential geometry and vector fields on manifolds [Agrachev and Sachkov 2004, Bloch 2003, Bullo and Lewis 2004, Isidori 1995, Jurdjevic 1997, Nijmeijer and van der Schaft 1990]. This general principle can go in many directions, so let us discuss a few of these.

Our presentation here is quite vague and not very careful. In the main body of the paper, we will be less vague and more careful.

Family of vector field models. Given that manifolds and vector fields are important, a first idea of what might comprise a control system is that it is a family of vector fields. For these models, trajectories are concatenations of integral curves of vector fields from the family. This is the model used in the development of the theory of accessibility of Sussmann and Jurdjevic [1972] and in the early work of Sussmann [1978] on local controllability. The work of Hermann and Krener [1977], while taking place in the setting of systems parameterised by control (such as we shall discuss in Section 1.1), uses the machinery of families of vector fields to study controllability and observability of nonlinear systems. Indeed, a good deal of the early work in control theory is developed in this sort of framework, and it is more or less sufficient when dealing with questions where piecewise constant controls are ample enough to handle the problems of interest. The theory is also highly satisfying in that it is very differential geometric, and the work utilising this approach is often characterised by a certain elegance.

However, the approach does have the drawback of not handling well some of the more important problems of control theory, such as feedback (where controls are specified as functions of state) and optimal control (where piecewise constant controls are often not a sufficiently rich class [cf. Fuller 1960]).

It is worth mentioning at this early stage in our presentation that one of the ingredients of our approach is a sort of fusion of the "family of vector fields" approach with the more common control parameterisation approach to whose description we now turn.

Models with control as a parameter. Given the limitations of the "family of vector fields" models for physical applications and also for a theory where merely measurable controls are needed, one feels as if one has to have the control as a parameter in the model, a parameter that one can vary in a quite general manner. These sorts of models are typically described by differential equations of the form

$$\dot{x}(t) = F(x(t), u(t)),$$

where $t \mapsto u(t)$ is the control and $t \mapsto x(t)$ is a corresponding trajectory. For us, the trajectory is a curve on a differentiable manifold M, but there can be some freedom in attributing properties to the control set $\mathcal{C}$ in which u takes its values, and on the properties of the system dynamics F. (In Section 7 we describe classes of such models in differential geometric terms.) This sort of model is virtually synonymous with "nonlinear control system" in the existing control literature. A common class of systems that are studied are control-affine systems, where

$$F(x, \boldsymbol{u}) = f_0(x) + \sum_{a=1}^{k} u^a f_a(x),$$

for vector fields $f_0, f_1, \dots, f_k$ on M, and where the control $\boldsymbol{u}$ takes values in a subset of $\mathbb{R}^k$. For control-affine systems, there is an extensively developed theory of controllability based on free Lie algebras [Bianchini and Stefani 1993, Kawski 1990a, Kawski 1999, Kawski 2006, Sussmann 1983, Sussmann 1987]. We will see in Section 7.3 that control-affine systems fit into our framework in a particularly satisfying way.

The above general model, and in particular the control-affine special case, are all examples where there is an explicit parameterisation of the control set, i.e., the control u lives in a particular set and the dynamics F is determined to depend on u in some particular way. It could certainly be the case, for instance, that one could have two different systems

$$\dot{x}(t) = F_1(x(t), u_1(t)), \quad \dot{x}(t) = F_2(x(t), u_2(t))$$

with exactly the same trajectories. This has led to an understanding that one should study equivalence classes of systems. A little precisely, if one has two systems

$$\dot{x}_1(t) = F_1(x_1(t), u_1(t)), \quad \dot{x}_2(t) = F_2(x_2(t), u_2(t)),$$

with $x_a(t) \in \mathsf{M}_a$ and $u_a(t) \in \mathcal{C}_a$, $a \in \{1, 2\}$, then there may exist a diffeomorphism $\Phi\colon \mathsf{M}_1 \to \mathsf{M}_2$ and a mapping $\kappa\colon \mathsf{M}_1 \times \mathcal{C}_1 \to \mathcal{C}_2$ (with some sort of regularity that we will not bother to mention) such that

1. $T_{x_1}\Phi \circ F_1(x_1, u_1) = F_2(\Phi(x_1), \kappa(x_1, u_1))$ and
2. the trajectories $t \mapsto x_1(t)$ for the first system are in 1–1 correspondence with those of the second system by $t \mapsto \Phi \circ x_1(t)$.[1]

Let us say a few words about this sort of "feedback equivalence." One can imagine it being useful in at least two ways.

1. First of all, one might use it as a kind of "acid test" on the viability of a control theoretic construction. That is, a control theoretic construction should make sense, not just for a system, but for the equivalence class of that system. This is somewhat akin to asking that constructions in differential geometry should be independent of coordinates. Indeed, in older presentations of differential geometry, this was often how constructions were defined: they were given in coordinates, and then demonstrated to behave properly under changes of coordinate. We shall illustrate in Example 1.1 below that many common constructions in control theory do not pass the "acid test" for viability as feedback-invariant constructions.
2. Feedback equivalence is also a device for classifying control systems, the prototypical example being "feedback linearisation," the determination of those systems that are linear systems in disguise [Jakubczyk and Respondek 1980]. In differential geometry, this is akin to the classification of geometric structures on manifolds, e.g., Riemannian, symplectic, etc.

In Section 8.7 we shall consider a natural notion of equivalence for systems of the sort we are introducing in this paper, and we will show that "feedback transformations" are vacuous in that they amount to being described by mappings between manifolds. This is good news, since the whole point of our framework is to eliminate control parameterisation from the picture and so eliminate the need for considering the effects of varying this parameterisation, cf. "coordinate-free" versus "coordinate-independent" in differential geometry.

[1] We understand that there are many ways of formulating system equivalence. But here we are content to be, not only vague, but far from comprehensive.

Thus the first of the preceding uses of feedback transformations simply does not come up for us: our framework is naturally feedback-invariant. The second use of feedback transformations, as will be seen in Section 8.7, amounts to the classification of families of vector fields under push-forward by diffeomorphisms. This is generally a completely hopeless undertaking, so we will have nothing to say about this. Studying this under severe restrictions using, for example, (1) the Cartan method of equivalence [e.g., Bryant and Gardner 1995, Gardner 1989], (2) the method of generalised transformations [e.g., Kang and Krener 1992, Kang and Krener 2006], (3) the study of singularities of vector fields and distributions [e.g., Jakubczyk and Respondek 1980, Pasillas-Lépine and Respondek 2002], one might expect that some results are possible.

Let us consider an example that shows how a classical control-theoretic construction, linearisation, is not invariant under even the very weak notion of equivalence where equivalent systems are those with the same trajectories.

1.1 Example: (Linearisation is not well-defined) We consider two control-affine systems

$$\begin{aligned} \dot{x}_1(t) &= x_2(t), & \dot{x}_1(t) &= x_2(t), \\ \dot{x}_2(t) &= x_3(t)u_1(t), & \dot{x}_2(t) &= x_3(t) + x_3(t)u_1(t), \\ \dot{x}_3(t) &= u_2(t), & \dot{x}_3(t) &= u_2(t), \end{aligned}$$

with $(x_1, x_2, x_3) \in \mathbb{R}^3$ and $(u_1, u_2) \in \mathbb{R}^2$. One can readily verify that these two systems have the same trajectories. If we linearise these two systems about the equilibrium point at $(0, 0, 0)$—in the usual sense of taking Jacobians with respect to state and control [Isidori 1995, page 172], [Khalil 2001, §12.2], [Nijmeijer and van der Schaft 1990, Proposition 3.3], [Sastry 1999, page 236], and [Sontag 1998, Definition 2.7.14]—then we get the two linear systems

$$\boldsymbol{A}_1 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad \boldsymbol{B}_1 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix}, \qquad \boldsymbol{A}_2 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}, \quad \boldsymbol{B}_2 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix},$$

respectively. The linearisation on the left is not controllable, while that on the right is.

The example suggests that (1) classical linearisation is not independent of parameterisation of controls and/or (2) the classical notion of linear controllability is not independent of parameterisation of controls. We shall see in Section 9.5 that both things, in fact, are true: neither classical linearisation nor the classical linear controllability test are feedback-invariant. This may come as a surprise to some. •

This example has been particularly chosen to provide probably the simplest illustration of the phenomenon of lack of feedback-invariance of common control theoretic constructions. Therefore, it should not be a surprise that an astute reader will notice that linearising the "uncontrollable" system about the control $(1, 0)$ rather than the control $(0, 0)$ will square things away as concerns the discrepancy between the two linearisations. But after doing this, the questions of, "What are the proper *definitions* of linearisation and linear controllability?" still remain. Moreover, one might expect that as one moves to constructions in control theory more advanced than mere linearisation, the dependence of these constructions on the parameterisation of controls becomes more pronounced. Thus the likelihood that a sophisticated construction, made using a specific control parameterisation, is feedback-invariant is quite small, and in any case would need proof to verify that it is. Such verification is not typically part of the standard development of methodologies in control theory. There are at least three reasons for this: (1) the importance of feedback-invariance is not universally recognised; (2) such verifications are generally extremely difficult, nearly impossible, in fact; (3) most methodologies will fail the verification, so it is hardly flattering to one's methodology to point this out. Some discussion of this is made by Lewis [2012].

But the bottom line is that our framework simply eliminates the need for any of this sort of verification. As long as one remains within the framework, feedback-invariance is guaranteed. One of the central goals of the paper is to provide the means by which one does not have to leave the framework to get things done. As we shall see, certain technical difficulties have to be overcome to achieve this.

Fibred manifold models. As we have tried to make clear in the discussion just preceding, the standard model for control theory has the unpleasant attribute of depending on parameterisation of controls. A natural idea to overcome this unwanted dependence is to do with controls as one does with states: regard them as taking values in a differentiable manifold. Moreover, the manner in which control enters the model should also be handled in an intrinsic manner. This leads to the "fibred manifold" picture of a control system which, as far as we can tell, originated in the papers of Brockett [1977] and Willems [1979], and was further developed by Nijmeijer and van der Schaft [1982]. This idea has been picked up on by many researchers in geometric control theory, and we point to the papers [Barbero-Liñán and Muñoz-Lecanda 2009, Bus 1984, Delgado-Téllez and Ibort 2003, Langerock 2003] as illustrative examples.

The basic idea is this. A control system is modelled by a fibred manifold $\pi\colon \mathsf{C} \to \mathsf{M}$ and a bundle map $F\colon \mathsf{C} \to \mathsf{TM}$ over id_{M}:

$$\begin{tikzcd} \mathsf{C} \arrow[r, "F"] \arrow[dr, "\pi"'] & \mathsf{TM} \arrow[d, "\pi_{\mathsf{TM}}"] \\ & \mathsf{M} \end{tikzcd}$$

One says that F is "a vector field over the bundle map π." Trajectories are then curves $t \mapsto x(t)$ in M satisfying $\dot{x}(t) = F(u(t))$ for some $t \mapsto u(t)$ satisfying $x(t) = \pi \circ u(t)$. When it is applicable, this is an elegant and profitable model for control theory. For example, for control models that arise in problems of differential geometry or the calculus of variations, this can be a useful model.

The difficulty with the model is that it is not always applicable, especially in physical system models. The problem that arises is the strong regularity of the control set and, implicitly, the controls: C is a manifold so it is naturally the codomain for smooth curves. In practice, control sets in physical models are seldom manifolds, as bounds on controls lead to boundaries of the control set. Moreover, the boundary sets are seldom smooth. Also, as we have mentioned above, controls cannot be restricted to be smooth or piecewise smooth; natural classes of controls are typically merely measurable. These matters become vital in optimal control theory where bounds on control sets lead to bang-bang extremals. When these considerations are overlaid on the fibred manifold picture, it becomes considerably less appealing and indeed problematic. One might try to patch up the model by generalising the structure, but at some point it ceases to be worthwhile; the framework is simply not well suited to certain problems of control theory.

Differential inclusion models. Another way to eliminate the control dependence seen in the models with fixed control parameterisation is to instead work with differential inclusions. A differential inclusion, roughly (we will be precise about differential inclusions in Section 7.4), assigns to each $x \in \mathsf{M}$ a subset $\mathscr{X}(x) \subseteq \mathsf{T}_x\mathsf{M}$, and trajectories are curves $t \mapsto x(t)$ satisfying $\dot{x}(t) \in \mathscr{X}(x(t))$. There is a well-developed theory for differential inclusions, and we refer to the literature for what is known, e.g., [Aubin and Cellina 1984, Filippov 1988, Smirnov 2002]. There are many appealing aspects to differential inclusions as far as our objectives here are concerned. In particular, differential inclusions do away with the explicit parameterisations of the admissible tangent vectors at a state $x \in \mathsf{M}$ by simply prescribing this set of admissible tangent vectors with no additional structure. Moreover, differential inclusions generalise the control-parameterised systems described above. Indeed, given such a control-parameterised system with dynamics F, we associate the differential inclusion

$$\mathscr{X}_F(x) = \{F(x,u) \mid \ u \in \mathcal{C}\}.$$

The trouble with differential inclusions is that their theory is quite difficult to understand if one just starts with differential inclusions coming "out of the blue." Indeed, it is immediately clear that one needs some sort of conditions on a differential inclusion to ensure that trajectories exist. Such conditions normally come in the form of some combination of compactness, convexity, and semicontinuity. However, the differential inclusions that arise in control theory are *highly* structured; certainly they are more regular than merely semicontinuous and they automatically possess many trajectories. Moreover, it is not clear how to develop an *independent* theory of differential inclusions, i.e., one not making reference to standard models for control theory, that captures the desired structure (in Example 8.13–4 we suggest a natural way of characterising a class of differential inclusions useful in geometric control theory). Also, differential inclusions do not themselves, i.e., without additional structure, capture the notion of a flow that is often helpful in the standard control-parameterised models, e.g., in the Maximum Principle of optimal control theory, cf. [Sussmann 2002]. However, differential inclusions *are* a useful tool for studying trajectories, and we include them in our development of our new framework in Section 8.

The "behavioural" approach. Starting with a series of papers [Willems 1986a, Willems 1986b, Willems 1987] and the often cited review [Willems 1991], Willems provides a framework for studying system theory, with an emphasis on linear systems. The idea in this approach is to provide a framework for dynamical systems as subsets of general functions of generalised time taking values in a set. The framework is also intended to provide a mathematical notion of interconnection as relations in a set. In this framework, the most general formulation is quite featureless, i.e., maps between sets and relations in sets. With this level of generality, the basic questions have a computer science flavour to them, in terms of formal languages. When one comes to making things more concrete, say by making the time-domain an interval in $\mathbb{R}$ for continuous-time systems, one ends up with differential-algebraic equations describing the behaviours and relations. For the most part, these ideas seem to have been only reasonably fully developed for linear models [Polderman and Willems 1998];

we are not aware of substantial work on nonlinear systems in the behavioural approach. It is also the case that the considerations of feedback-invariance, such as we discuss above, are not a part of the current landscape in behavioural models, although this is possible within the context of linear systems, cf. the beautiful book of [Wonham 1985].

Thus, while there are some idealogical similarities with our objectives and those of the behavioural approach, our thinking in this paper is in a quite specific and complementary direction to the existing work on the behavioural point of view.

1.2. Attributes of a modelling framework for geometric control systems. The preceding sections are meant to illustrate some standard frameworks for modelling control systems and the motivation for consideration of these, as well as pointing out their limitations. If one is going to propose a modelling framework, it is important to understand *a priori* just what it is that one hopes to be able to do in this framework. Here is a list of possible criteria, criteria that we propose to satisfy in our framework.

1. Models should provide for control parameterisation-independent constructions as discussed above.
2. We believe that being able to handle real analytic systems is essential to a useful theory. In practice, any smooth control system is also real analytic, and one wants to be able to make use of real analyticity to both strengthen conclusions, e.g., the real analytic version of Frobenius's Theorem [Nagano 1966], and to weaken hypotheses, e.g., the infinitesimal characterisation of invariant distributions [e.g., Agrachev and Sachkov 2004, Lemma 5.2].
3. The framework should be able to handle regularity in an internally consistent manner. This means, for example, that the conclusions should be consistent with hypotheses, e.g., smooth hypotheses with continuous conclusions suggest that the framework may not be perfectly natural or perfectly well-developed. The pursuit of this internal consistency in the real analytic case contributes to many of the difficulties we encounter in the paper.
4. The modelling framework should seamlessly deal with distinctions between local and global. Many notions in control theory are highly localised, e.g., local controllability of real analytic control systems. A satisfactory framework should include a systematic way of dealing with constructions in control theory that are of an inherently local nature. Moreover, the framework should allow a systematic means of understanding the passage from local to global in cases where this is possible and/or interesting. As we shall see, there are some simple instances of these phenomena that can easily go unnoticed if one is not looking for them.
5. Our interest is in *geometric* control theory, as we believe this is the right framework for studying nonlinear systems in general. A proper framework for geometric control theory should make it natural to use the tools of differential geometry.
6. While (we believe that) differential geometric methods are essential in nonlinear control theory, the quest for geometric elegance should not be carried out at the expense of a useful theory.

1.3. An outline of the paper. Let us discuss briefly the contents of the paper.

One of the essential elements of the paper is a characterisation of seminorms for the various topologies we use. Our definitions of these seminorms unify the presentation of the various degrees of regularity we consider—finitely differentiable, Lipschitz, smooth, holomorphic, and real analytic—making it so that, after the seminorms are in place, these various cases can be treated in very similar ways in many cases. The key to the construction of the seminorms that we use is the use of connections to decompose jet bundles into direct sums. In Section 2 we present these constructions. As we see in Section 5, in the real analytic case, some careful estimates must be performed to ensure that the geometric seminorms we use do, indeed, characterise the real analytic topology.

In Sections 3, 4, and 5 we describe topologies for spaces of finitely differentiable, Lipschitz, smooth, holomorphic, and real analytic vector fields. (While we do not have a *per se* interest in holomorphic systems, holomorphic geometry has an important part to play in real analytic geometry.) While these topologies are more or less classical in the smooth, finitely differentiable, and holomorphic cases, in the real analytic case the description we give is less well-known, and indeed many of our results here are new, or provide new and useful ways of understanding existing results.

Time-varying vector fields feature prominently in geometric control theory. In Section 6 we review some notions concerning such vector fields and develop a few not quite standard constructions and results for later use. In the smooth case, the ideas we present are probably contained in the work of Agrachev and Gamkrelidze [1978] (see also [Agrachev and Sachkov 2004]), but our presentation of the real analytic case is novel. For this reason, we present a rather complete treatment of the smooth case (with the finitely differentiable and Lipschitz cases following along similar lines) so as to provide a context for the more complicated real analytic case. We should point out that, even in the smooth case, we use properties of the topology that are not normally called upon, and we see that it is these deeper properties that really tie together the various regularity hypotheses we use. Indeed, what our presentation reveals is the connection between the standard pointwise—in time and state—conditions placed on time-varying vector fields and topological characterisations. This is, we believe, a fulfilling way of understanding the meaning of the usual pointwise conditions.

In Section 7 we review quite precisely a fairly general standard modelling framework in geometric control theory. While ultimately we wish to assert that there are some difficulties with this framework, understanding it clearly will give us some context for what will be, frankly, our rather abstract notion of a control system to follow. Also, we do wish to make sure that our proposed model does indeed generalise this more concrete and standard notion, so to prove this we need precise definitions. Additionally, as with time-varying vector fields, we show how natural pointwise regularity conditions are equivalent to topological characterisations of systems. Thus, while we do generalise the standard modelling framework for control theory, in doing so we arrive at a deeper understanding of this framework. For example, we introduce for the first time the notion of a "real analytic control system," which means that the real analytic structure is fully integrated into the structure of the control system; this is only made possible by understanding the topology for the space of real analytic vector fields. As a result, seemingly tautological statements like, "A real analytic control-affine system is a real analytic control system," now are theorems in our framework. Also, interestingly, we will show that, in many cases, our more general modelling framework can be cast in the standard framework, albeit in a non-obvious way; see Example 8.10–2.

In Section 8 we provide our modelling framework for geometric control systems, defining what we shall call "tautological control systems.[2]" After developing the background needed, we provide the definitions and then give the notion of a trajectory for these systems. We also show that our framework includes the standard framework of Section 7 as a special case. We carefully establish correspondences between our generalised models, the standard models, and differential inclusion models. Included in this correspondence is a description of the relationships between trajectories for these models. One feature of our framework that will appear strange initially is our use of presheaves and sheaves. These are the devices by which we can attempt to patch together local constructions to give global constructions. We understand that the use of this language will seem unnecessarily complicated initially. However, it will have its uses in the paper, e.g., our notion of transformations between tautological control systems is based on a standard construction in sheaf theory, and we will point out places where the reader may have unwittingly encountered some shadows of sheaf theory, even in familiar places in control theory.

We study the linearisation of tautological control systems in Section 9. The theory here has many satisfying elements attached to it. First of all, the framework naturally suggests *two* sorts of linearisation, one with respect to a reference trajectory and another with respect to a reference flow. This is an interesting distinction, and one that is, as far as we know, hitherto not made clear in the literature. Also, of course, our theory comprehends and rectifies the problems encountered in Example 1.1.

What is presented in this paper is the result of initial explorations of a modelling framework for geometric control theory. We certainly have not fully fleshed out all parts of this framework ourselves, despite the substantial length of the paper. In the closing section of the paper, Section 10, we outline places where there is obvious further work to be done.

1.4. Summary of contributions. This is a long and complex paper with many results, some significant, and some necessary for the foundations of the approach, but not necessarily significant *per se*. In order to facilitate the reading of the paper, we highlight the contributions that we feel are important. First we point out the more significant contributions.

1. The main contribution of the paper is the general feedback-invariant framework. This main contribution has with it a few novel components.
   (a) Our framework generalises the standard formulation and has some satisfying relationships with the standard theory and the theory of differential inclusions; see Proposition 8.11 and the trajectory equivalence results of Section 8.6. We conclude, for example, that our generalised formulation agrees with the standard formulation in two important cases: (i) for control-affine systems with arbitrary control sets (Theorem 8.37); (ii) for systems depending generally on the control with compact control sets (Theorem 8.35).
   (b) The framework relies in an essential and nontrivial way on topologies for spaces of vector fields. The full development of these topologies, and their integration into a theory for control systems, is fully executed here for the first time.
   (c) The formulation uses the theory of presheaves and sheaves in an essential way.

[2]The terminology "tautological" arises from two different attributes of our framework. First of all, when one makes the natural connection from our systems to standard control systems, we encounter the identity map (Example 8.10–2). Second, in our framework we prove that the only pure feedback transformation is the identity transformation (cf. Proposition 8.47).

  (d) Using a notion of morphism borrowed from sheaf theory, we *prove* that equivalence for our systems is simply diffeomorphism equivalence of vector fields; see Proposition 8.47. That is to say, we *prove* that our framework cannot involve any "feedback transformation" in the usual sense.

2. We provide, for the first time, a comprehensive treatment of real analytic time-varying vector fields and control systems. In particular,
   (a) we provide a concrete, usable, geometric characterisation of the real analytic topology by specifying a family of geometric seminorms (Theorem 5.5),
   (b) we provide conditions that ensure that a real analytic vector field with measurable time dependence will have a flow depending on initial conditions in a real analytic manner (Theorem 6.26),
   (c) we provide conditions that ensure that trajectories for a real analytic control system depend on initial conditions in a real analytic manner (Propositions 7.18 and 7.22), and
   (d) we show that real analytic vector fields depending measurably on time and/or continuously on a parameter can often be extended to holomorphic vector fields depending on time or parameter (Theorems 6.25 and 7.14).

   The last three results rely, sometimes in highly nontrivial ways, on the properties of the real analytic topology for vector fields.

3. We fully develop various "weak" formulations of properties such as continuity, boundedness, measurability, and integrability for spaces of finitely differentiable, Lipschitz, smooth, and real analytic vector fields. These weak formulations come in two forms, one for evaluations of vector fields on functions by Lie differentiation, which we call the "weak-$\mathscr{L}$" topology (see Theorems 3.5, 3.8, 3.14, and 5.8 and their corollaries), and one for evaluations in time and space (see Theorems 6.4, 6.10, and 6.22). These results use deep properties of the topologies for spaces of vector fields derived in Sections 3 and 5. In the existing literature, these weak formulations are often used without reference to their "strong" counterparts; here we make the (unsurprising, but sometimes nontrivial) link explicit.

4. In Section 9 we provide a coherent theory for linearisation of systems in our framework. The theory of linearisation that we develop is necessarily feedback-invariant, and as a consequence reveals some interesting structure that has previously been hidden by the standard treatment of linearisation which is *not* feedback-invariant, as we have seen in Example 1.1.

Along the way to these substantial definitions and results, we uncover a few minor, but still interesting, results and constructions.

5. We use to advantage some not entirely elementary geometric constructions to make elegant coordinate-free proofs. Here are some instances of this.
   (a) We provide a decomposition for jet bundles of sections of a vector bundle using the theory of connections; see Lemma 2.1. This decomposition is used to provide a concrete and useful collection of seminorms for the finitely differentiable, Lipschitz, and smooth compact-open topologies, and the real analytic topology. Indeed, without these seminorms, our descriptions of these topologies would be incomprehensible, as opposed to merely difficult as it already is in the real analytic case.

(b) We use our seminorms in an essential way to prove the equivalence of "weak-$\mathscr{L}$" and "strong" versions of the finitely differentiable, Lipschitz, and smooth compact-open topologies, and the real analytic topology for vector fields; see Theorems 3.5, 3.8, 3.14, and 5.8.

(c) These seminorms allow for relatively clean characterisations of the finitely differentiable, Lipschitz, and smooth compact-open, and real analytic topologies for vector fields on tangent bundles, using induced affine connections and Riemannian metrics on tangent bundles. These constructions appear in the proofs concerning linearisation; see Lemmata 9.2 and 9.7.

(d) The double vector bundle structure of the double tangent bundle TTM is used to provide a slick justification of our definition of linearisation, culminating in the formula (9.12).

6. We provide a "weak-$\mathscr{L}$" characterisation of the compact-open topology for holomorphic vector fields on a Stein manifold; see Theorem 4.5.

1.5. Notation, conventions, and background. In this section we overview what is needed to read the paper. We do use a lot of specialised material in essential ways, and we certainly do not review this comprehensively. Instead, we simply provide a few facts, the notation we shall use, and recommended sources. Throughout the paper we have tried to include precise references to material needed so that a reader possessing enthusiasm and lacking background can begin to chase down all of the ideas upon which we rely.

We shall use the slightly unconventional, but perfectly rational, notation of writing $A \subseteq B$ to denote set inclusion, and when we write $A \subset B$ we mean that $A \subseteq B$ and $A \neq B$. By id_A we denote the identity map on a set A. For a product $\prod_{i\in I} X_i$ of sets, $\mathrm{pr}_j\colon \prod_{i\in I} X_i \to X_j$ is the projection onto the jth component. For a subset $A \subseteq X$, we denote by χ_A the characteristic function of A, i.e.,

$$\chi_A(x) = \begin{cases} 1, & x \in A, \\ 0, & x \notin A. \end{cases}$$

By $\mathrm{card}(A)$ we denote the cardinality of a set A. By $\mathfrak{S}_k$ we denote the symmetric group on k symbols. We shall have occasion to talk about set-valued maps. If X and Y are sets and Φ is a set-valued map from X to Y, i.e., $\Phi(x)$ is a subset of Y, we shall write $\Phi\colon X \twoheadrightarrow Y$. By $\mathbb{Z}$ we denote the set of integers, with $\mathbb{Z}_{\geq 0}$ denoting the set of nonnegative integers and $\mathbb{Z}_{>0}$ denoting the set of positive integers. We denote by $\mathbb{R}$ and $\mathbb{C}$ the sets of real and complex numbers. By $\mathbb{R}_{\geq 0}$ we denote the set of nonnegative real numbers and by $\mathbb{R}_{>0}$ the set of positive real numbers. By $\overline{\mathbb{R}}_{\geq 0} = \mathbb{R}_{\geq 0} \cup \{\infty\}$ we denote the extended nonnegative real numbers. By δ_{jk}, $j,k \in \{1,\dots,n\}$, we denote the Kronecker delta.

We shall use constructions from algebra and multilinear algebra, referring to [Hungerford 1980], [Bourbaki 1989a, Chapter III], and [Bourbaki 1990, §IV.5]. If F is a field (for us, typically $\mathsf{F} \in \{\mathbb{R},\mathbb{C}\}$), if V is an F-vector space, and if $A \subseteq \mathsf{V}$, by $\mathrm{span}_{\mathsf{F}}(A)$ we denote the subspace generated by A. If F is a field and if U and V are F-vector spaces, by $\mathrm{Hom}_{\mathsf{F}}(\mathsf{U};\mathsf{V})$ we denote the set of linear maps from U to V. We denote $\mathrm{End}_{\mathsf{F}}(\mathsf{V}) = \mathrm{Hom}_{\mathsf{F}}(\mathsf{V};\mathsf{V})$ and $\mathsf{V}^* = \mathrm{Hom}_{\mathsf{F}}(\mathsf{V};\mathsf{F})$. If $\alpha \in \mathsf{V}^*$ and $v \in \mathsf{V}$, we may sometimes denote by $\langle \alpha; v\rangle \in \mathsf{F}$ the natural pairing. The k-fold tensor product of V with itself is denoted by $\mathrm{T}^k(\mathsf{V})$. Thus, if V is finite-dimensional, we identify $\mathrm{T}^k(\mathsf{V}^*)$ with the k-multilinear F-valued functions on V^k by

$$(\alpha^1 \otimes \cdots \otimes \alpha^k)(v_1, \dots, v_k) = \alpha^1(v_1) \cdots \alpha^k(v_k).$$

By $\mathrm{S}^k(\mathsf{V}^*)$ we denote the symmetric tensor algebra of degree k, which we identify with the symmetric k-multilinear F-valued functions on V^k, or polynomial functions of homogeneous degree k on V.

If $\mathbb{G}$ is an inner product on a $\mathbb{R}$-vector space V, we denote by $\mathbb{G}^\flat \in \mathrm{Hom}_{\mathbb{R}}(\mathsf{V}; \mathsf{V}^*)$ the associated mapping and by $\mathbb{G}^\sharp \in \mathrm{Hom}_{\mathbb{R}}(\mathsf{V}^*; \mathsf{V})$ the inverse of $\mathbb{G}^\flat$ when it is invertible.

For a topological space $\mathcal{X}$ and $A \subseteq \mathcal{X}$, $\mathrm{int}(A)$ denotes the interior of A and $\mathrm{cl}(A)$ denotes the closure of A. Neighbourhoods will always be open sets. The support of a continuous function f (or any other kind of object for which it makes sense to have a value "zero") is denoted by $\mathrm{supp}(f)$.

By $\mathsf{B}(r, \boldsymbol{x}) \subseteq \mathbb{R}^n$ we denote the open ball of radius r and centre $\boldsymbol{x}$. In like manner, $\overline{\mathsf{B}}(r, \boldsymbol{x})$ denotes the closed ball. If $r \in \mathbb{R}_{>0}$ and if $x \in \mathbb{F}$, $\mathbb{F} \in \{\mathbb{R}, \mathbb{C}\}$, we denote by

$$\mathsf{D}(r, x) = \{x' \in \mathbb{F} \mid |x' - x| < r\}$$

the disk of radius r centred at x. If $\boldsymbol{r} \in \mathbb{R}^n_{>0}$ and if $\boldsymbol{x} \in \mathbb{F}^n$, we denote by

$$\mathsf{D}(\boldsymbol{r}, \boldsymbol{x}) = \mathsf{D}(r_1, x_1) \times \cdots \times \mathsf{D}(r_n, x_n)$$

the polydisk with radius $\boldsymbol{r}$ centred at $\boldsymbol{x}$. In like manner, $\overline{\mathsf{D}}(\boldsymbol{r}, \boldsymbol{x})$ denotes the closed polydisk.

Elements of $\mathbb{F}^n$, $\mathbb{F} \in \{\mathbb{R}, \mathbb{C}\}$, are typically denoted with a bold font, e.g., "$\boldsymbol{x}$." The standard basis for $\mathbb{F}^n$ is denoted by $(\boldsymbol{e}_1, \dots, \boldsymbol{e}_n)$. By $\boldsymbol{I}_n$ we denote the $n \times n$ identity matrix. We denote by $\mathrm{L}(\mathbb{R}^n; \mathbb{R}^m)$ the set of linear maps from $\mathbb{R}^n$ to $\mathbb{R}^m$ (this is the same as $\mathrm{Hom}_{\mathbb{R}}(\mathbb{R}^n; \mathbb{R}^m)$, of course, but the more compact notation is sometimes helpful). The invertible linear maps on $\mathbb{R}^n$ we denote by $\mathsf{GL}(n; \mathbb{R})$. By $\mathrm{L}(\mathbb{R}^{n_1}, \dots, \mathbb{R}^{n_k}; \mathbb{R}^m)$ we denote the set of multilinear mappings from $\prod_{j=1}^k \mathbb{R}^{n_j}$ to $\mathbb{R}^m$. We abbreviate by $\mathrm{L}^k(\mathbb{R}^n; \mathbb{R}^m)$ the k-multilinear maps from $(\mathbb{R}^n)^k$ to $\mathbb{R}^m$. We denote by $\mathrm{L}^k_{\mathrm{sym}}(\mathbb{R}^n; \mathbb{R}^m)$ the set of symmetric k-multilinear maps from $(\mathbb{R}^n)^k$ to $\mathbb{R}^m$. With our notation above, $\mathrm{L}^k_{\mathrm{sym}}(\mathbb{R}^n; \mathbb{R}^m) \simeq \mathrm{S}^k((\mathbb{R}^n)^*) \otimes \mathbb{R}^m$, but, again, we prefer the slightly more compact notation in this special case.

If $\mathcal{U} \subseteq \mathbb{R}^n$ is open and if $\boldsymbol{\Phi} \colon \mathcal{U} \to \mathbb{R}^m$ is differentiable at $\boldsymbol{x} \in \mathcal{U}$, we denote its derivative by $\boldsymbol{D\Phi}(\boldsymbol{x})$. Higher-order derivatives, when they exist, are denoted by $\boldsymbol{D}^r\boldsymbol{\Phi}(\boldsymbol{x})$, r being the order of differentiation. We will also use the following partial derivative notation. Let $\mathcal{U}_j \subseteq \mathbb{R}^{n_j}$ be open, $j \in \{1, \dots, k\}$, and let $\boldsymbol{\Phi} \colon \mathcal{U}_1 \times \cdots \times \mathcal{U}_k \to \mathbb{R}^m$ be continuously differentiable. The derivative of the map

$$\boldsymbol{x}_j \mapsto \boldsymbol{\Phi}(\boldsymbol{x}_{1,0}, \dots, \boldsymbol{x}_j, \dots, \boldsymbol{x}_{k,0})$$

at $\boldsymbol{x}_{j,0}$ is denoted by $\boldsymbol{D}_j\boldsymbol{\Phi}(\boldsymbol{x}_{1,0}, \dots, \boldsymbol{x}_{k,0})$. Higher-order partial derivatives, when they exist, are denoted by $\boldsymbol{D}_j^r\boldsymbol{\Phi}(\boldsymbol{x}_{1,0}, \dots, \boldsymbol{x}_{k,0})$, r being the order of differentiation. We recall that if $\boldsymbol{\Phi} \colon \mathcal{U} \to \mathbb{R}^m$ is of class C^k, $k \in \mathbb{Z}_{>0}$, then $\boldsymbol{D}^k\boldsymbol{\Phi}(\boldsymbol{x})$ is symmetric. We shall sometimes find it convenient to use multi-index notation for derivatives. A ***multi-index*** with length n is an element of $\mathbb{Z}^n_{\geq 0}$, i.e., an n-tuple $I = (i_1, \dots, i_n)$ of nonnegative integers. If $\boldsymbol{\Phi} \colon \mathcal{U} \to \mathbb{R}^m$ is a smooth function, then we denote

$$\boldsymbol{D}^I\boldsymbol{\Phi}(\boldsymbol{x}) = \boldsymbol{D}_1^{i_1} \cdots \boldsymbol{D}_n^{i_n}\boldsymbol{\Phi}(\boldsymbol{x}).$$

We will use the symbol $|I| = i_1 + \cdots + i_n$ to denote the order of the derivative. Another piece of multi-index notation we shall use is

$$\boldsymbol{a}^I = a_1^{i_1} \cdots a_n^{i_n},$$

for $\boldsymbol{a} \in \mathbb{R}^n$ and $I \in \mathbb{Z}_{\geq 0}^n$. Also, we denote $I! = i_1! \cdots i_n!$.

If V is a $\mathbb{R}$-vector space and if $A \subseteq \mathsf{V}$, we denote by $\operatorname{conv}(A)$ the convex hull of A, by which we mean the set of all convex combinations of elements of A.

Our differential geometric conventions mostly follow [Abraham, Marsden, and Ratiu 1988]. Whenever we write "manifold," we mean "second-countable Hausdorff manifold." This implies, in particular, that manifolds are assumed to be metrisable [Abraham, Marsden, and Ratiu 1988, Corollary 5.5.13]. If we use the letter "n" without mentioning what it is, it is the dimension of the connected component of the manifold M with which we are working at that time. The tangent bundle of a manifold M is denoted by $\pi_{\mathsf{TM}} \colon \mathsf{TM} \to \mathsf{M}$ and the cotangent bundle by $\pi_{\mathsf{T}^*\mathsf{M}} \colon \mathsf{T}^*\mathsf{M} \to \mathsf{M}$. The derivative of a differentiable map $\Phi \colon \mathsf{M} \to \mathsf{N}$ is denoted by $T\Phi \colon \mathsf{TM} \to \mathsf{TN}$, with $T_x\Phi = T\Phi|\mathsf{T}_x\mathsf{M}$. If $I \subseteq \mathbb{R}$ is an interval and if $\xi \colon I \to \mathsf{M}$ is a curve that is differentiable at $t \in I$, we denote the tangent vector field to the curve at t by $\xi'(t) = T_t\xi(1)$. We use the symbols Φ^* and Φ_* for pull-back and push-forward. Precisely, if g is a function on N, $\Phi^*g = g \circ \Phi$, and if Φ is a diffeomorphism, if f is a function on M, if X is a vector field on M, and if Y is a vector field on N, we have $\Phi_*f = f \circ \Phi^{-1}$, $\Phi_*X = T\Phi \circ X \circ \Phi^{-1}$, and $\Phi^*Y = T\Phi^{-1} \circ Y \circ \Phi$. The flow of a vector field X is denoted by Φ_t^X, so $t \mapsto \Phi_t^X(x)$ is the integral curve of X passing through x at $t = 0$. We shall also use time-varying vector fields, but will develop the notation for the flows of these in the text.

If $\pi \colon \mathsf{E} \to \mathsf{M}$ is a vector bundle, we denote the fibre over $x \in \mathsf{M}$ by E_x and we sometimes denote by 0_x the zero vector in E_x. If $\mathsf{S} \subseteq \mathsf{M}$ is a submanifold, we denote by $\mathsf{E}|\mathsf{S}$ the restriction of E to S which we regard as a vector bundle over S. The ***vertical subbundle*** of E is the subbundle of TE defined by $\mathsf{VE} = \ker(T\pi)$. If $\mathbb{G}$ is a fibre metric on E, i.e., a smooth assignment of an inner product to each of the fibres of E, then $\|\cdot\|_{\mathbb{G}}$ denotes the norm associated with the inner product on fibres. If $\pi \colon \mathsf{E} \to \mathsf{M}$ is a vector bundle and if $\Phi \colon \mathsf{N} \to \mathsf{M}$ is a smooth map, then $\Phi^*\pi \colon \Phi^*\mathsf{E} \to \mathsf{N}$ denotes the pull-back of E to N [Kolář, Michor, and Slovák 1993, §III.9.5]. The dual of a vector bundle $\pi \colon \mathsf{E} \to \mathsf{M}$ is denoted by $\pi^* \colon \mathsf{E}^* \to \mathsf{M}$.

Generally we will try hard to avoid coordinate computations. However, they are sometimes unavoidable and we will use the Einstein summation convention when it is convenient to do so, but we will not do so slavishly.

We will work in both the smooth and real analytic categories, with occasional forays into the holomorphic category. We will also work with finitely differentiable objects, i.e., objects of class C^r for $r \in \mathbb{Z}_{\geq 0}$. (We will also work with Lipschitz objects, but will develop the notation for these in the text.) A good reference for basic real analytic analysis is [Krantz and Parks 2002], but we will need ideas going beyond those from this text, or any other text. Relatively recent work of e.g., [Domański 2012], [Vogt 2013], and [Domański and Vogt 2000] has shed a great deal of light on real analytic analysis, and we shall take advantage of this work. An analytic manifold or mapping will be said to be of ***class*** $\boldsymbol{\mathrm{C}^\omega}$. Let $r \in \mathbb{Z}_{\geq 0} \cup \{\infty, \omega\}$. The set of mappings of class C^r between manifolds M and N is denoted by $\mathrm{C}^r(\mathsf{M}; \mathsf{N})$. We abbreviate $\mathrm{C}^r(\mathsf{M}) = \mathrm{C}^r(\mathsf{M}; \mathbb{R})$. The set of sections of a vector bundle $\pi \colon \mathsf{E} \to \mathsf{M}$ of class C^r is denoted by $\Gamma^r(\mathsf{E})$. Thus, in particular, $\Gamma^r(\mathsf{TM})$ denotes the set of vector fields of class C^r. We shall think of $\Gamma^r(\mathsf{E})$ as a $\mathbb{R}$-vector space with the natural pointwise addition and scalar multiplication operations. If $f \in \mathrm{C}^r(\mathsf{M})$, $\mathrm{d}f \in \Gamma^r(\mathsf{T}^*\mathsf{M})$ denotes the differential of f. If $X \in \Gamma^r(\mathsf{TM})$ and $f \in \mathrm{C}^r(\mathsf{M})$, we denote the Lie derivative of f with respect to X by $\mathscr{L}_X f$.

We also work with holomorphic, i.e., complex analytic, manifolds and associated geometric constructions; real analytic geometry, at some level, seems to unavoidably rely on holomorphic geometry. A nice overview of holomorphic geometry, and some of its connections to real analytic geometry, is given in the book of Cieliebak and Eliashberg [2012]. There are many specialised texts on the subject of holomorphic geometry, including [Demailly 2012, Fritzsche and Grauert 2002, Gunning and Rossi 1965, Hörmander 1966] and the three volumes of Gunning [1990a], Gunning [1990b], and Gunning [1990c]. For our purposes, we shall just say the following things. By TM we denote the holomorphic tangent bundle of M. This is the object which, in complex differential geometry, is commonly denoted by $\mathsf{T}^{1,0}\mathsf{M}$. For holomorphic manifolds M and N, we denote by $\mathrm{C}^{\mathrm{hol}}(\mathsf{M};\mathsf{N})$ the set of holomorphic mappings from M to N, by $\mathrm{C}^{\mathrm{hol}}(\mathsf{M})$ the set of holomorphic functions on M (note that these functions are $\mathbb{C}$-valued, not $\mathbb{R}$-valued, of course), and by $\Gamma^{\mathrm{hol}}(\mathsf{E})$ the space of holomorphic sections of an holomorphic vector bundle $\pi\colon \mathsf{E} \to \mathsf{M}$. We shall use both the natural $\mathbb{C}$- and, by restriction, $\mathbb{R}$-vector space structures for $\Gamma^{\mathrm{hol}}(\mathsf{E})$.

We will make use of the notion of a "Stein manifold." For practical purposes, these can be taken to be holomorphic manifolds admitting a proper holomorphic embedding in complex Euclidean space.[3] Stein manifolds are characterised by having lots of holomorphic functions, distinguishing them from general holomorphic manifolds, e.g., compact holomorphic manifolds whose only holomorphic functions are those that are locally constant. There is a close connection between Stein manifolds and real analytic manifolds, and this explains our interest in Stein manifolds. We shall point out these connections as they arise in the text.

We shall occasionally make use of Cartan's Theorems A and B for Stein manifolds and real analytic manifolds; these are theorems about the cohomology of certain sheaves. In the holomorphic case, the original source is [Cartan 1951-52], but there are many good treatments in textbooks, including in [Taylor 2002]. For the real analytic case, the only complete reference seems to be the original work of Cartan [1957], although the short book of Guaraldo, Macrí, and Tancredi [1986] is also helpful. In using these theorems (and sometimes in other places where we use sheaves) we will use the following notation. Let $r \in \mathbb{Z}_{\geq 0} \cup \{\infty, \omega, \mathrm{hol}\}$ and let M be a smooth, real analytic, or holomorphic manifold, such as is demanded by r. By $\mathscr{C}^r_{\mathsf{M}}$ we denote the sheaf of functions of class C^r and by $\mathscr{C}^r_{x,\mathsf{M}}$ the set of germs of this sheaf at $x \in \mathsf{M}$. If $\pi\colon \mathsf{E} \to \mathsf{M}$ is a C^r-vector bundle, then $\mathscr{G}^r_{\mathsf{E}}$ denotes the sheaf of C^r-sections of E with $\mathscr{G}^r_{x,\mathsf{E}}$ the set of germs at x. The germ of a function (resp. section) at x will be denoted by $[f]_x$ (resp. $[\xi]_x$).

We will make use of jet bundles, and a standard reference is [Saunders 1989]. Appropriate sections of [Kolář, Michor, and Slovák 1993] (especially §12) are also useful. If $\pi\colon \mathsf{E} \to \mathsf{M}$ is a vector bundle and if $k \in \mathbb{Z}_{\geq 0}$, we denote by $\mathsf{J}^k\mathsf{E}$ the bundle of k-jets of E. For a section

[3] The equivalence of this to other characterisations of Stein manifolds is due to **RR:55**. A reader unfamiliar with holomorphic manifolds should note that, unlike in the smooth or real analytic cases, it is *not* generally true that an holomorphic manifold can be embedded in complex Euclidean space, even after the usual elimination of topological pathologies such as non-paracompactness. For example, compact holomorphic manifolds can never be holomorphically embedded in complex Euclidean space.

ξ of E, we denote by $j_k\xi$ the corresponding section of $\mathsf{J}^k\mathsf{E}$. The projection from $\mathsf{J}^k\mathsf{E}$ to $\mathsf{J}^l\mathsf{E}$, $l \leq k$, is denoted by π^k_l. If M and N are manifolds, we denote by $\mathsf{J}^k(\mathsf{M};\mathsf{N})$ the bundle of k jets of mappings from M to N. If $\Phi \in \mathrm{C}^\infty(\mathsf{M};\mathsf{N})$, $j_k\Phi$ denotes its k-jet, which is a mapping from M to $\mathsf{J}^k(\mathsf{M};\mathsf{N})$. In the proof of Theorem 6.6 we will briefly make use of jets of sections of fibred manifolds. We shall introduce there the notation we require, and the reader can refer to [Saunders 1989] to fill in the details.

We shall make use of connections, and refer to [Kolář, Michor, and Slovák 1993, §11, §17] for a comprehensive treatment of these, or to [Kobayashi and Nomizu 1963] for another comprehensive treatment and an alternative point of view.

We shall make reference to elementary ideas from sheaf theory; indeed we have already made reference to sheaves above. It will not be necessary to understand this theory deeply, at least not in the present paper. In particular, a comprehensive understanding of sheaf cohomology is not required, although, as indicated above, we do make use of Cartan's Theorems A and B in places. A nice introduction to the use of sheaves in smooth differential geometry can be found in the book of Ramanan [2005]. More advanced and comprehensive treatments include [Bredon 1997, Kashiwara and Schapira 1990], and the classic [Godement 1958]. The discussion of sheaf theory in (Stacks 2013) is also useful. For readers who are expert in sheaf theory, we comment that our reasons for using sheaves are not always the usual ones, so an adjustment of point of view may be required.

We shall make frequent and essential use of nontrivial facts about locally convex topological vector spaces, and refer to [Conway 1990, Grothendieck 1973, Horváth 1966, Jarchow 1981, Rudin 1991, Schaefer and Wolff 1999] for details. We shall also access the contemporary research literature on locally convex spaces, and will indicate this as we go along. We shall denote by $\mathrm{L}(\mathsf{U};\mathsf{V})$ the set of continuous linear maps from a locally convex space U to a locally convex space V. In particular, U' is the topological dual of U, meaning the continuous linear scalar-valued functions. We will break with the usual language one sees in the theory of locally convex spaces and call what are commonly called "inductive" and "projective" limits, instead "direct" and "inverse" limits, in keeping with the rest of category theory.

By λ we denote the Lebesgue measure on $\mathbb{R}$. We will talk about measurability of maps taking values in topological spaces. If $(\mathcal{T},\mathscr{M})$ is a measurable space and if $\mathcal{X}$ is a topological space, a mapping $\Psi\colon \mathcal{T} \to \mathcal{X}$ is ***Borel measurable*** if $\Psi^{-1}(\mathcal{O}) \in \mathscr{M}$ for every open set $\mathcal{O} \subseteq \mathcal{X}$. This is equivalent to requiring that $\Psi^{-1}(\mathcal{B}) \in \mathscr{M}$ for every Borel subset $\mathcal{B} \subseteq \mathcal{X}$.

One not completely standard topic we shall need to understand is integration of functions with values in locally convex spaces. There are multiple theories here,[4] so let us outline what we mean, following [Beckmann and Deitmar 2011]. We let $(\mathcal{T},\mathscr{M},\mu)$ be a finite measure space, let V be a locally convex topological vector space, and let $\Psi\colon \mathcal{T} \to \mathsf{V}$. Measurability of Ψ is Borel measurability mentioned above, and we note that there are other forms of measurability that arise for locally convex spaces (the comment made in footnote 4 applies to these multiple notions of measurability as well). The notion of the integral we use is the ***Bochner integral***. This is well understood for Banach spaces [Diestel and Uhl, Jr. 1977] and is often mentioned in an offhand manner as being "the same" for locally convex spaces [e.g., Schaefer and Wolff 1999, page 96]. A detailed textbook treatment does

[4]Most of the theories of integration in locally convex spaces coincide for the sorts of locally convex spaces we deal with.

not appear to exist, but fortunately this has been worked out in the note of [Beckmann and Deitmar 2011], to which we shall refer for details as needed. One has a notion of simple functions, meaning functions that are finite linear combinations, with coefficients in V, of characteristic functions of measurable sets. The ***integral*** of a simple function $\sigma = \sum_{j=1}^{k} v_j \chi_{A_j}$ is

$$\int_{\mathcal{T}} \sigma \, \mathrm{d}\mu = \sum_{j=1}^{k} \mu(A_j) v_j,$$

in the usual manner. A measurable function Ψ is ***Bochner approximable*** if it can be approximated with respect to any continuous seminorm by a net of simple functions. A Bochner approximable function Ψ is ***Bochner integrable*** if there is a net of simple functions approximating Ψ whose integrals converge in V to a unique value, which is called the ***integral*** of Ψ. If V is separable and complete, as will be the case for us in this paper, then a measurable function $\Psi \colon \mathcal{T} \to \mathsf{V}$ is Bochner integrable if and only if

$$\int_{\mathcal{T}} p \circ \Psi \, \mathrm{d}\mu < \infty$$

for every continuous seminorm p on V [Beckmann and Deitmar 2011, Theorems 3.2 and 3.3]. This construction of the integral clearly agrees with the standard construction of the Lebesgue integral for functions taking values in $\mathbb{R}$ or $\mathbb{C}$ (or any finite-dimensional vector space over $\mathbb{R}$ or $\mathbb{C}$, for that matter). If $A \subseteq \mathsf{V}$, by $\mathrm{L}^1(\mathcal{T}; A)$ we denote the space of Bochner integrable functions with values in A. The space $\mathrm{L}^1(\mathcal{T}; \mathsf{V})$ is itself a locally convex topological vector space with topology defined by the seminorms

$$\hat{p}(\Psi) = \int_{\mathcal{T}} p \circ \Psi \, \mathrm{d}\mu,$$

where p is a continuous seminorm for V [Schaefer and Wolff 1999, page 96]. In the case where $\mathcal{T} = I$ is an interval in $\mathbb{R}$, $\mathrm{L}^1_{\mathrm{loc}}(I; A)$ denotes the set of locally integrable functions, i.e., those functions whose restriction to any compact subinterval is integrable.

While it does not generally make sense to talk about integrability of measurable functions with values in a topological space, one *can* sensibly talk about essentially bounded functions. This means that one needs a notion of boundedness, this being supplied by a "bornology."[5] Bornologies are less popular than topologies, but a treatment in some generality can be found in [Hogbe-Nlend 1977]. There are two bornologies we consider in this paper. One is the ***compact bornology*** for a topological space $\mathcal{X}$ whose bounded sets are the relatively compact sets. The other is the ***von Neumann bornology*** for a locally convex topological vector space V whose bounded sets are those subsets $\mathcal{B} \subseteq \mathsf{V}$ for which, for any neighbourhood $\mathcal{N}$ of $0 \in \mathsf{V}$, there exists $\lambda \in \mathbb{R}_{>0}$ such that $\mathcal{B} \subseteq \lambda\mathcal{N}$. On any locally convex topological vector space we thus have these two bornologies, and generally they are not the same. Indeed, if V is an infinite-dimensional normed vector space, then the compact bornology is strictly contained in the von Neumann bornology. We will, in

[5] A ***bornology*** on a set $\mathcal{S}$ is a family $\mathscr{B}$ of subsets of $\mathcal{S}$, called ***bounded sets***, and satisfying the axioms:

1. $\mathcal{S}$ is covered by bounded sets, i.e., $\mathcal{S} = \cup_{B \in \mathscr{B}} B$;
2. subsets of bounded sets are bounded, i.e., if $B \in \mathscr{B}$ and if $A \subseteq B$, then $A \in \mathscr{B}$;
3. finite unions of bounded sets are bounded, i.e., if $B_1, \dots, B_k \in \mathscr{B}$, then $\cup_{j=1}^{k} B_j \in \mathscr{B}$.

fact, have occasion to use both of these bornologies, and shall make it clear which we mean. Now, if $(\mathcal{T}, \mathscr{M}, \mu)$ is a measure space and if $(\mathcal{X}, \mathscr{B})$ is a bornological space, i.e., a set $\mathcal{X}$ with a bornology $\mathscr{B}$, a measurable map $\Psi\colon \mathcal{T} \to \mathcal{X}$ is ***essentially bounded*** if there exists a bounded set $B \subseteq \mathcal{X}$ such that

$$\mu(\{t \in \mathcal{T} \mid \ \Psi(t) \notin B\}) = 0.$$

By $\mathrm{L}^\infty(\mathcal{T}; \mathcal{X})$ we denote the set of essentially bounded maps. If $\mathcal{T} = I$ is an interval in $\mathbb{R}$, a measurable map $\Psi\colon I \to \mathcal{X}$ is ***locally essentially bounded*** in the bornology $\mathscr{B}$ if $\Psi|J$ is essentially bounded in the bornology $\mathscr{B}$ for every compact subinterval $J \subseteq I$. By $\mathrm{L}^\infty_{\mathrm{loc}}(I; \mathcal{X})$ we denote the set of locally essentially bounded maps; thus the bornology is to be understood when we write expressions such as this.

Apologia. This is a paper about differential geometric control theory. It is, therefore, a paper touching upon two things, (1) differential geometry and (2) control theory.

It is our view that differential geometry *is* the language of nonlinear control theory. As such, our attitude toward the differential geometric aspects of what we do is unflinching in that our presentation relies, sometimes in nontrivial ways, on all of the tools of a differential geometer, including some that are not always a part of the nonlinear control theoretician's tool box, e.g., jet bundles, connections, locally convex topologies. In this paper, apart from presenting a new framework for control theory, we also hope to illustrate the value of differential geometric tools in analysing these systems, and, for that matter, any sort of geometric model in control theory. We have, therefore, eschewed the use of coordinates wherever possible, since it is our opinion that unfettered coordinate calculations are dangerous; they can lead one astray if one forgets for too long the necessity of developing definitions and results that do not depend on specific choices of coordinates. Also, overuse of coordinates has a tendency to mask structure, and it is structure that we are emphasising in this paper. We accept that our approach will make the paper difficult reading for some.

This is also a paper about control theory. And, as such, we wish to make the paper as faithful to the discipline as possible, within the confines of what we are doing. We are certainly not including in our modelling all of the elements that would be demanded by a practicing control engineer, e.g., no uncertainty, no robustness, no adaptive control, etc. And we are only considering our very limited class of models with ordinary differential equations on finite-dimensional manifolds, e.g., no partial differential equations, no discrete-time systems, no hybrid systems, etc. However, with respect to those elements of control theory that we *do* touch upon, we have tried to be sincere in making a framework that captures what one is likely to encounter in practice. This means, for example, that we assiduously refrain from imposing geometric structure that is not natural from the point of view of control theory. This tends to be a weakness of some purely differential geometric approaches to control theory, and it is a weakness that we have avoided duplicating.

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2. Fibre metrics for jet bundles

One of the principal devices we use in the paper are convenient seminorms for the various topologies we use for spaces of sections of vector bundles. Since such topologies rely on placing suitable norms on derivatives of sections, i.e., on jet bundles of vector bundles, in this section we present a means for defining such norms, using as our starting point a pair of connections, one for the base manifold, and one for the vector bundle. These allow us to provide a direct sum decomposition of the jet bundle into its component "derivatives," and so then a natural means of defining a fibre metric for jet bundles using metrics on the tangent bundle of the base manifold and the fibres of the vector bundle.

As we shall see, in the smooth case, these constructions are a convenience, whereas in the real analytic case, they provide a crucial ingredient in our global, coordinate-free description of seminorms for the topology of the space of real analytic sections of a vector bundle. For this reason, in this section we shall also consider the existence of, and some properties of, real analytic connections in vector bundles.

2.1. A decomposition for the jet bundles of a vector bundle. We let $\pi\colon \mathsf{E} \to \mathsf{M}$ be a smooth vector bundle with $\pi_m\colon \mathsf{J}^m\mathsf{E} \to \mathsf{M}$ its mth jet bundle. In a local trivialisation of $\mathsf{J}^m\mathsf{E}$, the fibres of this vector bundle are

$$\oplus_{j=0}^{m} \mathrm{L}^j_{\mathrm{sym}}(\mathbb{R}^n;\mathbb{R}^k),$$

with n the dimension of M and k the fibre dimension of E. This decomposition of the derivatives, order-by-order, that we see in the local trivialisation has no global analogue, but such a decomposition can be provided with the use of connections, and we describe how to do this.

We suppose that we have a linear connection ∇^0 on the vector bundle E and an affine connection ∇ on M. We then have a connection, that we also denote by ∇, on $\mathsf{T}^*\mathsf{M}$ defined by

$$\mathscr{L}_Y\langle \alpha; X\rangle = \langle \nabla_Y\alpha; X\rangle + \langle \alpha; \nabla_Y X\rangle.$$

For $\xi \in \Gamma^\infty(\mathsf{E})$ we then have $\nabla^0\xi \in \Gamma^\infty(\mathsf{T}^*\mathsf{M}\otimes\mathsf{E})$ defined by $\nabla^0\xi(X) = \nabla^0_X\xi$ for $X \in \Gamma^\infty(\mathsf{TM})$. The connections ∇^0 and ∇ extend naturally to a connection, that we denote by ∇^m, on $\mathrm{T}^m(\mathsf{T}^*\mathsf{M})\otimes\mathsf{E}$, $m \in \mathbb{Z}_{>0}$, by the requirement that

$$\begin{aligned}\nabla^m_X(\alpha^1\otimes\cdots\otimes\alpha^m\otimes\xi) \\ &= \sum_{j=1}^{m}(\alpha^1\otimes\cdots\otimes(\nabla_X\alpha_j)\otimes\cdots\otimes\alpha^m\otimes\xi) + \alpha^1\otimes\cdots\otimes\alpha^m\otimes(\nabla^0_X\xi)\end{aligned}$$

for $\alpha^1,\dots,\alpha^m \in \Gamma^\infty(\mathsf{T}^*\mathsf{M})$ and $\xi \in \Gamma^\infty(\mathsf{E})$. Note that

$$\nabla^{(m)}\xi \triangleq \nabla^m(\nabla^{m-1}\cdots(\nabla^1(\nabla^0\xi))) \in \Gamma^\infty(\mathrm{T}^{m+1}(\mathsf{T}^*\mathsf{M})\otimes\mathsf{E}). \tag{2.1}$$

Now, given $\xi \in \Gamma^\infty(\mathsf{E})$ and $m \in \mathbb{Z}_{\geq 0}$, we define

$$P^{m+1}_{\nabla,\nabla^0}(\xi) = \mathrm{Sym}_{m+1} \otimes \mathrm{id}_{\mathsf{E}}(\nabla^{(m)}\xi) \in \Gamma^\infty(\mathrm{S}^{m+1}(\mathsf{T}^*\mathsf{M}) \otimes \mathsf{E}),$$

where $\mathrm{Sym}_m \colon \mathrm{T}^m(\mathsf{V}) \to \mathrm{S}^m(\mathsf{V})$ is defined by

$$\mathrm{Sym}_m(v_1 \otimes \cdots \otimes v_m) = \frac{1}{m!} \sum_{\sigma \in \mathfrak{S}_m} v_{\sigma(1)} \otimes \cdots \otimes v_{\sigma(m)}.$$

We take the convention that $P^0_{\nabla,\nabla^0}(\xi) = \xi$.

The following lemma is then key for our presentation. While this lemma exists in the literature in various forms, often in the form of results concerning the extension of connections by "bundle functors" [e.g., Kolář, Michor, and Slovák 1993, Chapter X], we were unable to find the succinct statement we give here. Pohl [1966] gives existential results dual to what we give here, but stops short of giving an explicit formula such as we give below. For this reason, we give a complete proof of the lemma.

2.1 Lemma: (Decomposition of jet bundles) *The map*

$$\begin{aligned} S^m_{\nabla,\nabla^0} \colon \mathsf{J}^m\mathsf{E} &\to \oplus_{j=0}^m (\mathrm{S}^j(\mathsf{T}^*\mathsf{M}) \otimes \mathsf{E}) \\ j_m\xi(x) &\mapsto (\xi(x), P^1_{\nabla,\nabla^0}(\xi)(x), \dots, P^m_{\nabla,\nabla^0}(\xi)(x)) \end{aligned}$$

is an isomorphism of vector bundles, and, for each $m \in \mathbb{Z}_{>0}$, the diagram

$$\begin{tikzcd} \mathsf{J}^{m+1}\mathsf{E} \arrow[r, "S^{m+1}_{\nabla,\nabla^0}"] \arrow[d, "\pi^{m+1}_m"'] & \oplus_{j=0}^{m+1}(\mathrm{S}^j(\mathsf{T}^*\mathsf{M}) \otimes \mathsf{E}) \arrow[d, "\mathrm{pr}^{m+1}_m"] \\ \mathsf{J}^m\mathsf{E} \arrow[r, "S^m_{\nabla,\nabla^0}"'] & \oplus_{j=0}^m(\mathrm{S}^j(\mathsf{T}^*\mathsf{M}) \otimes \mathsf{E}) \end{tikzcd}$$

commutes, where pr^{m+1}_m is the obvious projection, stripping off the last component of the direct sum.

Proof: We prove the result by induction on m. For $m = 0$ the result is a tautology. For $m = 1$, as in [Kolář, Michor, and Slovák 1993, §17.1], we have a vector bundle mapping $S_{\nabla^0} \colon \mathsf{E} \to \mathsf{J}^1\mathsf{E}$ over id_M that determines the connection ∇^0 by

$$\nabla^0\xi(x) = j_1\xi(x) - S_{\nabla^0}(\xi(x)). \tag{2.2}$$

Let us show that S^1_{∇,∇^0} is well-defined. Thus let $\xi, \eta \in \Gamma^\infty(\mathsf{E})$ be such that $j_1\xi(x) = j_1\eta(x)$. Then, clearly, $\xi(x) = \eta(x)$, and the formula (2.2) shows that $\nabla\xi(x) = \nabla\eta(x)$, and so S^1_{∇,∇^0} is indeed well defined. It is clearly linear on fibres, so it remains to show that it is an isomorphism. This will follow from dimension counting if it is injective. However, if $S^1_{\nabla,\nabla^0}(j_1\xi(x)) = 0$ then $j_1\xi(x) = 0$ by (2.2).

For the induction step, we begin with a sublemma.

1 Sublemma: *Let F be a field and consider the following commutative diagram of finite-dimensional F-vector spaces with exact rows and columns:*

$$
\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & \mathsf{A}_1 \arrow[r, "\phi_1"] \arrow[d, "\iota_1"] & \mathsf{C}_1 \arrow[r, "\psi_1"] \arrow[d, "\iota_2"] \arrow[l, dashed, bend left, "p_1"] & \mathsf{B} \arrow[r] \arrow[d, equal] \arrow[l, dashed, bend left, "\gamma_1"] & 0 \\
0 \arrow[r] & \mathsf{A}_2 \arrow[r, "\phi_2"] \arrow[u, bend left, "\sigma_1"] & \mathsf{C}_2 \arrow[r, "\psi_2"] \arrow[l, bend left, "p_2"] & \mathsf{B} \arrow[r] \arrow[l, bend left, "\gamma_2"] & 0
\end{tikzcd}
$$

If there exists a mapping $\gamma_2 \in \mathrm{Hom}_{\mathsf{F}}(\mathsf{B}; \mathsf{C}_2)$ such that $\psi_2 \circ \gamma_2 = \mathrm{id}_{\mathsf{B}}$ (with $p_2 \in \mathrm{Hom}_{\mathsf{F}}(\mathsf{C}_2; \mathsf{A}_2)$ the corresponding projection), then there exists a unique mapping $\gamma_1 \in \mathrm{Hom}_{\mathsf{F}}(\mathsf{B}; \mathsf{C}_1)$ such that $\psi_1 \circ \gamma_1 = \mathrm{id}_{\mathsf{B}}$ and such that $\gamma_2 = \iota_2 \circ \gamma_1$. There is also induced a projection $p_1 \in \mathrm{Hom}_{\mathsf{F}}(\mathsf{C}_1; \mathsf{A}_1)$.

Moreover, if there additionally exists a mapping $\sigma_1 \in \mathrm{Hom}_{\mathsf{F}}(\mathsf{A}_2; \mathsf{A}_1)$ such that $\sigma_1 \circ \iota_1 = \mathrm{id}_{\mathsf{A}_1}$, then the projection p_1 is uniquely determined by the condition $p_1 = \sigma_1 \circ p_2 \circ \iota_2$.

Proof: We begin by extending the diagram to one of the form

$$
\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & \mathsf{A}_1 \arrow[r, "\phi_1"] \arrow[d, "\iota_1"] & \mathsf{C}_1 \arrow[r, "\psi_1"] \arrow[d, "\iota_2"] & \mathsf{B} \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & \mathsf{A}_2 \arrow[r, "\phi_2"] \arrow[d, "\kappa_1"] & \mathsf{C}_2 \arrow[r, "\psi_2"] \arrow[d, "\kappa_2"] & \mathsf{B} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \mathrm{coker}(\iota_1) \arrow[r, dashed, "\phi_3"] \arrow[d] & \mathrm{coker}(\iota_2) \arrow[r] \arrow[d] & 0 & \\
 & 0 & 0 & &
\end{tikzcd}
$$

also with exact rows and columns. We claim that there is a natural mapping ϕ_3 between the cokernels, as indicated by the dashed arrow in the diagram, and that ϕ_3 is, moreover, an isomorphism. Suppose that $u_2 \in \mathrm{image}(\iota_1)$ and let $u_1 \in \mathsf{A}_1$ be such that $\iota_1(u_1) = u_2$. By commutativity of the diagram, we have

$$\phi_2(u_2) = \phi_2 \circ \iota_1(u_1) = \iota_2 \circ \phi_1(u_1),$$

showing that $\phi_2(\mathrm{image}(\iota_1)) \subseteq \mathrm{image}(\iota_2)$. We thus have a well-defined homomorphism

$$
\begin{aligned}
\phi_3 \colon \mathrm{coker}(\iota_1) &\to \mathrm{coker}(\iota_2) \\
u_2 + \mathrm{image}(\iota_1) &\mapsto \phi_2(u_2) + \mathrm{image}(\iota_2).
\end{aligned}
$$

We now claim that ϕ_3 is injective. Indeed,

$$\phi_3(u_2 + \mathrm{image}(\iota_1)) = 0 \quad \Longrightarrow \quad \phi_2(u_2) \in \mathrm{image}(\iota_2).$$

Thus let $v_1 \in \mathsf{C}_1$ be such that $\phi_2(u_2) = \iota_2(v_1)$. Thus

$$\begin{aligned} & 0 = \psi_2 \circ \phi_2(u_2) = \psi_2 \circ \iota_2(v_1) = \psi_1(v_1) \\ \implies \quad & v_1 \in \ker(\psi_1) = \operatorname{image}(\phi_1). \end{aligned}$$

Thus $v_1 = \phi_1(u_1')$ for some $u_1' \in \mathsf{A}_1$. Therefore,

$$\phi_2(u_2) = \iota_2 \circ \phi_1(u_1') = \phi_2 \circ \iota_1(u_1'),$$

and injectivity of ϕ_2 gives $u_2 \in \operatorname{image}(\iota_1)$ and so $u_2 + \operatorname{image}(\iota_1) = 0 + \operatorname{image}(\iota_1)$, giving the desired injectivity of ϕ_3.

Now note that

$$\dim(\operatorname{coker}(\iota_1)) = \dim(\mathsf{A}_2) - \dim(\mathsf{A}_1)$$

by exactness of the left column. Also,

$$\dim(\operatorname{coker}(\iota_2)) = \dim(\mathsf{C}_2) - \dim(\mathsf{C}_1)$$

by exactness of the middle column. By exactness of the top and middle rows, we have

$$\dim(\mathsf{B}) = \dim(\mathsf{C}_2) - \dim(\mathsf{A}_2) = \dim(\mathsf{C}_1) - \dim(\mathsf{A}_1).$$

This proves that

$$\dim(\operatorname{coker}(\iota_1)) = \dim(\operatorname{coker}(\iota_2)).$$

Thus the homomorphism ϕ_3 is an isomorphism, as claimed.

Now we proceed with the proof, using the extended diagram, and identifying the bottom cokernels with the isomorphism ϕ_3. The existence of the stated homomorphism γ_2 means that the middle row in the diagram splits. Therefore, $\mathsf{C}_2 = \operatorname{image}(\phi_2) \oplus \operatorname{image}(\gamma_2)$. Thus there exists a well-defined projection $p_2 \in \operatorname{Hom}_{\mathsf{F}}(\mathsf{C}_2; \mathsf{A}_2)$ such that $p_2 \circ \phi_2 = \mathrm{id}_{\mathsf{A}_2}$ [Halmos 1974, Theorem 41.1].

We will now prove that $\operatorname{image}(\gamma_2) \subseteq \operatorname{image}(\iota_2)$. By commutativity of the diagram and since ψ_1 is surjective, if $w \in \mathsf{B}$ then there exists $v_1 \in \mathsf{C}_1$ such that $\psi_2 \circ \iota_2(v_1) = w$. Since $\psi_2 \circ \gamma_2 = \mathrm{id}_{\mathsf{B}}$, we have

$$\psi_2 \circ \iota_2(v_1) = \psi_2 \circ \gamma_2(w) \quad \implies \quad \iota_2(v_1) - \gamma_2(w) \in \ker(\psi_2) = \operatorname{image}(\phi_2).$$

Let $u_2 \in \mathsf{A}_2$ be such that $\phi_2(u_2) = \iota_2(v_1) - \gamma_2(w)$. Since $p_2 \circ \phi_2 = \mathrm{id}_{\mathsf{A}_2}$ we have

$$u_2 = p_2 \circ \iota_2(v_1) - p_2 \circ \gamma_2(w),$$

whence

$$\kappa_1(u_2) = \kappa_1 \circ p_2 \circ \iota_2(v_1) - \kappa_1 \circ p_2 \circ \gamma_2(w) = 0,$$

noting that (1) $\kappa_1 \circ p_2 = \kappa_2$ (by commutativity), (2) $\kappa_2 \circ \iota_2 = 0$ (by exactness), and (3) $p_2 \circ \gamma_2 = 0$ (by exactness). Thus $u_2 \in \ker(\kappa_1) = \operatorname{image}(\iota_1)$. Let $u_1 \in \mathsf{A}_1$ be such that $\iota_1(u_1) = u_2$. We then have

$$\iota_2(v_1) - \gamma_2(w) = \phi_2 \circ \iota_1(u_1) = \iota_2 \circ \phi_1(u_1),$$

which gives $\gamma_2(w) \in \operatorname{image}(\iota_2)$, as claimed.

Now we define $\gamma_1 \in \mathrm{Hom}_{\mathsf{F}}(\mathsf{B}; \mathsf{C}_1)$ by asking that $\gamma_1(w) \in \mathsf{C}_1$ have the property that $\iota_2 \circ \gamma_1(w) = \gamma_2(w)$, this making sense since we just showed that $\mathrm{image}(\gamma_2) \subseteq \mathrm{image}(\iota_2)$. Moreover, since ι_2 is injective, the definition uniquely prescribes γ_1. Finally we note that

$$\psi_1 \circ \gamma_1 = \psi_2 \circ \iota_2 \circ \gamma_1 = \psi_2 \circ \gamma_2 = \mathrm{id}_{\mathsf{B}},$$

as claimed.

To prove the final assertion, let us denote $\hat{p}_1 = \sigma_1 \circ p_2 \circ \iota_2$. We then have

$$\hat{p}_1 \circ \phi_1 = \sigma_1 \circ p_2 \circ \iota_2 \circ \phi_1 = \sigma_1 \circ p_2 \circ \phi_2 \circ \iota_1 = \sigma_1 \circ \iota_1 = \mathrm{id}_{\mathsf{A}_1},$$

using commutativity. We also have

$$\hat{p}_1 \circ \gamma_1 = \sigma_1 \circ p_2 \circ \iota_2 \circ \gamma_1 = \sigma_1 \circ p_2 \circ \gamma_2 = 0.$$

The two preceding conclusions show that $\hat{p}_1$ is the projection defined by the splitting of the top row of the diagram, i.e., $\hat{p}_1 = p_1$. ▼

Now suppose that the lemma is true for $m \in \mathbb{Z}_{>0}$. For any $k \in \mathbb{Z}_{>0}$ we have a short exact sequence

$$0 \longrightarrow \mathrm{S}^k(\mathsf{T}^*\mathsf{M}) \otimes \mathsf{E} \xrightarrow{\epsilon_k} \mathsf{J}^k\mathsf{E} \xrightarrow{\pi^k_{k-1}} \mathsf{J}^{k-1}\mathsf{E} \longrightarrow 0$$

for which we refer to [Saunders 1989, Theorem 6.2.9]. Recall from [Saunders 1989, Definition 6.2.25] that we have an inclusion $\iota_{1,m}$ of $\mathsf{J}^{m+1}\mathsf{E}$ in $\mathsf{J}^1(\mathsf{J}^m\mathsf{E})$ by $j_{m+1}\xi(x) \mapsto j_1(j_m\xi(x))$. We also have an induced injection

$$\hat{\iota}_{1,m} \colon \mathrm{S}^{m+1}(\mathsf{T}^*\mathsf{M}) \otimes \mathsf{E} \to \mathsf{T}^*\mathsf{M} \otimes \mathsf{J}^m\mathsf{E}$$

defined by the composition

$$\mathrm{S}^{m+1}(\mathsf{T}^*\mathsf{M}) \otimes \mathsf{E} \longrightarrow \mathsf{T}^*\mathsf{M} \otimes \mathrm{S}^m(\mathsf{T}^*\mathsf{M}) \otimes \mathsf{E} \xrightarrow{\mathrm{id} \otimes \epsilon_m} \mathsf{T}^*\mathsf{M} \otimes \mathsf{J}^m\mathsf{E}$$

Explicitly, the left arrow is defined by

$$\alpha^1 \odot \cdots \odot \alpha^{m+1} \otimes \xi \mapsto \sum_{j=1}^{m+1} \alpha^j \otimes \alpha^1 \odot \cdots \odot \alpha^{j-1} \odot \alpha^{j+1} \odot \cdots \odot \alpha^{m+1} \otimes \xi,$$

$\odot$ denoting the symmetric tensor product defined by

$$A \odot B = \sum_{\sigma \in \mathfrak{S}_{k,l}} \sigma(A \otimes B), \tag{2.3}$$

for $A \in \mathrm{S}^k(\mathsf{V})$ and $B \in \mathrm{S}^l(\mathsf{V})$, and with $\mathfrak{S}_{k,l}$ the subset of $\mathfrak{S}_{k+l}$ consisting of permutations σ satisfying

$$\sigma(1) < \cdots < \sigma(k), \quad \sigma(k+1) < \cdots < \sigma(k+l).$$

We thus have the following commutative diagram with exact rows and columns:

$$
\begin{tikzcd}
 & & 0 \arrow[d] & & 0 \arrow[d] & & 0 \arrow[d] & \\
0 \arrow[r] & & \mathrm{S}^{m+1}(\mathsf{T}^*\mathsf{M}) \otimes \mathsf{E} \arrow[rr, "\epsilon_{m+1}"] \arrow[d, shift left, "\hat{\iota}_{1,m}"] & & \mathsf{J}^{m+1}\mathsf{E} \arrow[rr, "\pi^{m+1}_m"] \arrow[ll, dashed, bend left, "P_{m+1}"] \arrow[d, "\iota_{1,m}"] & & \mathsf{J}^m\mathsf{E} \arrow[r] \arrow[ll, dashed, bend left, "\Gamma_{m+1}"] \arrow[d, equal] & 0 \\
0 \arrow[r] & & \mathsf{T}^*\mathsf{M} \otimes \mathsf{J}^m\mathsf{E} \arrow[rr, "\epsilon_{1,m}"] \arrow[u, dashed, shift left, "\lambda_{1,m}"] & & \mathsf{J}^1(\mathsf{J}^m\mathsf{E}) \arrow[rr, "(\pi_m)_1"] \arrow[ll, dashed, bend left, "P_{1,m}"] & & \mathsf{J}^m\mathsf{E} \arrow[r] \arrow[ll, dashed, bend left, "\Gamma_{1,m}"] & 0
\end{tikzcd}
\tag{2.4}
$$

We shall define a connection on $(\pi_m)_1 \colon \mathsf{J}^1(\mathsf{J}^m\mathsf{E}) \to \mathsf{J}^m\mathsf{E}$ which gives a splitting $\Gamma_{1,m}$ and $P_{1,m}$ of the lower row in the diagram. By the sublemma, this will give a splitting Γ_{m+1} and P_{m+1} of the upper row, and so give a projection from $\mathsf{J}^{m+1}\mathsf{E}$ onto $\mathrm{S}^{m+1}(\mathsf{T}^*\mathsf{M}) \otimes \mathsf{E}$, which will allow us to prove the induction step. To compute P_{m+1} from the sublemma, we shall also give a map $\lambda_{1,m}$ as in the diagram so that $\lambda_{1,m} \circ \hat{\iota}_{1,m}$ is the identity on $\mathrm{S}^{m+1}(\mathsf{T}^*\mathsf{M}) \otimes \mathsf{E}$.

We start, under the induction hypothesis, by making the identification

$$\mathsf{J}^m\mathsf{E} \simeq \oplus_{j=0}^m \mathrm{S}^j(\mathsf{T}^*\mathsf{M}) \otimes \mathsf{E},$$

and consequently writing a section of $\mathsf{J}^m\mathsf{E}$ as

$$x \mapsto (\xi(x), P^1_{\nabla,\nabla^0}(\xi(x)), \dots, P^m_{\nabla,\nabla^0}(\xi(x))).$$

We then have a connection $\overline{\nabla}^m$ on $\mathsf{J}^m\mathsf{E}$ given by

$$\overline{\nabla}^m_X(\xi, P^1_{\nabla,\nabla^0}(\xi), \dots, P^m_{\nabla,\nabla^0}(\xi)) = (\nabla^0_X\xi, \nabla^1_X P^1_{\nabla,\nabla^0}(\xi), \dots, \nabla^m_X P^m_{\nabla,\nabla^0}(\xi)).$$

Thus

$$\overline{\nabla}^m(\xi, P^1_{\nabla,\nabla^0}(\xi), \dots, P^m_{\nabla,\nabla^0}(\xi)) = (\nabla^0\xi, \nabla^1 P^1_{\nabla,\nabla^0}(\xi), \dots, \nabla^m P^m_{\nabla,\nabla^0}(\xi)),$$

which—according to the jet bundle characterisation of connections from [Kolář, Michor, and Slovák 1993, §17.1] and which we have already employed in (2.2)—gives the mapping $P_{1,m}$ in the diagram (2.4) as

$$P_{1,m}(j_1(\xi, P^1_{\nabla,\nabla^0}(\xi), \dots, P^m_{\nabla,\nabla^0}(\xi))) = (\nabla^0\xi, \nabla^1 P^1_{\nabla,\nabla^0}(\xi), \dots, \nabla^m P^m_{\nabla,\nabla^0}(\xi)).$$

Now we define a mapping $\lambda_{1,m}$ for which $\lambda_{1,m} \circ \hat{\iota}_{1,m}$ is the identity on $\mathrm{S}^{m+1}(\mathsf{T}^*\mathsf{M}) \otimes \mathsf{E}$. We continue to use the induction hypothesis in writing elements of $\mathsf{J}^m\mathsf{E}$, so that we consider elements of $\mathsf{T}^*\mathsf{M} \otimes \mathsf{J}^m\mathsf{E}$ of the form

$$(\alpha \otimes \xi, \alpha \otimes A_1, \dots, \alpha \otimes A_m),$$

for $\alpha \in \mathsf{T}^*\mathsf{M}$ and $A_k \in \mathrm{S}^k(\mathsf{T}^*\mathsf{M}) \otimes \mathsf{E}$, $k \in \{1, \dots, m\}$. We then define $\lambda_{1,m}$ by

$$
\begin{aligned}
\lambda_{1,m}(\alpha_0 \otimes \xi, \alpha_0 \otimes \alpha^1_1 \otimes \xi, \dots, \alpha_0 \otimes \alpha^1_m \odot \cdots \odot \alpha^m_m \otimes \xi) \\
= \mathrm{Sym}_{m+1} \otimes \mathrm{id}_{\mathsf{E}}(\alpha_0 \otimes \alpha^1_m \odot \cdots \odot \alpha^m_m \otimes \xi).
\end{aligned}
$$

Note that, with the form of $\mathsf{J}^m\mathsf{E}$ from the induction hypothesis, we have

$$\hat{\iota}_{1,m}(\alpha^1 \odot \cdots \odot \alpha^{m+1} \otimes \xi) = \Big(0, \dots, 0, \frac{1}{m+1} \sum_{j=1}^{m+1} \alpha^j \otimes \alpha^1 \odot \cdots \odot \alpha^{j-1} \odot \alpha^{j+1} \odot \cdots \odot \alpha^{m+1} \otimes \xi\Big).$$

We then directly verify that $\lambda_{1,m} \circ \hat{\iota}_{1,m}$ is indeed the identity.

We finally claim that

$$P_{m+1}(j_{m+1}\xi(x)) = P^{m+1}_{\nabla,\nabla^0}(\xi), \tag{2.5}$$

which will establish the lemma. To see this, first note that it suffices to define P_{m+1} on $\operatorname{image}(\epsilon_{m+1})$ since

1. $\mathsf{J}^{m+1}\mathsf{E} \simeq (\mathrm{S}^{m+1}(\mathsf{T}^*\mathsf{M}) \otimes \mathsf{E}) \oplus \mathsf{J}^m\mathsf{E}$,
2. P_{m+1} is zero on $\mathsf{J}^m\mathsf{E} \subseteq \mathsf{J}^{m+1}\mathsf{E}$ (thinking of the inclusion arising from the connection-induced isomorphism from the preceding item), and
3. $P_{m+1} \circ \epsilon_{m+1}$ is the identity map on $\mathrm{S}^{m+1}(\mathsf{T}^*\mathsf{M}) \otimes \mathsf{E}$.

In order to connect the algebra and the geometry, let us write elements of $\mathrm{S}^{m+1}(\mathsf{T}^*\mathsf{M}) \otimes \mathsf{E}$ in a particular way. We let $x \in \mathsf{M}$ and let $f^1, \dots, f^{m+1}$ be smooth functions contained in the maximal ideal of $\mathrm{C}^\infty(\mathsf{M})$ at x, i.e., $f^j(x) = 0$, $j \in \{1, \dots, m+1\}$. Let ξ be a smooth section of E. We then can work with elements of $\mathrm{S}^{m+1}(\mathsf{T}^*\mathsf{M}) \otimes \mathsf{E}$ of the form

$$\mathrm{d}f^1(x) \odot \cdots \odot \mathrm{d}f^{m+1}(x) \otimes \xi(x).$$

We then have

$$\epsilon_{m+1}(\mathrm{d}f^1(x) \odot \cdots \odot \mathrm{d}f^{m+1}(x) \otimes \xi(x)) = j_{m+1}(f^1 \cdots f^{m+1}\xi)(x);$$

this is easy to see using the Leibniz Rule [cf. Goldschmidt 1967, Lemma 2.1]. (See [Abraham, Marsden, and Ratiu 1988, Supplement 2.4A] for a description of the higher-order Leibniz Rule.) Now, using the last part of the sublemma, we compute

$$\begin{aligned}
&P_{m+1}(j_{m+1}(f^1 \cdots f^{m+1}\xi)(x)) \\
&\quad = \lambda_{1,m} \circ P_{1,m} \circ \iota_{1,m}(j_{m+1}(f^1 \cdots f^{m+1}\xi)(x)) \\
&\quad = \lambda_{1,m} \circ P_{1,m}(j_1(f^1 \cdots f^{m+1}\xi, P^1_{\nabla,\nabla^0}(f^1 \cdots f^{m+1}\xi), \dots, P^m_{\nabla,\nabla^0}(f^1 \cdots f^{m+1}\xi))(x)) \\
&\quad = \lambda_{1,m}(\nabla^0(f^1 \cdots f^{m+1}\xi)(x), \nabla^1 P^1_{\nabla,\nabla^0}(f^1 \cdots f^{m+1}\xi)(x), \dots, \nabla^m P^m_{\nabla,\nabla^0}(f^1 \cdots f^{m+1}\xi)(x)) \\
&\quad = \mathrm{Sym}_{m+1} \otimes \mathrm{id}_{\mathsf{E}}(\nabla^m P^m_{\nabla,\nabla^0}(f^1 \cdots f^{m+1}\xi)(x)) \\
&\quad = P^{m+1}_{\nabla,\nabla_0}(f^1 \cdots f^{m+1}\xi)(x),
\end{aligned}$$

which shows that, with P_{m+1} defined as in (2.5), $P_{m+1} \circ \epsilon_{m+1}$ is indeed the identity on $\mathrm{S}^{m+1}(\mathsf{T}^*\mathsf{M}) \otimes \mathsf{E}$.

The commuting of the diagram in the statement of the lemma follows directly from the recursive nature of the constructions. ■

2.2. Fibre metrics using jet bundle decompositions. We also require the following result concerning inner products on tensor products.

2.2 Lemma: (Inner products on tensor products) *Let U and V be finite-dimensional $\mathbb{R}$-vector spaces and let $\mathbb{G}$ and $\mathbb{H}$ be inner products on U and V, respectively. Then the element $\mathbb{G}\otimes\mathbb{H}$ of $\mathrm{T}^2(\mathsf{U}^*\otimes\mathsf{V}^*)$ defined by*

$$\mathbb{G}\otimes\mathbb{H}(u_1\otimes v_1, u_2\otimes v_2) = \mathbb{G}(u_1,u_2)\mathbb{H}(v_1,v_2)$$

is an inner product on $\mathsf{U}\otimes\mathsf{V}$.

Proof: Let $(e_1,\dots,e_m)$ and $(f_1,\dots,f_n)$ be orthonormal bases for U and V, respectively. Then

$$\{e_a\otimes f_j \mid a\in\{1,\dots,m\},\ j\in\{1,\dots,n\}\} \tag{2.6}$$

is a basis for $\mathsf{U}\otimes\mathsf{V}$. Note that

$$\mathbb{G}\otimes\mathbb{H}(e_a\otimes f_j, e_b\otimes f_k) = \mathbb{G}(e_a,e_b)\mathbb{H}(f_j,f_k) = \delta_{ab}\delta_{jk},$$

which shows that $\mathbb{G}\otimes\mathbb{H}$ is indeed an inner product, as (2.6) is an orthonormal basis. ■

Now, we let $\mathbb{G}_0$ be a fibre metric on E and let $\mathbb{G}$ be a Riemannian metric on M. Let us denote by $\mathbb{G}^{-1}$ the associated fibre metric on $\mathsf{T}^*\mathsf{M}$ defined by

$$\mathbb{G}^{-1}(\alpha_x,\beta_x) = \mathbb{G}(\mathbb{G}^\sharp(\alpha_x),\mathbb{G}^\sharp(\beta_x)).$$

By induction using the preceding lemma, we have a fibre metric $\mathbb{G}_j$ on $\mathrm{T}^j(\mathsf{T}^*\mathsf{M})\otimes\mathsf{E}$ induced by $\mathbb{G}^{-1}$ and $\mathbb{G}_0$. By restriction, this gives a fibre metric on $\mathrm{S}^j(\mathsf{T}^*\mathsf{M})\otimes\mathsf{E}$. We can thus define a fibre metric $\overline{\mathbb{G}}_m$ on $\mathsf{J}^m\mathsf{E}$ given by

$$\overline{\mathbb{G}}_m(j_m\xi(x), j_m\eta(x)) = \sum_{j=0}^m \mathbb{G}_j\Big(\frac{1}{j!}P^j_{\nabla,\nabla^0}(\xi)(x), \frac{1}{j!}P^j_{\nabla,\nabla^0}(\eta)(x)\Big),$$

with the convention that $\nabla^{(-1)}\xi = \xi$. Associated to this inner product on fibres is the norm on fibres, which we denote by $\|\cdot\|_{\overline{\mathbb{G}}_m}$. We shall use these fibre norms continually in our descriptions of our various topologies below.

2.3. Real analytic connections. The fibre metrics from the preceding section will be used to define seminorms for spaces of sections of vector bundles. In the finitely differentiable and smooth cases, the particular fibre metrics we define above are not really required to give seminorms for the associated topologies: any fibre metrics on the jet bundles will suffice. Indeed, as long as one is only working with finitely many derivatives at one time, the choice of fibre norms on jet bundles is of no consequence, since different choices will be equivalent on compact subsets of M, cf. Section 3.1. However, when we work with the real analytic topology, we are no longer working only with finitely many derivatives, but with the infinite jet of a section. For this reason, different choices of fibre metric for jet bundles may give rise to different topologies for the space of real analytic sections, unless the behaviour of the fibre metrics is compatible as the order of derivatives goes to infinity. In this section we give a fundamental inequality for our fibre metrics of Section 2.2 in the real analytic case that ensures that they, in fact, describe the real analytic topology.

First let us deal with the matter of existence of real analytic data defining these fibre metrics.

2.3 Lemma: (Existence of real analytic connections and fibre metrics) *If* $\pi\colon \mathsf{E} \to \mathsf{M}$ *is a real analytic vector bundle, then there exist*

(i) a real analytic linear connection on E*,*

(ii) a real analytic affine connection on M*,*

(iii) a real analytic fibre metric on E*, and*

(iv) a real analytic Riemannian metric on M*.*

Proof: By [Grauert 1958, Theorem 3], there exists a proper real analytic embedding ι_{E} of E in $\mathbb{R}^N$ for some $N \in \mathbb{Z}_{>0}$. There is then an induced proper real analytic embedding ι_{M} of M in $\mathbb{R}^N$ by restricting ι_{E} to the zero section of E. Let us take the subbundle $\hat{\mathsf{E}}$ of $\mathsf{T}\mathbb{R}^N|\iota_{\mathsf{M}}(\mathsf{M})$ whose fibre at $\iota_{\mathsf{M}}(x) \in \iota_{\mathsf{M}}(\mathsf{M})$ is

$$\hat{\mathsf{E}}_{\iota_{\mathsf{M}}(x)} = T_{0_x}\iota_{\mathsf{E}}(\mathsf{V}_{0_x}\mathsf{E}).$$

Now recall that $\mathsf{E} \simeq \zeta^*\mathsf{VE}$, where $\zeta\colon \mathsf{M} \to \mathsf{E}$ is the zero section [Kolář, Michor, and Slovák 1993, page 55]. Let us abbreviate $\hat{\iota}_{\mathsf{E}} = T\iota_{\mathsf{E}}|\zeta^*\mathsf{VE}$. We then have the following diagram

$$\begin{tikzcd}
\mathsf{E} \simeq \zeta^*\mathsf{VE} \arrow[r, "\hat{\iota}_{\mathsf{E}}"] \arrow[d, "\pi"'] & \mathbb{R}^N \times \mathbb{R}^N \arrow[d, "\mathrm{pr}_2"] \\
\mathsf{M} \arrow[r, "\iota_{\mathsf{M}}"'] & \mathbb{R}^N
\end{tikzcd} \tag{2.7}$$

describing a monomorphism of real analytic vector bundles over the proper embedding ι_{M}, with the image of $\hat{\iota}_{\mathsf{E}}$ being $\hat{\mathsf{E}}$.

Among the many ways to prescribe a linear connection on the vector bundle E, we will take the prescription whereby one defines a mapping $K\colon \mathsf{TE} \to \mathsf{E}$ such that the two diagrams

$$\begin{tikzcd}
\mathsf{TE} \arrow[r, "K"] \arrow[d, "T\pi"'] & \mathsf{E} \arrow[d, "\pi"] \\
\mathsf{TM} \arrow[r, "\pi_{\mathsf{TM}}"'] & \mathsf{M}
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathsf{TE} \arrow[r, "K"] \arrow[d, "\pi_{\mathsf{TE}}"'] & \mathsf{E} \arrow[d, "\pi"] \\
\mathsf{E} \arrow[r, "\pi"'] & \mathsf{M}
\end{tikzcd} \tag{2.8}$$

define vector bundle mappings [Kolář, Michor, and Slovák 1993, §11.11]. We define K as follows. For $e_x \in \mathsf{E}_x$ and $X_{e_x} \in \mathsf{T}_{e_x}\mathsf{E}$ we have

$$T_{e_x}\hat{\iota}_{\mathsf{E}}(X_{e_x}) \in \mathsf{T}_{\hat{\iota}_{\mathsf{E}}(e_x)}(\mathbb{R}^N \times \mathbb{R}^N) \simeq \mathbb{R}^N \oplus \mathbb{R}^N,$$

and we define K so that

$$\hat{\iota}_{\mathsf{E}} \circ K(X_{e_x}) = \mathrm{pr}_2 \circ T_{e_x}\hat{\iota}_{\mathsf{E}}(X_{e_x});$$

this uniquely defines K by injectivity of $\hat{\iota}_{\mathsf{E}}$, and amounts to using on E the connection induced on image$(\hat{\iota}_{\mathsf{E}})$ by the trivial connection on $\mathbb{R}^N \times \mathbb{R}^N$. In particular, this means that we think of $\hat{\iota}_{\mathsf{E}} \circ K(X_{e_x})$ as being an element of the fibre of the trivial bundle $\mathbb{R}^N \times \mathbb{R}^N$ at $\iota_{\mathsf{M}}(x)$.

If $v_x \in \mathsf{TM}$, if $e, e' \in \mathsf{E}$, and if $X \in \mathsf{T}_e\mathsf{E}$ and $X' \in \mathsf{T}_{e'}\mathsf{E}$ satisfy $X, X' \in T\pi^{-1}(v_x)$, then note that

$$\begin{aligned} T_e\pi(X) = T_{e'}\pi(X') &\implies T_e(\iota_{\mathsf{M}} \circ \pi)(X) = T_{e'}(\iota_{\mathsf{M}} \circ \pi)(X') \\ &\implies T_e(\mathrm{pr}_2 \circ \hat{\iota}_{\mathsf{E}})(X) = T_{e'}(\mathrm{pr}_2 \circ \hat{\iota}_{\mathsf{E}})(X') \\ &\implies T_{\iota_{\mathsf{M}}(x)} \mathrm{pr}_2 \circ T_e\hat{\iota}_{\mathsf{E}}(X) = T_{\iota_{\mathsf{M}}(x)} \mathrm{pr}_2 \circ T_{e'}\hat{\iota}_{\mathsf{E}}(X'). \end{aligned}$$

Thus we can write

$$T_e\hat{\iota}_{\mathsf{E}}(X) = (\boldsymbol{x}, \boldsymbol{e}, \boldsymbol{u}, \boldsymbol{v}), \quad T_{e'}\hat{\iota}_{\mathsf{E}}(X) = (\boldsymbol{x}, \boldsymbol{e}', \boldsymbol{u}, \boldsymbol{v}')$$

for suitable $\boldsymbol{x}, \boldsymbol{u}, \boldsymbol{e}, \boldsymbol{e}', \boldsymbol{v}, \boldsymbol{v}' \in \mathbb{R}^N$. Therefore,

$$\hat{\iota}_{\mathsf{E}} \circ K(X) = (\boldsymbol{x}, \boldsymbol{v}), \quad \hat{\iota}_{\mathsf{E}} \circ K(X') = (\boldsymbol{x}, \boldsymbol{v}'), \quad \hat{\iota}_{\mathsf{E}} \circ K(X + X') = (\boldsymbol{x}, \boldsymbol{v} + \boldsymbol{v}'),$$

from which we immediately conclude that, for addition in the vector bundle $T\pi \colon \mathsf{TE} \to \mathsf{TM}$, we have

$$\hat{\iota}_{\mathsf{E}} \circ K(X + X') = \hat{\iota}_{\mathsf{E}} \circ K(X) + \hat{\iota}_{\mathsf{E}} \circ K(X'),$$

showing that the diagram on the left in (2.8) makes K a vector bundle mapping.

On the other hand, if $e_x \in \mathsf{E}$ and if $X, X' \in \mathsf{T}_{e_x}\mathsf{E}$, then we have, using vector bundle addition in $\pi_{\mathsf{TE}} \colon \mathsf{TE} \to \mathsf{E}$,

$$\begin{aligned} \hat{\iota}_{\mathsf{E}} \circ K(X + X') &= \mathrm{pr}_2 \circ T_{e_x}\hat{\iota}_{\mathsf{E}}(X + X') \\ &= \mathrm{pr}_2 \circ T_{e_x}\hat{\iota}_{\mathsf{E}}(X) + \mathrm{pr}_2 \circ T_{e_x}\hat{\iota}_{\mathsf{E}}(X') \\ &= \hat{\iota}_{\mathsf{E}} \circ K(X) + \hat{\iota}_{\mathsf{E}} \circ K(X'), \end{aligned}$$

giving that the diagram on the right in (2.8) makes K a vector bundle mapping. Since K is real analytic, this defines a real analytic linear connection ∇^0 on E as in [Kolář, Michor, and Slovák 1993, §11.11].

The existence of $\mathbb{G}_0$, $\mathbb{G}$, and ∇ are straightforward. Indeed, we let $\mathbb{G}_{\mathbb{R}^N}$ be the Euclidean metric on $\mathbb{R}^N$, and define $\mathbb{G}_0$ and $\mathbb{G}$ by

$$\mathbb{G}_0(e_x, e'_x) = \mathbb{G}_{\mathbb{R}^N}(\hat{\iota}_{\mathsf{E}}(e_x), \hat{\iota}_{\mathsf{E}}(e'_x))$$

and

$$\mathbb{G}(v_x, v'_x) = \mathbb{G}_{\mathbb{R}^N}(T_x\iota_{\mathsf{M}}(v_x), T_x\iota_{\mathsf{M}}(v'_x)).$$

The affine connection ∇ can be taken to be the Levi-Civita connection of $\mathbb{G}$. ■

The existence of a real analytic linear connection in a real analytic vector bundle is asserted at the bottom of page 302 in [Kriegl and Michor 1997], and we fill in the blanks in the preceding proof.

Now let us provide a fundamental relationship between the geometric fibre norms of Section 2.2 and norms constructed in local coordinate charts.

2.4 Lemma: (A fundamental estimate for fibre norms) *Let $\mathcal{U} \subseteq \mathbb{R}^n$ be open, denote $\mathbb{R}^k_{\mathcal{U}} = \mathcal{U} \times \mathbb{R}^k$, let $K \subseteq \mathcal{U}$ be compact, and consider the trivial vector bundle $\mathrm{pr}_1 \colon \mathbb{R}^k_{\mathcal{U}} \to \mathcal{U}$. Let $\mathbb{G}$ be a Riemannian metric on $\mathcal{U}$, let $\mathbb{G}_0$ be a vector bundle metric on $\mathbb{R}^k_{\mathcal{U}}$, let ∇ be an affine connection on $\mathcal{U}$, and let ∇^0 be a vector bundle connection on $\mathbb{R}^k_{\mathcal{U}}$, with all of these being real analytic. Then there exist $C, \sigma \in \mathbb{R}_{>0}$ such that*

$$\frac{\sigma^m}{C}\|j_m\boldsymbol{\xi}(\boldsymbol{x})\|_{\overline{\mathbb{G}}_m} \leq \sup\left\{\frac{1}{I!}|\boldsymbol{D}^I\xi^a(\boldsymbol{x})| \;\middle|\; |I| \leq m,\ a \in \{1,\dots,k\}\right\} \leq \frac{C}{\sigma^m}\|j_m\boldsymbol{\xi}(\boldsymbol{x})\|_{\overline{\mathbb{G}}_m}$$

for every $\boldsymbol{\xi} \in \Gamma^\infty(\mathbb{R}^k_{\mathcal{U}})$, $\boldsymbol{x} \in K$, and $m \in \mathbb{Z}_{\geq 0}$.

Proof: We begin the proof with a series of sublemmata of a fairly technical nature. From these the lemma will follow in a more or less routine manner.

Let us first prove a result which gives a useful local trivialisation of a vector bundle and a corresponding Taylor expansion for real analytic sections.

1 Sublemma: *Let $\pi \colon \mathsf{E} \to \mathsf{M}$ be a real analytic vector bundle, let ∇^0 be a real analytic linear connection on E, and let ∇ be a real analytic affine connection on M. Let $x \in \mathsf{M}$, and let $\mathcal{N} \subseteq \mathsf{T}_x\mathsf{M}$ be a convex neighbourhood of 0_x and $\mathcal{V} \subseteq \mathsf{M}$ be a neighbourhood of x such that the exponential map $\exp_x$ corresponding to ∇ is a real analytic diffeomorphism from $\mathcal{N}$ to $\mathcal{V}$. For $y \in \mathcal{V}$, let $\tau_{xy} \colon \mathsf{E}_x \to \mathsf{E}_y$ be parallel transport along the geodesic $t \mapsto \exp_x(t\exp_x^{-1}(y))$. Define*

$$\begin{aligned}\kappa_x \colon \mathcal{N} \times \mathsf{E}_x &\to \mathsf{E}|\mathcal{V}\\ (v, e_x) &\mapsto \tau_{x,\exp_x(v)}(e_x).\end{aligned}$$

Then

(i) κ_x is a real analytic vector bundle isomorphism over $\exp_x$ and

(ii) if $\xi \in \Gamma^\omega(\mathsf{E}|\mathcal{V})$, then

$$\kappa_x^{-1} \circ \xi \circ \exp_x(v) = \sum_{m=0}^{\infty} \frac{1}{m!}\nabla^{(m-1)}\xi(x)(\underbrace{v, \cdots, v}_{m \text{ times}})$$

for v in a sufficiently small neighbourhood of $0_x \in \mathsf{T}_x\mathsf{M}$.

Proof: (i) Consider the vector field X_{∇,∇^0} on the Whitney sum $\mathsf{TM} \oplus \mathsf{E}$ defined by

$$X_{\nabla,\nabla^0}(v_x, e_x) = \mathrm{hlft}(v_x, v_x) \oplus \mathrm{hlft}_0(e_x, v_x),$$

where $\mathrm{hlft}(v_x, u_x)$ is the horizontal lift of $u_x \in \mathsf{T}_x\mathsf{M}$ to $\mathsf{T}_{v_x}\mathsf{TM}$ and $\mathrm{hlft}_0(e_x, u_x)$ is the horizontal lift of $u_x \in \mathsf{T}_x\mathsf{M}$ to $\mathsf{T}_{e_x}\mathsf{E}$. Note that, since

$$T\pi_{\mathsf{TM}}(\mathrm{hlft}(v_x, v_x)) = T\pi(\mathrm{hlft}_0(e_x, v_x)),$$

this is indeed a vector field on $\mathsf{TM} \oplus \mathsf{E}$. Moreover, the integral curve of X_{∇,∇^0} through (v_x, e_x) is $t \mapsto \gamma'(t) \oplus \tau(t)$, where γ is the geodesic with initial condition $\gamma'(0) = v_x$ and where $t \mapsto \tau(t)$ is parallel transport of e_x along γ. This is a real analytic vector field, and so the flow depends in a real analytic manner on initial condition [Sontag 1998, Proposition C.3.12]. In particular, it depends in a real analytic manner on initial conditions lying in $\mathcal{N} \times \mathsf{E}_x$. But, in this case, the map from initial condition to value at $t = 1$ is exactly κ_x. This shows that κ_x is indeed real analytic. Moreover, it is clearly fibre preserving over $\exp_x$ and is linear on fibres, and so is a vector bundle map [cf. Abraham, Marsden, and Ratiu 1988, Proposition 3.4.12(iii)].

(ii) For $v \in \mathcal{N}$, let γ_v be the geodesic satisfying $\gamma_v'(0) = v$. Then, for $t \in \mathbb{R}_{>0}$ satisfying $|t| \leq 1$, define

$$\alpha_v(t) = \kappa_x^{-1} \circ \xi(\gamma_v(t)) = \tau_{x,\gamma_v(t)}^{-1}(\xi(\gamma_v(t)).$$

We compute derivatives of α_v as follows, by induction and using the fact that $\nabla_{\gamma_v'(t)}\gamma_v'(t) = 0$:

$$\begin{aligned}
\boldsymbol{D}\alpha_v(t) &= \tau_{x,\gamma_v(t)}^{-1}(\nabla^0\xi(\gamma_v'(t))) \\
\boldsymbol{D}^2\alpha_v(t) &= \tau_{x,\gamma_v(t)}^{-1}(\nabla^{(1)}\xi(\gamma_v'(t), \gamma_v'(t))) \\
&\ \ \vdots \\
\boldsymbol{D}^m\alpha_v(t) &= \tau_{x,\gamma_v(t)}^{-1}(\nabla^{(m-1)}\xi(\underbrace{\gamma_v'(t), \dots, \gamma_v'(t)}_{m \text{ times}})).
\end{aligned}$$

By these computations, we have

$$\left.\frac{\mathrm{d}^m}{\mathrm{d}t^m}\right|_{t=0} (\kappa_x^{-1} \circ \xi(\exp_x(tv)) = \nabla^{(m-1)}\xi(\underbrace{v, \dots, v}_{m \text{ times}}),$$

and so

$$\kappa_x^{-1} \circ \xi(\exp_x(tv)) = \sum_{m=0}^{\infty} \frac{t^m}{m!}\nabla^{(m-1)}\xi(\underbrace{v, \dots, v}_{m \text{ times}}),$$

which is the desired result upon letting $t = 1$ and supposing that v is in a sufficiently small neighbourhood of $0_x \in \mathsf{T}_x\mathsf{M}$. ▼

Next we introduce some notation in the general setting of the preceding sublemma that will be useful later. We fix $x \in \mathsf{M}$. We let $\mathcal{N}_x \subseteq \mathsf{T}_x\mathsf{M}$ and $\mathcal{V}_x \subseteq \mathsf{M}$ be neighbourhoods of 0_x and x, respectively, such that $\exp_x \colon \mathcal{N}_x \to \mathcal{V}_x$ is a diffeomorphism. For $y \in \mathcal{V}_x$ we then define

$$\begin{aligned}
I'_{xy} \colon \mathcal{N}'_{xy} \times \mathsf{E}_x &\to \mathsf{E}|\mathcal{V}'_{xy} \\
(v, e_x) &\mapsto \tau_{x,\exp_x(v+\exp_x^{-1}(y))}(e_x)
\end{aligned}$$

for neighbourhoods $\mathcal{N}'_{xy} \subseteq \mathsf{T}_x\mathsf{M}$ of $0_x \in \mathsf{T}_x\mathsf{M}$ and $\mathcal{V}'_{xy} \subseteq \mathsf{M}$ of y. We note that I'_{xy} is a real analytic vector bundle isomorphism over the diffeomorphism

$$\begin{aligned}
i'_{xy} \colon \mathcal{N}'_{xy} &\to \mathcal{V}'_{xy} \\
v &\mapsto \exp_x(v + \exp_x^{-1}(y)).
\end{aligned}$$

Thus $I_{xy} \triangleq I'_{xy} \circ \kappa_x^{-1}$ is a real analytic vector bundle isomorphism from $\mathsf{E}|\mathcal{U}'_{xy}$ to $\mathsf{E}|\mathcal{V}'_{xy}$ for appropriate neighbourhoods $\mathcal{U}'_{xy} \subseteq \mathsf{M}$ of x and $\mathcal{V}'_{xy} \subseteq \mathsf{M}$ of y. If we define $i_{xy} \colon \mathcal{U}'_{xy} \to \mathcal{V}'_{xy}$ by $i_{xy} = i'_{xy} \circ \exp_x^{-1}$, then I_{xy} is a vector bundle mapping over i_{xy}. Along similar lines, $\hat{I}_{xy} \triangleq \kappa_y^{-1} \circ I'_{xy}$ is a vector bundle isomorphism between the trivial bundles $\mathcal{O}'_{xy} \times \mathsf{E}_x$ and $\mathcal{N}'_{xy} \times \mathsf{E}_y$ for appropriate neighbourhoods $\mathcal{O}'_{xy} \subseteq \mathsf{T}_x\mathsf{M}$ and $\mathcal{N}'_{xy} \subseteq \mathsf{T}_y\mathsf{M}$ of the origin. If we define $\hat{i}_{xy} \colon \mathcal{O}'_{xy} \to \mathcal{N}'_{xy}$ by $\hat{i}_{xy} = \exp_y^{-1} \circ i'_{xy}$, then $\hat{I}_{xy}$ is a vector bundle map over $\hat{i}_{xy}$.

The next sublemma indicates that the neighbourhoods $\mathcal{U}'_{xy}$ of x and $\mathcal{O}'_{xy}$ of 0_x can be uniformly bounded from below.

2 Sublemma: *The neighbourhood $\mathcal{V}_x$ and the neighbourhoods $\mathcal{U}'_{xy}$ and $\mathcal{O}'_{xy}$ above may be chosen so that*

$$\operatorname{int}(\cap_{y\in\mathcal{V}_x}\mathcal{U}'_{xy}) \neq \varnothing, \quad \operatorname{int}(\cap_{y\in\mathcal{V}_x}\mathcal{O}'_{xy}) \neq \varnothing.$$

Proof: By [Kobayashi and Nomizu 1963, Theorem III.8.7] we can choose $\mathcal{V}_x$ so that, if $y \in \mathcal{V}_x$, then there is a normal coordinate neighbourhood $\mathcal{V}_y$ of y containing $\mathcal{V}_x$. Taking $\mathcal{V}'_{xy} = \mathcal{V}_x \cap \mathcal{V}_y$ and $\mathcal{U}'_{xy} = \mathcal{V}_x$ gives the sublemma. ▼

We shall always assume $\mathcal{V}_x$ chosen as in the preceding sublemma, and we let $\mathcal{U}'_x \subseteq \mathsf{M}$ be a neighbourhood of x and $\mathcal{O}'_x \subseteq \mathsf{T}_x\mathsf{M}$ be a neighbourhood of 0_x such that

$$\mathcal{U}'_x \subseteq \operatorname{int}(\cap_{y\in\mathcal{V}_x}\mathcal{U}'_{xy}), \quad \mathcal{O}'_x \subseteq \operatorname{int}(\cap_{y\in\mathcal{V}_x}\mathcal{O}'_{xy}).$$

These constructions can be "bundled together" as one to include the dependence on $y \in \mathcal{V}_x$ in a clearer manner. Since this will be useful for us, we explain it here. Let us denote $\mathcal{D}_x = \mathcal{V}_x \times \mathcal{U}'_x$, let $\mathrm{pr}_2\colon \mathcal{D}_x \to \mathcal{U}'_x$ be the projection onto the second factor, and denote

$$\begin{aligned} i_x\colon \mathcal{D}_x &\to \mathsf{M} \\ (y, x') &\mapsto i_{xy}(x'). \end{aligned}$$

Consider the pull-back bundle $\mathrm{pr}_2^*\pi\colon\ \mathrm{pr}_2^*\mathsf{E}|\mathcal{U}'_x \to \mathcal{D}_x$. Thus

$$\mathrm{pr}_2^*\mathsf{E}|\mathcal{U}'_x = \{((y,x'), e_{y'}) \in \mathcal{D}_x \times \mathsf{E}|\mathcal{U}'_x \mid\ y' = x'\}.$$

We then have a real analytic vector bundle mapping

$$\begin{aligned} I_x\colon \mathrm{pr}_2^*\mathsf{E}|\mathcal{U}'_x &\to \mathsf{E} \\ ((y,x'), e_{x'}) &\mapsto I_{xy}(e_{x'}) \end{aligned}$$

which is easily verified to be defined over i_x and is isomorphic on fibres. Given $\xi \in \Gamma^\infty(\mathsf{E})$, we define $I_x^*\xi \in \Gamma^\infty(\mathrm{pr}_2^*\mathsf{E}|\mathcal{U}'_x)$ by

$$I_x^*\xi(y,x') = (I_x)^{-1}_{(y,x')} \circ \xi \circ i_x(y,x') = I_{xy}^{-1} \circ \xi \circ i_{xy}(x').$$

For $y \in \mathcal{V}_x$ fixed, we denote by $I_{xy}^*\xi \in \Gamma^\infty(\mathsf{E}|\mathcal{U}'_x)$ the section given by

$$I_{xy}^*\xi(x') = I_x^*\xi(y,x') = I_{xy}^{-1} \circ \xi \circ i_{xy}(x').$$

A similar construction can be made in the local trivialisations. Here we denote $\hat{\mathcal{D}}_x = \mathcal{V}_x \times \mathcal{O}_x$, let $\mathrm{pr}_2\colon \hat{\mathcal{D}}_x \to \mathcal{O}_x$ be the projection onto the second factor, and consider the map

$$\begin{aligned} \hat{i}_x\colon \hat{\mathcal{D}}_x &\to \mathsf{TM} \\ (y, v_x) &\mapsto \hat{i}_{xy}(v_x). \end{aligned}$$

Denote by $\pi_{\mathsf{TM}}^*\pi\colon \pi_{\mathsf{TM}}^*\mathsf{E} \to \mathsf{TM}$ the pull-back bundle and also define the pull-back bundle

$$\mathrm{pr}_2^*\pi_{\mathsf{TM}}^*\pi\colon\ \mathrm{pr}_2^*\pi_{\mathsf{TM}}^*\mathsf{E} \to \hat{\mathcal{D}}_x.$$

Note that

$$\mathrm{pr}_2^*\pi_{\mathsf{TM}}^*\mathsf{E} = \{((y,v_x),(u_y,e_y)) \in \hat{\mathcal{D}}_x \times \pi_{\mathsf{TM}}^*\mathsf{E} \mid\ x = y\}.$$

We then define the real analytic vector bundle map

$$\begin{aligned}\hat{I}_x\colon \mathrm{pr}_2^*\,\pi_{\mathsf{TM}}^*\mathsf{E} &\to \pi_{\mathsf{TM}}^*\mathsf{E}\\ ((y,v_x),(u_x,e_x)) &\mapsto (v_x,\hat{I}_{xy}(v_x,e_x)).\end{aligned}$$

Given a local section $\eta \in \Gamma^\infty(\pi_{\mathsf{TM}}^*\mathsf{E})$ defined in a neighbourhood of the zero section, define a local section $\hat{I}_x^*\eta \in \Gamma^\infty(\mathrm{pr}_2^*\,\pi_{\mathsf{TM}}^*\mathsf{E})$ in a neighbourhood of the zero section of $\hat{\mathcal{D}}_x$ by

$$\hat{I}_x^*\eta(y,v_x) = (\hat{I}_x)^{-1}_{(y,v_x)} \circ \eta \circ \hat{\imath}_x(y,v_x) = \hat{I}_{xy}^{-1} \circ \eta \circ i_x(y,v_x).$$

For $y \in \mathcal{V}_x$ fixed, we denote by η_y the restriction of η to a neighbourhood of $0_y \in \mathsf{T}_y\mathsf{M}$. We then denote by

$$\hat{I}_{xy}^*\eta_y(v_x) = \hat{I}_x^*\eta(y,v_x) = \hat{I}_{xy}^{-1} \circ \eta_y \circ \hat{\imath}_{xy}(v_x)$$

the element of $\Gamma^\infty(\mathcal{O}'_x \times \mathsf{E}_x)$.

The following simple lemma ties the preceding two constructions together.

3 Sublemma: *Let $\xi \in \Gamma^\infty(\mathsf{E})$ and let $\hat{\xi} \in \Gamma^\infty(\pi_{\mathsf{TM}}^*\mathsf{E})$ be defined in a neighbourhood of the zero section by*

$$\hat{\xi} = \kappa_y^{-1} \circ \xi \circ \exp_y.$$

Then, for each $y \in \mathcal{V}_x$,

$$\hat{I}_{xy}^*\hat{\xi}_y = \kappa_x^{-1} \circ I_{xy}^*\xi \circ \exp_x.$$

Proof: We have

$$\begin{aligned}\hat{I}_{xy}^*\hat{\xi}(v_x) &= \hat{I}_{xy}^{-1} \circ \hat{\xi} \circ \hat{\imath}_{xy}(v_x)\\ &= (I'_{xy})^{-1} \circ \kappa_y \circ \hat{\xi} \circ \exp_y^{-1} \circ i'_{xy}(v_x)\\ &= \kappa_x^{-1} \circ I_{xy}^{-1} \circ \kappa_y \circ \hat{\xi} \circ \exp_y^{-1} \circ i_{xy} \circ \exp_x(v_x)\\ &= \kappa_x^{-1} \circ I_{xy}^{-1} \circ \xi \circ i_{xy} \circ \exp_x(v_x)\\ &= \kappa_x^{-1} I_{xy}^*\xi \circ \exp_x(v_x).\end{aligned}$$

as claimed. ▼

Let us leave these general vector bundle considerations and proceed to local estimates. We shall consider estimates associated with local vector bundle maps. First we consider an estimate arising from multiplication.

4 Sublemma: *If $\mathcal{U} \subseteq \mathbb{R}^n$ is open, if $f \in \mathrm{C}^\omega(\mathcal{U})$, and if $K \subseteq \mathcal{U}$ is compact, then there exist $C,\sigma \in \mathbb{R}_{>0}$ such that*

$$\sup\left\{\frac{1}{I!}\boldsymbol{D}^I(fg)(\boldsymbol{x}) \;\middle|\; |I| \le m\right\} \le C\sigma^{-m}\sup\left\{\frac{1}{I!}\boldsymbol{D}^I g(\boldsymbol{x}) \;\middle|\; |I| \le m\right\}$$

for every $g \in \mathrm{C}^\infty(\mathcal{U})$, $\boldsymbol{x} \in K$, and $m \in \mathbb{Z}_{\ge 0}$.

Proof: For multi-indices $I, J \in \mathbb{Z}_{\ge 0}^n$, let us write $J \le I$ if $I - J \in \mathbb{Z}_{\ge 0}^n$. For $I \in \mathbb{Z}_{\ge 0}^n$ we have

$$\frac{1}{I!}\boldsymbol{D}^I(fg)(\boldsymbol{x}) = \sum_{J \le I} \frac{\boldsymbol{D}^J g(\boldsymbol{x})}{J!}\frac{\boldsymbol{D}^{I-J}f(\boldsymbol{x})}{(I-J)!},$$

by the Leibniz Rule. By [Krantz and Parks 2002, Lemma 2.1.3], the number of multi-indices in n variables of order at most $|I|$ is $\frac{(n+|I|)!}{n!|I|!}$. Note that, by the binomial theorem,

$$(a_1+a_2)^{n+|I|} = \sum_{j=0}^{n+|I|} \frac{(n+|I|)!}{(n+|I|-j)!j!} a_1^j a_2^{n+|I|-j}.$$

Evaluating at $a_1 = a_2 = 1$ and considering the summand corresponding to $j = |I|$, this gives

$$\frac{(n+|I|)!}{n!|I|!} \leq 2^{n+|I|}.$$

Using this inequality we derive

$$\begin{aligned}\frac{1}{I!}|\boldsymbol{D}^I(fg)(\boldsymbol{x})| &\leq \sum_{|J|\leq|I|} \sup\left\{\frac{|\boldsymbol{D}^J f(\boldsymbol{x})|}{J!} \;\middle|\; |J|\leq|I|\right\} \sup\left\{\frac{|\boldsymbol{D}^J g(\boldsymbol{x})|}{J!} \;\middle|\; |J|\leq|I|\right\}\\ &\leq \frac{(n+|I|)!}{n!|I|!} \sup\left\{\frac{|\boldsymbol{D}^J f(\boldsymbol{x})|}{J!} \;\middle|\; |J|\leq|I|\right\} \sup\left\{\frac{|\boldsymbol{D}^J g(\boldsymbol{x})|}{J!} \;\middle|\; |J|\leq|I|\right\}\\ &\leq 2^{n+|I|} \sup\left\{\frac{|\boldsymbol{D}^J f(\boldsymbol{x})|}{J!} \;\middle|\; |J|\leq|I|\right\} \sup\left\{\frac{|\boldsymbol{D}^J g(\boldsymbol{x})|}{J!} \;\middle|\; |J|\leq|I|\right\}.\end{aligned}$$

By [Krantz and Parks 2002, Proposition 2.2.10], there exist $B, r \in \mathbb{R}_{>0}$ such that

$$\frac{1}{J!}|\boldsymbol{D}^J f(\boldsymbol{x})| \leq B r^{-|J|}, \qquad J \in \mathbb{Z}^n_{\geq 0},\ \boldsymbol{x} \in K.$$

We can suppose, without loss of generality, that $r < 1$ so that we have

$$\frac{1}{I!}|\boldsymbol{D}^I(fg)(\boldsymbol{x})| \leq 2^n B\Big(\frac{2}{r}\Big)^{|I|} \sup\left\{\frac{|\boldsymbol{D}^J g(\boldsymbol{x})|}{J!} \;\middle|\; |J|\leq|I|\right\}, \qquad \boldsymbol{x} \in K.$$

We conclude, therefore, that if $|I| \leq m$ we have

$$\frac{1}{I!}|\boldsymbol{D}^I(fg)(\boldsymbol{x})| \leq 2^n B\Big(\frac{2}{r}\Big)^{m} \sup\left\{\frac{|\boldsymbol{D}^J g(\boldsymbol{x})|}{J!} \;\middle|\; |J|\leq m\right\}, \qquad \boldsymbol{x} \in K,$$

which is the result upon taking $C = 2^n B$ and $\sigma = \frac{2}{r}$. ▼

Next we give an estimate for derivatives of compositions of mappings, one of which is real analytic. Thus we have a real analytic mapping $\boldsymbol{\Phi}\colon \mathcal{U} \to \mathcal{V}$ between open sets $\mathcal{U} \subseteq \mathbb{R}^n$ and $\mathcal{V} \subseteq \mathbb{R}^k$ and $f \in \mathrm{C}^\infty(\mathcal{V})$. By the higher-order Chain Rule [e.g., Constantine and Savits 1996], we can write

$$\boldsymbol{D}^I(f\circ\boldsymbol{\Phi})(\boldsymbol{x}) = \sum_{\substack{H\in\mathbb{Z}^m_{\geq 0}\\ |H|\leq|I|}} A_{I,H}(\boldsymbol{x})\boldsymbol{D}^H f(\boldsymbol{\Phi}(\boldsymbol{x}))$$

for $\boldsymbol{x} \in \mathcal{U}$ and for some real analytic functions $A_{I,H} \in \mathrm{C}^\omega(\mathcal{U})$. The proof of the next sublemma gives estimates for the $A_{I,H}$'s, and is based on computations of Thilliez [1997] in the proof of his Proposition 2.5.

5 Sublemma: *Let $\mathcal{U} \subseteq \mathbb{R}^n$ and $\mathcal{V} \subseteq \mathbb{R}^k$ be open, let $\boldsymbol{\Phi} \in \mathrm{C}^\omega(\mathcal{U};\mathcal{V})$, and let $K \subseteq \mathcal{U}$ be compact. Then there exist $C, \sigma \in \mathbb{R}_{>0}$ such that*

$$|\boldsymbol{D}^J A_{I,H}(\boldsymbol{x})| \le C\sigma^{-(|I|+|J|)}(|I|+|J|-|H|)!$$

for every $\boldsymbol{x} \in K$, $I, J \in \mathbb{Z}^n_{\ge 0}$, and $H \in \mathbb{Z}^k_{\ge 0}$.

Proof: First we claim that, for $j_1, \dots, j_r \in \{1, \dots, n\}$,

$$\frac{\partial^r (f \circ \boldsymbol{\Phi})}{\partial x^{j_1} \cdots \partial x^{j_r}}(\boldsymbol{x}) = \sum_{s=1}^{r} \sum_{a_1,\dots,a_s=1}^{k} B^{a_1\cdots a_s}_{j_1\cdots j_r}(\boldsymbol{x}) \frac{\partial^s f}{\partial y^{a_1} \cdots \partial y^{a_s}}(\boldsymbol{\Phi}(\boldsymbol{x})),$$

where the real analytic functions $B^{a_1\cdots a_s}_{j_1\cdots j_r}$, $a_1, \dots, a_s \in \{1, \dots, k\}$, $j_1, \dots, j_r \in \{1, \dots, n\}$, $r, s \in \mathbb{Z}_{>0}$, $s \le r$, are defined by the following recursion, starting with $B^a_j = \frac{\partial \Phi^a}{\partial x^j}$:

1. $B^a_{j_1\cdots j_r} = \dfrac{\partial B^a_{j_2\cdots j_r}}{\partial x^{j_1}}$;
2. $B^{a_1\cdots a_s}_{j_1\cdots j_r} = \dfrac{\partial B^{a_1\cdots a_s}_{j_2\cdots j_r}}{\partial x^{j_1}} + \dfrac{\partial \Phi^{a_1}}{\partial x^{j_1}} B^{a_2\cdots a_s}_{j_2\cdots j_r}$, $r \ge 2$, $s \in \{2, \dots, r-1\}$;
3. $B^{a_1\cdots a_r}_{j_1\cdots j_r} = \dfrac{\partial \Phi^{a_1}}{\partial x^{j_1}} B^{a_2\cdots a_r}_{j_2\cdots j_r}$.

This claim we prove by induction on r. It is clear for $r = 1$, so suppose the assertion true up to $r-1$. By the induction hypothesis we have

$$\frac{\partial^{r-1} (f \circ \boldsymbol{\Phi})}{\partial x^{j_2} \cdots \partial x^{j_r}}(\boldsymbol{x}) = \sum_{s=1}^{r-1} \sum_{a_1,\dots,a_s=1}^{k} B^{a_1\cdots a_s}_{j_2\cdots j_r}(\boldsymbol{x}) \frac{\partial^s f}{\partial y^{a_1} \cdots \partial y^{a_s}}(\boldsymbol{\Phi}(\boldsymbol{x})).$$

We then compute

$$\begin{aligned}
&\frac{\partial}{\partial x^{j_1}} \frac{\partial^{r-1} (f \circ \boldsymbol{\Phi})}{\partial x^{j_2} \cdots \partial x^{j_r}}(\boldsymbol{x}) \\
&= \sum_{s=1}^{r-1} \sum_{a_1,\dots,a_s=1}^{k} \Big(\frac{\partial B^{a_1,\dots,a_s}_{j_2\cdots j_r}}{\partial x^{j_1}}(\boldsymbol{x}) \frac{\partial^s f}{\partial y^{a_1} \cdots \partial y^{a_s}}(\boldsymbol{\Phi}(\boldsymbol{x})) \\
&\quad + \sum_{b=1}^{k} B^{a_1\cdots a_s}_{j_2\cdots j_r}(\boldsymbol{x}) \frac{\partial \Phi^b}{\partial x^{j_1}}(\boldsymbol{x}) \frac{\partial^{s+1} f}{\partial y^b \partial y^{a_1} \cdots \partial y^{a_s}}(\boldsymbol{\Phi}(\boldsymbol{x})) \Big) \\
&= \sum_{s=1}^{r-1} \sum_{a_1,\dots,a_s=1}^{k} \frac{\partial B^{a_1\cdots a_s}_{j_2\cdots j_r}}{\partial x^{j_1}}(\boldsymbol{x}) \frac{\partial^s f}{\partial y^{a_1} \cdots \partial y^{a_s}}(\boldsymbol{\Phi}(\boldsymbol{x})) \\
&\quad + \sum_{s=2}^{r} \sum_{a_1,\dots,a_s=1}^{k} B^{a_2\cdots a_s}_{j_2\cdots j_r}(\boldsymbol{x}) \frac{\partial \Phi^{a_1}}{\partial x^{j_1}}(\boldsymbol{x}) \frac{\partial^s f}{\partial y^{a_1} \cdots \partial y^{a_s}}(\boldsymbol{\Phi}(\boldsymbol{x})) \\
&= \sum_{a=1}^{k} \frac{\partial B^a_{j_2\cdots j_r}}{\partial x^{j_1}}(\boldsymbol{x}) \frac{\partial f}{\partial y^a}(\boldsymbol{\Phi}(\boldsymbol{x})) \\
&\quad + \sum_{s=2}^{r-1} \sum_{a_1,\dots,a_s=1}^{k} \Big(\frac{\partial B^{a_1\cdots a_s}_{j_2\cdots j_r}}{\partial x^{j_1}}(\boldsymbol{x}) + \frac{\partial \Phi^{a_1}}{\partial x^{j_1}}(\boldsymbol{x}) B^{a_2\cdots a_s}_{j_2\cdots j_r}(\boldsymbol{x}) \Big) \frac{\partial^s f}{\partial y^{a_1} \cdots \partial y^{a_s}}(\boldsymbol{\Phi}(\boldsymbol{x}))
\end{aligned}$$

$$+\sum_{a_1,\dots,a_r=1}^{k}\frac{\partial\Phi^{a_1}}{\partial x^{j_1}}(\boldsymbol{x})B^{a_2\cdots a_r}_{j_2\cdots j_r}(\boldsymbol{x})\frac{\partial^s f}{\partial y^{a_1}\cdots\partial y^{a_r}}(\boldsymbol{\Phi}(\boldsymbol{x})),$$

from which our claim follows.

Next we claim that there exist $A,\rho,\alpha,\beta\in\mathbb{R}_{>0}$ such that

$$|\boldsymbol{D}^J B^{a_1\cdots a_s}_{j_1\cdots j_r}(\boldsymbol{x})|\le (A\alpha)^r\Big(\frac{\beta}{\rho}\Big)^{r+|J|-s}(r+|J|-s)!$$

for every $\boldsymbol{x}\in K$, $J\in\mathbb{Z}^n_{\ge 0}$, $a_1,\dots,a_s\in\{1,\dots,k\}$, $j_1,\dots,j_k\in\{1,\dots,n\}$, $r,s\in\mathbb{Z}_{>0}$, $s\le r$. This we prove by induction on r once again. First let $\beta\in\mathbb{R}_{>0}$ be sufficiently large that

$$\sum_{I\in\mathbb{Z}^n_{\ge 0}}\beta^{-|I|}<\infty,$$

and denote this value of this sum by S. Then let $\alpha=2S$. By [Krantz and Parks 2002, Proposition 2.2.10] there exist $A,\rho\in\mathbb{R}_{>0}$ such that

$$|\boldsymbol{D}^J\boldsymbol{D}^j\Phi^a(\boldsymbol{x})|\le AJ!\rho^{-|J|}$$

for every $\boldsymbol{x}\in K$, $J\in\mathbb{Z}^n_{\ge 0}$, $j\in\{1,\dots,n\}$, and $a\in\{1,\dots,k\}$. This gives the claim for $r=1$. So suppose the claim true up to $r-1$. Then, for any $a_1,\dots,a_s\in\{1,\dots,k\}$ and $j_1,\dots,j_r\in\{1,\dots,n\}$, $s\le r$, $B^{a_1\cdots a_s}_{j_1\cdots j_r}$ has one of the three forms listed above in the recurrent definition. These three forms are themselves sums of terms of the form

$$\underbrace{\frac{\partial B^{a_1\cdots a_s}_{j_2\cdots j_r}}{\partial x^{j_1}}}_{P},\qquad\underbrace{\frac{\partial\Phi^{a_1}}{\partial x^{j_1}}B^{a_2\cdots a_s}_{j_2\cdots j_r}}_{Q}.$$

Let us, therefore, estimate derivatives of these terms, abbreviated by P and Q as above.

We directly have, by the induction hypothesis,

$$\begin{aligned}|\boldsymbol{D}^J P(\boldsymbol{x})|&\le (A\alpha)^r\Big(\frac{\beta}{\rho}\Big)^{r+|J|-s}(r+|J|-s)!\\&\le A^r\alpha^{r-1}S\Big(\frac{\beta}{\rho}\Big)^{r+|J|-s}(r+|J|-s)!,\end{aligned}$$

noting that $\alpha=2S$. By the Leibniz Rule we have

$$\boldsymbol{D}^J Q(\boldsymbol{x})=\sum_{J_1+J_2=J}\frac{J!}{J_1!J_2!}\boldsymbol{D}^{J_1}\boldsymbol{D}^{j_1}\Phi^{a_1}(\boldsymbol{x})\boldsymbol{D}^{J_2}B^{a_2\cdots a_s}_{j_2\cdots j_r}(\boldsymbol{x}).$$

By the induction hypothesis we have

$$|\boldsymbol{D}^{J_2}B^{a_2\cdots a_s}_{j_2\cdots j_r}(\boldsymbol{x})|\le (A\alpha)^{r-1}\Big(\frac{\beta}{\rho}\Big)^{r+|J_2|-s}(r+|J_2|-s)!$$

for every $\boldsymbol{x}\in K$ and $J_2\in\mathbb{Z}_{\ge 0}$. Therefore,

$$|\boldsymbol{D}^J Q(\boldsymbol{x})|\le\sum_{J_1+J_2=J}\frac{J!}{J_2!}A(A\alpha)^{r-1}\Big(\frac{\beta}{\rho}\Big)^{r+|J|-s}\beta^{-|J_1|}(r+|J_2|-s)!$$

for every $\boldsymbol{x} \in K$ and $J \in \mathbb{Z}_{\geq 0}^n$. Now note that, for any $a, b, c \in \mathbb{Z}_{>0}$ with $b < c$, we have

$$\frac{(a+b)!}{b!} = (1+b)\cdots(a+b) < (1+c)\cdots(a+c) = \frac{(a+c)!}{c!}.$$

Thus, if $L, J \in \mathbb{Z}_{\geq 0}^n$ satisfy $L < J$ (meaning that $J - L \in \mathbb{Z}_{\geq 0}^n$), then we have

$$l_k \leq j_k \quad \Longrightarrow \quad \frac{(a+l_k)!}{l_k!} \leq \frac{(a+j_k)!}{j_k!} \quad \Longrightarrow \quad \frac{j_k!}{l_k!} \leq \frac{(a+j_k)!}{(a+l_k)!}$$

for every $a \in \mathbb{Z}_{>0}$ and $k \in \{1, \dots, n\}$. Therefore,

$$\frac{(j_1 + \dots + j_{n-1} + j_n)!}{(j_1 + \dots + j_{n-1} + l_n)!} \geq \frac{j_n!}{l_n!}$$

and

$$\begin{aligned}\frac{(j_1 + \dots + j_{n-2} + j_{n-1} + j_n)!}{(j_1 + \dots + j_{n-2} + l_{n-1} + l_n)!}& \\ &= \frac{(j_1 + \dots + j_{n-1} + j_n)!}{(j_1 + \dots + j_{n-1} + l_n)!}\frac{(j_1 + \dots + j_{n-2} + j_{n-1} + l_n)!}{(j_1 + \dots + j_{n-2} + l_{n-1} + l_n)!} \geq \frac{j_{n-1}!}{l_{n-1}!}\frac{j_n!}{l_n!}.\end{aligned}$$

Continuing in this way, we get

$$\frac{J!}{L!} \leq \frac{|J|!}{|L|!}.$$

We also have

$$\frac{(r + |J_2| - s)!}{|J_2|!} \leq \frac{(r + |J| - s)!}{|J|!}.$$

Thus we have

$$\begin{aligned}|\boldsymbol{D}^J Q(\boldsymbol{x})| &\leq \sum_{J_1 + J_2 = J} \frac{J!}{J_2!} A (A\alpha)^{r-1} \Big(\frac{\beta}{\rho}\Big)^{r+|J|-s} \beta^{-|J_1|} (r + |J_2| - s)! \\ &\leq A (A\alpha)^{r-1} \Big(\frac{\beta}{\rho}\Big)^{r+|J|-s} (r + |J| - s)! \sum_{J_1 + J_2 = J} \beta^{-|J_1|} \\ &\leq AS (A\alpha)^{r-1} \Big(\frac{\beta}{\rho}\Big)^{r+|J|-s} (r + |J| - s)!\end{aligned}$$

Combining the estimates for P and Q to give an estimate for their sum, and recalling that $\alpha = 2S$, gives our claim that there exist $A, \rho, \alpha, \beta \in \mathbb{R}_{>0}$ such that

$$|\boldsymbol{D}^J B^{a_1 \cdots a_s}_{j_1 \cdots j_r}(\boldsymbol{x})| \leq (A\alpha)^r \Big(\frac{\beta}{\rho}\Big)^{r+|J|-s} (r + |J| - s)!$$

for every $\boldsymbol{x} \in K$, $J \in \mathbb{Z}_{\geq 0}^n$, $a_1, \dots, a_s \in \{1, \dots, k\}$, and $j_1, \dots, j_r \in \{1, \dots, n\}$, $r, s \in \mathbb{Z}_{>0}$, $s \leq r$.

To conclude the proof of the lemma, note that given an index $\boldsymbol{j} = (j_1, \dots, j_r) \in \{1, \dots, n\}^r$ we define a multi-index $I(\boldsymbol{j}) = (i_1, \dots, i_n) \in \mathbb{Z}_{\geq 0}^n$ by asking that i_l be the number of times l appears in the list $\boldsymbol{j}$. Similarly an index $\boldsymbol{a} = (a_1, \dots, a_s) \in \{1, \dots, k\}^s$ gives rise to a multi-index $H(\boldsymbol{a}) \in \mathbb{Z}_{\geq 0}^k$. Moreover, by construction we have

$$B_{j_1 \cdots j_r}^{a_1 \cdots a_s} = A_{I(\boldsymbol{j}), H(\boldsymbol{a})}.$$

Let $C = 1$ and $\sigma^{-1} = \max\{A\alpha, \frac{\beta}{\rho}\}$ and suppose, without loss of generality, that $\sigma \leq 1$. Then

$$(A\alpha)^{|I|} \leq \sigma^{-(|I|+|J|)}, \quad \left(\frac{\beta}{\rho}\right)^{r+|J|-s} \leq \sigma^{-(|I|+|J|)}$$

for every $I, J \in \mathbb{Z}_{\geq 0}^n$. Thus we have

$$|\boldsymbol{D}^J A_{I,H}(\boldsymbol{x})| \leq C\sigma^{-(|I|+|J|)}(|I| + |J| - |H|)!$$

as claimed. ▼

Next we consider estimates for derivatives arising from composition.

6 Sublemma: *Let $\mathcal{U} \subseteq \mathbb{R}^n$ and $\mathcal{V} \subseteq \mathbb{R}^k$ be open, let $\boldsymbol{\Phi} \in \mathrm{C}^\omega(\mathcal{U}; \mathcal{V})$, and let $K \subseteq \mathcal{U}$ be compact. Then there exist $C, \sigma \in \mathbb{R}_{>0}$ such that*

$$\sup\left\{\frac{1}{I!}|\boldsymbol{D}^I(f \circ \boldsymbol{\Phi})(\boldsymbol{x})| \;\middle|\; |I| \leq m\right\} \leq C\sigma^{-m} \sup\left\{\frac{1}{I!}|\boldsymbol{D}^H f(\boldsymbol{\Phi}(\boldsymbol{x}))| \;\middle|\; |H| \leq m\right\}$$

for every $f \in \mathrm{C}^\infty(\mathcal{V})$, $\boldsymbol{x} \in K$, and $m \in \mathbb{Z}_{\geq 0}$.

Proof: As we denoted preceding the statement of Sublemma 5 above, let us write

$$\boldsymbol{D}^I(f \circ \boldsymbol{\Phi})(\boldsymbol{x}) = \sum_{\substack{H \in \mathbb{Z}_{\geq 0}^m \\ |H| \leq |I|}} A_{I,H}(\boldsymbol{x}) \boldsymbol{D}^H f(\boldsymbol{\Phi}(\boldsymbol{x}))$$

for $\boldsymbol{x} \in \mathcal{U}$ and for some real analytic functions $A_{I,H} \in \mathrm{C}^\omega(\mathcal{U})$. By Sublemma 5, let $A, r \in \mathbb{R}_{>0}$ be such that

$$|\boldsymbol{D}^J A_{I,H}(\boldsymbol{x})| \leq Ar^{-(|I|+|J|)}(|I| + |J| - |H|)!$$

for $\boldsymbol{x} \in K$. By the multinomial theorem [Krantz and Parks 2002, Theorem 1.3.1] we can write

$$(a_1 + \cdots + a_n)^{|I|} = \sum_{|J|=|I|} \frac{|J|!}{J!} \boldsymbol{a}^J$$

for every $I \in \mathbb{Z}_{\geq 0}^n$. Setting $a_1 = \cdots = a_n = 1$ gives $\frac{|I|!}{I!} \leq n^{|I|}$ for every $I \in \mathbb{Z}_{\geq 0}^n$. As in the proof of Sublemma 4 we have that the number of multi-indices of length k and degree at most $|I|$ is bounded above by $2^{k+|I|}$. Also, by a similar binomial theorem argument, if $|H| \leq |I|$, then we have

$$\frac{(|I| - |H|)!|H|!}{|I|!} \leq 2^{|I|}.$$

Putting this together yields

$$\begin{aligned}
\frac{1}{I!}|\boldsymbol{D}^I(f \circ \boldsymbol{\Phi})(\boldsymbol{x})| &\leq An^{|I|}r^{-|I|} \sum_{|H| \leq |I|} \frac{(|I| - |H|)!|H|!}{|I|!} \frac{1}{H!} |\boldsymbol{D}^H f(\boldsymbol{\Phi}(\boldsymbol{x}))| \\
&\leq An^{|I|} 2^{k+|I|} 2^{|I|} r^{-|I|} \sup\left\{\frac{1}{H!}|\boldsymbol{D}^H f(\boldsymbol{\Phi}(\boldsymbol{x}))| \;\middle|\; |H| \leq |I|\right\} \\
&= 2^k A(4nr^{-1})^{|I|} \sup\left\{\frac{1}{H!}|\boldsymbol{D}^H f(\boldsymbol{\Phi}(\boldsymbol{x}))| \;\middle|\; |H| \leq |I|\right\}
\end{aligned}$$

whenever $\boldsymbol{x} \in K$. Let us denote $C = 2^k A$ and $\sigma^{-1} = 4nr^{-1}$ and take r so that $4nr^{-1} \geq 1$, without loss of generality. We then have

$$\sup\left\{\frac{1}{I!}|\boldsymbol{D}^I(f \circ \boldsymbol{\Phi})(\boldsymbol{x})| \;\middle|\; |I| \leq m\right\} \leq C\sigma^{-1} \sup\left\{\frac{1}{H!}|\boldsymbol{D}^H f(\boldsymbol{\Phi}(\boldsymbol{x}))| \;\middle|\; |H| \leq m\right\}$$

for every $f \in \mathrm{C}^\infty(\mathcal{U}_2)$, $\boldsymbol{x} \in K$, and $m \in \mathbb{Z}_{\geq 0}$, as claimed. ▼

Now we can state the following estimate for vector bundle mappings which is essential for our proof.

7 Sublemma: *Let $\mathcal{U} \subseteq \mathbb{R}^n$ and $\mathcal{V} \subseteq \mathbb{R}^k$ be open, let $l \in \mathbb{Z}_{>0}$, and consider the trivial vector bundles $\mathbb{R}^l_{\mathcal{U}}$ and $\mathbb{R}^l_{\mathcal{V}}$. Let $\boldsymbol{\Phi} \in \mathrm{C}^\omega(\mathcal{U}; \mathcal{V})$, let $\boldsymbol{A} \in \mathrm{C}^\omega(\mathcal{U}; \mathsf{GL}(l; \mathbb{R}))$, and let $K \subseteq \mathcal{U}$ be compact. Then there exist $C, \sigma \in \mathbb{R}_{>0}$ such that*

$$\sup\left\{\frac{1}{I!}|\boldsymbol{D}^I(\boldsymbol{A}^{-1} \cdot (\boldsymbol{\xi} \circ \boldsymbol{\Phi}))^b(\boldsymbol{x})| \;\middle|\; |I| \leq m,\ b \in \{1, \dots, l\}\right\}$$
$$\leq C\sigma^{-m} \sup\left\{\frac{1}{H!}|\boldsymbol{D}^H \xi^a(\boldsymbol{\Phi}(\boldsymbol{x}))| \;\middle|\; |H| \leq m,\ a \in \{1, \dots, l\}\right\},$$

for every $\boldsymbol{\xi} \in \Gamma^\infty(\mathbb{R}^l_{\mathcal{V}})$, $\boldsymbol{x} \in K$, and $m \in \mathbb{Z}_{\geq 0}$.

Proof: By Sublemma 6 there exist $C_1, \sigma_1 \in \mathbb{R}_{>0}$ such that

$$\sup\left\{\frac{1}{I!}|\boldsymbol{D}^I(\boldsymbol{\xi} \circ \boldsymbol{\Phi})^a(\boldsymbol{x})| \;\middle|\; |I| \leq m,\ a \in \{1, \dots, l\}\right\}$$
$$\leq C_1\sigma_1^{-m} \sup\left\{\frac{1}{H!}|\boldsymbol{D}^H \xi^a(\boldsymbol{\Phi}(\boldsymbol{x}))| \;\middle|\; |H| \leq m,\ a \in \{1, \dots, l\}\right\}$$

for every $\boldsymbol{\xi} \in \Gamma^\infty(\mathbb{R}^l_{\mathcal{V}})$, $\boldsymbol{x} \in K$, and $m \in \mathbb{Z}_{\geq 0}$.

Now let $\boldsymbol{\eta} \in \Gamma^\infty(\mathbb{R}^l_{\mathcal{U}})$. Let $B^b_a \in \mathrm{C}^\omega(\mathcal{U})$, $a \in \{1, \dots, l\}$, $b \in \{1, \dots, l\}$, be the components of $\boldsymbol{A}^{-1}$. By Sublemma 4, there exist $C_2, \sigma_2 \in \mathbb{R}_{>0}$ such that

$$\sup\left\{\frac{1}{I!}|\boldsymbol{D}^I(B^b_a(\boldsymbol{x})\eta^a(\boldsymbol{x}))| \;\middle|\; |I| \leq m,\ a, b \in \{1, \dots, l\}\right\}$$
$$\leq C_2\sigma_2^{-m} \sup\left\{\frac{1}{I!}|\boldsymbol{D}^I \eta^a(\boldsymbol{x})| \;\middle|\; |I| \leq m,\ a \in \{1, \dots, l\}\right\}$$

for every $\boldsymbol{x} \in K$ and $m \in \mathbb{Z}_{\geq 0}$. (There is no implied sum over "a" in the preceding formula.) Therefore, by the triangle inequality,

$$\sup\left\{\frac{1}{I!}|\boldsymbol{D}^I(\boldsymbol{A}^{-1} \cdot \boldsymbol{\eta})^b(\boldsymbol{x})| \;\middle|\; |I| \leq m,\ b \in \{1, \dots, l\}\right\}$$
$$\leq lC_2\sigma_2^{-m} \sup\left\{\frac{1}{I!}|\boldsymbol{D}^I \eta^a(\boldsymbol{x})| \;\middle|\; |I| \leq m,\ a \in \{1, \dots, l\}\right\}$$

for every $\boldsymbol{x} \in K$ and $m \in \mathbb{Z}_{\geq 0}$.

Combining the estimates from the preceding two paragraphs gives

$$\sup\left\{\frac{1}{I!}|\boldsymbol{D}^I(\boldsymbol{\xi} \circ \boldsymbol{\Phi})^b(\boldsymbol{x})| \;\middle|\; |I| \leq m,\ b \in \{1, \dots, l\}\right\}$$
$$\leq lC_1C_2(\sigma_1\sigma_2)^{-m} \sup\left\{\frac{1}{H!}|\boldsymbol{D}^H \xi^a(\boldsymbol{\Phi}(\boldsymbol{x}))| \;\middle|\; |H| \leq m,\ a \in \{1, \dots, l\}\right\}$$

for every $\boldsymbol{\xi} \in \Gamma^\infty(\mathbb{R}^l_{\mathcal{V}})$, $\boldsymbol{x} \in K$, and $m \in \mathbb{Z}_{\geq 0}$, which is the desired result after taking $C = lC_1C_2$ and $\sigma = \sigma_1\sigma_2$. ▼

Now we begin to provide some estimates that closely resemble those in the statement of the lemma. We begin by establishing an estimate resembling that of the required form for a fixed $\boldsymbol{x} \in \mathcal{U}$.

8 Sublemma: *Let $\mathcal{U} \subseteq \mathbb{R}^n$ be open, denote $\mathbb{R}^k_{\mathcal{U}} = \mathcal{U} \times \mathbb{R}^k$, and consider the trivial vector bundle $\mathrm{pr}_1 \colon \mathbb{R}^k_{\mathcal{U}} \to \mathcal{U}$. Let $\mathbb{G}$ be a Riemannian metric on $\mathcal{U}$, let $\mathbb{G}_0$ be a vector bundle metric on $\mathbb{R}^k_{\mathcal{U}}$, let ∇ be an affine connection on $\mathcal{U}$, and let ∇^0 be a vector bundle connection on $\mathbb{R}^k_{\mathcal{U}}$, with all of these being real analytic. For $\boldsymbol{\xi} \in \Gamma^\infty(\mathbb{R}^k_{\mathcal{U}})$ and $\boldsymbol{x} \in \mathcal{U}$, denote by $\hat{\boldsymbol{\xi}}_{\boldsymbol{x}}$ the corresponding section of $\mathcal{N}_{\boldsymbol{x}} \times \mathbb{R}^k$ defined by the isomorphism $\kappa_{\boldsymbol{x}}$ of Sublemma 1. For $K \subseteq \mathcal{U}$ compact, there exist $C, \sigma \in \mathbb{R}_{>0}$ such that the following inequalities hold for each $\boldsymbol{\xi} \in \Gamma^\infty(\mathbb{R}^k_{\mathcal{U}})$, $\boldsymbol{x} \in K$, and $m \in \mathbb{Z}_{\geq 0}$:*

(i) $\|j_m\boldsymbol{\xi}(\boldsymbol{x})\|_{\overline{\mathbb{G}}_m} \leq C\sigma^{-m} \sup\left\{\frac{1}{I!}|\boldsymbol{D}^I \hat{\xi}^a_{\boldsymbol{x}}(\boldsymbol{0})| \;\middle|\; |I| \leq m,\ a \in \{1,\dots,k\}\right\}$;

(ii) $\left\{\frac{1}{I!}|\boldsymbol{D}^I \hat{\xi}^a_{\boldsymbol{x}}(\boldsymbol{0})| \;\middle|\; |I| \leq m,\ a \in \{1,\dots,k\}\right\} \leq C\sigma^{-m}\|j_m\boldsymbol{\xi}(\boldsymbol{x})\|_{\overline{\mathbb{G}}_m}$.

Proof: By Sublemma 1 we have

$$\hat{\boldsymbol{\xi}}_{\boldsymbol{x}}(\boldsymbol{v}) = \sum_{m=0}^\infty \frac{1}{m!}\nabla^{(m-1)}\boldsymbol{\xi}(\boldsymbol{x})(\underbrace{\boldsymbol{v},\dots,\boldsymbol{v}}_{m\ \text{times}})$$

for $\boldsymbol{v}$ in some neighbourhood of $\boldsymbol{0} \in \mathbb{R}^n$. We also have

$$\hat{\boldsymbol{\xi}}_{\boldsymbol{x}}(\boldsymbol{v}) = \sum_{m\in\mathbb{Z}_{\geq 0}} \frac{1}{m!}\boldsymbol{D}^m\hat{\boldsymbol{\xi}}_{\boldsymbol{x}}(\boldsymbol{0})(\underbrace{\boldsymbol{v},\dots,\boldsymbol{v}}_{m\ \text{times}})$$

for every $\boldsymbol{v}$ in some neighbourhood of $\boldsymbol{0} \in \mathbb{R}^n$. As the relation

$$\sum_{m=0}^\infty \frac{1}{m!}\nabla^{(m-1)}\boldsymbol{\xi}(\boldsymbol{x})(\underbrace{\boldsymbol{v},\dots,\boldsymbol{v}}_{m\ \text{times}}) = \sum_{m\in\mathbb{Z}_{\geq 0}} \frac{1}{m!}\boldsymbol{D}^m\hat{\boldsymbol{\xi}}_{\boldsymbol{x}}(\boldsymbol{0})(\underbrace{\boldsymbol{v},\dots,\boldsymbol{v}}_{m\ \text{times}})$$

holds for every $\boldsymbol{v} \in \mathbb{R}^n$, it follows that

$$P^m_{\nabla,\nabla^0}(\boldsymbol{\xi})(\boldsymbol{x}) = \boldsymbol{D}^m\hat{\boldsymbol{\xi}}_{\boldsymbol{x}}(\boldsymbol{0})$$

for every $m \in \mathbb{Z}_{\geq 0}$. Take $m \in \mathbb{Z}_{\geq 0}$. We have

$$\sum_{r=0}^m \frac{1}{(r!)^2}\|P^r_{\nabla,\nabla^0}(\boldsymbol{\xi})(\boldsymbol{x})\|^2_{\mathbb{G}_r} \leq \sum_{r=0}^m \frac{A'A^r}{(r!)^2}\|\boldsymbol{D}^r\hat{\boldsymbol{\xi}}_{\boldsymbol{x}}(\boldsymbol{0})\|^2,$$

where $A' \in \mathbb{R}_{>0}$ depends on $\mathbb{G}_0$, $A \in \mathbb{R}_{>0}$ depends on $\mathbb{G}$, and where $\|\cdot\|$ denotes the 2-norm, i.e., the square root of the sum of squares of components. We can, moreover, assume without loss of generality that $A \geq 1$ so that we have

$$\sum_{r=0}^m \frac{1}{(r!)^2}\|P^r_{\nabla,\nabla^0}(\boldsymbol{\xi})(\boldsymbol{x})\|^2_{\mathbb{G}_r} \leq A'A^m\sum_{r=0}^m \frac{1}{(r!)^2}\|\boldsymbol{D}^r\hat{\boldsymbol{\xi}}_{\boldsymbol{x}}(\boldsymbol{0})\|^2.$$

By [Krantz and Parks 2002, Lemma 2.1.3],

$$\mathrm{card}(\{I \in \mathbb{Z}^n_{\geq 0} \mid |I| \leq m\}) = \frac{(n+m)!}{n!m!}.$$

Note that the 2-norm for $\mathbb{R}^N$ is related to the ∞-norm for $\mathbb{R}^N$ by $\|\boldsymbol{a}\|_2 \leq \sqrt{N}\|\boldsymbol{a}\|_\infty$ so that

$$\sum_{r=0}^{m} \frac{1}{(r!)^2}\|\boldsymbol{D}^r\hat{\boldsymbol{\xi}}_{\boldsymbol{x}}(\boldsymbol{0})\|^2 \leq k\frac{(n+m)!}{n!m!}\Big(\sup\Big\{\frac{1}{r!}|\boldsymbol{D}^I\hat{\xi}^a_{\boldsymbol{x}}(\boldsymbol{0})| \;\Big|\; |I| \leq m,\ a \in \{1,\dots,k\}\Big\}\Big)^2.$$

By the binomial theorem, as in the proof of Sublemma 4,

$$\frac{(n+m)!}{n!m!} \leq 2^{n+m}.$$

Thus

$$\|j_m\boldsymbol{\xi}(\boldsymbol{x})\|_{\overline{\mathbb{G}}_m} \leq \sqrt{kA'2^n}(\sqrt{2A})^m \sup\Big\{\frac{1}{I!}|\boldsymbol{D}^I\hat{\xi}^a_{\boldsymbol{x}}(\boldsymbol{0})| \;\Big|\; |I| \leq m,\ a \in \{1,\dots,k\}\Big\} \tag{2.9}$$

for every $m \in \mathbb{Z}_{\geq 0}$. The above computations show that this inequality is satisfied for a real analytic section $\boldsymbol{\xi}$. However, it also is satisfied if $\boldsymbol{\xi}$ is a smooth section. This we argue as follows. Let $\boldsymbol{\xi} \in \Gamma^\infty(\mathbb{R}^k_{\mathcal{U}})$ and, for $m \in \mathbb{Z}_{>0}$, let $\boldsymbol{\xi}_m \in \Gamma^\omega(\mathbb{R}^k_{\mathcal{U}})$ be the section whose coefficients are polynomial functions of degree at most m and such that $j_m\boldsymbol{\xi}_m(\boldsymbol{x}) = j_m\boldsymbol{\xi}(\boldsymbol{x})$. Also let $\hat{\boldsymbol{\xi}}_{\boldsymbol{x},m}$ be the corresponding section of $\mathcal{N}_{\boldsymbol{x}} \times \mathbb{R}^k$. We then have

$$j_m\boldsymbol{\xi}_m(\boldsymbol{x}) = j_m\boldsymbol{\xi}(\boldsymbol{x}), \quad \boldsymbol{D}^I\hat{\boldsymbol{\xi}}_{\boldsymbol{x},m}(\boldsymbol{0}) = \boldsymbol{D}^I\hat{\boldsymbol{\xi}}_{\boldsymbol{x}}(\boldsymbol{0}),$$

for every $I \in \mathbb{Z}^n_{\geq 0}$ satisfying $|I| \leq m$, the latter by the formula for the higher-order Chain Rule [Abraham, Marsden, and Ratiu 1988, Supplement 2.4A]. Since $\boldsymbol{\xi}_m$ is real analytic, this shows that (2.9) is also satisfied for every $m \in \mathbb{Z}_{\geq 0}$ if $\boldsymbol{\xi}$ is smooth.

To establish the other estimate asserted in the sublemma, let $\boldsymbol{x} \in K$ and, using the notation of Sublemma 1, let $\mathcal{N}_{\boldsymbol{x}}$ be a relatively compact neighbourhood of $\boldsymbol{0} \in \mathbb{R}^n \simeq \mathsf{T}_{\boldsymbol{x}}\mathbb{R}^n$ and $\mathcal{V}_{\boldsymbol{x}} \subseteq \mathcal{U}$ be a relatively compact neighbourhood of $\boldsymbol{x}$ such that $\kappa_{\boldsymbol{x}} \colon \mathcal{N}_{\boldsymbol{x}} \times \mathbb{R}^k \to \mathcal{V}_{\boldsymbol{x}} \times \mathbb{R}^k$ is a real analytic vector bundle isomorphism. Let $\boldsymbol{\xi} \in \Gamma^\omega(\mathbb{R}^k_{\mathcal{V}_{\boldsymbol{x}}})$ and let $\hat{\boldsymbol{\xi}}_{\boldsymbol{x}} \in \Gamma^\omega(\mathbb{R}^k_{\mathcal{N}_{\boldsymbol{x}}})$ be defined by $\hat{\boldsymbol{\xi}}_{\boldsymbol{x}}(\boldsymbol{v}) = \kappa_{\boldsymbol{x}}^{-1} \circ \boldsymbol{\xi}(\exp_{\boldsymbol{x}}(\boldsymbol{v}))$. As in the first part of the estimate, we have

$$\boldsymbol{D}^m\hat{\boldsymbol{\xi}}_{\boldsymbol{x}}(\boldsymbol{0}) = P^m_{\nabla,\nabla^0}(\boldsymbol{\xi})(\boldsymbol{x})$$

for every $m \in \mathbb{Z}_{\geq 0}$. For indices $\boldsymbol{j} = (j_1,\dots,j_m) \in \{1,\dots,n\}^m$ we define $I(\boldsymbol{j}) = (i_1,\dots,i_n) \in \mathbb{Z}^n_{\geq 0}$ by asking that i_j be the number of times "j" appears in the list $\boldsymbol{j}$. We then have

$$\begin{aligned}\sup\Big\{\frac{1}{I!}|\boldsymbol{D}^I\hat{\xi}^a_{\boldsymbol{x}}(\boldsymbol{0})| \;\Big|\; |I| \leq m,\ a \in \{1,\dots,k\}\Big\} = \\ \sup\Big\{\frac{1}{I(\boldsymbol{j})!}|(P^r_{\nabla,\nabla^0}(\boldsymbol{\xi})(\boldsymbol{x}))^a_{j_1\cdots j_r}| \;\Big| \\ j_1,\dots,j_r \in \{1,\dots,n\},\ r \in \{0,1,\dots,m\},\ a \in \{1,\dots,k\}\Big\}.\end{aligned}$$

By an application of the multinomial theorem as in the proof of Sublemma 6, we have $\frac{|I|!}{I!} \leq n^{|I|}$ for every $I \in \mathbb{Z}^n_{\geq 0}$. We then have

$$\frac{1}{I(\boldsymbol{j})!}|(P^r_{\nabla,\nabla^0}(\boldsymbol{\xi})(\boldsymbol{x}))^a_{j_1\cdots j_r}| \leq \frac{n^r}{r!}|(P^r_{\nabla,\nabla^0}(\boldsymbol{\xi})(\boldsymbol{x}))^a_{j_1\cdots j_r}|$$

for every $j_1,\dots,j_r \in \{1,\dots,n\}$ and $a \in \{1,\dots,k\}$. Using the fact that the ∞-norm for $\mathbb{R}^N$ is related to the 2-norm for $\mathbb{R}^N$ by $\|\boldsymbol{a}\|_\infty \le \|\boldsymbol{a}\|_2$, we have

$$\sup\left\{\frac{1}{I!}|\boldsymbol{D}^I\hat{\xi}^a_{\boldsymbol{x}}(\boldsymbol{0})| \;\middle|\; |I|\le m,\ a\in\{1,\dots,k\}\right\} \le \left(\sum_{r=0}^m \left(\frac{n^r}{r!}\right)^2 B'B^r \|P^r_{\nabla,\nabla^0}(\boldsymbol{\xi})(\boldsymbol{x})\|^2_{\mathbb{G}_r}\right)^{1/2},$$

where $B' \in \mathbb{R}_{>0}$ depends on $\mathbb{G}_0$ and $B \in \mathbb{R}_{>0}$ depends on $\mathbb{G}$. We may, without loss of generality, suppose that $B \ge 1$ so that we have

$$\sup\left\{\frac{1}{I!}|\boldsymbol{D}^I\hat{\xi}^a_{\boldsymbol{x}}(\boldsymbol{0})| \;\middle|\; |I|\le m,\ a\in\{1,\dots,k\}\right\} \le \sqrt{B'}(n\sqrt{B})^m \|j_m\boldsymbol{\xi}(\boldsymbol{x})\|_{\overline{\mathbb{G}}_m}$$

for every $m \in \mathbb{Z}_{\ge 0}$. As in the first part of the proof, while we have demonstrated the preceding inequality for $\boldsymbol{\xi}$ real analytic, it can also be demonstrated to hold for $\boldsymbol{\xi}$ smooth.

The sublemma follows by taking

$$C = \max\{\sqrt{kA'2^n}, \sqrt{B'}\}, \quad \sigma^{-1} = \max\{\sqrt{2}A, n\sqrt{B}\}. \qquad ▼$$

The next estimates we consider will allow us to expand the pointwise estimate from the preceding sublemma to a local estimate of the same form. The construction makes use of the vector bundle isomorphisms I_{xy} and $\hat{I}_{xy}$ defined after Sublemma 1. In the statement and proof of the following sublemma, we make free use of the notation we introduced where these mappings were defined.

9 Sublemma: *Let $\mathcal{U} \subseteq \mathbb{R}^n$ be open, denote $\mathbb{R}^k_{\mathcal{U}} = \mathcal{U}\times\mathbb{R}^k$, and consider the trivial vector bundle $\mathrm{pr}_1\colon \mathbb{R}^k_{\mathcal{U}} \to \mathcal{U}$. Let $\mathbb{G}$ be a Riemannian metric on $\mathcal{U}$, let $\mathbb{G}_0$ be a vector bundle metric on $\mathbb{R}^k_{\mathcal{U}}$, let ∇ be an affine connection on $\mathcal{U}$, and let ∇^0 be a vector bundle connection on $\mathbb{R}^k_{\mathcal{U}}$, with all of these being real analytic. For each $\boldsymbol{x}\in\mathcal{U}$ there exist a neighbourhood $\mathcal{V}_{\boldsymbol{x}}$ and $C_{\boldsymbol{x}},\sigma_{\boldsymbol{x}}\in\mathbb{R}_{>0}$ such that we have the following inequalities for each $\boldsymbol{\xi}\in\Gamma^\infty(\mathbb{R}^k_{\mathcal{U}})$, $m\in\mathbb{Z}_{\ge0}$, and $\boldsymbol{y}\in\mathcal{V}_{\boldsymbol{x}}$:*

(i) $\sup\left\{\frac{1}{I!}|\boldsymbol{D}^I\hat{\xi}^a_{\boldsymbol{y}}(\boldsymbol{0})| \;\middle|\; |I|\le m,\ a\in\{1,\dots,k\}\right\}$
$\le C_{\boldsymbol{x}}\sigma_{\boldsymbol{x}}^{-1}\sup\left\{\frac{1}{I!}|\boldsymbol{D}^I((\hat{I}^*_{\boldsymbol{xy}})^{-1}\hat{\boldsymbol{\xi}}_{\boldsymbol{y}})^a(\boldsymbol{0})| \;\middle|\; |I|\le m,\ a\in\{1,\dots,k\}\right\}$;

(ii) $\sup\left\{\frac{1}{I!}|\boldsymbol{D}^I(\hat{I}^*_{\boldsymbol{xy}}\hat{\boldsymbol{\xi}}_{\boldsymbol{y}})^a|(\boldsymbol{0}) \;\middle|\; |I|\le m,\ a\in\{1,\dots,k\}\right\}$
$\le C_{\boldsymbol{x}}\sigma_{\boldsymbol{x}}^{-1}\sup\left\{\frac{1}{I!}|\boldsymbol{D}^I\hat{\xi}^a_{\boldsymbol{y}}(\boldsymbol{0})| \;\middle|\; |I|\le m,\ a\in\{1,\dots,k\}\right\}$;

(iii) $\|j_m\boldsymbol{\xi}(\boldsymbol{y})\|_{\overline{\mathbb{G}}_m} \le C_{\boldsymbol{x}}\sigma_{\boldsymbol{x}}^{-1}\|j_m((I^*_{\boldsymbol{xy}})^{-1}\boldsymbol{\xi})(\boldsymbol{x})\|_{\overline{\mathbb{G}}_m}$;

(iv) $\|j_m(I^*_{\boldsymbol{xy}}\boldsymbol{\xi})(\boldsymbol{x})\|_{\overline{\mathbb{G}}_m} \le C_{\boldsymbol{x}}\sigma_{\boldsymbol{x}}^{-1}\|j_m\boldsymbol{\xi}(\boldsymbol{y})\|_{\overline{\mathbb{G}}_m}$.

Proof: We begin the proof with an observation. Suppose that we have an open subset $\mathcal{U}\subseteq\mathbb{R}^n\times\mathbb{R}^k$ and $f\in\mathrm{C}^\omega(\mathcal{U})$. We wish to think of f as a function of $\boldsymbol{x}\in\mathbb{R}^n$ depending on a parameter $\boldsymbol{p}\in\mathbb{R}^k$ in a jointly real analytic manner. We note that, for $K\subseteq\mathcal{U}$ compact, we have $C,\sigma\in\mathbb{R}_{>0}$ such that the partial derivatives satisfy a bound

$$|\boldsymbol{D}^I_1 f(\boldsymbol{x},\boldsymbol{p})| \le CI!\sigma^{-|I|}$$

for every $(\boldsymbol{x},\boldsymbol{p})\in K$ and $I\in\mathbb{Z}^n_{\ge0}$. This is a mere specialisation of [Krantz and Parks 2002, Proposition 2.2.10] to partial derivatives. The point is that the bound for the partial derivatives is uniform in the parameter $\boldsymbol{p}$. With this in mind, we note that the following are easily checked:

1. the estimate of Sublemma 4 can be extended to the case where f depends in a jointly real analytic manner on a parameter, and the estimate is uniform in the parameter over compact sets;
2. the estimate of Sublemma 5 can be extended to the case where $\boldsymbol{\Phi}$ depends in a jointly real analytic manner on a parameter, and the estimate is uniform in the parameter over compact sets;
3. as a consequence of the preceding fact, the estimate of Sublemma 6 can be extended to the case where $\boldsymbol{\Phi}$ depends in a jointly real analytic manner on a parameter, and the estimate is uniform in the parameter over compact sets;
4. as a consequence of the preceding three facts, the estimate of Sublemma 7 can be extended to the case where $\boldsymbol{\Phi}$ and $\boldsymbol{A}$ depend in a jointly real analytic manner on a parameter, and the estimate is uniform in the parameter over compact sets.

Now let us proceed with the proof.

We take $\mathcal{V}_{\boldsymbol{x}}$ as in the discussion following Sublemma 1. Let us introduce coordinate notation for all maps needed. We have

$$\begin{aligned}
\hat{\boldsymbol{\xi}}_{\boldsymbol{y}}(\boldsymbol{u}) &= \hat{\boldsymbol{\xi}}(\boldsymbol{y},\boldsymbol{u}) = \boldsymbol{\xi}\circ\exp_{\boldsymbol{y}}(\boldsymbol{u}),\\
I^*_{\boldsymbol{xy}}\boldsymbol{\xi}(\boldsymbol{x}') &= \boldsymbol{A}(\boldsymbol{y},\boldsymbol{x}')\cdot(\boldsymbol{\xi}\circ i_{\boldsymbol{xy}}(\boldsymbol{x}')),\\
\hat{I}^*_{\boldsymbol{xy}}\hat{\boldsymbol{\xi}}_{\boldsymbol{y}}(\boldsymbol{v}) &= \hat{\boldsymbol{A}}(\boldsymbol{y},\boldsymbol{v})\cdot(\hat{\boldsymbol{\xi}}_{\boldsymbol{y}}\circ\hat{i}_{\boldsymbol{xy}}(\boldsymbol{v})),\\
(I^*_{\boldsymbol{xy}})^{-1}\boldsymbol{\xi}(\boldsymbol{y}') &= \boldsymbol{A}^{-1}(\boldsymbol{y},i^{-1}_{\boldsymbol{xy}}(\boldsymbol{y}')\cdot(\boldsymbol{\xi}\circ i^{-1}_{\boldsymbol{xy}}(\boldsymbol{y}')),\\
(\hat{I}^*_{\boldsymbol{xy}})^{-1}\hat{\boldsymbol{\xi}}_{\boldsymbol{y}}(\boldsymbol{v}) &= \hat{\boldsymbol{A}}^{-1}(\boldsymbol{y},\hat{i}^{-1}_{\boldsymbol{xy}}(\boldsymbol{u}))\cdot(\hat{\boldsymbol{\xi}}_{\boldsymbol{y}}\circ\hat{i}^{-1}_{\boldsymbol{xy}}(\boldsymbol{v})),
\end{aligned}$$

for appropriate real analytic mappings $\boldsymbol{A}$ and $\hat{\boldsymbol{A}}$ taking values in $\mathsf{GL}(k;\mathbb{R})$. Note that, for every $I\in\mathbb{Z}^n_{\geq 0}$,

$$\boldsymbol{D}^I(\hat{I}^*_{\boldsymbol{xy}}\hat{\boldsymbol{\xi}}_{\boldsymbol{y}})(\boldsymbol{0}) = \boldsymbol{D}^I_2(\hat{I}^*_{\boldsymbol{x}}\hat{\boldsymbol{\xi}})(\boldsymbol{y},\boldsymbol{0}),$$

and similarly for $\boldsymbol{D}^I((\hat{I}^*_{\boldsymbol{xy}})^{-1}\hat{\boldsymbol{\xi}}_{\boldsymbol{y}})(\boldsymbol{0})$. The observation made at the beginning of the proof shows that parts (i) and (ii) follow immediately from Sublemma 7. Parts (iii) and (iv) follow from the first two parts after an application of Sublemma 8. ▼

By applications of (a) Sublemma 9, (b) Sublemmata 3 and 8, (c) Sublemma 9 again, and (d) Sublemma 7, there exist

$$A_{1,\boldsymbol{x}},A_{2,\boldsymbol{x}},A_{3,\boldsymbol{x}},A_{4,\boldsymbol{x}},r_{1,\boldsymbol{x}},r_{2,\boldsymbol{x}},r_{3,\boldsymbol{x}},r_{4,\boldsymbol{x}}\in\mathbb{R}_{>0}$$

and a relatively compact neighbourhood $\mathcal{V}_{\boldsymbol{x}}\subseteq\mathcal{U}$ of $\boldsymbol{x}$ such that

$$\begin{aligned}
\|j_m\boldsymbol{\xi}(\boldsymbol{y})\|_{\overline{\mathbb{G}}_m} &\leq A_{1,\boldsymbol{x}}r^{-m}_{1,\boldsymbol{x}}\|j_m((I^*_{\boldsymbol{xy}})^{-1}\boldsymbol{\xi})(\boldsymbol{x})\|_{\overline{\mathbb{G}}_m}\\
&\leq A_{2,\boldsymbol{x}}r^{-m}_{2,\boldsymbol{x}}\sup\left\{\frac{1}{I!}|\boldsymbol{D}^I((\hat{I}^*_{\boldsymbol{xy}})^{-1}\hat{\boldsymbol{\xi}}_{\boldsymbol{y}})^a(\boldsymbol{0})|\ \middle|\ |I|\leq m,\ a\in\{1,\dots,k\}\right\}\\
&\leq A_{3,\boldsymbol{x}}r^{-m}_{3,\boldsymbol{x}}\sup\left\{\frac{1}{I!}|\boldsymbol{D}^I\hat{\xi}^a_{\boldsymbol{y}}(\boldsymbol{0})|\ \middle|\ |I|\leq m,\ a\in\{1,\dots,k\}\right\}\\
&\leq A_{4,\boldsymbol{x}}r^{-m}_{4,\boldsymbol{x}}\sup\left\{\frac{1}{I!}|\boldsymbol{D}^I\xi^a(\boldsymbol{y})|\ \middle|\ |I|\leq m,\ a\in\{1,\dots,k\}\right\}
\end{aligned}$$

for every $\boldsymbol{\xi} \in \Gamma^\infty(\mathbb{R}^k_{\mathcal{U}})$, $m \in \mathbb{Z}_{\geq 0}$, and $\boldsymbol{y} \in \mathcal{V}_{\boldsymbol{x}}$. Take $\boldsymbol{x}_1, \dots, \boldsymbol{x}_k \in K$ such that $K \subseteq \cup_{j=1}^k \mathcal{V}_{\boldsymbol{x}_j}$ and define

$$C_1 = \max\{A_{4,\boldsymbol{x}_1}, \dots, A_{4,\boldsymbol{x}_k}\}, \quad \sigma_1 = \min\{r_{4,\boldsymbol{x}_1}, \dots, r_{4,\boldsymbol{x}_k}\},$$

so that

$$\|j_m\boldsymbol{\xi}(\boldsymbol{x})\|_{\overline{\mathbb{G}}_m} \leq C_1\sigma_1^{-m} \sup\left\{\frac{1}{I!}|\boldsymbol{D}^I\xi^a(\boldsymbol{x})| \;\middle|\; |I| \leq m,\ a \in \{1, \dots, k\}\right\}$$

for every $\boldsymbol{\xi} \in \Gamma^\infty(\mathbb{R}^k_{\mathcal{U}})$, $m \in \mathbb{Z}_{\geq 0}$, and $\boldsymbol{x} \in K$. This gives one half of the estimate in the lemma.

For the other half of the estimate in the lemma, we apply (a) Sublemma 7, (b) Sublemma 9, (c) Sublemmata 3 and 8, and (d) Sublemma 9 again to assert the existence of

$$A_{1,\boldsymbol{x}}, A_{2,\boldsymbol{x}}, A_{3,\boldsymbol{x}}, A_{4,\boldsymbol{x}}, r_{1,\boldsymbol{x}}, r_{2,\boldsymbol{x}}, r_{3,\boldsymbol{x}}, r_{4,\boldsymbol{x}} \in \mathbb{R}_{>0}$$

and a relatively compact neighbourhood $\mathcal{V}_{\boldsymbol{x}} \subseteq \mathcal{U}$ of $\boldsymbol{x}$ such that

$$\begin{aligned}\sup&\left\{\frac{1}{I!}|\boldsymbol{D}^I\xi^a(\boldsymbol{y})| \;\middle|\; |I| \leq m,\ a \in \{1, \dots, k\}\right\}\\ &\leq A_{1,\boldsymbol{x}}r_{1,\boldsymbol{x}}^{-m} \sup\left\{\frac{1}{I!}|\boldsymbol{D}^I\hat{\xi}^a_{\boldsymbol{y}}(\boldsymbol{0})| \;\middle|\; |I| \leq m,\ a \in \{1, \dots, k\}\right\}\\ &\leq A_{2,\boldsymbol{x}}r_{2,\boldsymbol{x}}^{-m} \sup\left\{\frac{1}{I!}|\boldsymbol{D}^I((\hat{I}^*_{\boldsymbol{xy}})^{-1}\hat{\boldsymbol{\xi}}_{\boldsymbol{y}})^a(\boldsymbol{0})| \;\middle|\; |I| \leq m,\ a \in \{1, \dots, k\}\right\}\\ &\leq A_{3,\boldsymbol{x}}r_{3,\boldsymbol{x}}^{-m}\|j_m((I^*_{\boldsymbol{xy}})^{-1}\boldsymbol{\xi})(\boldsymbol{x})\|_{\overline{\mathbb{G}}_m} \leq A_{4,\boldsymbol{x}}r_{4,\boldsymbol{x}}^{-m}\|j_m\boldsymbol{\xi}(\boldsymbol{y})\|_{\overline{\mathbb{G}}_m}\end{aligned}$$

for every $\boldsymbol{\xi} \in \Gamma^\infty(\mathbb{R}^k_{\mathcal{U}})$, $m \in \mathbb{Z}_{\geq 0}$, and $\boldsymbol{y} \in \mathcal{V}_{\boldsymbol{x}}$. As we argued above using a standard compactness argument, there exist $C_2, \sigma_2 \in \mathbb{R}_{>0}$ such that

$$\sup\left\{\frac{1}{I!}|\boldsymbol{D}^I\xi^a(\boldsymbol{x})| \;\middle|\; |I| \leq m,\ a \in \{1, \dots, k\}\right\} \leq C_2\sigma_2^{-m}\|j_m\boldsymbol{\xi}(\boldsymbol{x})\|_{\overline{\mathbb{G}}_m}$$

for every $\boldsymbol{\xi} \in \Gamma^\infty(\mathbb{R}^k_{\mathcal{U}})$, $m \in \mathbb{Z}_{\geq 0}$, and $\boldsymbol{x} \in K$. Taking $C = \max\{C_1, C_2\}$ and $\sigma = \min\{\sigma_1, \sigma_2\}$ gives the lemma. ■

The preceding lemma will come in handy on a few crucial occasions. To illustrate how it can be used, we give the following characterisation of real analytic sections, referring to Section 3 below for the definition of the seminorm $p^\infty_{K,m}$ used in the statement.

2.5 Lemma: (Characterisation of real analytic sections) *Let $\pi\colon \mathsf{E} \to \mathsf{M}$ be a real analytic vector bundle and let $\xi \in \Gamma^\infty(\mathsf{E})$. Then the following statements are equivalent:*

(i) $\xi \in \Gamma^\omega(\mathsf{E})$;

(ii) for every compact set $K \subseteq \mathsf{M}$, there exist $C, r \in \mathbb{R}_{>0}$ such that $p^\infty_{K,m}(\xi) \leq Cr^{-m}$ for every $m \in \mathbb{Z}_{\geq 0}$.

Proof: (i) $\Longrightarrow$ (ii) Let $K \subseteq \mathsf{M}$ be compact, let $x \in K$, and let $(\mathcal{V}_x, \psi_x)$ be a vector bundle chart for E with $(\mathcal{U}_x, \phi_x)$ the corresponding chart for M. Let $\boldsymbol{\xi}\colon \phi(\mathcal{U}_x) \to \mathbb{R}^k$ be the local representative of ξ. By [Krantz and Parks 2002, Proposition 2.2.10], there exist a neighbourhood $\mathcal{U}'_x \subseteq \mathcal{U}_x$ of x and $B_x, \sigma_x \in \mathbb{R}_{>0}$ such that

$$|\boldsymbol{D}^I\xi^a(\boldsymbol{x}')| \leq B_xI!\sigma_x^{-|I|}$$

for every $a \in \{1,\dots,k\}$, $\boldsymbol{x}' \in \mathrm{cl}(\mathcal{U}'_x)$, and $I \in \mathbb{Z}^n_{\geq 0}$. We can suppose, without loss of generality, that $\sigma_x \in (0,1)$. In this case, if $|I| \leq m$,

$$\frac{1}{I!}|\boldsymbol{D}^I\xi^a(\boldsymbol{x}')| \leq B_x\sigma_x^{-m}$$

for every $a \in \{1,\dots,k\}$ and $\boldsymbol{x}' \in \mathrm{cl}(\mathcal{U}'_x)$. By Lemma 2.4, there exist $C_x, r_x \in \mathbb{R}_{>0}$ such that

$$\|j_m\xi(x')\|_{\overline{\mathbb{G}}_m} \leq C_x r_x^{-m}, \qquad x' \in \mathrm{cl}(\mathcal{U}'_x),\ m \in \mathbb{Z}_{\geq 0}.$$

Let $x_1,\dots,x_k \in K$ be such that $K \subseteq \cup_{j=1}^k \mathcal{U}'_{x_j}$ and let $C = \max\{C_{x_1},\dots,C_{x_k}\}$ and $r = \min\{r_{x_1},\dots,r_{x_k}\}$. Then, if $x \in K$, we have $x \in \mathcal{U}'_{x_j}$ for some $j \in \{1,\dots,k\}$ and so

$$\|j_m\xi(x)\|_{\overline{\mathbb{G}}_m} \leq C_{x_j} r_{x_j}^{-m} \leq C r^{-m},$$

as desired.

(ii) $\implies$ (ii) Let $x \in \mathsf{M}$ and let $(\mathcal{V},\psi)$ be a vector bundle chart for E such that the associated chart $(\mathcal{U},\phi)$ for M is a relatively compact coordinate chart about x. Let $\boldsymbol{\xi}\colon \phi(\mathcal{U}) \to \mathbb{R}^k$ be the local representative of ξ. By hypothesis, there exist $C, r \in \mathbb{R}_{>0}$ such that $\|j_m\xi(x')\|_{\overline{\mathbb{G}}_m} \leq Cr^{-m}$ for every $m \in \mathbb{Z}_{\geq 0}$ and $x' \in \mathcal{U}$. Let $\mathcal{U}'$ be a relatively compact neighbourhood of x such that $\mathrm{cl}(\mathcal{U}') \subseteq \mathcal{U}$. By Lemma 2.4, there exist $B, \sigma \in \mathbb{R}_{>0}$ such that

$$|\boldsymbol{D}^I\xi^a(\boldsymbol{x}')| \leq BI!\sigma^{-|I|}$$

for every $a \in \{1,\dots,k\}$, $\boldsymbol{x}' \in \mathrm{cl}(\mathcal{U}')$, and $I \in \mathbb{Z}^n_{\geq 0}$. We conclude real analyticity of ξ in a neighbourhood of x by [Krantz and Parks 2002, Proposition 2.2.10]. ■

3. The compact-open topologies for the spaces of finitely differentiable, Lipschitz, and smooth vector fields

In Sections 6 and 7 we will look carefully at two related things: (1) time-varying vector fields and (2) control systems. In doing so, we focus on structure that allows us to prove useful properties such as regular dependence of flows on initial conditions. Also, in our framework of tautological control systems in Section 8, we will need to impose structure on systems where we have carefully eliminated the usual structure of a control parameterisation. To do this, we use the topological structure of sets of vector fields in an essential way. In this and the subsequent two sections we describe appropriate topologies for finitely differentiable, Lipschitz, smooth, holomorphic, and real analytic vector fields. The topology we use in this section in the smooth case (and the easily deduced finitely differentiable case) is classical, and is described, for example, in [Agrachev and Sachkov 2004, §2.2]; see also [Michor 1980, Chapter 4]. What we do that is original is provide a characterisation of the seminorms for this topology using the jet bundle fibre metrics from Section 2.2. The fruits of the effort expended in the next three sections is harvested in the remainder of the paper, where our concrete definitions of seminorms permit a relatively unified analysis in Sections 6 and 7 of time-varying vector fields and control systems. Also, the treatment of our new class of systems in Section 8 is made relatively simple by our descriptions of topologies for spaces of vector fields.

One facet of our presentation that is novel is that we flesh out completely the "weak-$\mathscr{L}$" characterisations of topologies for vector fields. These topologies characterise vector fields by how they act on functions through Lie differentiation. The use of such "weak" characterisations is commonplace [e.g., Agrachev and Sachkov 2004, Sussmann 1998], although the equivalence with strong characterisation is not typically proved; indeed, we know of no existing proofs of our Theorems 3.5, 3.8, 3.14, and 5.8. We show that, for the issues that come up in this paper, the weak characterisations for vector field topologies agree with the direct "strong" characterisations. This requires some detailed knowledge of the topologies we use.

While our primary interest is in vector fields, i.e., sections of the tangent bundle, it is advantageous to work instead with topologies for sections of general vector bundles, and then specialise to vector fields. We will also work with topologies for functions, but this falls out easily from the general vector bundle treatment.

3.1. General smooth vector bundles. We let $\pi\colon \mathsf{E} \to \mathsf{M}$ be a smooth vector bundle with ∇^0 a linear connection on E, ∇ an affine connection on M, $\mathbb{G}_0$ a fibre metric on E, and $\mathbb{G}$ a Riemannian metric on M. This gives us, as in Section 2.2, fibre metrics $\overline{\mathbb{G}}_m$ on the jet bundles $\mathsf{J}^m\mathsf{E}$, $m \in \mathbb{Z}_{\geq 0}$, and corresponding fibre norms $\|\cdot\|_{\overline{\mathbb{G}}_m}$.

For a compact set $K \subseteq \mathsf{M}$ we now define a seminorm $p^\infty_{K,m}$ on $\Gamma^\infty(\mathsf{E})$ by

$$p^\infty_{K,m}(\xi) = \sup\{\|j_m\xi(x)\|_{\overline{\mathbb{G}}_m} \mid x \in K\}.$$

The locally convex topology on $\Gamma^\infty(\mathsf{TM})$ defined by the family of seminorms $p^\infty_{K,m}$, $K \subseteq \mathsf{M}$ compact, $m \in \mathbb{Z}_{\geq 0}$, is called the ***smooth compact open*** or $\mathbf{CO^\infty}$***-topology*** for $\Gamma^\infty(\mathsf{E})$.

We comment that the seminorms depend on the choices of ∇, ∇^0, $\mathbb{G}$, and $\mathbb{G}_0$, but the CO^∞-topology is independent of these choices. We will constantly throughout the paper use these seminorms, and in doing so we will automatically be assuming that we have selected the linear connection ∇^0, the affine connection ∇, the fibre metric $\mathbb{G}_0$, and the Riemannian metric $\mathbb{G}$. We will do this often without explicit mention of these objects having been chosen.

3.2. Properties of the $\mathbf{CO^\infty}$-topology. Let us say a few words about the CO^∞-topology, referring to references for details. The locally convex CO^∞-topology has the following attributes.

CO^∞-1. It is Hausdorff: [Michor 1980, page 4.3.1].

CO^∞-2. It is complete: [Michor 1980, page 4.3.2].

CO^∞-3. It is metrisable: [Michor 1980, page 4.3.1].

CO^∞-4. It is separable: We could not find this stated anywhere, but here's a sketch of a proof. By embedding E in Euclidean space $\mathbb{R}^N$ and, using an argument like that for real analytic vector bundles in the proof of Lemma 2.3, we regard E as a subbundle of a trivial bundle over the submanifold $\mathsf{M} \subseteq \mathbb{R}^N$. In this case, we can reduce our claim of separability of the CO^∞-topology to that for smooth functions on submanifolds of $\mathbb{R}^N$. Here we can argue as follows. If $K \subseteq \mathsf{M}$ is compact, it can be contained in a compact cube C in $\mathbb{R}^N$. Then we can use a cutoff function to take any smooth function on M and leave it untouched on a neighbourhood of K, but have it and all of its derivatives vanish outside a compact set contained in $\mathrm{int}(C)$. Then we can use Fourier series to approximate in the CO$^\infty$-topology [Stein and Weiss 1971, Theorem VII.2.11(b)]. Since there are countably many Fourier basis functions, this gives the desired separability.

CO$^\infty$-5. It is nuclear:[6] [Jarchow 1981, Theorem 21.6.6].

CO$^\infty$-6. It is Suslin:[7] This follows since $\Gamma^\infty(\mathsf{TM})$ is a Polish space (see footnote 7), as we have already seen.

Some of these attributes perhaps seem obscure, but we will, in fact, use all of them!

Since the CO$^\infty$-topology is metrisable, it is exactly characterised by its convergent sequences, so let us describe these. A sequence $(\xi_k)_{k\in\mathbb{Z}_{>0}}$ in $\Gamma^\infty(\mathsf{E})$ converges to $\xi\in\Gamma^\infty(\mathsf{E})$ if and only if, for each compact set $K\subseteq\mathsf{M}$ and for each $m\in\mathbb{Z}_{\geq 0}$, the sequence $(j_m\xi_k|K)_{k\in\mathbb{Z}_{>0}}$ converges uniformly to $j_m\xi|K$, cf. combining [Munkres 2000, Theorem 46.8] and [Michor 1980, Lemma 4.2].

Since the topology is nuclear, it follows that subsets of $\Gamma^\infty(\mathsf{TM})$ are compact if and only if they are closed and von Neumann bounded [Pietsch 1969, Proposition 4.47]. That is to say, in a nuclear locally convex space, the compact bornology and the von Neumann bornology agree, according to the terminology introduced in Section 1.5. It is then interesting to characterise von Neumann bounded subsets of $\Gamma^\infty(\mathsf{E})$. One can show that a subset $\mathcal{B}$ is bounded in the von Neumann bornology if and only if every continuous seminorm on V is a bounded function when restricted to $\mathcal{B}$ [Rudin 1991, Theorem 1.37(b)]. Therefore, to characterise von Neumann bounded subsets, we need only characterise subsets on which each of the seminorms $p^\infty_{K,m}$ is a bounded function. This obviously gives the following characterisation.

3.1 Lemma: (Bounded subsets in the CO$^\infty$-topology) *A subset $\mathcal{B}\subseteq\Gamma^\infty(\mathsf{E})$ is bounded in the von Neumann bornology if and only if the following property holds: for any compact set $K\subseteq\mathsf{M}$ and any $m\in\mathbb{Z}_{\geq 0}$, there exists $C\in\mathbb{R}_{>0}$ such that $p^\infty_{K,m}(\xi)\leq C$*

[6] There are several ways of characterising nuclear spaces. Here is one. A continuous linear mapping $L\colon\mathsf{E}\to\mathsf{F}$ between Banach spaces is ***nuclear*** if there exist sequences $(v_j)_{j\in\mathbb{Z}_{>0}}$ in F and $(\alpha_j)_{j\in\mathbb{Z}_{>0}}$ in E' such that $\sum_{j\in\mathbb{Z}_{>0}}\|\alpha_j\|\|v_j\|<\infty$ and such that

$$L(u)=\sum_{j=1}^\infty \alpha_j(u)v_j,$$

the sum converging in the topology of V. Now suppose that V is a locally convex space and p is a continuous seminorm on V. We denote by $\overline{\mathsf{V}}_p$ the completion of

$$\mathsf{V}/\{v\in\mathsf{V}\mid p(v)=0\};$$

thus $\overline{\mathsf{V}}_p$ is a Banach space. The space V is ***nuclear*** if, for any continuous seminorm p, there exists a continuous seminorm q satisfying $q\leq p$ such that the mapping

$$\begin{aligned} i_{p,q}\colon \overline{\mathsf{V}}_p &\to \overline{\mathsf{V}}_q\\ v+\{v'\in\mathsf{V}\mid p(v)=0\} &\mapsto v+\{v'\in\mathsf{V}\mid q(v)=0\}\end{aligned}$$

is nuclear. It is to be understood that this definition is essentially meaningless at a first encounter, so we refer to [Hogbe-Nlend and Moscatelli 1981, Pietsch 1969] and relevant sections of [Jarchow 1981] to begin understanding the notion of a nuclear space. The only attribute of nuclear spaces of interest to us here is that their relatively compact subsets are exactly the von Neumann bounded subsets [Pietsch 1969, Proposition 4.47].

[7] A ***Polish space*** is a complete separable metrisable space. A ***Suslin space*** is a continuous image of a Polish space. A good reference for the basic properties of Suslin spaces is [Bogachev 2007, Chapter 6].

for every $\xi \in \mathcal{B}$.

Let us give a coordinate characterisation of the smooth compact-open topology, just for concreteness and so that the reader can see that our constructions agree with perhaps more familiar things. If we have a smooth vector bundle $\pi\colon \mathsf{E} \to \mathsf{M}$, we let $(\mathcal{V}, \psi)$ be a vector bundle chart for E inducing a chart $(\mathcal{U}, \phi)$ for M. For $\xi \in \Gamma^\infty(\mathsf{E})$, the local representative of ξ has the form

$$\mathbb{R}^n \supseteq \phi(\mathcal{U}) \ni \boldsymbol{x} \mapsto (\boldsymbol{x}, \boldsymbol{\xi}(\boldsymbol{x})) \in \phi(\mathcal{U}) \times \mathbb{R}^k.$$

Thus we have an associated map $\boldsymbol{\xi}\colon \phi(\mathcal{U}) \to \mathbb{R}^k$ that describes the section locally. A ***CO*** $^{\boldsymbol{\infty}}$***-subbasic neighbourhood*** is a subset $\mathcal{B}_\infty(\xi, \mathcal{V}, K, \epsilon, m)$ of $\Gamma^\infty(\mathsf{E})$, where

1. $\xi \in \Gamma^\infty(\mathsf{E})$,
2. $(\mathcal{V}, \psi)$ is a vector bundle chart for E with associated chart $(\mathcal{U}, \phi)$ for M,
3. $K \subseteq \mathcal{U}$ is compact,
4. $\epsilon \in \mathbb{R}_{>0}$,
5. $m \in \mathbb{Z}_{\geq 0}$, and
6. $\eta \in \mathcal{B}_\infty(\xi, \mathcal{V}, K, \epsilon, m)$ if and only if

$$\|\boldsymbol{D}^l\boldsymbol{\eta}(\boldsymbol{x}) - \boldsymbol{D}^l\boldsymbol{\xi}(\boldsymbol{x})\| < \epsilon, \qquad \boldsymbol{x} \in \phi(K),\ l \in \{0, 1, \dots, m\},$$

where $\boldsymbol{\xi}, \boldsymbol{\eta}\colon \phi(\mathcal{U}) \to \mathbb{R}^k$ are the local representatives.

One can show that the CO$^\infty$-topology is that topology having as a subbase the CO$^\infty$-subbasic neighbourhoods. This is the definition used by [Hirsch 1976], for example. To show that this topology agrees with our intrinsic characterisation is a straightforward bookkeeping chore, and the interested reader can refer to Lemma 2.4 to see how this is done in the more difficult real analytic case. This more concrete characterisation using vector bundle charts can be useful should one ever wish to verify some properties in examples. It can also be useful in general arguments in emergencies when one does not have the time to flesh out coordinate-free constructions.

3.3. The weak-$\mathscr{L}$ topology for smooth vector fields. The CO$^\infty$-topology for smooth sections of a vector bundle, merely by specialisation, gives a locally convex topology on the set $\Gamma^\infty(\mathsf{TM})$ of smooth vector fields and the set $\mathrm{C}^\infty(\mathsf{M})$ of smooth functions (noting that a smooth function is obviously identified with a section of the trivial vector bundle $\mathsf{M}\times\mathbb{R}$). The only mildly interesting thing in these cases is that one does not need a separate linear connection in the vector bundles or a separate fibre metric. Indeed, TM is already assumed to have a linear connection (the affine connection on M) and a fibre metric (the Riemannian metric on M), and the trivial bundle has the canonical flat linear connection defined by $\nabla_X f = \mathscr{L}_X f$ and the standard fibre metric induced by absolute value on the fibres.

We wish to see another way of describing the CO$^\infty$-topology on $\Gamma^\infty(\mathsf{TM})$ by noting that a vector field defines a linear map, indeed a derivation, on $\mathrm{C}^\infty(\mathsf{M})$ by Lie differentiation: $f \mapsto \mathscr{L}_X f$. The topology we describe for $\Gamma^\infty(\mathsf{TM})$ is a sort of weak topology arising from the CO$^\infty$-topology on $\mathrm{C}^\infty(\mathsf{M})$ and Lie differentiation. To properly set the stage for the fact that we will repeat this construction for our other topologies, it is most clear to work in a general setting for a moment, and then specialise in each subsequent case.

The general setup is provided by the next definition.

3.2 Definition: (Weak boundedness, continuity, measurability, and integrability) Let $\mathbb{F} \in \{\mathbb{R}, \mathbb{C}\}$ and let U and V be $\mathbb{F}$-vector spaces with V locally convex. Let $\mathscr{A} \subseteq \mathrm{Hom}_{\mathbb{R}}(\mathsf{U};\mathsf{V})$ and let the ***weak-$\mathscr{A}$ topology*** on U be the weakest topology for which A is continuous for every $A \in \mathscr{A}$ [Horváth 1966, §2.11].

Also let $(\mathcal{X}, \mathscr{O})$ be a topological space, let $(\mathcal{T}, \mathscr{M})$ be a measurable space, and let $\mu\colon \mathscr{M} \to \overline{\mathbb{R}}_{\geq 0}$ be a finite measure. We have the following notions:

(i) a subset $\mathcal{B} \subseteq \mathsf{U}$ is ***weak-$\mathscr{A}$ bounded in the von Neumann bornology*** if $A(\mathcal{B})$ is bounded in the von Neumann bornology for every $A \in \mathscr{A}$;

(ii) a map $\Phi\colon \mathcal{X} \to \mathsf{U}$ is ***weak-$\mathscr{A}$ continuous*** if $A \circ \Phi$ is continuous for every $A \in \mathscr{A}$;

(iii) a map $\Psi\colon \mathcal{T} \to \mathsf{U}$ is ***weak-$\mathscr{A}$ measurable*** if $A \circ \Psi$ is measurable for every $A \in \mathscr{A}$;

(iv) a map $\Psi\colon \mathcal{T} \to \mathsf{U}$ is ***weak-$\mathscr{A}$ Bochner integrable*** with respect to μ if $A \circ \Psi$ is Bochner integrable with respect to μ for every $A \in \mathscr{A}$. •

As can be seen in Section 2.11 of [Horváth 1966], the weak-$\mathscr{A}$ topology is a locally convex topology, and a subbase for open sets in this topology is

$$\{A^{-1}(\mathcal{O}) \mid A \in \mathscr{A},\ \mathcal{O} \subseteq \mathsf{V} \text{ open}\}.$$

Equivalently, the weak-$\mathscr{A}$ topology is defined by the seminorms

$$u \mapsto q(A(u)), \qquad A \in \mathscr{A},\ q \text{ a continuous seminorm for } \mathsf{V}.$$

This is a characterisation of the weak-$\mathscr{A}$ topology we will use often.

We now have the following result which gives conditions for the equivalence of "weak-$\mathscr{A}$" notions with the usual notions. We call a subset $\mathscr{A} \subseteq \mathrm{Hom}_{\mathbb{F}}(\mathsf{U};\mathsf{V})$ ***point separating*** if, given distinct $u_1, u_2 \in \mathsf{U}$, there exists $A \in \mathscr{A}$ such that $A(u_1) \neq A(u_2)$.

3.3 Lemma: (Equivalence of weak-$\mathscr{A}$ and locally convex notions for general locally convex spaces) *Let $\mathbb{F} \in \{\mathbb{R}, \mathbb{C}\}$ and let U and V be locally convex $\mathbb{F}$-vector spaces. Let $\mathscr{A} \subseteq \mathrm{Hom}_{\mathbb{R}}(\mathsf{U};\mathsf{V})$ and suppose that the weak-$\mathscr{A}$ topology agrees with the locally convex topology for U. Let $(\mathcal{X}, \mathscr{O})$ be a topological space, let $(\mathcal{T}, \mathscr{M})$ be a measurable space, and let $\mu\colon \mathscr{M} \to \overline{\mathbb{R}}_{\geq 0}$ be a finite measure. Then the following statements hold:*

(i) a subset $\mathcal{B} \subseteq \mathsf{U}$ is bounded in the von Neumann bornology if and only if it is weak-$\mathscr{A}$ bounded in the von Neumann bornology;

(ii) a map $\Phi\colon \mathcal{X} \to \mathsf{U}$ is continuous if and only if it is weak-$\mathscr{A}$ continuous;

(iii) for a map $\Psi\colon \mathcal{T} \to \mathsf{U}$,

(a) if Ψ is measurable, then it is weak-$\mathscr{A}$ measurable;

(b) if U and V are Hausdorff Suslin spaces, if $\mathscr{A}$ contains a countable point separating subset, and if Ψ is weak-$\mathscr{A}$ measurable, then Ψ is measurable;

(iv) if U is complete and separable, a map $\Psi\colon \mathcal{T} \to \mathsf{U}$ is Bochner integrable with respect to μ if and only if it is weak-$\mathscr{A}$ Bochner integrable with respect to μ.

Proof: (i) and (ii): Both of these assertions follows directly from the fact that the locally convex topology of U agrees with the weak-$\mathscr{A}$ topology. Indeed, the equivalence of these topologies implies that (a) if p is a continuous seminorm for the locally convex topology of U, then there exist continuous seminorms $q_1, \dots, q_k$ for V and $A_1, \dots, A_k \in \mathscr{A}$ such that

$$p(u) \leq q_1(A_1(u)) + \dots + q_k(A_k(u)), \qquad u \in \mathsf{U}, \tag{3.1}$$

and (b) if q is a continuous seminorm for V and if $A \in \mathscr{A}$, then there exists a continuous seminorm p for the locally convex topology for U such that

$$q(A(u)) \leq p(u), \qquad u \in \mathsf{U}. \tag{3.2}$$

(iii) First suppose that Ψ is measurable and let $A \in \mathscr{A}$. Since the locally convex topology of U agrees with the weak-$\mathscr{A}$ topology, A is continuous in the locally convex topology of U. Therefore, if Ψ is measurable, it follows immediately by continuity of A that $A \circ \Psi$ is measurable.

Next suppose that U and V are Suslin, that $\mathscr{A}$ contains a countable point separating subset, and that Ψ is weak-$\mathscr{A}$ measurable. Without loss of generality, let us suppose that $\mathscr{A}$ is itself countable. By $\mathsf{V}^{\mathscr{A}}$ we denote the mappings from $\mathscr{A}$ to V, with the usual pointwise vector space structure. A typical element of $\mathsf{V}^{\mathscr{A}}$ we denote by ϕ. By [Bogachev 2007, Lemma 6.6.5(iii)], $\mathsf{V}^{\mathscr{A}}$ is a Suslin space. Let us define a mapping $\iota_{\mathscr{A}} \colon \mathsf{U} \to \mathsf{V}^{\mathscr{A}}$ by $\iota_{\mathscr{A}}(u)(A) = A(u)$. Since $\mathscr{A}$ is point separating, we easily verify that $\iota_{\mathscr{A}}$ is injective, and so we have U as a subspace of the countable product $\mathsf{V}^{\mathscr{A}}$. For $A \in \mathscr{A}$ let $\mathrm{pr}_A \colon \mathsf{V}^{\mathscr{A}} \to \mathsf{V}$ be the projection defined by $\mathrm{pr}_A(\phi) = \phi(A)$. Since V is Suslin, it is hereditary Lindelöf [Bogachev 2007, Lemma 6.6.4]. Thus the Borel σ-algebra of $\mathsf{V}^{\mathscr{A}}$ is the same as the initial Borel σ-algebra defined by the projections pr_A, $A \in \mathscr{A}$, i.e., the smallest σ-algebra for which the projections are measurable [Bogachev 2007, Lemma 6.4.2]. By hypothesis, $(A \circ \Psi)^{-1}(\mathcal{B})$ is measurable for every $A \in \mathscr{A}$ and every Borel set $\mathcal{B} \subseteq \mathsf{V}$. Now we note that $\mathrm{pr}_A \circ \iota_{\mathscr{A}}(v) = A(v)$, from which we deduce that

$$(A \circ \Psi)^{-1}(\mathcal{B}) = (\iota_{\mathscr{A}} \circ \Psi)^{-1}(\mathrm{pr}_A^{-1}(\mathcal{B}))$$

is measurable for every $A \in \mathscr{A}$ and every Borel set $\mathcal{B} \subseteq \mathsf{V}$. Thus $\iota_{\mathscr{A}} \circ \Psi$ is measurable.

Since U is Suslin, by definition there is a Polish space $\mathcal{P}$ and a continuous surjection $\sigma \colon \mathcal{P} \to \mathsf{U}$. If $\mathcal{C} \subseteq \mathsf{U}$ is a Borel set, then $\sigma^{-1}(\mathcal{C}) \subseteq \mathcal{P}$ is a Borel set. Note that $\iota_{\mathscr{A}}$ is continuous (since $\mathrm{pr}_A \circ \iota_{\mathscr{A}}$ is continuous for every $A \in \mathscr{A}$) and so is a Borel mapping. By [Fremlin 2006, Theorem 423I], we have that $\iota_{\mathscr{A}} \circ \sigma(\sigma^{-1}(\mathcal{C})) \subseteq \mathsf{V}$ is Borel. Since σ is surjective, this means that $\iota_{\mathscr{A}}(\mathcal{C}) \subseteq \mathsf{V}$ is Borel. Finally, since

$$\Psi^{-1}(\mathcal{C}) = (\iota_{\mathscr{A}} \circ \Psi)^{-1}(\iota_{\mathscr{A}}(\mathcal{C})),$$

measurability of Ψ follows.

(iv) Since U is separable and complete, by Beckmann and Deitmar [2011, Theorems 3.2 and 3.3] Bochner integrability of Ψ is equivalent to integrability, in the sense of Lebesgue, of $t \mapsto p \circ \Psi(t)$ for any continuous seminorm p. Thus, Ψ is Bochner integrable with respect to the locally convex topology of U if and only if $t \mapsto p \circ \Psi(t)$ is integrable, and Ψ is weak-$\mathscr{A}$ Bochner integrable if and only if $t \mapsto q_A(\Psi(t))$ is integrable for every $A \in \mathscr{A}$. This part of the proof now follows from the inequalities (3.1) and (3.2) that characterise the equivalence of the locally convex and weak-$\mathscr{A}$ topologies for U. ■

The proof of the harder direction in part (iii) is an adaptation of [Thomas 1975, Theorem 1] to our more general setting. We will revisit this idea again when we talk about measurability of time-varying vector fields in Section 6.

For $f \in \mathrm{C}^\infty(\mathsf{M})$, let us define

$$\begin{aligned} \mathscr{L}_f \colon \Gamma^\infty(\mathsf{TM}) &\to \mathrm{C}^\infty(\mathsf{M}) \\ X &\mapsto \mathscr{L}_X f. \end{aligned}$$

The topology for $\Gamma^\infty(\mathsf{TM})$ we now define corresponds to the general case of Definition 3.2 by taking $\mathsf{U} = \Gamma^\infty(\mathsf{TM})$, $\mathsf{V} = \mathrm{C}^\infty(\mathsf{M})$, and $\mathscr{A} = \{\mathscr{L}_f \mid f \in \mathrm{C}^\infty(\mathsf{M})\}$. To this end, we make the following definition.

3.4 Definition: (Weak-$\mathscr{L}$ topology for space of smooth vector fields) For a smooth manifold M, the ***weak-$\mathscr{L}$ topology*** for $\Gamma^\infty(\mathsf{TM})$ is the weakest topology for which $\mathscr{L}_f$ is continuous for every $f \in \mathrm{C}^\infty(\mathsf{M})$, if $\mathrm{C}^\infty(\mathsf{M})$ has the CO^∞-topology. •

We now have the following result.

3.5 Theorem: (Weak-$\mathscr{L}$ characterisation of CO$^\infty$-topology for smooth vector fields) *For a smooth manifold, the following topologies for $\Gamma^\infty(\mathsf{TM})$ agree:*

(i) the CO^∞-topology;

(ii) the weak-$\mathscr{L}$ topology.

Proof: (i)$\subseteq$(ii) For this part of the proof, we assume that M has a well-defined dimension. The proof is easily modified by additional notation to cover the case where this may not hold. Let $K \subseteq \mathsf{M}$ be compact and let $m \in \mathbb{Z}_{\geq 0}$. Let $x \in K$ and let $(\mathcal{U}_x, \phi_x)$ be a coordinate chart for M about x with coordinates denoted by $(x^1, \dots, x^n)$. Let $\boldsymbol{X} \colon \phi_x(\mathcal{U}_x) \to \mathbb{R}^n$ be the local representative of $X \in \Gamma^\infty(\mathsf{TM})$. For $j \in \{1, \dots, n\}$ let $f_x^j \in \mathrm{C}^\infty(\mathsf{M})$ have the property that, for some relatively compact neighbourhood $\mathcal{V}_x$ of x with $\mathrm{cl}(\mathcal{V}_x) \subseteq \mathcal{U}_x$, $f_x^j = x^j$ in some neighbourhood of $\mathrm{cl}(\mathcal{V}_x)$. (This is done using standard extension arguments for smooth functions, cf. [Abraham, Marsden, and Ratiu 1988, Proposition 5.5.8].) Then, in a neighbourhood of $\mathrm{cl}(\mathcal{V}_x)$ in $\mathcal{U}_x$, we have $\mathscr{L}_X f_x^j = X^j$. Therefore, for each $y \in \mathrm{cl}(\mathcal{V}_x)$,

$$j_m X(y) \mapsto \sum_{j=1}^n \|j_m(\mathscr{L}_X f_x^j)(y)\|_{\overline{\mathbb{G}}_m}$$

is a norm on the fibre $\mathsf{J}_y^m \mathsf{E}$. Therefore, there exists $C_x \in \mathbb{R}_{>0}$ such that

$$\|j_m X(y)\|_{\overline{\mathbb{G}}_m} \leq C_x \sum_{j=1}^n \|j_m(\mathscr{L}_X f_x^j)(y)\|_{\overline{\mathbb{G}}_m}, \qquad y \in \mathrm{cl}(\mathcal{V}_x).$$

Since K is compact, let $x_1, \dots, x_k \in K$ be such that $K \subseteq \cup_{a=1}^k \mathcal{V}_{x_a}$. Let

$$C = \max\{C_{x_1}, \dots, C_{x_r}\}.$$

Then, if $y \in K$ we have $y \in \mathcal{V}_{x_a}$ for some $a \in \{1, \dots, r\}$, and so

$$\|j_m X(y)\|_{\overline{\mathbb{G}}_m} \leq C \sum_{j=1}^n \|j_m(\mathscr{L}_X f_{x_a}^j)(y)\|_{\overline{\mathbb{G}}_m} \leq C \sum_{a=1}^r \sum_{j=1}^n \|j_m(\mathscr{L}_X f_{x_a}^j)(y)\|_{\overline{\mathbb{G}}_m}.$$

Taking supremums over $y \in K$ gives

$$p_{K,m}^\infty(X) \leq C \sum_{a=1}^r \sum_{j=1}^n p_{K,m}^\infty(\mathscr{L}_X f_{x_a}^j),$$

This part of the theorem then follows since the weak-$\mathscr{L}$ topology, as we indicated following Definition 3.2 above, is defined by the seminorms

$$X \mapsto p^\infty_{K,m}(\mathscr{L}_X f), \qquad K \subseteq \mathsf{M} \text{ compact},\ m \in \mathbb{Z}_{\geq 0},\ f \in \mathrm{C}^\infty(\mathsf{M}).$$

(ii)⊆(i) As per (2.1), let us abbreviate

$$\nabla^j(\dots(\nabla^1(\nabla^0 A))) = \nabla^{(j)} A,$$

where A can be either a vector field or one-form, in what we will need. Since covariant differentials commute with contractions [Dodson and Poston 1991, Theorem 7.03(F)], an elementary induction argument gives the formula

$$\nabla^{(m-1)}(\mathrm{d}f(X)) = \sum_{j=0}^m \binom{m}{j} C_{1,m-j+1}((\nabla^{(m-j-1)}X) \otimes (\nabla^{(j-1)}\mathrm{d}f)), \tag{3.3}$$

where $C_{1,m-j+1}$ is the contraction defined by

$$\begin{aligned} C_{1,m-j+1}(v \otimes \alpha^1 \otimes \dots \otimes \alpha^{m-j} \otimes \alpha^{m-j+1} \otimes \alpha^{m-j+2} \otimes \dots \otimes \alpha^{m+1}) \\ = (\alpha^{m-j+1}(v))(\alpha^1 \otimes \dots \otimes \alpha^{m-j} \otimes \alpha^{m-j+2} \otimes \dots \otimes \alpha^{m+1}). \end{aligned}$$

In writing (3.3) we use the convention $\nabla^{(-1)}X = X$ and $\nabla^{(-1)}(\mathrm{d}f) = \mathrm{d}f$. Next we claim that $\mathscr{L}_f$ is continuous for every $f \in \mathrm{C}^\infty(\mathsf{M})$ if $\Gamma^\infty(\mathsf{TM})$ is provided with CO$^\infty$-topology. Indeed, let $K \subseteq \mathsf{M}$, let $m \in \mathbb{Z}_{>0}$, and let $f \in \mathrm{C}^\infty(\mathsf{M})$. By (3.3) (after a few moments of thought), we have, for some suitable $M_0, M_1 \dots, M_m \in \mathbb{R}_{>0}$,

$$p^\infty_{K,m}(\mathscr{L}_X f) \leq \sum_{j=0}^m M_{m-j} p^\infty_{K,m-j}(X) p^\infty_{K,j+1}(f) \leq \sum_{j=0}^m M'_j p^\infty_{K,j}(X).$$

This gives continuity of the identity map, if we provide the domain with the CO$^\infty$-topology and the codomain with the weak-$\mathscr{L}$ topology, cf. [Schaefer and Wolff 1999, §III.1.1]. Thus open sets in the weak-$\mathscr{L}$ topology are contained in the CO$^\infty$-topology. ■

With respect to the concepts of interest to us, this gives the following result.

3.6 Corollary: (Weak-$\mathscr{L}$ characterisations of boundedness, continuity, measurability, and integrability for the CO$^\infty$-topology) *Let* M *be a smooth manifold, let* $(\mathcal{X}, \mathscr{O})$ *be a topological space, let* $(\mathcal{T}, \mathscr{M})$ *be a measurable space, and let* $\mu\colon \mathscr{M} \to \overline{\mathbb{R}}_{\geq 0}$ *be a finite measure. The following statements hold:*

(i) *a subset* $\mathcal{B} \subseteq \Gamma^\infty(\mathsf{TM})$ *is bounded in the von Neumann bornology if and only if it is weak-*$\mathscr{L}$ *bounded in the von Neumann bornology;*

(ii) *a map* $\Phi\colon \mathcal{X} \to \Gamma^\infty(\mathsf{TM})$ *is continuous if and only if it is weak-*$\mathscr{L}$ *continuous;*

(iii) *a map* $\Psi\colon \mathcal{T} \to \Gamma^\infty(\mathsf{TM})$ *is measurable if and only if it is weak-*$\mathscr{L}$ *measurable;*

(iv) *a map* $\Psi\colon \mathcal{T} \to \Gamma^\infty(\mathsf{TM})$ *is Bochner integrable if and only if it is weak-*$\mathscr{L}$ *Bochner integrable.*

Proof: We first claim that $\mathscr{A} \triangleq \{\mathscr{L}_f \mid f \in \mathrm{C}^\infty(\mathsf{M})\}$ has a countable point separating subset. This is easily proved as follows. For notational simplicity, suppose that M has a well-defined dimension. Let $x \in \mathsf{M}$ and note that there exist a neighbourhood $\mathcal{U}_x$ of x and $f_x^1, \dots, f_x^n \in \mathrm{C}^\infty(\mathsf{M})$ such that

$$\mathsf{T}_y^*\mathsf{M} = \operatorname{span}_{\mathbb{R}}(\mathrm{d}f^1(y), \dots, \mathrm{d}f^n(y)), \qquad y \in \mathcal{U}_x.$$

Since M is second countable it is Lindelöf [Willard 1970, Theorem 16.9]. Therefore, there exists $(x_j)_{j\in\mathbb{Z}_{>0}}$ such that $\mathsf{M} = \cup_{j\in\mathbb{Z}_{>0}}\mathcal{U}_{x_j}$. The countable collection of linear mappings $\mathscr{L}_{f_{x_j}^k}$, $k \in \{1, \dots, n\}$, $j \in \mathbb{Z}_{>0}$, is then point separating. Indeed, if $X, Y \in \Gamma^\infty(\mathsf{TM})$ are distinct, then there exists $x \in \mathsf{M}$ such that $X(x) \neq Y(x)$. Let $j \in \mathbb{Z}_{>0}$ be such that $x \in \mathcal{U}_{x_j}$ and note that we must have $\mathscr{L}_{f_{x_j}^k}(X)(x) \neq \mathscr{L}_{f_{x_j}^k}(Y)(x)$ for some $k \in \{1, \dots, n\}$, giving our claim.

The result is now a direct consequence of Lemma 3.3, noting that the CO^∞-topology on $\Gamma^\infty(\mathsf{TM})$ is complete, separable, and Suslin (we also need that the CO^∞-topology on $\mathrm{C}^\infty(\mathsf{M})$ is Suslin, which it is), as we have seen above in properties CO^∞-2, CO^∞-4, and CO^∞-6. ■

3.4. Topologies for finitely differentiable vector fields. The constructions of this section so far are easily adapted to the case where objects are only finitely differentiable. We sketch here how this can be done. We let $\pi\colon \mathsf{E} \to \mathsf{M}$ be a smooth vector bundle, and we suppose that we have a linear connection ∇^0 on E, an affine connection ∇ on M, a fibre metric $\mathbb{G}_0$ on E, and a Riemannian metric $\mathbb{G}$ on M. Let $r \in \mathbb{Z}_{\geq 0} \cup \{\infty\}$ and let $m \in \mathbb{Z}_{\geq 0}$ with $m \leq r$. By $\Gamma^r(\mathsf{E})$ we denote the space of C^r-sections of E. We define seminorms p_K^m, $K \subseteq \mathsf{M}$ compact, on $\Gamma^r(\mathsf{E})$ by

$$p_K^m(\xi) = \sup\{\|j_m\xi(x)\|_{\overline{\mathbb{G}}_m} \mid x \in K\},$$

and these seminorms define a locally convex topology that we call the ***CO^m-topology***. Let us list some of the attributes of this topology.

CO^m-1. It is Hausdorff: [Michor 1980, page 4.3.1].

CO^m-2. It is complete if and only if $m = r$: [Michor 1980, page 4.3.2].

CO^m-3. It is metrisable: [Michor 1980, page 4.3.1].

CO^m-4. It is separable: This can be shown to follow by an argument similar to that given above for the CO^∞-topology.

CO^m-5. It is probably not nuclear: In case M is compact, note that p_{M}^m is a norm that characterises the CO^m-topology. A normed vector space is nuclear if and only if it is finite-dimensional [Pietsch 1969, Theorem 4.4.14], so the CO^m-topology cannot be nuclear when M is compact except in cases of degenerate dimension. But, even when M is not compact, the CO^m-topology is not likely nuclear, although we have neither found a reference nor proved this.

CO^m-6. It is Suslin when $m = r$: This follows since $\Gamma^m(\mathsf{TM})$ is a Polish space, as we have already seen.

CO^m-7. The CO^m-topology is weaker than the CO^r-topology: This is more or less clear from the definitions.

From the preceding, we point out two places where one must take care in using the CO^m-topology, $m \in \mathbb{Z}_{\geq 0}$, contrasted with the CO^∞-topology. First of all, the topology, if used on $\Gamma^r(\mathsf{E})$, $r > m$, is not complete, so convergence arguments must be modified appropriately. Second, it is no longer the case that bounded sets are relatively compact. Instead, relatively compact subsets will be described by an appropriate version of the Arzelà–Ascoli Theorem, cf. [Jost 2005, Theorem 5.21]. Therefore, we need to specify for these spaces whether we will be using the von Neumann bornology or the compact bornology when we use the word "bounded." These caveats notwithstanding, it is oftentimes appropriate to use these weaker topologies.

Of course, the preceding can be specialised to vector fields and functions, and one can define the weak-$\mathscr{L}$ topologies corresponding to the topologies for finitely differentiable sections. In doing this, we apply the general construction of Definition 3.2 with $\mathsf{U} = \Gamma^r(\mathsf{TM})$, $\mathsf{V} = \mathrm{C}^r(\mathsf{M})$ (with the CO^m-topology), and $\mathscr{A} = \{\mathscr{L}_f \mid f \in \mathrm{C}^\infty(\mathsf{M})\}$, where

$$\begin{aligned} \mathscr{L}_f \colon \Gamma^r(\mathsf{TM}) &\to \mathrm{C}^r(\mathsf{M}) \\ X &\mapsto \mathscr{L}_X f. \end{aligned}$$

This gives the following definition.

3.7 Definition: (Weak-$\mathscr{L}$ topology for space of finitely differentiable vector fields) Let M be a smooth manifold, let $m \in \mathbb{Z}_{\geq 0}$, and let $r \in \mathbb{Z}_{\geq 0} \cup \{\infty\}$ have the property that $r \geq m$. The ***weak-$(\mathscr{L}, m)$ topology*** for $\Gamma^r(\mathsf{TM})$ is the weakest topology for which $\mathscr{L}_f$ is continuous for each $f \in \mathrm{C}^\infty(\mathsf{M})$, where $\mathrm{C}^r(\mathsf{M})$ is given the CO^m-topology. •

We can show that the weak-$(\mathscr{L}, m)$ topology agrees with the CO^m-topology.

3.8 Theorem: (Weak-$\mathscr{L}$ topology for finitely differentiable vector fields) *Let M be a smooth manifold, let $m \in \mathbb{Z}_{\geq 0}$, and let $r \in \mathbb{Z}_{\geq 0} \cup \{\infty\}$ have the property that $r \geq m$. Then the following two topologies for $\Gamma^r(\mathsf{TM})$ agree:*

(i) the CO^m-topology;

(ii) the weak-$(\mathscr{L}, m)$-topology.

Proof: Let us first show that the CO^m-topology is weaker than the weak-$(\mathscr{L}, m)$ topology. Just as in the corresponding part of the proof of Theorem 3.5, we can show that, for $K \subseteq \mathsf{M}$ compact, there exist $f^1, \dots, f^r \in \mathrm{C}^\infty(\mathsf{M})$, compact $K_1, \dots, K_r \subseteq \mathsf{M}$, and $C_1, \dots, C_r \in \mathbb{R}_{>0}$ such that

$$p_K^m(X) \leq C_1 p_{K_1}^m(\mathscr{L}_X f^1) + \dots + C_r p_{K_r}^m(\mathscr{L}_X f^r)$$

for every $X \in \Gamma^r(\mathsf{TM})$. This estimate gives this part of the theorem.

To prove that the weak $(\mathscr{L}, m)$-topology is weaker than the CO^m-topology, it suffices to show that $\mathscr{L}_f$ is continuous if $\Gamma^r(\mathsf{TM})$ and $\mathrm{C}^r(\mathsf{M})$ are given the CO^m-topology. This can be done just as in Theorem 3.5, with suitable modifications since we only have to account for m derivatives. ■

We also have the corresponding relationships between various attributes and their weak counterparts.

3.9 Corollary: (Weak-$\mathscr{L}$ characterisations of boundedness, continuity, measurability, and integrability for the CO^m-topology) *Let M be a smooth manifold, let $m \in \mathbb{Z}_{\geq 0}$, and let $r \in \mathbb{Z}_{\geq 0} \cup \{\infty\}$ have the property that $r \geq m$. Let $(\mathcal{X}, \mathscr{O})$ be a topological space, let $(\mathcal{T}, \mathscr{M})$ be a measurable space, and let $\mu \colon \mathscr{M} \to \overline{\mathbb{R}}_{\geq 0}$ be a finite measure. The following statements hold:*

(i) *a subset* $\mathcal{B} \subseteq \Gamma^r(\mathsf{TM})$ *is* CO^m*-bounded in the von Neumann bornology if and only if it is weak-*$(\mathscr{L}, m)$ *bounded in the von Neumann bornology;*

(ii) *a map* $\Phi\colon \mathcal{X} \to \Gamma^r(\mathsf{TM})$ *is* CO^m*-continuous if and only if it is weak-*$(\mathscr{L}, m)$ *continuous;*

(iii) *a map* $\Psi\colon \mathcal{T} \to \Gamma^m(\mathsf{TM})$ *is* CO^m*-measurable if and only if it is weak-*$(\mathscr{L}, m)$ *measurable;*

(iv) *a map* $\Psi\colon \mathcal{T} \to \Gamma^m(\mathsf{TM})$ *is Bochner integrable if and only if it is weak-*$(\mathscr{L}, m)$ *Bochner integrable.*

Proof: In the proof of Corollary 3.6 we established that $\{\mathscr{L}_f \mid f \in \mathrm{C}^\infty(\mathsf{M})\}$ was point separating as a family of linear mappings with domain $\Gamma^\infty(\mathsf{TM})$. The same proof is valid if the domain is $\Gamma^m(\mathsf{TM})$. The result is then a direct consequence of Lemma 3.3, taking care to note that the CO^m-topology on $\Gamma^r(\mathsf{TM})$ is separable, and is also complete and Suslin when $r = m$ (and $\mathrm{C}^r(\mathsf{M})$ is Suslin when $r = m$), as we have seen in properties CO^m-2, CO^m-4, and CO^m-6 above. ■

3.5. Topologies for Lipschitz vector fields. It is also possible to characterise Lipschitz sections, so let us indicate how this is done in geometric terms. Throughout our discussion of the Lipschitz case, we make the assumption that the affine connection ∇ on M is the Levi-Civita connection for $\mathbb{G}$ and that the linear connection ∇^0 on E is $\mathbb{G}_0$-orthogonal, by which we mean that parallel translation consists of isometries. The existence of such a connection is ensured by the reasoning of Kobayashi and Nomizu [1963] following the proof of their Proposition III.1.5. We suppose that M is connected, for simplicity. If it is not, then one has to allow the metric we are about to define to take infinite values. This is not problematic [Burago, Burago, and Ivanov 2001, Exercise 1.1.2], but we wish to avoid the more complicated accounting procedures. The ***length*** of a piecewise differentiable curve $\gamma\colon [a,b] \to \mathsf{M}$ is

$$\ell_{\mathbb{G}}(\gamma) = \int_a^b \sqrt{\mathbb{G}(\gamma'(t), \gamma'(t))}\, \mathrm{d}t.$$

One easily shows that the length of the curve γ depends only on image(γ), and not on the particular parameterisation. We can, therefore, restrict ourselves to curves defined on $[0,1]$. In this case, for $x_1, x_2 \in \mathsf{M}$, we define the ***distance*** between x_1 and x_2 to be

$$\begin{aligned}\mathrm{d}_{\mathbb{G}}(x_1, x_2) = \inf\{\ell_{\mathbb{G}}(\gamma) |\ \gamma\colon [0,1] \to \mathsf{M} \text{ is a piecewise}\\ \text{differentiable curve for which } \gamma(0) = x_1 \text{ and } \gamma(1) = x_2\}.\end{aligned}$$

It is relatively easy to show that $(\mathsf{M}, \mathrm{d}_{\mathbb{G}})$ is a metric space [Abraham, Marsden, and Ratiu 1988, Proposition 5.5.10].

Now we define a canonical Riemannian metric on the total space E of a vector bundle $\pi\colon \mathsf{E} \to \mathsf{M}$, following the construction of Sasaki [1958] for tangent bundles. The linear connection ∇^0 gives a splitting $\mathsf{TE} \simeq \pi^*\mathsf{TM} \oplus \pi^*\mathsf{E}$ [Kolář, Michor, and Slovák 1993, §11.11]. The second component of this decomposition is the vertical component so $T_{e_x}\pi$ restricted to the first component is an isomorphism onto $\mathsf{T}_x\mathsf{M}$, i.e., the first component is "horizontal." Let us denote by hor: $\mathsf{TE} \to \pi^*\mathsf{TM}$ and ver: $\mathsf{TE} \to \pi^*\mathsf{E}$ the projections onto the first and second components of the direct sum decomposition. This then gives the Riemannian metric $\mathbb{G}_{\mathsf{E}}$ on E defined by

$$\mathbb{G}_{\mathsf{E}}(X_{e_x}, Y_{e_x}) = \mathbb{G}(\mathrm{hor}(X_{e_x}), \mathrm{hor}(Y_{e_x})) + \mathbb{G}_0(\mathrm{ver}(X_{e_x}), \mathrm{ver}(Y_{e_x})).$$

Now let us consider various ways of characterising Lipschitz sections. To this end, we let $\xi\colon \mathsf{M} \to \mathsf{E}$ be such that $\xi(x) \in \mathsf{E}_x$ for every $x \in \mathsf{M}$. For compact $K \subseteq \mathsf{M}$ we then define

$$L_K(\xi) = \sup\left\{ \frac{\mathrm{d}_{\mathbb{G}_{\mathsf{E}}}(\xi(x_1), \xi(x_2))}{\mathrm{d}_{\mathbb{G}}(x_1, x_2)} \;\middle|\; x_1, x_2 \in K,\ x_1 \neq x_2 \right\}.$$

This is the ***K-dilatation*** of ξ. For a piecewise differentiable curve $\gamma\colon [0,T] \to \mathsf{M}$, we denote by $\tau_{\gamma,t}\colon \mathsf{E}_{\gamma(0)} \to \mathsf{E}_{\gamma(t)}$ the isomorphism of parallel translation along γ for each $t \in [0,T]$. We then define

$$l_K(\xi) = \sup\left\{ \frac{\|\tau_{\gamma,1}^{-1}(\xi\circ\gamma(1)) - \xi\circ\gamma(0)\|_{\mathbb{G}_0}}{\ell_{\mathbb{G}}(\gamma)} \;\middle|\; \gamma\colon [0,1] \to \mathsf{M},\ \gamma(0), \gamma(1) \in K,\ \gamma(0) \neq \gamma(1) \right\}, \tag{3.4}$$

which is the ***K-sectional dilatation*** of ξ. Finally, we define

$$\begin{aligned} \mathrm{Dil}\,\xi\colon \mathsf{M} &\to \mathbb{R}_{\geq 0} \\ x &\mapsto \inf\{L_{\mathrm{cl}(\mathcal{U})}(\xi) \mid \mathcal{U} \text{ is a relatively compact neighbourhood of } x\}, \end{aligned}$$

and

$$\begin{aligned} \mathrm{dil}\,\xi\colon \mathsf{M} &\to \mathbb{R}_{\geq 0} \\ x &\mapsto \inf\{l_{\mathrm{cl}(\mathcal{U})}(\xi) \mid \mathcal{U} \text{ is a relatively compact neighbourhood of } x\}, \end{aligned}$$

which are the ***local dilatation*** and ***local sectional dilatation***, respectively, of ξ. Following [Weaver 1999, Proposition 1.5.2] one can show that

$$L_K(\xi+\eta) \leq L_K(\xi) + L_K(\eta), \quad l_K(\xi+\eta) \leq l_K(\xi) + l_K(\eta), \qquad K \subseteq \mathsf{M} \text{ compact},$$

and

$$\mathrm{Dil}\,(\xi+\eta)(x) \leq \mathrm{Dil}\,\xi(x) + \mathrm{Dil}\,\eta(x), \quad \mathrm{dil}\,(\xi+\eta)(x) \leq \mathrm{dil}\,\xi(x) + \mathrm{dil}\,\eta(x), \qquad x \in \mathsf{M}.$$

The following lemma connects the preceding notions.

3.10 Lemma: (Characterisations of Lipschitz sections) *Let $\pi\colon \mathsf{E} \to \mathsf{M}$ be a smooth vector bundle and let $\xi\colon \mathsf{M} \to \mathsf{E}$ be such that $\xi(x) \in \mathsf{E}_x$ for every $x \in \mathsf{M}$. Then the following statements are equivalent:*

(i) $L_K(\xi) < \infty$ *for every compact* $K \subseteq \mathsf{M}$*;*
(ii) $l_K(\xi) < \infty$ *for every compact* $K \subseteq \mathsf{M}$*;*
(iii) $\mathrm{Dil}\,\xi(x) < \infty$ *for every* $x \in \mathsf{M}$*;*
(iv) $\mathrm{dil}\,\xi(x) < \infty$ *for every* $x \in \mathsf{M}$*.*

Moreover, we have the equalities

$$L_K(\xi) = \sqrt{l_K(\xi)^2 + 1}, \quad \mathrm{Dil}\,\xi(x) = \sqrt{\mathrm{dil}\,\xi(x)^2 + 1}$$

for every compact $K \subseteq \mathsf{M}$ and every $x \in \mathsf{M}$.

Proof: The equivalence of (i) and (ii), along with the equality $L_K = \sqrt{l_K^2 + 1}$, follows from the arguments of Canary, Epstein, and Marden [2006, Lemma II.A.2.4]. This also implies the equality $\mathrm{Dil}\,\xi(x) = \sqrt{\mathrm{dil}\,\xi(x)^2 + 1}$ when both $\mathrm{Dil}\,\xi(x)$ and $\mathrm{dil}\,\xi(x)$ are finite.

(i) $\Longrightarrow$ (iii) If $x \in \mathsf{M}$ and if $\mathcal{U}$ is a relatively compact neighbourhood of x, then $L_{\mathrm{cl}(\mathcal{U})}(\xi) < \infty$ and so $\mathrm{Dil}\,\xi(x) < \infty$.

(ii) $\Longrightarrow$ (iv) This follows just as does the preceding part of the proof.

(iii) $\Longrightarrow$ (i) Suppose that $\mathrm{Dil}\,\xi(x) < \infty$ for every $x \in \mathsf{M}$ and that there exists a compact set $K \subseteq \mathsf{M}$ such that $L_K(\xi) \not< \infty$. Then there exist sequences $(x_j)_{j\in\mathbb{Z}_{>0}}$ and $(y_j)_{j\in\mathbb{Z}_{>0}}$ in K such that $x_j \neq y_j$, $j \in \mathbb{Z}_{>0}$, and

$$\lim_{j\to\infty} \frac{\mathrm{d}_{\mathbb{G}_{\mathsf{E}}}(\xi(x_j), \xi(y_j))}{\mathrm{d}_{\mathbb{G}}(x_j, y_j)} = \infty.$$

Since $\mathrm{Dil}\,\xi(x) < \infty$ for every $x \in \mathsf{M}$, it follows directly that ξ is continuous and so $\xi(K)$ is bounded in the metric $\mathbb{G}_{\mathsf{E}}$. Therefore, there exists $C \in \mathbb{R}_{>0}$ such that

$$\mathrm{d}_{\mathbb{G}_{\mathsf{E}}}(\xi(x_j), \xi(y_j)) \leq C, \qquad j \in \mathbb{Z}_{>0},$$

and so we must have $\lim_{j\to\infty} \mathrm{d}_{\mathbb{G}}(x_j, y_j) = 0$. Let $(x_{j_k})_{k\in\mathbb{Z}_{>0}}$ be a subsequence converging to $x \in K$ and note that $(y_{j_k})_{k\in\mathbb{Z}_{>0}}$ then also converges to x. This implies that $\mathrm{Dil}\,\xi(x) \not< \infty$, which proves the result.

(iv) $\Longrightarrow$ (ii) This follows just as the preceding part of the proof. ■

With the preceding, we can define what we mean by a locally Lipschitz section of a vector bundle, noting that, if $\mathrm{dil}\,\xi(x) < \infty$ for every $x \in \mathsf{M}$, ξ is continuous. Our definition is in the general situation where sections are of class C^m with the mth derivative being, not just continuous, but Lipschitz.

3.11 Definition: (Locally Lipschitz section) For a smooth vector bundle $\pi\colon \mathsf{E} \to \mathsf{M}$ and for $m \in \mathbb{Z}_{\geq 0}$, $\xi \in \Gamma^m(\mathsf{E})$ is of ***class*** $\mathbf{C}^{m+\mathrm{lip}}$ if $j_m\xi\colon \mathsf{M} \to \mathsf{J}^m\mathsf{E}$ satisfies any of the four equivalent conditions of Lemma 3.10. If ξ is of class $\mathrm{C}^{0+\mathrm{lip}}$ then we say it is ***locally Lipschitz***. By $\Gamma^{\mathrm{lip}}(\mathsf{E})$ we denote the space of locally Lipschitz sections of E. For $m \in \mathbb{Z}_{\geq 0}$, by $\Gamma^{m+\mathrm{lip}}(\mathsf{E})$ we denote the space of sections of E of class $\mathrm{C}^{m+\mathrm{lip}}$. •

It is straightforward, if tedious, to show that a section is of class $\mathrm{C}^{m+\mathrm{lip}}$ if and only if, in any coordinate chart, the section is m-times continuously differentiable with the mth derivative being locally Lipschitz in the usual Euclidean sense. The essence of the argument is that, in any sufficiently small neighbourhood of a point in M, the distance functions $\mathrm{d}_{\mathbb{G}}$ and $\mathrm{d}_{\mathbb{G}_{\mathsf{E}}}$ are equivalent to the Euclidean distance functions defined in coordinates.

The following characterisation of the local sectional dilatation is useful.

3.12 Lemma: (Local sectional dilatation using derivatives) *For a smooth vector bundle $\pi\colon \mathsf{E} \to \mathsf{M}$ and for $\xi \in \Gamma^{\mathrm{lip}}(\mathsf{E})$, we have*

$$\mathrm{dil}\,\xi(x) = \inf\{\sup\{\|\nabla_{v_y}\xi\|_{\mathbb{G}_0} \mid y \in \mathrm{cl}(\mathcal{U}),\ \|v_y\|_{\mathbb{G}} = 1,\ \xi \textit{ differentiable at } y\} \mid \mathcal{U} \textit{ is a relatively compact neighbourhood of } x\}.$$

Proof: As per [Kobayashi and Nomizu 1963, Proposition IV.3.4], let $\mathcal{U}$ be a geodesically convex, relatively compact open set. We claim that

$$l_{\mathrm{cl}(\mathcal{U})}(\xi) = \sup\{\|\nabla^0_{v_y}\xi\|_{\mathbb{G}_0} \mid y \in \mathrm{cl}(\mathcal{U}),\ \|v_y\|_{\mathbb{G}} = 1,\ \xi \text{ differentiable at } y\}.$$

By [Canary, Epstein, and Marden 2006, Lemma II.A.2.4], to determine $l_{\mathrm{cl}(\mathcal{U})}(\xi)$, it suffices in the formula (3.4) to use only length minimising geodesics whose images are contained in $\mathrm{cl}(\mathcal{U})$. Let $x \in \mathcal{U}$, let $v_x \in \mathsf{T}_x\mathsf{M}$ have unit length, and let $\gamma\colon [0,T] \to \mathrm{cl}(\mathcal{U})$ be a minimal length geodesic such that $\gamma'(0) = v_x$. If x is a point of differentiability for ξ, then

$$\lim_{t\to 0} \frac{\|\tau_{\gamma,t}^{-1}(\xi \circ \gamma(t)) - \xi \circ \gamma(0)\|_{\mathbb{G}_0}}{t} = \|\nabla^0_{v_y}\xi\|_{\mathbb{G}_0}.$$

From this we conclude that

$$l_{\mathrm{cl}(\mathcal{U})}(\xi) \geq \sup\{\|\nabla^0_{v_x}\xi\|_{\mathbb{G}_0} \mid\ x \in \mathrm{cl}(\mathcal{U}),\ \|v_x\|_{\mathbb{G}} = 1,\ \xi \text{ differentiable at } y\}.$$

Suppose the opposite inequality does not hold. Then there exist $x_1, x_2 \in \mathrm{cl}(\mathcal{U})$ such that, if $\gamma\colon [0,T] \to \mathsf{M}$ is the arc-length parameterised minimal length geodesic from x_1 to x_2, then

$$\frac{\|\tau_{\gamma,T}^{-1}(\xi \circ \gamma(T)) - \xi \circ \gamma(0)\|}{T} > \|\nabla^0_{v_x}\xi\|_{\mathbb{G}_0} \tag{3.5}$$

for every $x \in \mathrm{cl}(\mathcal{U})$ for which ξ is differentiable at x and every $v_x \in \mathsf{T}_x\mathsf{M}$ of unit length. Note that $\alpha\colon t \mapsto \tau_{\gamma,t}^{-1}(\xi \circ \gamma(t))$ is a Lipschitz curve in $\mathsf{T}_{x_1}\mathsf{M}$. By Rademacher's Theorem [Federer 1969, Theorem 3.1.5], this curve is almost everywhere differentiable. If α is differentiable at t we have

$$\alpha'(t) = \tau_{\gamma,t}^{-1}(\nabla^0_{\gamma'(t)}\xi).$$

Therefore, also by Rademacher's Theorem and since ∇^0 is $\mathbb{G}_0$-orthogonal, we have

$$\sup\left\{\frac{\|\tau_{\gamma,t}^{-1}(\xi \circ \gamma(t)) - \xi \circ \gamma(0)\|_{\mathbb{G}_0}}{t} \;\middle|\; t \in [0,T]\right\} \\ = \sup\{\|\nabla^0_{\gamma'(t)}\xi\|_{\mathbb{G}_0} \mid\ t \in [0,T],\ \xi \text{ is differentiable at } \gamma(t)\}.$$

This, however, contradicts (3.5), and so our claim holds.

Now let $x \in \mathsf{M}$ and let $(\mathcal{U}_j)_{j\in\mathbb{Z}_{>0}}$ be a sequence of relatively compact, geodesically convex neighbourhood of x such that $\cap_{j\in\mathbb{Z}_{>0}}\mathcal{U}_j = \{x\}$. Then

$$\mathrm{dil}\,\xi(x) = \lim_{j\to\infty} l_{\mathrm{cl}(\mathcal{U}_j)}(\xi)$$

and

$$\inf\{\sup\{\|\nabla^0_{v_y}\xi\|_{\mathbb{G}_0} \mid\ y \in \mathrm{cl}(\mathcal{U}),\ \|v_y\|_{\mathbb{G}} = 1,\ \xi \text{ differentiable at } y\}| \\ \mathcal{U} \text{ is a relatively compact neighbourhood of } x\} \\ = \lim_{j\to\infty} \sup\{\|\nabla^0_{v_y}\xi\|_{\mathbb{G}_0} \mid\ y \in \mathrm{cl}(\mathcal{U}_j),\ \|v_y\|_{\mathbb{G}} = 1,\ \xi \text{ differentiable at } y\}.$$

The lemma now follows from the claim in the opening paragraph. ■

Let us see how to topologise spaces of locally Lipschitz sections. Lemma 3.10 gives us four possibilities for doing this. In order to be as consistent as possible with our other definitions of seminorms, we use the "locally sectional" characterisation of Lipschitz seminorms. Thus, for $\xi \in \Gamma^{\mathrm{lip}}(\mathsf{E})$ and $K \subseteq \mathsf{M}$ compact, let us define

$$\lambda_K(\xi) = \sup\{\mathrm{dil}\,\xi(x) \mid\ x \in K\}$$

and then define a seminorm p^{lip}_K, $K \subseteq \mathsf{M}$ compact, on $\Gamma^{\mathrm{lip}}(\mathsf{E})$ by

$$p^{\mathrm{lip}}_K(\xi) = \max\{\lambda_K(\xi), p^0_K(\xi)\}.$$

The seminorms p^{lip}_K, $K \subseteq \mathsf{M}$ compact, give the ***CO$^{\mathrm{lip}}$-topology*** on $\Gamma^r(\mathsf{E})$ for $r \in \mathbb{Z}_{>0} \cup \{\infty\}$. To topologise $\Gamma^{m+\mathrm{lip}}(\mathsf{E})$, note that the CO$^{\mathrm{lip}}$-topology on $\Gamma^{\mathrm{lip}}(\mathsf{J}^m\mathsf{E})$ induces a topology on $\Gamma^{m+\mathrm{lip}}(\mathsf{E})$ that we call the ***CO$^{m+\mathrm{lip}}$-topology***. The seminorms for this locally convex topology are

$$p^{m+\mathrm{lip}}_K(\xi) = \max\{\lambda^m_K(\xi), p^m_K(\xi)\}, \qquad K \subseteq \mathsf{M} \text{ compact},$$

where

$$\lambda^m_K(\xi) = \sup\{\operatorname{dil} j_m\xi(x) \mid x \in K\}.$$

Note that $\operatorname{dil} j_m\xi$ is unambiguously defined. Let us briefly explain why. If the connections ∇ and ∇^0 are metric connections for $\mathbb{G}$ and $\mathbb{G}_0$, as we are assuming, then the induced connection ∇^m on $\mathrm{T}^k(\mathsf{T}^*\mathsf{M}) \otimes \mathsf{E}$ is also metric with respect to the induced metric determined from Lemma 2.2. It then follows from Lemma 2.1 that the dilatation for sections of $\mathsf{J}^m\mathsf{E}$ can be defined just as for sections of E.

Note that $\Gamma^{\mathrm{lip}}(\mathsf{E}) \subseteq \Gamma^0(\mathsf{E})$ and $\Gamma^r(\mathsf{E}) \subseteq \Gamma^{\mathrm{lip}}(\mathsf{E})$ for $r \in \mathbb{Z}_{>0}$. Thus we adopt the convention that $0 < \mathrm{lip} < 1$ for the purposes of ordering degrees of regularity. Let $m \in \mathbb{Z}_{\geq 0}$, and let $r \in \mathbb{Z}_{\geq 0} \cup \{\infty\}$ and $r' \in \{0, \mathrm{lip}\}$ be such that $r + r' \geq m + \mathrm{lip}$. We adopt the obvious convention that $\infty + \mathrm{lip} = \infty$. The seminorms $p^{m+\mathrm{lip}}_K$, $K \subseteq \mathsf{M}$ compact, can then be defined on $\Gamma^{r+r'}(\mathsf{E})$.

Let us record some properties of the CO$^{m+\mathrm{lip}}$-topology for $\Gamma^{r+r'}(\mathsf{E})$. This topology is not extensively studied like the other differentiable topologies, but we can nonetheless enumerate its essential properties.

CO$^{m+\mathrm{lip}}$-1. It is Hausdorff: This is clear.

CO$^{m+\mathrm{lip}}$-2. It is complete if and only if $r + r' = m + \mathrm{lip}$: This is more or less because, for a compact metric space, the space of Lipschitz functions is a Banach space [Weaver 1999, Proposition 1.5.2]. Since $\Gamma^{m+\mathrm{lip}}(\mathsf{E})$ is the inverse limit of the Banach spaces $\Gamma^{m+\mathrm{lip}}(\mathsf{E}|K_j)$,[8] $j \in \mathbb{Z}_{>0}$, for a compact exhaustion $(K_j)_{j\in\mathbb{Z}_{>0}}$ of M, and since the inverse limit of complete locally convex spaces is complete [Horváth 1966, Proposition 2.11.3], we conclude the stated assertion.

CO$^{m+\mathrm{lip}}$-3. It is metrisable: This is argued as follows. First of all, it is a countable inverse limit of Banach spaces. Inverse limits are closed subspaces of the direct product [Robertson and Robertson 1980, Proposition V.19]. The direct product of metrisable spaces, in particular Banach spaces, is metrisable [Willard 1970, Theorem 22.3].

CO$^{m+\mathrm{lip}}$-4. It is separable: This is a consequence of the result of Greene and Wu [1979, Theorem 1.2$'$] which says that Lipschitz functions on Riemannian manifolds can be approximated in the CO$^{\mathrm{lip}}$-topology by smooth functions, and by the separability of the space of smooth functions.

CO$^{m+\mathrm{lip}}$-5. It is probably not nuclear: For compact base manifolds, $\Gamma^{m+\mathrm{lip}}(\mathsf{E})$ is an infinite-dimensional normed space, and so not nuclear [Pietsch 1969, Theorem 4.4.14]. But, even when M is not compact, the CO$^{m+\mathrm{lip}}$-topology is not likely nuclear, although we have neither found a reference nor proved this.

[8] To be clear, by $\Gamma^{m+\mathrm{lip}}(\mathsf{E}|K)$ we denote the space of sections of class $m+\mathrm{lip}$ defined on a neighbourhood of K.

CO$^{m+\mathrm{lip}}$-6. It is Suslin when $m+\mathrm{lip} = r+r'$: This follows since $\Gamma^{m+\mathrm{lip}}(\mathsf{E})$ is a Polish space, as we have already seen.

Of course, the preceding can be specialised to vector fields and functions, and one can define the weak-$\mathscr{L}$ topologies corresponding to the above topologies. To do this, we apply the general construction of Definition 3.2 with $\mathsf{U} = \Gamma^{r+r'}(\mathsf{TM})$, $\mathsf{V} = \mathrm{C}^{r+r'}(\mathsf{M})$ (with the CO^m-topology), and $\mathscr{A} = \{\mathscr{L}_f \mid f \in \mathrm{C}^\infty(\mathsf{M})\}$, where

$$\begin{aligned}\mathscr{L}_f\colon \Gamma^{r+r'}(\mathsf{TM}) &\to \mathrm{C}^{r+r'}(\mathsf{M})\\ X &\mapsto \mathscr{L}_X f.\end{aligned}$$

We then have the following definition.

3.13 Definition: (Weak-$\mathscr{L}$ topology for space of Lipschitz vector fields) Let M be a smooth manifold, let $m \in \mathbb{Z}_{\geq 0}$, and let $r \in \mathbb{Z}_{\geq 0} \cup \{\infty\}$ and $r' \in \{0, \mathrm{lip}\}$ have the property that $r + r' \geq m + \mathrm{lip}$. The ***weak-$(\mathscr{L}, m+\mathrm{lip})$ topology*** for $\Gamma^{r+r'}(\mathsf{TM})$ is the weakest topology for which $\mathscr{L}_f$ is continuous for each $f \in \mathrm{C}^\infty(\mathsf{M})$, where $\mathrm{C}^{r+r'}(\mathsf{M})$ is given the CO$^{m+\mathrm{lip}}$-topology. •

We can show that the weak-$(\mathscr{L}, m+\mathrm{lip})$ topology agrees with the CO$^{m+\mathrm{lip}}$-topology.

3.14 Theorem: (Weak-$\mathscr{L}$ topology for Lipschitz vector fields) *Let M be a smooth manifold, let $m \in \mathbb{Z}_{\geq 0}$, and let $r \in \mathbb{Z}_{\geq 0} \cup \{\infty\}$ and $r' \in \{0, \mathrm{lip}\}$ have the property that $r + r' \geq m + \mathrm{lip}$. Then the following two topologies for $\Gamma^{r+r'}(\mathsf{E})$ agree:*

(i) the CO$^{m+\mathrm{lip}}$-topology;

(ii) the weak-$(\mathscr{L}, m+\mathrm{lip})$-topology.

Proof: We prove the theorem only for the case $m = 0$, since the general case follows from this in combination with Theorem 3.8.

Let us first show that the CO$^{\mathrm{lip}}$-topology is weaker than the weak-$(\mathscr{L}, \mathrm{lip})$ topology. Let $K \subseteq \mathsf{M}$ be compact and for $x \in \mathsf{M}$ choose a coordinate chart $(\mathcal{U}_x, \phi_x)$ and functions $f_x^1, \dots, f_x^n \in \mathrm{C}^\infty(\mathsf{M})$ agreeing with the coordinate functions in a neighbourhood of a geodesically convex relatively compact neighbourhood $\mathcal{V}_x$ of x [Kobayashi and Nomizu 1963, Proposition IV.3.4]. We denote by $\boldsymbol{X}\colon \phi_x(\mathcal{U}_x) \to \mathbb{R}^n$ the local representative of X. Since $\mathscr{L}_X f_x^j = X^j$ on a neighbourhood of $\mathcal{V}_x$, there exists $C_x \in \mathbb{R}_{>0}$ such that

$$\|\tau_{\gamma,1}^{-1}(X(x_1)) - X(x_2)\|_{\mathbb{G}} \leq C_x \sum_{j=1}^n |\mathscr{L}_X f_x^j(x_1) - \mathscr{L}_X f_x^j(x_2)|$$

for every distinct $x_1, x_2 \in \mathrm{cl}(\mathcal{V}_x)$, where γ is the unique minimal length geodesic from x_2 to x_1 (the inequality is a consequence of the fact that the ℓ^1 norm for $\mathbb{R}^n$ is equivalent to any other norm). This gives an inequality

$$\mathrm{dil}\, X(y) \leq C_x(\mathrm{dil}\, \mathscr{L}_X f_x^1(y) + \cdots + \mathrm{dil}\, \mathscr{L}_X f_x^n(y))$$

for every $y \in \mathcal{V}_x$. Now let $x_1, \dots, x_k \in K$ be such that $K \subseteq \cup_{j=1}^k \mathcal{V}_{x_j}$. From this point, it is a bookkeeping exercise, exactly like that in the corresponding part of the proof of Theorem 3.5, to arrive at the inequality

$$\lambda_K(X) \leq C_1 \lambda_K(\mathscr{L}_X f^1) + \cdots + C_r \lambda_K(\mathscr{L}_X f^r).$$

From the proof of Theorem 3.8 we also have

$$p^0_K(X) \leq C'_1 p^0_K(\mathscr{L}_X f^1) + \cdots + C'_r p^0_K(\mathscr{L}_X f^r),$$

and this gives the result.

To prove that the weak $(\mathscr{L}, \mathrm{lip})$-topology is weaker than the $\mathrm{CO}^{\mathrm{lip}}$-topology, it suffices to show that $\mathscr{L}_f$ is continuous for every $f \in \mathrm{C}^\infty(\mathsf{M})$ if $\Gamma^{r+r'}(\mathsf{TM})$ and $\mathrm{C}^{r+r'}(\mathsf{M})$ are given the $\mathrm{CO}^{m+\mathrm{lip}}$-topology. Thus let $K \subseteq \mathsf{M}$ be compact and let $f \in \mathrm{C}^\infty(\mathsf{M})$. We choose a relatively compact geodesically convex chart $(\mathcal{U}_x, \phi_x)$ about $x \in K$ and compute, for distinct $x_1, x_2 \in \mathcal{U}_x$,

$$\begin{aligned}
&|\mathscr{L}_X f(x_1) - \mathscr{L}_X f(x_2)| \\
&\quad \leq \sum_{j=1}^n \Big| X^j(x_1) \frac{\partial f}{\partial x^j}(x_1) - X^j(x_2)\frac{\partial f}{\partial x^j}(x_2)\Big| \\
&\quad \leq \sum_{j=1}^n \Big(|X^j(x_1)| \Big|\frac{\partial f}{\partial x^j}(x_1) - \frac{\partial f}{\partial x^j}(x_2)\Big| + |X^j(x_1) - X^j(x_2)| \Big|\frac{\partial f}{\partial x^j}(x_2)\Big|\Big) \\
&\quad \leq \sum_{j=1}^n \Big(A_x p^0_{\mathrm{cl}(\mathcal{U}_x)}(X) \frac{\partial f}{\partial x^j}(y) \mathrm{d}_{\mathbb{G}}(x_1, x_2)\Big) + B_x \|\tau^{-1}_{\gamma,1} X(x_1) - X(x_2)\|_{\mathbb{G}},
\end{aligned}$$

for some $y \in \mathcal{U}_x$, using the mean value theorem [Abraham, Marsden, and Ratiu 1988, Proposition 2.4.8], and where γ is the unique length minimising geodesic from x_2 to x_1. Thus we have an inequality

$$\lambda_{\mathrm{cl}(\mathcal{U}_x)}(\mathscr{L}_X f) \leq A_x p^0_{\mathrm{cl}(\mathcal{U}_x)}(X) + B_x \lambda_{\mathrm{cl}(\mathcal{U}_x)}(X),$$

for a possibly different A_x. Letting $x_1, \dots, x_k \in K$ be such that $K \subseteq \cup_{j=1}^k \mathcal{U}_x$, some more bookkeeping like that in the first part of the proof of Theorem 3.5 gives

$$\lambda_K(\mathscr{L}_X f) \leq \sum_{j=1}^k (A_j p^0_{\mathrm{cl}(\mathcal{U}_{x_j})}(X) + B_j \lambda_{\mathrm{cl}(\mathcal{U}_{x_j})}(X))$$

for suitable constants $A_j, B_j \in \mathbb{R}_{>0}$, $j \in \{1, \dots, k\}$. Since, from the proof of Theorem 3.8, we also have

$$p^0_K(\mathscr{L}_X f) \leq \sum_{j=1}^k C_j p^0_{\mathrm{cl}(\mathcal{U}_{x_j})}(X)$$

for suitable constants $C_1, \dots, C_k \in \mathbb{R}_{>0}$, the result follows. ■

We also have the corresponding relationships between various attributes and their weak counterparts.

3.15 Corollary: (Weak-$\mathscr{L}$ characterisations of boundedness, continuity, measurability, and integrability for the $\mathrm{CO}^{m+\mathrm{lip}}$-topology) *Let M be a smooth manifold, let $m \in \mathbb{Z}_{\geq 0}$, and let $r \in \mathbb{Z}_{\geq 0} \cup \{\infty, \mathrm{lip}\}$ and $r' \in \{0, \mathrm{lip}\}$ have the property that $r + r' \geq m + \mathrm{lip}$. Let $(\mathcal{X}, \mathscr{O})$ be a topological space, let $(\mathcal{T}, \mathscr{M})$ be a measurable space, and let $\mu\colon \mathscr{M} \to \overline{\mathbb{R}}_{\geq 0}$ be a finite measure. The following statements hold:*

(i) *a subset* $\mathcal{B} \subseteq \Gamma^{r+r'}(\mathsf{TM})$ *is* $\mathrm{CO}^{m+\mathrm{lip}}$*-bounded in the von Neumann bornology if and only if it is weak-*$(\mathscr{L}, m+\mathrm{lip})$ *bounded in the von Neumann bornology;*

(ii) *a map* $\Phi\colon \mathcal{X} \to \Gamma^{r+r'}(\mathsf{TM})$ *is* $\mathrm{CO}^{m+\mathrm{lip}}$*-continuous if and only if it is weak-*$(\mathscr{L}, m+\mathrm{lip})$ *continuous;*

(iii) *a map* $\Psi\colon \mathcal{T} \to \Gamma^{m+\mathrm{lip}}(\mathsf{TM})$ *is* $\mathrm{CO}^{m+\mathrm{lip}}$*-measurable if and only if it is weak-*$(\mathscr{L}, m+\mathrm{lip})$ *measurable;*

(iv) *a map* $\Psi\colon \mathcal{T} \to \Gamma^{m+\mathrm{lip}}(\mathsf{TM})$ *is Bochner integrable if and only if it is weak-*$(\mathscr{L}, m+\mathrm{lip})$ *Bochner integrable.*

Proof: In the proof of Corollary 3.6 we established that $\{\mathscr{L}_f \mid f \in \mathrm{C}^\infty(\mathsf{M})\}$ was point separating as a family of linear mappings with domain $\Gamma^\infty(\mathsf{TM})$. The same proof is valid if the domain is $\Gamma^{m+\mathrm{lip}}(\mathsf{TM})$. The result is then a direct consequence of Lemma 3.3, noting that the $\mathrm{CO}^{m+\mathrm{lip}}$-topology on $\Gamma^{r+r'}(\mathsf{TM})$ is separable, and is also complete and Suslin when $r+r' = m+\mathrm{lip}$ (and $\mathrm{C}^{r+r'}(\mathsf{M})$ is Suslin when $r+r' = m+\mathrm{lip}$), as we have seen above in properties $\mathrm{CO}^{m+\mathrm{lip}}$-2, $\mathrm{CO}^{m+\mathrm{lip}}$-4, and $\mathrm{CO}^{m+\mathrm{lip}}$-6. ■

3.16 Notation: $(m+m')$ In order to try to compactify the presentation of the various degrees of regularity we consider, we will frequently speak of the class "$m+m'$" where $m \in \mathbb{Z}_{\geq 0}$ and $m' \in \{0, \mathrm{lip}\}$. This allows us to include the various Lipschitz cases alongside the finitely differentiable cases. Thus, whenever the reader sees "$m+m'$," this is what they should have in mind. •

4. The CO$^{\mathrm{hol}}$-topology for the space of holomorphic vector fields

While in this paper we have no *per se* interest in holomorphic vector fields, it is the case that an understanding of certain constructions for real analytic vector fields rely in an essential way on their holomorphic extensions. Also, as we shall see, we will arrive at a description of the real analytic topology that, while often easy to use in general arguments, is not well suited for verifying hypotheses in examples. In these cases, it is often most convenient to extend from real analytic to holomorphic, where things are easier to verify.

Thus in this section we overview the holomorphic case. We begin with vector bundles, as in the smooth case.

4.1. General holomorphic vector bundles.

We let $\pi\colon \mathsf{E} \to \mathsf{M}$ be an holomorphic vector bundle with $\Gamma^{\mathrm{hol}}(\mathsf{E})$ the set of holomorphic sections. We let $\mathbb{G}$ be an Hermitian fibre metric on E, and, for $K \subseteq \mathsf{M}$ compact, define a seminorm p_K^{hol} on $\Gamma^{\mathrm{hol}}(\mathsf{E})$ by

$$p_K^{\mathrm{hol}}(\xi) = \sup\{\|\xi(z)\|_{\mathbb{G}} \mid z \in K\}.$$

The $\mathbf{CO^{hol}}$***-topology*** for $\Gamma^{\mathrm{hol}}(\mathsf{E})$ is the locally convex topology defined by the family of seminorms p_K^{hol}, $K \subseteq \mathsf{M}$ compact.

We shall have occasion to make use of bounded holomorphic sections. Thus we let $\pi\colon \mathsf{E} \to \mathsf{M}$ be an holomorphic vector bundle with Hermitian fibre metric $\mathbb{G}$. We denote by $\Gamma^{\mathrm{hol}}_{\mathrm{bdd}}(\mathsf{E})$ the sections of E that are bounded, and on $\Gamma^{\mathrm{hol}}_{\mathrm{bdd}}(\mathsf{E})$ we define a norm

$$p_\infty^{\mathrm{hol}}(\xi) = \sup\{\|\xi(z)\|_{\mathbb{G}} \mid z \in \mathsf{M}\}.$$

If we wish to draw attention to the domain of the section, we will write the norm as $p^{\mathrm{hol}}_{\mathsf{M},\infty}$. This will occur when we have sections defined on an open subset of the manifold.

The following lemma makes an assertion of which we shall make use.

4.1 Lemma: (The topology of $\Gamma^{\mathrm{hol}}_{\mathrm{bdd}}(\mathsf{E})$) *Let $\pi\colon \mathsf{E}\to\mathsf{M}$ be an holomorphic vector bundle. The subspace topology on $\Gamma^{\mathrm{hol}}_{\mathrm{bdd}}(\mathsf{E})$, induced from the $\mathrm{CO}^{\mathrm{hol}}$-topology, is weaker than the norm topology induced by the norm $p^{\mathrm{hol}}_{\infty}$. Moreover, $\Gamma^{\mathrm{hol}}_{\mathrm{bdd}}(\mathsf{E})$ is a Banach space. Also, if $\mathcal{U}\subseteq\mathsf{M}$ is a relatively compact open set with $\mathrm{cl}(\mathcal{U})\subset\mathsf{M}$, then the restriction map from $\Gamma^{\mathrm{hol}}(\mathsf{E})$ to $\Gamma^{\mathrm{hol}}_{\mathrm{bdd}}(\mathsf{E}|\mathcal{U})$ is continuous.*

Proof: It suffices to show that a sequence $(\xi_j)_{j\in\mathbb{Z}_{>0}}$ in $\Gamma^{\mathrm{hol}}_{\mathrm{bdd}}(\mathsf{E})$ converges to $\xi\in\Gamma^{\mathrm{hol}}_{\mathrm{bdd}}(\mathsf{E})$ uniformly on compact subsets of M if it converges in norm. This, however, is obvious. It remains to prove completeness of $\Gamma^{\mathrm{hol}}_{\mathrm{bdd}}(\mathsf{E})$ in the norm topology. By [Hewitt and Stromberg 1975, Theorem 7.9], a Cauchy sequence $(\xi_j)_{j\in\mathbb{Z}_{>0}}$ in $\Gamma^{\mathrm{hol}}_{\mathrm{bdd}}(\mathsf{E})$ converges to a bounded continuous section ξ of E. That ξ is also holomorphic follows since uniform limits of holomorphic sections are holomorphic [Gunning 1990a, page 5]. For the final assertion, since the topology of $\Gamma^{\mathrm{hol}}(\mathsf{E})$ is metrisable (see $\mathrm{CO}^{\mathrm{hol}}$-3 below), it suffices to show that the restriction of a convergent sequence in $\Gamma^{\mathrm{hol}}(\mathsf{E})$ to $\mathcal{U}$ converges uniformly. This, however, follows since $\mathrm{cl}(\mathcal{U})$ is compact. ■

One of the useful attributes of holomorphic geometry is that properties of higher derivatives can be deduced from the mapping itself. To make this precise, we first make the following observations.

1. Hermitian inner products on $\mathbb{C}$-vector spaces give inner products on the underlying $\mathbb{R}$-vector space.
2. By Lemma 2.3, there exist a real analytic affine connection ∇ on M and a real analytic vector bundle connection ∇^0 on E.

Therefore, the seminorms defined in Section 3.1 can be made sense of for holomorphic sections.

4.2 Proposition: (Cauchy estimates for vector bundles) *Let $\pi\colon\mathsf{E}\to\mathsf{M}$ be an holomorphic vector bundle, let $K\subseteq\mathsf{M}$ be compact, and let $\mathcal{U}$ be a relatively compact neighbourhood of K. Then there exist $C,r\in\mathbb{R}_{>0}$ such that*

$$p^{\infty}_{K,m}(\xi)\le Cr^{-m}p^{\mathrm{hol}}_{\mathcal{U},\infty}(\xi)$$

for every $m\in\mathbb{Z}_{\ge0}$ and $\xi\in\Gamma^{\mathrm{hol}}_{\mathrm{bdd}}(\mathsf{E}|\mathcal{U})$.

Moreover, if $(\mathcal{U}_j)_{j\in\mathbb{Z}_{>0}}$ is a sequence of relatively compact neighbourhoods of K such that (i) $\mathrm{cl}(\mathcal{U}_j)\subseteq\mathcal{U}_{j+1}$ and (ii) $K=\cap_{j\in\mathbb{Z}_{>0}}\mathcal{U}_j$, and if $C_j,r_j\in\mathbb{R}_{>0}$ are such that

$$p^{\infty}_{K,m}(\xi)\le C_j r_j^{-m}p^{\mathrm{hol}}_{\mathcal{U}_j,\infty}(\xi),\qquad m\in\mathbb{Z}_{\ge0},\ \xi\in\Gamma^{\mathrm{hol}}_{\mathrm{bdd}}(\mathsf{E}|\mathcal{U}_j),$$

then $\lim_{j\to\infty}r_j=0$.

Proof: Let $z\in K$ and let $(\mathcal{W}_z,\psi_z)$ be an holomorphic vector bundle chart about z with $(\mathcal{U}_z,\phi_z)$ the associated chart for M, supposing that $\mathcal{U}_z\subseteq\mathcal{U}$. Let $k\in\mathbb{Z}_{>0}$ be such that $\psi_z(\mathcal{W}_z)=\phi_z(\mathcal{U}_z)\times\mathbb{C}^k$. Let $\boldsymbol{z}=\phi_z(z)$ and let $\boldsymbol{\xi}\colon\phi_z(\mathcal{U}_z)\to\mathbb{C}^k$ be the local representative of $\xi \in \Gamma^{\mathrm{hol}}_{\mathrm{bdd}}(\mathsf{E}|\mathcal{U})$. Note that when taking real derivatives of $\boldsymbol{\xi}$ with respect to coordinates, we can think of taking derivatives with respect to

$$\frac{\partial}{\partial z^j} = \frac{1}{2}\Big(\frac{\partial}{\partial x^j} - \mathrm{i}\frac{\partial}{\partial y^j}\Big), \quad \frac{\partial}{\partial \bar{z}^j} = \frac{1}{2}\Big(\frac{\partial}{\partial x^j} + \mathrm{i}\frac{\partial}{\partial y^j}\Big), \qquad j \in \{1, \dots, n\}.$$

Since $\boldsymbol{\xi}$ is holomorphic, the $\frac{\partial}{\partial \bar{z}^j}$ derivatives will vanish [Krantz 1992, page 27]. Thus, for the purposes of the multi-index calculations, we consider multi-indices of length n (not $2n$). In any case, applying the usual Cauchy estimates [Krantz 1992, Lemma 2.3.9], there exists $r \in \mathbb{R}_{>0}$ such that

$$|\boldsymbol{D}^I \xi^a(\boldsymbol{z})| \leq I! r^{-|I|} \sup\{|\xi^a(\boldsymbol{\zeta})| \mid \boldsymbol{\zeta} \in \overline{\mathsf{D}}(\boldsymbol{r}, \boldsymbol{z})\}$$

for every $a \in \{1, \dots, k\}$, $I \in \mathbb{Z}^n_{\geq 0}$, and $\xi \in \Gamma^{\mathrm{hol}}_{\mathrm{bdd}}(\mathsf{E}|\mathcal{U})$. We may choose $r \in (0,1)$ such that $\overline{\mathsf{D}}(\boldsymbol{r}, \boldsymbol{z})$ is contained in $\phi_z(\mathcal{U}_z)$, where $\boldsymbol{r} = (r, \dots, r)$. Denote $\mathcal{V}_z = \phi_z^{-1}(\mathsf{D}(\boldsymbol{r}, \boldsymbol{z}))$. There exists a neighbourhood $\mathcal{V}'_z$ of z such that $\mathrm{cl}(\mathcal{V}'_z) \subseteq \mathcal{V}_z$ and such that

$$|\boldsymbol{D}^I \xi^a(\boldsymbol{z}')| \leq 2 I! r^{-|I|} \sup\{|\xi^a(\boldsymbol{\zeta})| \mid \boldsymbol{\zeta} \in \overline{\mathsf{D}}(\boldsymbol{r}, \boldsymbol{z})\}$$

for every $\boldsymbol{z}' \in \phi_z(\mathcal{V}'_z)$, $\xi \in \Gamma^{\mathrm{hol}}_{\mathrm{bdd}}(\mathsf{E}|\mathcal{U})$, $a \in \{1, \dots, k\}$, and $I \in \mathbb{Z}^n_{\geq 0}$. If $|I| \leq m$ then, since we are assuming that $r < 1$, we have

$$\frac{1}{I!}|\boldsymbol{D}^I \xi^a(\boldsymbol{z}')| \leq 2 r^{-m} \sup\{|\xi^a(\boldsymbol{\zeta})| \mid \boldsymbol{\zeta} \in \overline{\mathsf{D}}(\boldsymbol{r}, \boldsymbol{z})\}$$

for every $a \in \{1, \dots, k\}$, $\boldsymbol{z}' \in \phi_z(\mathcal{V}'_z)$, and $\xi \in \Gamma^{\mathrm{hol}}_{\mathrm{bdd}}(\mathsf{E}|\mathcal{U})$. By Lemma 2.4, it follows that there exist $C_z, r_z \in \mathbb{R}_{>0}$ such that

$$\|j_m\xi(z)\|_{\overline{\mathbb{G}}_m} \leq C_z r_z^{-m} p^{\mathrm{hol}}_{\mathcal{V}_z,\infty}(\xi)$$

for all $z \in \mathcal{V}'_z$, $m \in \mathbb{Z}_{\geq 0}$, and $\xi \in \Gamma^{\mathrm{hol}}_{\mathrm{bdd}}(\mathsf{E}|\mathcal{U})$. Let $z_1, \dots, z_k \in K$ be such that $K \subseteq \cup_{j=1}^k \mathcal{V}'_{z_j}$, and let $C = \max\{C_{z_1}, \dots, C_{z_k}\}$ and $r = \min\{r_{z_1}, \dots, r_{z_k}\}$. If $z \in K$, then $z \in \mathcal{V}'_{z_j}$ for some $j \in \{1, \dots, k\}$ and so we have

$$\|j_m\xi(z)\|_{\overline{\mathbb{G}}_m} \leq C_{z_j} r_{z_j}^{-m} p^{\mathrm{hol}}_{\mathcal{V}_{z_j},\infty}(\xi) \leq C r^{-m} p^{\mathrm{hol}}_{\mathcal{U},\infty}(\xi),$$

and taking supremums over $z \in K$ on the left gives the result.

The final assertion of the proposition immediately follows by observing in the preceding construction how "r" was defined, namely that it had to be chosen so that polydisks of radius r in the coordinate charts remained in $\mathcal{U}$. ■

4.2. Properties of the CO$^{\mathrm{hol}}$-topology. The CO$^{\mathrm{hol}}$-topology for $\Gamma^{\mathrm{hol}}(\mathsf{E})$ has the following attributes.

CO$^{\mathrm{hol}}$-1. It is Hausdorff: [Kriegl and Michor 1997, Theorem 8.2].

CO$^{\mathrm{hol}}$-2. It is complete: [Kriegl and Michor 1997, Theorem 8.2].

CO$^{\mathrm{hol}}$-3. It is metrisable: [Kriegl and Michor 1997, Theorem 8.2].

CO$^{\mathrm{hol}}$-4. It is separable: This follows since $\Gamma^{\mathrm{hol}}(\mathsf{E})$ is a closed subspace of $\Gamma^{\infty}(\mathsf{E})$ by [Kriegl and Michor 1997, Theorem 8.2] and since subspaces of separable metric spaces are separable [Willard 1970, Theorems 16.2, 16.9, and 16.11].

CO$^{\mathrm{hol}}$-5. It is nuclear: [Kriegl and Michor 1997, Theorem 8.2]. Note that, when M is compact, $p^{\mathrm{hol}}_{\mathsf{M}}$ is a norm for the C$^{\mathrm{hol}}$-topology. A consequence of this is that $\Gamma^{\mathrm{hol}}(\mathsf{E})$ must be finite-dimensional in these cases since the only nuclear normed vector spaces are those that are finite-dimensional [Pietsch 1969, Theorem 4.4.14].

CO$^{\mathrm{hol}}$-6. It is Suslin: This follows since $\Gamma^{\mathrm{hol}}(\mathsf{E})$ is a Polish space, as we have seen above.

Being metrisable, it suffices to describe the CO$^{\mathrm{hol}}$-topology by describing its convergent sequences; these are more or less obviously the sequences that converge uniformly on every compact set.

As with spaces of smooth sections, we are interested in the fact that nuclearity of $\Gamma^{\mathrm{hol}}(\mathsf{E})$ implies that compact sets are exactly those sets that are closed and von Neumann bounded. The following result is obvious in the same way that Lemma 3.1 is obvious once one understands Theorem 1.37(b) from [Rudin 1991].

4.3 Lemma: (Bounded subsets in the CO$^{\mathrm{hol}}$-topology) *A subset $\mathcal{B} \subseteq \Gamma^{\mathrm{hol}}(\mathsf{E})$ is bounded in the von Neumann bornology if and only if the following property holds: for any compact set $K \subseteq \mathsf{M}$, there exists $C \in \mathbb{R}_{>0}$ such that $p^{\mathrm{hol}}_K(\xi) \leq C$ for every $\xi \in \mathcal{B}$.*

4.3. The weak-$\mathscr{L}$ topology for holomorphic vector fields. As in the smooth case, one simply specialises the constructions for general vector bundles to get the CO$^{\mathrm{hol}}$-topology for the space $\Gamma^{\mathrm{hol}}(\mathsf{TM})$ of holomorphic vector fields and the space C$^{\mathrm{hol}}(\mathsf{M})$ of holomorphic functions, noting that an holomorphic function is obviously identified with a section of the trivial holomorphic vector bundle $\mathsf{M} \times \mathbb{C}$.

As with smooth vector fields, for holomorphic vector fields we can seek a weak-$\mathscr{L}$ characterisation of the CO$^{\mathrm{hol}}$-topology. To begin, we need to understand the Lie derivative in the holomorphic case. Thinking of $\mathrm{C}^{\mathrm{hol}}(\mathsf{M}) \subseteq \mathrm{C}^{\infty}(\mathsf{M}) \otimes \mathbb{C}$ and using the Wirtinger formulae,

$$\frac{\partial}{\partial z^j} = \frac{1}{2}\Big(\frac{\partial}{\partial x^j} - \mathrm{i}\frac{\partial}{\partial y^j}\Big), \quad \frac{\partial}{\partial \bar{z}^j} = \frac{1}{2}\Big(\frac{\partial}{\partial x^j} + \mathrm{i}\frac{\partial}{\partial y^j}\Big), \qquad j \in \{1, \dots, n\},$$

in an holomorphic chart, one sees that the usual differential of a $\mathbb{C}$-valued function can be decomposed as $\mathrm{d}_{\mathbb{C}} f = \partial f + \bar{\partial} f$, the first term on the right corresponding to "$\frac{\partial}{\partial z}$" and the second to "$\frac{\partial}{\partial \bar{z}}$." For holomorphic functions, the Cauchy–Riemann equations [Krantz 1992, page 27] imply that $\mathrm{d}_{\mathbb{C}} f = \partial f$. Thus we define the Lie derivative of an holomorphic function f with respect to an holomorphic vector field X by $\mathscr{L}_X f = \langle \partial f; X \rangle$. Fortunately, in coordinates this assumes the expected form:

$$\mathscr{L}_X f = \sum_{j=1}^{n} X^j \frac{\partial f}{\partial z^j}.$$

It is *not* the case that on a general holomorphic manifold there is a correspondence between derivations of the $\mathbb{C}$-algebra C$^{\mathrm{hol}}(\mathsf{M})$ and holomorphic vector fields by Lie differentiation.[9]

[9]For example, on a compact holomorphic manifold, the only holomorphic functions are locally constant [Fritzsche and Grauert 2002, Corollary IV.1.3], and so the only derivation is the zero derivation. However, the $\mathbb{C}$-vector space of holomorphic vector fields, while not large, may have positive dimension. For example, the space of holomorphic vector fields on the Riemann sphere has $\mathbb{C}$-dimension three [Ilyashenko and Yakovenko 2008, Problem 17.9].

However, for a certain class of holomorphic manifolds, those known as "Stein manifolds," the exact correspondence between derivations of the $\mathbb{C}$-algebra $\mathrm{C}^{\mathrm{hol}}(\mathsf{M})$ and holomorphic vector fields under Lie differentiation *does* hold [Grabowski 1981]. This is good news for us, since Stein manifolds are intimately connected with real analytic manifolds, as we shall see in the next section.

With the preceding discussion in mind, we can move ahead with Definition 3.2 with $\mathsf{U} = \Gamma^{\mathrm{hol}}(\mathsf{TM})$, $\mathsf{V} = \mathrm{C}^{\mathrm{hol}}(\mathsf{M})$ (with the $\mathrm{CO}^{\mathrm{hol}}$-topology), and $\mathscr{A} = \{\mathscr{L}_f \mid f \in \mathrm{C}^{\mathrm{hol}}(\mathsf{M})\}$, where

$$\begin{aligned} \mathscr{L}_f\colon \Gamma^{\mathrm{hol}}(\mathsf{TM}) &\to \mathrm{C}^{\mathrm{hol}}(\mathsf{M}) \\ X &\mapsto \mathscr{L}_X f. \end{aligned}$$

We make the following definition.

4.4 Definition: (Weak-$\mathscr{L}$ topology for space of holomorphic vector fields) For an holomorphic manifold M, the ***weak-$\mathscr{L}$ topology*** for $\Gamma^{\mathrm{hol}}(\mathsf{TM})$ is the weakest topology for which $\mathscr{L}_f$ is continuous for every $f \in \mathrm{C}^{\mathrm{hol}}(\mathsf{M})$, if $\mathrm{C}^{\mathrm{hol}}(\mathsf{M})$ has the $\mathrm{CO}^{\mathrm{hol}}$-topology. •

We then have the following result.

4.5 Theorem: (Weak-$\mathscr{L}$ characterisation of $\mathrm{CO}^{\mathrm{hol}}$-topology for holomorphic vector fields on Stein manifolds) *For a Stein manifold M, the following topologies for $\Gamma^{\mathrm{hol}}(\mathsf{TM})$ agree:*

(i) the $\mathrm{CO}^{\mathrm{hol}}$-topology;

(ii) the weak-$\mathscr{L}$ topology.

Proof: (i)⊆(ii) As we argued in the proof of the corresponding assertion of Theorem 3.5, it suffices to show that

$$p_K^{\mathrm{hol}}(X) \le C_1 p_{K_1}^{\mathrm{hol}}(\mathscr{L}_X f^1) + \cdots + C_r p_{K_r}^{\mathrm{hol}}(\mathscr{L}_X f^r)$$

for some $C_1, \dots, C_r \in \mathbb{R}_{>0}$, some $K_1, \dots, K_r \subseteq \mathsf{M}$ compact, and some $f^1, \dots, f^r \in \mathrm{C}^{\mathrm{hol}}(\mathsf{M})$.

Let $K \subseteq \mathsf{M}$ be compact. For simplicity, we assume that M is connected and so has a well-defined dimension n. If not, then the arguments are easily modified by change of notation to account for this. Since M is a Stein manifold, for every $z \in K$ there exists a coordinate chart $(\mathcal{U}_z, \phi_z)$ with coordinate functions $z^1, \dots, z^n\colon \mathcal{U}_z \to \mathbb{C}$ that are restrictions to $\mathcal{U}_z$ of globally defined holomorphic functions on M. Depending on your source, this is either a theorem or part of the definition of a Stein manifold [Fritzsche and Grauert 2002, Hörmander 1966]. Thus, for $j \in \{1, \dots, n\}$, let $f_z^j \in \mathrm{C}^{\mathrm{hol}}(\mathsf{M})$ be the holomorphic function which, when restricted to $\mathcal{U}_z$, gives the coordinate function z^j. Clearly, $\mathscr{L}_X f_z^j = X^j$ on $\mathcal{U}_z$. Also, there exists $C_z \in \mathbb{R}_{>0}$ such that

$$\|X(\zeta)\|_{\mathbb{G}} \le C_z(|X^1(\zeta)| + \cdots + |X^n(\zeta)|), \qquad \zeta \in \mathrm{cl}(\mathcal{V}_z),$$

for some relatively compact neighbourhood $\mathcal{V}_z \subseteq \mathcal{U}_z$ of z (this follows from the fact that all norms are equivalent to the ℓ^1 norm for $\mathbb{C}^n$). Thus

$$\|X(\zeta)\|_{\mathbb{G}} \le C_z(|\mathscr{L}_X f_z^1(\zeta)| + \cdots + |\mathscr{L}_X f_z^n(\zeta)|), \qquad \zeta \in \mathrm{cl}(\mathcal{V}_z).$$

Let $z_1, \dots, z_k \in K$ be such that $K \subseteq \cup_{j=1}^k \mathcal{V}_{z_j}$. Let $f^1, \dots, f^{kn}$ be the list of globally defined holomorphic functions

$$f_{z_1}^1, \dots, f_{z_1}^n, \dots, f_{z_k}^1, \dots, f_{z_k}^n$$

and let $C_1, \dots, C_{kn}$ be the list of coefficients

$$\underbrace{C_{z_1}, \dots, C_{z_1}}_{n \text{ times}}, \dots, \underbrace{C_{z_k}, \dots, C_{z_k}}_{n \text{ times}}.$$

If $z \in K$, then $z \in \mathcal{V}_{z_j}$ for some $j \in \{1, \dots, k\}$ and so

$$\|X(z)\|_{\mathbb{G}} \le C_1 |\mathscr{L}_X f^1(z)| + \dots + C_{kn} |\mathscr{L}_X f^{kn}(z)|,$$

which gives

$$p_K^{\mathrm{hol}}(X) \le C_1 p_K^{\mathrm{hol}}(\mathscr{L}_X f^1) + \dots + C_{kn} p_K^{\mathrm{hol}}(\mathscr{L}_X f^{kn}),$$

as needed.

(ii)$\subseteq$(i) We claim that $\mathscr{L}_f$ is continuous for every $f \in \mathrm{C}^{\mathrm{hol}}(\mathsf{M})$ if $\Gamma^{\mathrm{hol}}(\mathsf{TM})$ has the $\mathrm{CO}^{\mathrm{hol}}$-topology. Let $K \subseteq \mathsf{M}$ be compact and let $\mathcal{U}$ be a relatively compact neighbourhood of K in M. Note that, for $f \in \mathrm{C}^{\mathrm{hol}}(\mathsf{M})$,

$$p_K^{\mathrm{hol}}(\mathscr{L}_X f) \le C p_{K,1}^{\infty}(f) p_K^{\mathrm{hol}}(X) \le C' p_K^{\mathrm{hol}}(X),$$

using Proposition 4.2, giving continuity of the identity map if we provide the domain with the $\mathrm{CO}^{\mathrm{hol}}$-topology and the codomain with the weak-$\mathscr{L}$ topology, cf. [Schaefer and Wolff 1999, §III.1.1]. Thus open sets in the weak-$\mathscr{L}$ topology are contained in the $\mathrm{CO}^{\mathrm{hol}}$-topology. ■

As in the smooth case, we shall use the theorem according to the following result.

4.6 Corollary: (Weak-$\mathscr{L}$ characterisations of boundedness, continuity, measurability, and integrability for the $\mathrm{CO}^{\mathrm{hol}}$-topology) *Let M be a Stein manifold, let $(\mathcal{X}, \mathscr{O})$ be a topological space, let $(\mathcal{T}, \mathscr{M})$ be a measurable space, and let $\mu \colon \mathscr{M} \to \overline{\mathbb{R}}_{\ge 0}$ be a finite measure. The following statements hold:*

(i) a subset $\mathcal{B} \subseteq \Gamma^{\mathrm{hol}}(\mathsf{TM})$ is bounded in the von Neumann bornology if and only if it is weak-$\mathscr{L}$ bounded in the von Neumann bornology;

(ii) a map $\Phi \colon \mathcal{X} \to \Gamma^{\mathrm{hol}}(\mathsf{TM})$ is continuous if and only if it is weak-$\mathscr{L}$ continuous;

(iii) a map $\Psi \colon \mathcal{T} \to \Gamma^{\mathrm{hol}}(\mathsf{TM})$ is measurable if and only if it is weak-$\mathscr{L}$ measurable;

(iv) a map $\Psi \colon \mathcal{T} \to \Gamma^{\mathrm{hol}}(\mathsf{TM})$ is Bochner integrable if and only if it is weak-$\mathscr{L}$ Bochner integrable.

Proof: As in the proof of Corollary 3.6, we need to show that $\{\mathscr{L}_f \mid f \in \mathrm{C}^{\mathrm{hol}}(\mathsf{M})\}$ has a countable point separating subset. The argument here follows that in the smooth case, except that here we have to use the properties of Stein manifolds, cf. the proof of the first part of Theorem 4.5 above, to assert the existence, for each $z \in \mathsf{M}$, of a neighbourhood on which there are globally defined holomorphic functions whose differentials span the cotangent space at each point. Since $\Gamma^{\mathrm{hol}}(\mathsf{TM})$ is complete, separable, and Suslin, and since $\mathrm{C}^{\mathrm{hol}}(\mathsf{M})$ is Suslin by properties $\mathrm{CO}^{\mathrm{hol}}$-2, $\mathrm{CO}^{\mathrm{hol}}$-4 and $\mathrm{CO}^{\mathrm{hol}}$-6 above, the corollary follows from Lemma 3.3. ■

5. The C^ω-topology for the space of real analytic vector fields

In this section we examine a topology on the set of real analytic vector fields. As we shall see, this requires some considerable effort. Agrachev and Gamkrelidze [1978] consider the real analytic case by considering bounded holomorphic extensions to neighbourhoods of $\mathbb{R}^n$ of fixed width in $\mathbb{C}^n$. Our approach is more general, more geometric, and global, using a natural real analytic topology described, for example, in the work of Martineau [1966]. This allows us to dramatically broaden the class of real analytic systems that we can handle to include "all" analytic systems.

The first observation we make is that $\Gamma^\omega(\mathsf{E})$ is not a closed subspace of $\Gamma^\infty(\mathsf{E})$ in the CO^∞-topology. To see this, consider the following. Take a smooth but not real analytic function on $\mathbb{S}^1$. The Fourier series of this function gives rise, by taking partial sums, to a sequence of real analytic functions. Standard harmonic analysis [Stein and Weiss 1971, Theorem VII.2.11(b)] shows that this sequence and all of its derivatives converge uniformly, and so in the CO^∞-topology, to the original function. Thus we have a Cauchy sequence in $\mathrm{C}^\omega(\mathbb{S}^1)$ that does not converge, with respect to the CO^∞-topology, in $\mathrm{C}^\omega(\mathbb{S}^1)$.

The second observation we make is that a plain restriction of the topology for holomorphic objects is not sufficient. The reason for this is that, upon complexification (a process we describe in detail below) there will not be a uniform neighbourhood to which all real analytic objects can be extended. Let us look at this for an example, where "object" is "function." For $r \in \mathbb{R}_{>0}$ we consider the real analytic function $f_r\colon \mathbb{R} \to \mathbb{R}$ defined by $f_r(x) = \frac{r^2}{r^2+x^2}$. We claim that there is no neighbourhood $\overline{\mathcal{U}}$ of $\mathbb{R}$ in $\mathbb{C}$ to which all of the functions f_r, $r \in \mathbb{R}_{>0}$, can be extended. Indeed, take some such neighbourhood $\overline{\mathcal{U}}$ and let $r \in \mathbb{R}_{>0}$ be sufficiently small that $\overline{\mathsf{D}}(r,0) \subseteq \overline{\mathcal{U}}$. To see that f_r cannot be extended to an holomorphic function $\overline{f}_r$ on $\overline{\mathcal{U}}$, let $\overline{f}_r$ be such an holomorphic extension. Then $\overline{f}_r(z)$ must be equal to $\frac{r^2}{r^2+z^2}$ for $z \in \mathsf{D}(r,0)$ by uniqueness of holomorphic extensions [Cieliebak and Eliashberg 2012, Lemma 5.40]. But this immediately prohibits $\overline{f}_r$ from being holomorphic on any neighbourhood of $\overline{\mathsf{D}}(r,0)$, giving our claim.

Therefore, to topologise the space of real analytic vector fields, we will need to do more than either (1) restrict the CO^∞-topology or (2) use the $\mathrm{CO}^{\mathrm{hol}}$-topology in an "obvious" way. Note that it is the "obvious" use of the $\mathrm{CO}^{\mathrm{hol}}$-topology for holomorphic objects that is employed by Agrachev and Gamkrelidze [1978] in their study of time-varying real analytic vector fields. Moreover, Agrachev and Gamkrelidze [1978] also restrict to *bounded* holomorphic extensions. What we propose is an improvement on this in that it works far more generally, and is also more natural to a geometric treatment of the real analytic setting. We comment at this point that we shall see in Theorems 6.25 and 7.14 below that the consideration of bounded holomorphic extensions to fixed neighbourhoods in the complexification is sometimes sufficient locally. But conclusions such as this become hard theorems with precise hypotheses in our approach, not starting points for the theory.

As in the smooth and holomorphic cases, we begin by considering a general vector bundle.

5.1. A natural direct limit topology. We let $\pi\colon \mathsf{E} \to \mathsf{M}$ be a real analytic vector bundle. We shall extend E to an holomorphic vector bundle that will serve an an important device for all of our constructions.

Complexifications. Let us take some time to explain how holomorphic extensions can be constructed. The following two paragraphs distill out important parts of about forty years of intensive development of complex analysis, culminating in the paper of Grauert [1958].

For simplicity, let us assume that M is connected and so has pure dimension, and so the fibres of E also have a fixed dimension. As in Section 2.3, we suppose that we have a real analytic affine connection ∇ on M, a real analytic vector bundle connection ∇^0 on E, a real analytic Riemannian metric $\mathbb{G}$ on M, and a real analytic fibre metric $\mathbb{G}_0$ on E. We also assume the data required to make the diagram (2.7) giving $\pi\colon \mathsf{E} \to \mathsf{M}$ as the image of a real analytic vector bundle monomorphism in the trivial vector bundle $\mathbb{R}^N \times \mathbb{R}^N$ for some suitable $N \in \mathbb{Z}_{>0}$.

Now we complexify. Recall that, if V is a $\mathbb{C}$-vector space, then multiplication by $\sqrt{-1}$ induces a $\mathbb{R}$-linear map $J \in \mathrm{End}_{\mathbb{R}}(\mathsf{V})$. A $\mathbb{R}$-subspace U of V is ***totally real*** if $\mathsf{U} \cap J(\mathsf{U}) = \{0\}$. A submanifold of an holomorphic manifold, thinking of the latter as a smooth manifold, is ***totally real*** if its tangent spaces are totally real subspaces. By [Whitney and Bruhat 1959, Proposition 1], for a real analytic manifold M there exists a complexification $\overline{\mathsf{M}}$ of M, i.e., an holomorphic manifold having M as a totally real submanifold and where $\overline{\mathsf{M}}$ has the same $\mathbb{C}$-dimension as the $\mathbb{R}$-dimension of M. As shown by Grauert [1958, §3.4], for any neighbourhood $\overline{\mathcal{U}}$ of M in $\overline{\mathsf{M}}$, there exists a Stein neighbourhood $\overline{\mathcal{S}}$ of M contained in $\overline{\mathcal{U}}$. By arguments involving extending convergent real power series to convergent complex power series (the conditions on coefficients for convergence are the same for both real and complex power series), one can show that there is an holomorphic extension of ι_{M} to $\iota_{\overline{\mathsf{M}}}\colon \overline{\mathsf{M}} \to \mathbb{C}^N$, possibly after shrinking $\overline{\mathsf{M}}$ [Cieliebak and Eliashberg 2012, Lemma 5.40]. By applying similar reasoning to the transition maps for the real analytic vector bundle E, one obtains an holomorphic vector bundle $\overline{\pi}\colon \overline{\mathsf{E}} \to \overline{\mathsf{M}}$ for which the diagram

$$\begin{tikzcd}
\overline{\mathsf{E}} \arrow[rrr] \arrow[ddd, "\overline{\pi}"'] & & & \mathbb{C}^N \times \mathbb{C}^N \arrow[ddd, "\mathrm{pr}_2"] \\
& \mathsf{E} \arrow[ul] \arrow[r, "\hat{\iota}_{\mathsf{E}}"] \arrow[d, "\pi"'] & \mathbb{R}^N \times \mathbb{R}^N \arrow[ur] \arrow[d, "\mathrm{pr}_2"] & \\
& \mathsf{M} \arrow[r, "\iota_{\mathsf{M}}"'] \arrow[dl] & \mathbb{R}^N \arrow[dr] & \\
\overline{\mathsf{M}} \arrow[rrr, "\iota_{\overline{\mathsf{M}}}"'] & & & \mathbb{C}^N
\end{tikzcd}$$

commutes, where all diagonal arrows are complexification and where the inner diagram is as defined in the proof of Lemma 2.3. One can then define an Hermitian fibre metric $\overline{\mathbb{G}}_0$ on $\overline{\mathsf{E}}$ induced from the standard Hermitian metric on the fibres of the vector bundle $\mathbb{C}^N \times \mathbb{C}^N$ and an Hermitian metric $\overline{\mathbb{G}}$ on $\overline{\mathsf{M}}$ induced from the standard Hermitian metric on $\mathbb{C}^N$.

In the remainder of this section, we assume that the preceding constructions have been done and fixed once and for all.

Germs of holomorphic sections over subsets of a real analytic manifold. In two different places, we will need to consider germs of holomorphic sections. In this section we organise the methodology for doing this to unify the notation.

Let $A \subseteq \mathsf{M}$ and let $\mathscr{N}_A$ be the set of neighbourhoods of A in the complexification $\overline{\mathsf{M}}$. For $\overline{\mathcal{U}}, \overline{\mathcal{V}} \in \mathscr{N}_A$, and for $\overline{\xi} \in \Gamma^{\mathrm{hol}}(\overline{\mathsf{E}}|\overline{\mathcal{U}})$ and $\overline{\eta} \in \Gamma^{\mathrm{hol}}(\overline{\mathsf{E}}|\overline{\mathcal{V}})$, we say that $\overline{\xi}$ is ***equivalent*** to $\overline{\eta}$ if there exist $\overline{\mathcal{W}} \in \mathscr{N}_A$ and $\overline{\zeta} \in \Gamma^{\mathrm{hol}}(\overline{\mathsf{E}}|\overline{\mathcal{W}})$ such that $\overline{\mathcal{W}} \subseteq \overline{\mathcal{U}} \cap \overline{\mathcal{V}}$ and such that

$$\overline{\xi}|\overline{\mathcal{W}} = \overline{\eta}|\overline{\mathcal{W}} = \overline{\zeta}.$$

By $\mathscr{G}^{\mathrm{hol}}_{A,\overline{\mathsf{E}}}$ we denote the set of equivalence classes, which we call the set of germs of sections of $\overline{\mathsf{E}}$ over A. By $[\overline{\xi}]_A$ we denote the equivalence class of $\overline{\xi} \in \Gamma^{\mathrm{hol}}(\overline{\mathsf{E}}|\overline{\mathcal{U}})$ for some $\overline{\mathcal{U}} \in \mathscr{N}_A$.

Now, for $x \in \mathsf{M}$, E_x is a totally real subspace of $\overline{\mathsf{E}}_x$ with half the real dimension, and so it follows that

$$\overline{\mathsf{E}}_x = \mathsf{E}_x \oplus J(\mathsf{E}_x),$$

where J is the complex structure on the fibres of $\overline{\mathsf{E}}$. For $\overline{\mathcal{U}} \in \mathscr{N}_A$, denote by $\Gamma^{\mathrm{hol},\mathbb{R}}(\overline{\mathsf{E}}|\overline{\mathcal{U}})$ those holomorphic sections $\overline{\xi}$ of $\overline{\mathsf{E}}|\overline{\mathcal{U}}$ such that $\overline{\xi}(x) \in \mathsf{E}_x$ for $x \in \overline{\mathcal{U}} \cap \mathsf{M}$. We think of this as being a locally convex topological $\mathbb{R}$-vector space with the seminorms $p^{\mathrm{hol}}_{\overline{K}}$, $\overline{K} \subseteq \overline{\mathcal{U}}$ compact, defined by

$$p^{\mathrm{hol}}_{\overline{K}}(\overline{\xi}) = \sup\{\|\overline{\xi}(\overline{x})\|_{\overline{\mathbb{G}}_0} \mid \overline{x} \in \overline{K}\},$$

i.e., we use the locally convex structure induced from the usual $\mathrm{CO}^{\mathrm{hol}}$-topology on $\Gamma^{\mathrm{hol}}(\overline{\mathsf{E}}|\overline{\mathcal{U}})$.

5.1 Remark: (Closedness of "real" sections) We note that $\Gamma^{\mathrm{hol},\mathbb{R}}(\overline{\mathsf{E}}|\overline{\mathcal{U}})$ is a closed $\mathbb{R}$-subspace of $\Gamma^{\mathrm{hol}}(\overline{\mathsf{E}})$ in the $\mathrm{CO}^{\mathrm{hol}}$-topology, i.e., the restriction of requiring "realness" on M is a closed condition. This is easily shown, and we often assume it often without mention. •

Denote by $\mathscr{G}^{\mathrm{hol},\mathbb{R}}_{A,\overline{\mathsf{E}}}$ the set of germs of sections from $\Gamma^{\mathrm{hol},\mathbb{R}}(\overline{\mathsf{E}}|\overline{\mathcal{U}})$, $\overline{\mathcal{U}} \in \mathscr{N}_A$. If $\overline{\mathcal{U}}_1, \overline{\mathcal{U}}_2 \in \mathscr{N}_A$ satisfy $\overline{\mathcal{U}}_1 \subseteq \overline{\mathcal{U}}_2$, then we have the restriction mapping

$$\begin{aligned} r_{\overline{\mathcal{U}}_2,\overline{\mathcal{U}}_1}\colon \Gamma^{\mathrm{hol},\mathbb{R}}(\overline{\mathsf{E}}|\overline{\mathcal{U}}_2) &\to \Gamma^{\mathrm{hol},\mathbb{R}}(\overline{\mathsf{E}}|\overline{\mathcal{U}}_1) \\ \overline{\xi} &\mapsto \overline{\xi}|\overline{\mathcal{U}}_1. \end{aligned}$$

This restriction is continuous since, for any compact set $\overline{K} \subseteq \overline{\mathcal{U}}_1 \subseteq \overline{\mathcal{U}}_2$ and any $\overline{\xi} \in \Gamma^{\mathrm{hol},\mathbb{R}}(\overline{\mathsf{E}}|\overline{\mathcal{U}}_2)$, we have $p^{\mathrm{hol}}_{\overline{K}}(r_{\overline{\mathcal{U}}_2,\overline{\mathcal{U}}_1}(\overline{\xi})) \le p^{\mathrm{hol}}_{\overline{K}}(\overline{\xi})$ (in fact we have equality, but the inequality emphasises what is required for our assertion to be true [Schaefer and Wolff 1999, §III.1.1]). We also have maps

$$\begin{aligned} r_{\overline{\mathcal{U}},A}\colon \Gamma^{\mathrm{hol},\mathbb{R}}(\overline{\mathsf{E}}|\overline{\mathcal{U}}) &\to \mathscr{G}^{\mathrm{hol},\mathbb{R}}_{A,\overline{\mathsf{E}}} \\ \overline{\xi} &\mapsto [\overline{\xi}]_A. \end{aligned}$$

Note that $\mathscr{N}_A$ is a directed set by inclusion; that is, $\overline{\mathcal{U}}_2 \preceq \overline{\mathcal{U}}_1$ if $\overline{\mathcal{U}}_1 \subseteq \overline{\mathcal{U}}_2$. Thus we have the directed system $(\Gamma^{\mathrm{hol},\mathbb{R}}(\mathsf{T}\overline{\mathcal{U}}))_{\overline{\mathcal{U}} \in \mathscr{N}_A}$, along with the mappings $r_{\overline{\mathcal{U}}_2,\overline{\mathcal{U}}_1}$, in the category of locally convex topological $\mathbb{R}$-vector spaces. The usual notion of direct limit in the category of $\mathbb{R}$-vector spaces gives $\mathscr{G}^{\mathrm{hol},\mathbb{R}}_{A,\overline{\mathsf{E}}}$, along with the linear mappings $r_{\overline{\mathcal{U}},A}$, $\overline{\mathcal{U}} \in \mathscr{N}_A$, as the direct limit of this directed system [cf. Lang 2005, Theorem III.10.1]. This vector space then has the finest locally convex topology making the maps $r_{\overline{\mathcal{U}},A}$, $\overline{\mathcal{U}} \in \mathscr{N}_A$, continuous, i.e., the direct limit in the category of locally convex topological vector spaces. We refer to this as the ***direct limit topology*** for $\mathscr{G}^{\mathrm{hol},\mathbb{R}}_{A,\overline{\mathsf{E}}}$.

The direct limit topology. We shall describe four topologies (or more, depending on which descriptions you regard as being distinct) for the space of real analytic sections of a real analytic vector bundle. The first is quite direct, involving an application of the construction above to the case of $A = \mathsf{M}$. In this case, the following lemma is key to our constructions.

5.2 Lemma: (Real analytic sections as holomorphic germs) *There is a natural* $\mathbb{R}$-*vector space isomorphism between* $\Gamma^\omega(\mathsf{E})$ *and* $\mathscr{G}^{\mathrm{hol},\mathbb{R}}_{\mathsf{M},\overline{\mathsf{E}}}$.

Proof: Let $\xi \in \Gamma^\omega(\mathsf{E})$. As in [Cieliebak and Eliashberg 2012, Lemma 5.40], there is an extension of ξ to a section $\overline{\xi} \in \Gamma^{\mathrm{hol},\mathbb{R}}(\overline{\mathsf{E}}|\overline{\mathcal{U}})$ for some $\overline{\mathcal{U}} \in \mathscr{N}_{\mathsf{M}}$. We claim that the map $i_{\mathsf{M}}\colon \Gamma^\omega(\mathsf{E}) \to \mathscr{G}^{\mathrm{hol},\mathbb{R}}_{\mathsf{M},\overline{\mathsf{E}}}$ defined by $i_{\mathsf{M}}(\xi) = [\overline{\xi}]_{\mathsf{M}}$ is the desired isomorphism. That i_{M} is independent of the choice of extension $\overline{\xi}$ is a consequence of the fact that the extension to $\overline{\xi}$ is unique inasmuch as any two such extensions agree on some neighbourhood contained in their intersection; this is the uniqueness assertion of [Cieliebak and Eliashberg 2012, Lemma 5.40]. This fact also ensures that i_{M} is injective. For surjectivity, let $[\overline{\xi}]_{\mathsf{M}} \in \mathscr{G}^{\mathrm{hol},\mathbb{R}}_{\mathsf{M},\overline{\mathsf{E}}}$ and let us define $\xi\colon \mathsf{M} \to \mathsf{E}$ by $\xi(x) = \overline{\xi}(x)$ for $x \in \mathsf{M}$. Note that the restriction of $\overline{\xi}$ to M is real analytic because the values of $\overline{\xi}|\mathsf{M}$ at points in a neighbourhood of $x \in \mathsf{M}$ are given by the restriction of the (necessarily convergent) $\mathbb{C}$-Taylor series of $\overline{\xi}$ to M. Obviously, $i_{\mathsf{M}}(\xi) = [\overline{\xi}]_{\mathsf{M}}$. ■

Now we use the direct limit topology on $\mathscr{G}^{\mathrm{hol},\mathbb{R}}_{\mathsf{M},\overline{\mathsf{E}}}$ described above, along with the preceding lemma, to immediately give a locally convex topology for $\Gamma^\omega(\mathsf{E})$ that we refer to as the ***direct C$^\omega$-topology***.

Let us make an important observation about the direct C$^\omega$-topology. Let us denote by $\mathscr{S}_{\mathsf{M}}$ the set of all Stein neighbourhoods of M in $\overline{\mathsf{M}}$. As shown by Grauert [1958, §3.4], if $\overline{\mathcal{U}} \in \mathscr{N}_{\mathsf{M}}$ then there exists $\overline{\mathcal{S}} \in \mathscr{S}_{\mathsf{M}}$ with $\overline{\mathcal{S}} \subseteq \overline{\mathcal{U}}$. Therefore, $\mathscr{S}_{\mathsf{M}}$ is cofinal in $\mathscr{N}_{\mathsf{M}}$ and so the directed systems $(\Gamma^{\mathrm{hol}}(\mathsf{E}|\overline{\mathcal{U}}))_{\overline{\mathcal{U}}\in\mathscr{N}_{\mathsf{M}}}$ and $(\Gamma^{\mathrm{hol}}(\mathsf{E}|\overline{\mathcal{S}}))_{\overline{\mathcal{S}}\in\mathscr{S}_{\mathsf{M}}}$ induce the same final topology on $\Gamma^\omega(\mathsf{E})$ [Grothendieck 1973, page 137].

5.2. Topologies for germs of holomorphic functions about compact sets.

In the preceding section, we gave a more or less direct description of a topology for the space of real analytic sections. This description has a benefit of being the one that one might naturally arrive at after some thought. However, there is not a lot that one can do with this description of the topology. In this section we develop the means by which one can consider alternative descriptions of this topology that, for example, lead to explicit seminorms for the topology on the space of real analytic sections. These seminorms will be an essential part of our developing a useful theory for time-varying real analytic vector fields and real analytic control systems.

The direct limit topology for the space of germs about a compact set. We continue with the notation from Section 5.1. For $K \subseteq \mathsf{M}$ compact, we have the direct limit topology, described above for general subsets $A \subseteq \mathsf{M}$, on $\mathscr{G}^{\mathrm{hol},\mathbb{R}}_{K,\overline{\mathsf{E}}}$. We seem to have gained nothing, since we have yet another direct limit topology. However, the direct limit can be shown to be of a friendly sort as follows. Unlike the general situation, since K is compact there is a *countable* family $(\overline{\mathcal{U}}_{K,j})_{j\in\mathbb{Z}_{>0}}$ from $\mathscr{N}_K$ with the property that $\mathrm{cl}(\overline{\mathcal{U}}_{K,j+1}) \subseteq \overline{\mathcal{U}}_{K,j}$ and $K = \cap_{j\in\mathbb{Z}_{>0}}\overline{\mathcal{U}}_{K,j}$. Moreover, the sequence $(\overline{\mathcal{U}}_{K,j})_{j\in\mathbb{Z}_{>0}}$ is cofinal in $\mathscr{N}_K$, i.e., if $\overline{\mathcal{U}} \in \mathscr{N}_K$, then there exists $j \in \mathbb{Z}_{>0}$ with $\overline{\mathcal{U}}_{K,j} \subseteq \overline{\mathcal{U}}$. Let us fix such a family of neighbourhoods. Let us fix $j \in \mathbb{Z}_{>0}$ for a moment. Let $\Gamma^{\mathrm{hol},\mathbb{R}}_{\mathrm{bdd}}(\overline{\mathsf{E}}|\overline{\mathcal{U}}_{K,j})$ be the set of bounded sections from $\Gamma^{\mathrm{hol},\mathbb{R}}(\overline{\mathsf{E}}|\overline{\mathcal{U}}_{K,j})$, boundedness being taken relative to the Hermitian fibre metric $\overline{\mathbb{G}}_0$. As we have seen in Lemma 4.1, if we define a norm on $\Gamma^{\mathrm{hol},\mathbb{R}}_{\mathrm{bdd}}(\overline{\mathsf{E}}|\overline{\mathcal{U}}_{K,j})$ by

$$p^{\mathrm{hol}}_{\overline{\mathcal{U}}_{K,j},\infty}(\overline{\xi}) = \sup\{\|\overline{\xi}(\overline{x})\|_{\overline{\mathbb{G}}_0} \mid \overline{x} \in \overline{\mathcal{U}}_{K,j}\},$$

then this makes $\Gamma^{\mathrm{hol},\mathbb{R}}_{\mathrm{bdd}}(\overline{\mathcal{U}}_{K,j})$ into a Banach space, a closed subspace of the Banach space of bounded continuous sections of $\overline{\mathsf{E}}|\overline{\mathcal{U}}_{K,j}$. Now, no longer fixing j, we have a sequence of inclusions

$$\begin{aligned}\Gamma^{\mathrm{hol},\mathbb{R}}_{\mathrm{bdd}}(\overline{\mathsf{E}}|\overline{\mathcal{U}}_{K,1}) \subseteq \Gamma^{\mathrm{hol},\mathbb{R}}(\overline{\mathsf{E}}|\overline{\mathcal{U}}_{K,1}) \subseteq \Gamma^{\mathrm{hol},\mathbb{R}}_{\mathrm{bdd}}(\overline{\mathsf{E}}|\overline{\mathcal{U}}_{K,2}) \subseteq& \\ \cdots \subseteq \Gamma^{\mathrm{hol},\mathbb{R}}(\overline{\mathsf{E}}|\overline{\mathcal{U}}_{K,j}) \subseteq \Gamma^{\mathrm{hol},\mathbb{R}}_{\mathrm{bdd}}(\overline{\mathsf{E}}|\overline{\mathcal{U}}_{K,j+1}) \subseteq \cdots .&\end{aligned}$$

The inclusion $\Gamma^{\mathrm{hol},\mathbb{R}}(\overline{\mathcal{U}}_{K,j}) \subseteq \Gamma^{\mathrm{hol},\mathbb{R}}_{\mathrm{bdd}}(\overline{\mathcal{U}}_{K,j+1})$, $j \in \mathbb{Z}_{>0}$, is by restriction from $\overline{\mathcal{U}}_{K,j}$ to the smaller $\overline{\mathcal{U}}_{K,j+1}$, keeping in mind that $\mathrm{cl}(\overline{\mathcal{U}}_{K,j+1}) \subseteq \overline{\mathcal{U}}_{K,j}$. By Lemma 4.1, all inclusions are continuous. For $j \in \mathbb{Z}_{>0}$ define

$$\begin{aligned} r_{K,j} \colon \Gamma^{\mathrm{hol},\mathbb{R}}_{\mathrm{bdd}}(\overline{\mathsf{E}}|\overline{\mathcal{U}}_{K,j}) &\to \mathscr{G}^{\mathrm{hol},\mathbb{R}}_{K,\overline{\mathsf{E}}} \\ \overline{\xi} &\mapsto [\overline{\xi}]_K. \end{aligned} \tag{5.1}$$

Now one can show that the direct limit topologies induced on $\mathscr{G}^{\mathrm{hol},\mathbb{R}}_{K,\overline{\mathsf{E}}}$ by the directed system $(\Gamma^{\mathrm{hol},\mathbb{R}}(\overline{\mathsf{E}}|\overline{\mathcal{U}}))_{\overline{\mathcal{U}}\in\mathscr{N}_K}$ of Fréchet spaces and by the directed system $(\Gamma^{\mathrm{hol},\mathbb{R}}_{\mathrm{bdd}}(\overline{\mathsf{E}}|\overline{\mathcal{U}}_{K,j}))_{j\in\mathbb{Z}_{>0}}$ of Banach spaces agree [Kriegl and Michor 1997, Theorem 8.4]. We refer to [Bierstedt 1988], starting on page 63, for a fairly comprehensive discussion of the topology we have just described in the context of germs of holomorphic functions about a compact subset $K \subseteq \mathbb{C}^n$.

A weighted direct limit topology for sections of bundles of infinite jets. Here we provide a direct limit topology for a subspace of the space of continuous sections of the infinite jet bundle of a vector bundle. Below we shall connect this direct limit topology to the direct limit topology described above for germs of holomorphic sections about a compact set. The topology we give here has the advantage of providing explicit seminorms for the topology of germs, and subsequently for the space of real analytic sections.

For this description, we work with infinite jets, so let us introduce the notation we will use for this, referring to [Saunders 1989, Chapter 7] for details. Let us denote by $\mathsf{J}^\infty\mathsf{E}$ the bundle of infinite jets of a vector bundle $\pi \colon \mathsf{E} \to \mathsf{M}$, this being the inverse limit (in the category of sets, for the moment) of the inverse system $(\mathsf{J}^m\mathsf{E})_{m\in\mathbb{Z}_{\geq 0}}$ with mappings π^{m+1}_m, $m \in \mathbb{Z}_{\geq 0}$. Precisely,

$$\mathsf{J}^\infty\mathsf{E} = \Big\{\phi \in \prod_{m\in\mathbb{Z}_{\geq 0}} \mathsf{J}^m\mathsf{E} \;\Big|\; \pi^k_l \circ \phi(k) = \phi(l),\ k,l \in \mathbb{Z}_{\geq 0},\ k \geq l\Big\}.$$

We let $\pi^\infty_m \colon \mathsf{J}^\infty\mathsf{E} \to \mathsf{J}^m\mathsf{E}$ be the projection defined by $\pi^\infty_m(\phi) = \phi(m)$. For $\xi \in \Gamma^\infty(\mathsf{E})$ we let $j_\infty\xi \colon \mathsf{M} \to \mathsf{J}^\infty\mathsf{E}$ be defined by $\pi^\infty_m \circ j_\infty\xi(x) = j_m\xi(x)$. By a theorem of Borel [1895], if $\phi \in \mathsf{J}^\infty\mathsf{E}$, there exist $\xi \in \Gamma^\infty(\mathsf{E})$ and $x \in \mathsf{M}$ such that $j_\infty\xi(x) = \phi$. We can define sections of $\mathsf{J}^\infty\mathsf{E}$ in the usual manner: a section is a map $\Xi\colon \mathsf{M}\to\mathsf{J}^\infty\mathsf{E}$ satisfying $\pi_0^\infty\circ\Xi(x)=x$ for every $x\in\mathsf{M}$. We shall equip $\mathsf{J}^\infty\mathsf{E}$ with the initial topology so that a section Ξ is continuous if and only if $\pi_m^\infty\circ\Xi$ is continuous for every $m\in\mathbb{Z}_{\geq 0}$. We denote the space of continuous sections of $\mathsf{J}^\infty\mathsf{E}$ by $\Gamma^0(\mathsf{J}^\infty\mathsf{E})$. Since we are only dealing with continuous sections, we can talk about sections defined on any subset $A\subseteq\mathsf{M}$, using the relative topology on A. The continuous sections defined on $A\subseteq\mathsf{M}$ will be denoted by $\Gamma^0(\mathsf{J}^\infty\mathsf{E}|A)$.

Now let $K\subseteq\mathsf{M}$ be compact and, for $j\in\mathbb{Z}_{>0}$, denote

$$\mathscr{E}_j(K)=\{\Xi\in\Gamma^0(\mathsf{J}^\infty\mathsf{E}|K)\mid\ \sup\{j^{-m}\|\pi_m^\infty\circ\Xi(x)\|_{\overline{\mathbb{G}}_m}\mid\ m\in\mathbb{Z}_{\geq 0},\ x\in K\}<\infty\},$$

and on $\mathscr{E}_j(K)$ we define a norm $p_{K,j}$ by

$$p_{K,j}(\Xi)=\sup\{j^{-m}\|\pi_m^\infty\circ\Xi(x)\|_{\overline{\mathbb{G}}_m}\mid\ m\in\mathbb{Z}_{\geq 0},\ x\in K\}.$$

One readily verifies that, for each $j\in\mathbb{Z}_{>0}$, $(\mathscr{E}_j(K),p_{K,j})$ is a Banach space. Note that $\mathscr{E}_j(K)\subseteq\mathscr{E}_{j+1}(K)$ and that $p_{K,j+1}(\Xi)\leq p_{K,j}(\Xi)$ for $\Xi\in\mathscr{E}_j(K)$, and so the inclusion of $\mathscr{E}_j(K)$ in $\mathscr{E}_{j+1}(K)$ is continuous. We let $\mathscr{E}(K)$ be the direct limit of the directed system $(\mathscr{E}_j(K))_{j\in\mathbb{Z}_{>0}}$.

We shall subsequently explore more closely the relationship between the direct limit topology for $\mathscr{E}(K)$ and the topology for $\mathscr{G}^{\mathrm{hol},\mathbb{R}}_{K,\overline{\mathsf{E}}}$. For now, we merely observe that the direct limit topology for $\mathscr{E}(K)$ admits a characterisation by seminorms. To state the result, let us denote by $\mathrm{c}_{\downarrow 0}(\mathbb{Z}_{\geq 0};\mathbb{R}_{>0})$ the set of nonincreasing sequences $(a_m)_{m\in\mathbb{Z}_{\geq 0}}$ in $\mathbb{R}_{>0}$ that converge to 0. Let us abbreviate such a sequence by $\boldsymbol{a}=(a_m)_{m\in\mathbb{Z}_{\geq 0}}$. The following result is modelled after [Vogt 2013, Lemma 1].

5.3 Lemma: (Seminorms for $\mathscr{E}(K)$) *The direct limit topology for $\mathscr{E}(K)$ is defined by the seminorms*

$$p_{K,\boldsymbol{a}}=\sup\{a_0a_1\cdots a_m\|\pi_m^\infty\circ\Xi(x)\|_{\overline{\mathbb{G}}_m}\mid\ m\in\mathbb{Z}_{\geq 0},\ x\in K\},$$

for $\boldsymbol{a}\in\mathrm{c}_{\downarrow 0}(\mathbb{Z}_{\geq 0};\mathbb{R}_{>0})$.

Proof: First we show that the seminorms $p_{K,\boldsymbol{a}}$, $\boldsymbol{a}\in\mathrm{c}_{\downarrow 0}(\mathbb{Z}_{\geq 0};\mathbb{R}_{>0})$, are continuous on $\mathscr{E}(K)$. It suffices to show that $p_{K,\boldsymbol{a}}|\mathscr{E}_j(K)$ is continuous for each $j\in\mathbb{Z}_{>0}$ [Conway 1990, Proposition IV.5.7]. Thus, since $\mathscr{E}_j(K)$ is a Banach space, it suffices to show that, if $(\Xi_k)_{k\in\mathbb{Z}_{>0}}$ is a sequence in $\mathscr{E}_j(K)$ converging to zero, then $\lim_{k\to\infty}p_{K,\boldsymbol{a}}(\Xi_k)=0$. Let $N\in\mathbb{Z}_{\geq 0}$ be such that $a_N<\frac{1}{j}$. Let $C\geq 1$ be such that

$$a_0a_1\cdots a_m\leq Cj^{-m},\qquad m\in\{0,1,\dots,N\},$$

this being possible since there are only finitely many inequalities to satisfy. Therefore, for any $m\in\mathbb{Z}_{\geq 0}$, we have $a_0a_1\cdots a_m\leq Cj^{-m}$. Then, for any $\Xi\in\Gamma^0(\mathsf{J}^\infty\mathsf{E}|K)$,

$$a_0a_1\cdots a_m\|\pi_m^\infty\circ\Xi(x)\|_{\overline{\mathbb{G}}_m}\leq Cj^{-m}\|\pi_m^\infty\circ\Xi(x)\|_{\overline{\mathbb{G}}_m}$$

for every $x\in K$ and $m\in\mathbb{Z}_{\geq 0}$. From this we immediately have $\lim_{k\to\infty}p_{K,\boldsymbol{a}}(\Xi_k)=0$, as desired. This shows that the direct limit topology on $\mathscr{E}(K)$ is stronger than the topology defined by the family of seminorms $p_{K,\boldsymbol{a}}$, $\boldsymbol{a}\in\mathrm{c}_{\downarrow 0}(\mathbb{Z}_{\geq 0};\mathbb{R}_{>0})$.

For the converse, we show that every neighbourhood of $0 \in \mathscr{E}(K)$ in the direct limit topology contains a neighbourhood of zero in the topology defined by the seminorms $p_{K,\boldsymbol{a}}$, $\boldsymbol{a} \in \mathrm{c}_{\downarrow 0}(\mathbb{Z}_{\geq 0}; \mathbb{R}_{>0})$. Let $\mathcal{B}_j$ denote the unit ball in $\mathscr{E}_j(K)$. A neighbourhood of 0 in the direct limit topology contains a union of balls $\epsilon_j \mathcal{B}_j$ for some $\epsilon_j \in \mathbb{R}_{>0}$, $j \in \mathbb{Z}_{>0}$, (see [Schaefer and Wolff 1999, page 54]) and we can assume, without loss of generality, that $\epsilon_j \in (0,1)$ for each $j \in \mathbb{Z}_{>0}$. We define an increasing sequence $(m_j)_{j \in \mathbb{Z}_{>0}}$ in $\mathbb{Z}_{\geq 0}$ as follows. Let $m_1 = 0$. Having defined $m_1, \dots, m_j$, define $m_{j+1} > m_j$ by requiring that $j < \epsilon_{j+1}^{1/m_{j+1}}(j+1)$. For $m \in \{m_j, \dots, m_{j+1} - 1\}$, define $a_m \in \mathbb{R}_{>0}$ by $a_m^{-1} = \epsilon_j^{1/m_j} j$. Note that, for $m \in \{m_j, \dots, m_{j+1} - 1\}$, we have

$$a_m^{-m} = \epsilon_j^{m/m_j} j^m \leq \epsilon_j j^m.$$

Note that $\lim_{m \to \infty} a_m = 0$. If $\Xi \in \Gamma^0(\mathsf{J}^\infty \mathsf{E}|K)$ satisfies $p_{K,\boldsymbol{a}}(\Xi) \leq 1$ then, for $m \in \{m_j, \dots, m_{j+1} - 1\}$, we have

$$\begin{aligned} j^{-m} \|\pi_m^\infty \circ \Xi(x)\|_{\overline{\mathbb{G}}_m} &\leq a_m^m \epsilon_j \|\pi_m^\infty \circ \Xi(x)\|_{\overline{\mathbb{G}}_m} \\ &\leq a_0 a_1 \cdots a_m \epsilon_j \|\pi_m^\infty \circ \Xi(x)\|_{\overline{\mathbb{G}}_m} \leq \epsilon_j \end{aligned}$$

for $x \in K$. Thus, if $\Xi \in \Gamma^0(\mathsf{J}^\infty \mathsf{E}|K)$ satisfies $p_{K,\boldsymbol{a}}(\Xi) \leq 1$ then, for $m \in \{m_j, \dots, m_{j+1} - 1\}$, we have $\pi_m^\infty \circ \Xi \in \epsilon_j \mathcal{B}_j$. Therefore, $\Xi \in \cup_{j \in \mathbb{Z}_{>0}} \epsilon_j \mathcal{B}_j$, and this shows that, for $\boldsymbol{a}$ as constructed above,

$$\{\Xi \in \Gamma^0(\mathsf{J}^\infty \mathsf{E}|K) \mid p_{K,\boldsymbol{a}}(\Xi) \leq 1\} \subseteq \cup_{j \in \mathbb{Z}_{>0}} \epsilon_j \mathcal{B}_j,$$

giving the desired conclusion. ■

The following attribute of the direct limit topology for $\mathscr{E}(K)$ will also be useful.

5.4 Lemma: ($\mathscr{E}(K)$ is a regular direct limit) *The direct limit topology for $\mathscr{E}(K)$ is regular, i.e., if $\mathcal{B} \subseteq \mathscr{E}(K)$ is von Neumann bounded, then there exists $j \in \mathbb{Z}_{>0}$ such that $\mathcal{B}$ is contained in and von Neumann bounded in $\mathscr{E}_j(K)$.*

Proof: Let $\mathcal{B}_j \subseteq \mathscr{E}_j(K)$, $j \in \mathbb{Z}_{>0}$, be the closed unit ball with respect to the norm topology. We claim that $\mathcal{B}_j$ is closed in the direct limit topology of $\mathscr{E}(K)$. To prove this, we shall prove that $\mathcal{B}_j$ is closed in a topology that is weaker than the direct limit topology.

The weaker topology we use is the topology induced by the topology of pointwise convergence in $\Gamma^0(\mathsf{J}^\infty \mathsf{E}|K)$. To be precise, let $\mathscr{E}'_j(K)$ be the vector space $\mathscr{E}_j(K)$ with the topology defined by the seminorms

$$p_{x,j}(\Xi) = \sup\{j^{-m} \|\pi_m^\infty \circ \Xi(x)\|_{\overline{\mathbb{G}}_m} \mid m \in \mathbb{Z}_{\geq 0}\}, \qquad x \in K.$$

Clearly the identity map from $\mathscr{E}_j(K)$ to $\mathscr{E}'_j(K)$ is continuous, and so the topology of $\mathscr{E}'_j(K)$ is weaker than the usual topology of $\mathscr{E}(K)$. Now let $\mathscr{E}'(K)$ be the direct limit of the directed system $(\mathscr{E}'_j(K))_{j \in \mathbb{Z}_{>0}}$. Note that, algebraically, $\mathscr{E}'(K) = \mathscr{E}(K)$, but the spaces have different topologies, the topology for $\mathscr{E}'(K)$ being weaker than that for $\mathscr{E}(K)$.

We will show that $\mathcal{B}_j$ is closed in $\mathscr{E}'(K)$. Let $(I, \preceq)$ be a directed set and let $(\Xi_i)_{i \in I}$ be a convergent net in $\mathcal{B}_j$ in the topology of $\mathscr{E}'(K)$. Thus we have a map $\Xi\colon K \to \mathsf{J}^\infty \mathsf{E}|K$ such that, for each $x \in K$, $\lim_{i \in I} \Xi_i(x) = \Xi(x)$. If $\Xi \notin \mathcal{B}_j$ then there exists $x \in K$ such that

$$\sup\{j^{-m} \|\pi_m^\infty \circ \Xi(x)\|_{\overline{\mathbb{G}}_m} \mid m \in \mathbb{Z}_{\geq 0}\} > 1.$$

Let $\epsilon \in \mathbb{R}_{>0}$ be such that

$$\sup\{j^{-m}\|\pi^\infty_m \circ \Xi(x)\|_{\overline{\mathbb{G}}_m} \mid m \in \mathbb{Z}_{\geq 0}\} > 1 + \epsilon$$

and let $i_0 \in I$ be such that

$$\sup\{j^{-m}\|\pi^\infty \circ \Xi_i(x) - \pi^\infty_m \circ \Xi(x)\|_{\overline{\mathbb{G}}_m} \mid m \in \mathbb{Z}_{\geq 0}\} < \epsilon$$

for $i_0 \preceq i$, this by pointwise convergence. We thus have, for all $i_0 \preceq i$,

$$\begin{aligned}\epsilon <\ & \sup\{j^{-m}\|\pi^\infty_m \circ \Xi(x)\|_{\overline{\mathbb{G}}_m} \mid m \in \mathbb{Z}_{\geq 0}\} - \sup\{j^{-m}\|\pi^\infty_m \circ \Xi_i(x)\|_{\overline{\mathbb{G}}_m} \mid m \in \mathbb{Z}_{\geq 0}\} \\ \leq\ & \sup\{j^{-m}\|\pi^\infty_m \circ \Xi_i(x) - \pi^\infty_m \circ \Xi(x)\|_{\overline{\mathbb{G}}_m} \mid m \in \mathbb{Z}_{\geq 0}\} < \epsilon,\end{aligned}$$

which contradiction gives the conclusion that $\Xi \in \mathcal{B}_j$.

Since $\mathcal{B}_j$ has been shown to be closed in $\mathscr{E}(K)$, the lemma now follows from [Bierstedt 1988, Corollary 7]. ■

Seminorms for the topology of spaces of holomorphic germs. Let us define seminorms $p^\omega_{K,\boldsymbol{a}}$, $K \subseteq \mathsf{M}$ compact, $\boldsymbol{a} \in \mathrm{c}_{\downarrow 0}(\mathbb{Z}_{\geq 0}; \mathbb{R}_{>0})$, for $\mathscr{G}^{\mathrm{hol},\mathbb{R}}_{K,\overline{\mathsf{E}}}$ by

$$p^\omega_{K,\boldsymbol{a}}([\overline{\xi}]_K) = \sup\{a_0 a_1 \cdots a_m \|j_m\overline{\xi}(x)\|_{\overline{\mathbb{G}}_m} \mid x \in K,\ m \in \mathbb{Z}_{\geq 0}\}.$$

We can (and will) also think of $p^\omega_{K,\boldsymbol{a}}$ as being a seminorm on $\Gamma^\omega(\mathsf{E})$ defined by the same formula.

Let us prove that the seminorms $p^\omega_{K,\boldsymbol{a}}$, $K \subseteq \mathsf{M}$ compact, $\boldsymbol{a} \in \mathrm{c}_{\downarrow 0}(\mathbb{Z}_{\geq 0}; \mathbb{R}_{>0})$, can be used to define the direct limit topology on $\mathscr{G}^{\mathrm{hol},\mathbb{R}}_{K,\overline{\mathsf{E}}}$.

5.5 Theorem: (Seminorms for $\mathscr{G}^{\mathrm{hol},\mathbb{R}}_{K,\overline{\mathsf{E}}}$) *Let $\pi\colon \mathsf{E} \to \mathsf{M}$ be a real analytic vector bundle and let $K \subseteq \mathsf{M}$ be compact. Then the family of seminorms $p^\omega_{K,\boldsymbol{a}}$, $\boldsymbol{a} \in \mathrm{c}_{\downarrow 0}(\mathbb{Z}_{\geq 0}; \mathbb{R}_{>0})$, defines a locally convex topology on $\mathscr{G}^{\mathrm{hol},\mathbb{R}}_{K,\overline{\mathsf{E}}}$ agreeing with the direct limit topology.*

Proof: Let $K \subseteq \mathsf{M}$ be compact and let $(\overline{\mathcal{U}}_j)_{j \in \mathbb{Z}_{>0}}$ be a sequence of neighbourhoods of K in $\overline{\mathsf{M}}$ such that $\mathrm{cl}(\overline{\mathcal{U}}_{j+1}) \subseteq \overline{\mathcal{U}}_j$, $j \in \mathbb{Z}_{>0}$, and such that $K = \cap_{j \in \mathbb{Z}_{>0}} \overline{\mathcal{U}}_j$. We have mappings

$$\begin{aligned} r_{\overline{\mathcal{U}}_j,K}\colon \Gamma^{\mathrm{hol},\mathbb{R}}_{\mathrm{bdd}}(\overline{\mathsf{E}}|\overline{\mathcal{U}}_j) &\to \mathscr{G}^{\mathrm{hol},\mathbb{R}}_{K,\overline{\mathsf{E}}} \\ \overline{\xi} &\mapsto [\overline{\xi}]_K. \end{aligned}$$

The maps $r_{\overline{\mathcal{U}}_j,K}$ can be assumed to be injective without loss of generality, by making sure that each open set $\overline{\mathcal{U}}_j$ consists of disconnected neighbourhoods of the connected components of K. Since M is Hausdorff and the connected components of K are compact, this can always be done by choosing the initial open set $\overline{\mathcal{U}}_1$ sufficiently small. In this way, $\Gamma^{\mathrm{hol},\mathbb{R}}_{\mathrm{bdd}}(\overline{\mathsf{E}}|\overline{\mathcal{U}}_j)$, $j \in \mathbb{Z}_{>0}$, are regarded as subspaces of $\mathscr{G}^{\mathrm{hol},\mathbb{R}}_{K,\overline{\mathsf{E}}}$. It is convenient to be able to do this.

We will work with the locally convex space $\mathscr{E}(K)$ introduced in Section 5.2, and define a mapping $L_K\colon \mathscr{G}^{\mathrm{hol},\mathbb{R}}_{K,\overline{\mathsf{E}}} \to \mathscr{E}(K)$ by $L_K([\overline{\xi}]_K) = j_\infty\xi|K$. Let us prove that this mapping is well-defined, i.e., show that, if $[\overline{\xi}]_K \in \mathscr{G}^{\mathrm{hol},\mathbb{R}}_{K,\overline{\mathsf{E}}}$, then $L_K([\overline{\xi}]_K) \in \mathscr{E}_j(K)$ for some $j \in \mathbb{Z}_{>0}$.

Let $\overline{\mathcal{U}}$ be a neighbourhood of K in $\overline{\mathsf{M}}$ on which the section $\overline{\xi}$ is defined, holomorphic, and bounded. Then $\xi|(\mathsf{M} \cap \overline{\mathcal{U}})$ is real analytic and so, by Lemma 2.5, there exist $C, r \in \mathbb{R}_{>0}$ such that

$$\|j_m\xi(x)\|_{\overline{\mathbb{G}}_m} \leq Cr^{-m}, \qquad x \in K,\ m \in \mathbb{Z}_{\geq 0}.$$

If $j > r^{-1}$ it immediately follows that

$$\sup\{j^{-m}\|j_m\xi(x)\|_{\overline{\mathbb{G}}_m} \mid x \in K,\ m \in \mathbb{Z}_{\geq 0}\} < \infty,$$

i.e., $L_K([\overline{\xi}]_K) \in \mathscr{E}_j(K)$.

The following lemma records the essential feature of L_K.

1 Lemma: *The mapping L_K is a continuous, injective, open mapping, and so an homeomorphism onto its image.*

Proof: To show that L_K is continuous, it suffices to show that $L_K|\Gamma^{\mathrm{hol},\mathbb{R}}_{\mathrm{bdd}}(\overline{\mathsf{E}}|\overline{\mathcal{U}}_j)$ is continuous for each $j \in \mathbb{Z}_{\geq 0}$. We will show this by showing that, for each $j \in \mathbb{Z}_{>0}$, there exists $j' \in \mathbb{Z}_{>0}$ such that $L_K(\Gamma^{\mathrm{hol}}_{\mathrm{bdd}}(\overline{\mathsf{E}}|\overline{\mathcal{U}}_j)) \subseteq \mathscr{E}_{j'}(K)$ and such that L_K is continuous as a map from $\Gamma^{\mathrm{hol}}_{\mathrm{bdd}}(\overline{\mathsf{E}}|\overline{\mathcal{U}}_j)$ to $\mathscr{E}_{j'}(K)$. Since $\mathscr{E}_{j'}(K)$ is continuously included in $\mathscr{E}(K)$, this will give the continuity of L_K. First let us show that $L_K(\Gamma^{\mathrm{hol}}_{\mathrm{bdd}}(\overline{\mathsf{E}}|\overline{\mathcal{U}}_j)) \subseteq \mathscr{E}_{j'}(K)$ for some $j' \in \mathbb{Z}_{>0}$. By Proposition 4.2, there exist $C, r \in \mathbb{R}_{>0}$ such that

$$\|j_m\xi(x)\|_{\overline{\mathbb{G}}_m} \leq Cr^{-m}p^{\mathrm{hol}}_{\overline{\mathcal{U}}_j,\infty}(\overline{\xi})$$

for every $m \in \mathbb{Z}_{\geq 0}$ and $\overline{\xi} \in \Gamma^{\mathrm{hol}}_{\mathrm{bdd}}(\overline{\mathsf{E}}|\overline{\mathcal{U}}_j)$. Taking $j' \in \mathbb{Z}_{>0}$ such that $j' \geq r^{-1}$ we have $L_K(\Gamma^{\mathrm{hol}}_{\mathrm{bdd}}(\overline{\mathsf{E}}|\overline{\mathcal{U}}_j)) \subseteq \mathscr{E}_{j'}(K)$, as claimed. To show that L_K is continuous as a map from $\Gamma^{\mathrm{hol}}_{\mathrm{bdd}}(\overline{\mathsf{E}}|\overline{\mathcal{U}}_j)$ to $\mathscr{E}_{j'}(K)$, let $([\xi_k]_K)_{k\in\mathbb{Z}_{>0}}$ be a sequence in $\Gamma^{\mathrm{hol}}_{\mathrm{bdd}}(\overline{\mathsf{E}}|\overline{\mathcal{U}}_j)$ converging to zero. We then have

$$\lim_{k\to\infty} \sup\{(j')^{-m}\|j_m\xi_k(x)\|_{\overline{\mathbb{G}}} \mid x \in K,\ m \in \mathbb{Z}_{\geq 0}\} \leq \lim_{k\to\infty} C\sup\{\|\overline{\xi}_k(z)\|_{\overline{\mathbb{G}}} \mid z \in \overline{\mathcal{U}}_j\} = 0,$$

giving the desired continuity.

Since germs of holomorphic sections are uniquely determined by their infinite jets, injectivity of L_K follows.

We claim that, if $\mathcal{B} \subseteq \mathscr{E}(K)$ is von Neumann bounded, then $L_K^{-1}(\mathcal{B})$ is also von Neumann bounded. By Lemma 5.4, if $\mathcal{B} \subseteq \mathscr{E}(K)$ is bounded, then $\mathcal{B}$ is contained and bounded in $\mathscr{E}_j(K)$ for some $j \in \mathbb{Z}_{>0}$. Therefore, there exists $C \in \mathbb{R}_{>0}$ such that, if $L_K([\overline{\xi}]_K) \subseteq \mathcal{B}$, then

$$\|j_m\xi(x)\|_{\overline{\mathbb{G}}_m} \leq Cj^{-m}, \qquad x \in K,\ m \in \mathbb{Z}_{\geq 0}.$$

Let $x \in K$ and let $(\mathcal{V}_x, \psi_x)$ be a vector bundle chart for E about x with corresponding chart $(\mathcal{U}_x, \phi_x)$ for M. Suppose the fibre dimension of E over $\mathcal{U}_x$ is k and that ϕ_x takes values in $\mathbb{R}^n$. Let $\mathcal{U}'_x \subseteq \mathcal{U}_x$ be a relatively compact neighbourhood of x such that $\mathrm{cl}(\mathcal{U}'_x) \subseteq \mathcal{U}_x$. Denote $K_x = K \cap \mathrm{cl}(\mathcal{U}'_x)$. By Lemma 2.4, there exist $C_x, r_x \in \mathbb{R}_{>0}$ such that, if $L_K([\overline{\xi}]_K) \subseteq \mathcal{B}$, then

$$|\boldsymbol{D}^I\xi^a(\boldsymbol{x})| \leq C_x I! r_x^{-|I|}, \qquad \boldsymbol{x} \in \phi_x(K_x),\ I \in \mathbb{Z}^n_{\geq 0},\ a \in \{1, \dots, k\},$$

where $\boldsymbol{\xi}$ is the local representative of ξ. Note that this implies the following for each $[\overline{\xi}]_K$ such that $L_K([\overline{\xi}]_K) \subseteq \mathcal{B}$ and for each $a \in \{1, \dots, k\}$:

1. $\overline{\xi}^a$ admits a convergent power series expansion to an holomorphic function on the polydisk $\mathsf{D}(\boldsymbol{\sigma}_x, \phi_x(x))$ for $\sigma_x < r_x$;
2. on the polydisk $\mathsf{D}(\boldsymbol{\sigma}_x, \phi_x(x))$, $\overline{\xi}^a$ satisfies $|\overline{\xi}^a| \leq (\frac{1}{1-\sigma_x})^n$.

It follows that, if $L_K([\overline{\xi}]_K) \in \mathcal{B}$, then $\overline{\xi}$ has a bounded holomorphic extension in some coordinate polydisk around each $x \in K$. By a standard compactness argument and since $\cap_{j\in\mathbb{Z}_{\geq 0}}\overline{\mathcal{U}}_j = K$, there exists $j' \in \mathbb{Z}_{>0}$ such that $\overline{\xi} \in \Gamma^{\mathrm{hol},\mathbb{R}}_{\mathrm{bdd}}(\overline{\mathsf{E}}|\overline{\mathcal{U}}_{j'})$ for each $[\overline{\xi}]_K$ such that $L_K([\overline{\xi}]_K) \in \mathcal{B}$, and that the set of such sections of $\overline{\mathsf{E}}|\overline{\mathcal{U}}_{j'}$ is von Neumann bounded, i.e., norm bounded. Thus $L_K^{-1}(\mathcal{B})$ is von Neumann bounded, as claimed.

Note also that $\mathscr{E}(K)$ is a DF-space since Banach spaces are DF-spaces [Jarchow 1981, Corollary 12.4.4] and countable direct limits of DF-spaces are DF-spaces [Jarchow 1981, Theorem 12.4.8]. Therefore, by the open mapping lemma from §2 of Baernstein [1971], the result follows. ▼

From the lemma, it follows that the direct limit topology of $\mathscr{G}^{\mathrm{hol},\mathbb{R}}_{K,\overline{\mathsf{E}}}$ agrees with that induced by its image in $\mathscr{E}(K)$. Since the seminorms $p_{K,\boldsymbol{a}}$, $\boldsymbol{a} \in \mathrm{c}_{\downarrow 0}(\mathbb{Z}_{\geq 0}; \mathbb{R}_{>0})$, define the locally convex topology of $\mathscr{E}(K)$ by Lemma 5.3, it follows that the seminorms $p^{\omega}_{K,\boldsymbol{a}}$, $\boldsymbol{a} \in \mathrm{c}_{\downarrow 0}(\mathbb{Z}_{\geq 0}; \mathbb{R}_{>0})$, define the direct limit topology of $\mathscr{G}^{\mathrm{hol},\mathbb{R}}_{K,\overline{\mathsf{E}}}$. ■

The problem of providing seminorms for the direct limit topology of $\mathscr{G}^{\mathrm{hol},\mathbb{R}}_{K,\overline{\mathsf{E}}}$ is a nontrivial one, so let us provide a little history for what led to the preceding theorem. First of all, the first concrete characterisation of seminorms for germs of holomorphic functions about compact subsets of $\mathbb{C}^n$ comes in [Mujica 1984]. Mujica provides seminorms having two parts, one very much resembling the seminorms we use, and another part that is more complicated. These seminorms specialise to the case where the compact set lies in $\mathbb{R}^n \subseteq \mathbb{C}^n$, and the first mention of this we have seen in the research literature is in the notes of Domański [2012]. The first full proof that the seminorms analogous to those we define are, in fact, the seminorms for the space of real analytic functions on open subsets of $\mathbb{R}^n$ appears in the recent note of Vogt [2013]. Our presentation is an adaptation, not quite trivial as it turns out, of Vogt's constructions. One of the principal difficulties is Lemma 2.4 which is essential in showing that our jet bundle fibre metrics $\|\cdot\|_{\overline{\mathbb{G}}_m}$ are suitable for defining the seminorms for the real analytic topology. Note that one cannot use arbitrary fibre metrics, since one needs to have the behaviour of these metrics be regulated to the real analytic topology as the order of jets goes to infinity. Because our fibre metrics are constructed by differentiating objects defined at low order, i.e., the connections ∇ and ∇^0, we can ensure that the fibre metrics are compatible with real analytic growth conditions on derivatives.

An inverse limit topology for the space of real analytic sections. In the preceding three sections we provided three topologies for the space $\mathscr{G}^{\mathrm{hol},\mathbb{R}}_{K,\overline{\mathsf{E}}}$ of holomorphic sections about a compact subset K of a real analytic manifold: (1) the "standard" direct limit topology; (2) the topology induced by the direct limit topology on $\mathscr{E}(K)$; (3) the topology defined by the seminorms $p^{\omega}_{K,\boldsymbol{a}}$, $K \subseteq \mathsf{M}$ compact, $\boldsymbol{a} \in \mathrm{c}_{\downarrow 0}(\mathbb{Z}_{\geq 0}; \mathbb{R}_{>0})$. We showed in Lemma 5.3 and Theorem 5.5 that these three topologies agree. Now we shall use these constructions to easily arrive at (1) a topology on $\Gamma^{\omega}(\mathsf{E})$ induced by the locally convex topologies on the spaces $\mathscr{G}^{\mathrm{hol},\mathbb{R}}_{K,\overline{\mathsf{E}}}$, $K \subseteq \mathsf{M}$ compact, and (2) seminorms for the topology of $\Gamma^{\omega}(\mathsf{E})$.

For a compact set $K \subseteq \mathsf{M}$ we have an inclusion $i_K \colon \Gamma^\omega(\mathsf{E}) \to \mathscr{G}^{\mathrm{hol},\mathbb{R}}_{K,\overline{\mathsf{E}}}$ defined as follows. If $\xi \in \Gamma^\omega(\mathsf{E})$, then ξ admits an holomorphic extension $\overline{\xi}$ defined on a neighbourhood $\overline{\mathcal{U}} \subseteq \overline{\mathsf{M}}$ of M [Cieliebak and Eliashberg 2012, Lemma 5.40]. Since $\overline{\mathcal{U}} \in \mathscr{N}_K$ we define $i_K(\xi) = [\overline{\xi}]_K$. Now we have a compact exhaustion $(K_j)_{j\in\mathbb{Z}_{>0}}$ of M. Since $\mathscr{N}_{K_{j+1}} \subseteq \mathscr{N}_{K_j}$ we have a projection

$$\begin{aligned} \pi_j \colon \mathscr{G}^{\mathrm{hol},\mathbb{R}}_{K_{j+1},\overline{\mathsf{E}}} &\to \mathscr{G}^{\mathrm{hol},\mathbb{R}}_{K_j,\overline{\mathsf{E}}} \\ [\overline{\xi}]_{K_{j+1}} &\mapsto [\overline{\xi}]_{K_j}. \end{aligned}$$

One can check that, as $\mathbb{R}$-vector spaces, the inverse limit of the inverse family $(\mathscr{G}^{\mathrm{hol},\mathbb{R}}_{K_j,\overline{\mathsf{E}}})_{j\in\mathbb{Z}_{>0}}$ is isomorphic to $\mathscr{G}^{\mathrm{hol},\mathbb{R}}_{\mathsf{M},\overline{\mathsf{E}}}$, the isomorphism being given explicitly by the inclusions

$$\begin{aligned} i_j \colon \mathscr{G}^{\mathrm{hol},\mathbb{R}}_{\mathsf{M},\overline{\mathsf{E}}} &\to \mathscr{G}^{\mathrm{hol},\mathbb{R}}_{K_j,\overline{\mathsf{E}}} \\ [\overline{\xi}]_{\mathsf{M}} &\mapsto [\overline{\xi}]_{K_j}. \end{aligned}$$

Keeping in mind Lemma 5.2, we then have the inverse limit topology on $\Gamma^\omega(\mathsf{E})$ induced by the mappings i_j, $j \in \mathbb{Z}_{>0}$. The topology so defined we call the ***inverse*** $\mathbf{C}^\omega$***-topology*** for $\Gamma^\omega(\mathsf{E})$.

It is now a difficult theorem of Martineau [1966, Theorem 1.2(a)] that the direct C^ω-topology of Section 5.1 agrees with the inverse C^ω-topology. Therefore, we call the resulting topology the $\mathbf{C}^\omega$***-topology***. It is clear from Theorem 5.5 and the preceding inverse limit construction that the seminorms $p^\omega_{K,\boldsymbol{a}}$, $K \subseteq \mathsf{M}$ compact, $\boldsymbol{a} \in \mathrm{c}_{\downarrow 0}(\mathbb{Z}_{\geq 0}; \mathbb{R}_{>0})$, define the C^ω-topology.

5.3. Properties of the $\mathbf{C}^\omega$-topology. To say some relevant things about the C^ω-topology, let us first consider the direct limit topology for $\mathscr{G}^{\mathrm{hol},\mathbb{R}}_{K,\overline{\mathsf{E}}}$, $K \subseteq \mathsf{M}$ compact, as this is an important building block for the C^ω-topology. First, we recall that a ***strict direct limit*** of locally convex spaces consists of a sequence $(\mathsf{V}_j)_{j\in\mathbb{Z}_{>0}}$ of locally convex spaces that are subspaces of some vector space V, and which have the nesting property $\mathsf{V}_j \subseteq \mathsf{V}_{j+1}$, $j \in \mathbb{Z}_{>0}$. In defining the direct limit topology for $\mathscr{G}^{\mathrm{hol},\mathbb{R}}_{K,\overline{\mathsf{E}}}$ we defined it as a strict direct limit of Banach spaces. Moreover, the restriction mappings from $\Gamma^{\mathrm{hol},\mathbb{R}}_{\mathrm{bdd}}(\overline{\mathsf{E}}|\overline{\mathcal{U}}_j)$ to $\Gamma^{\mathrm{hol},\mathbb{R}}_{\mathrm{bdd}}(\overline{\mathsf{E}}|\overline{\mathcal{U}}_{j+1})$ can be shown to be compact [Kriegl and Michor 1997, Theorem 8.4]. Direct limits such as these are known as "Silva spaces" or "DFS spaces." Silva spaces have some nice properties, and these provide some of the following attributes for the direct limit topology for $\mathscr{G}^{\mathrm{hol},\mathbb{R}}_{K,\overline{\mathsf{E}}}$.

$\mathscr{G}^{\mathrm{hol},\mathbb{R}}$-1. It is Hausdorff: [Narici and Beckenstein 2010, Theorem 12.1.3].

$\mathscr{G}^{\mathrm{hol},\mathbb{R}}$-2. It is complete: [Narici and Beckenstein 2010, Theorem 12.1.10].

$\mathscr{G}^{\mathrm{hol},\mathbb{R}}$-3. It is not metrisable: [Narici and Beckenstein 2010, Theorem 12.1.8].

$\mathscr{G}^{\mathrm{hol},\mathbb{R}}$-4. It is regular: [Kriegl and Michor 1997, Theorem 8.4]. This means that every von Neumann bounded subset of $\mathscr{G}^{\mathrm{hol},\mathbb{R}}_{K,\overline{\mathsf{E}}}$ is contained and von Neumann bounded in $\Gamma^{\mathrm{hol},\mathbb{R}}(\overline{\mathsf{E}}|\overline{\mathcal{U}}_j)$ for some $j \in \mathbb{Z}_{>0}$.

$\mathscr{G}^{\mathrm{hol},\mathbb{R}}$-5. It is reflexive: [Kriegl and Michor 1997, Theorem 8.4].

$\mathscr{G}^{\mathrm{hol},\mathbb{R}}$-6. Its strong dual is a nuclear Fréchet space: [Kriegl and Michor 1997, Theorem 8.4]. Combined with reflexivity, this means that $\mathscr{G}^{\mathrm{hol},\mathbb{R}}_{K,\overline{\mathsf{E}}}$ is the strong dual of a nuclear Fréchet space.

$\mathscr{G}^{\mathrm{hol},\mathbb{R}}$-7. It is nuclear: [Schaefer and Wolff 1999, Corollary III.7.4].

$\mathscr{G}^{\mathrm{hol},\mathbb{R}}$-8. It is Suslin: This follows from [Fernique 1967, Théorème I.5.1(b)] since $\mathscr{G}^{\mathrm{hol},\mathbb{R}}_{K,\overline{\mathsf{E}}}$ is a strict direct limit of separable Fréchet spaces.

These attributes for the spaces $\mathscr{G}^{\mathrm{hol},\mathbb{R}}_{K,\overline{\mathsf{E}}}$ lead, more or less, to the following attributes of $\Gamma^\omega(\mathsf{E})$.

C^ω-1. It is Hausdorff: It is a union of Hausdorff topologies.

C^ω-2. It is complete: [Horváth 1966, Corollary to Proposition 2.11.3].

C^ω-3. It is not metrisable: It is a union of non-metrisable topologies.

C^ω-4. It is separable: [Domański 2012, Theorem 16].

C^ω-5. It is nuclear: [Schaefer and Wolff 1999, Corollary III.7.4].

C^ω-6. It is Suslin: Here we note that a countable direct product of Suslin spaces is Suslin [Bogachev 2007, Lemma 6.6.5(iii)]. Next we note that the inverse limit is a closed subspace of the direct product [Robertson and Robertson 1980, Proposition V.19]. Next, closed subspaces of Suslin spaces are Suslin spaces [Bogachev 2007, Lemma 6.6.5(ii)]. Therefore, since $\Gamma^\omega(\mathsf{E})$ is the inverse limit of the Suslin spaces $\mathscr{G}^{\mathrm{hol},\mathbb{R}}_{K_j,\overline{\mathsf{E}}}$, $j \in \mathbb{Z}_{>0}$, we conclude that $\Gamma^\omega(\mathsf{E})$ is Suslin.

As we have seen with the CO^∞- and $\mathrm{CO}^{\mathrm{hol}}$-topologies for $\Gamma^\infty(\mathsf{E})$ and $\Gamma^{\mathrm{hol}}(\mathsf{E})$, nuclearity of the C^ω-topology implies that compact subsets of $\Gamma^\omega(\mathsf{E})$ are exactly those that are closed and von Neumann bounded. For von Neumann boundedness, we have the following characterisation.

5.6 Lemma: (Bounded subsets in the $\mathbf{C}^\omega$-topology) *A subset $\mathcal{B} \subseteq \Gamma^\omega(\mathsf{E})$ is bounded in the von Neumann bornology if and only if the following property holds: for any compact set $K \subseteq \mathsf{M}$ and any $\boldsymbol{a} \in \mathrm{c}_{\downarrow 0}(\mathbb{Z}_{\geq 0}; \mathbb{R}_{>0})$, there exists $C \in \mathbb{R}_{>0}$ such that $p^\omega_{K,\boldsymbol{a}}(\xi) \leq C$ for every $\xi \in \mathcal{B}$.*

5.4. The weak-$\mathscr{L}$ topology for real analytic vector fields. As in the finitely differentiable, Lipschitz, smooth, and holomorphic cases, the above constructions for general vector bundles can be applied to the tangent bundle and the trivial vector bundle $\mathsf{M} \times \mathbb{R}$ to give the ***$\mathbf{C}^\omega$-topology*** on the space $\Gamma^\omega(\mathsf{TM})$ of real analytic vector fields and the space $\mathrm{C}^\omega(\mathsf{M})$ of real analytic functions. As we have already done in these other cases, we wish to provide a weak characterisation of the C^ω-topology for $\Gamma^\omega(\mathsf{TM})$. First of all, if $X \in \Gamma^\omega(\mathsf{TM})$, then $f \mapsto \mathscr{L}_X f$ is a derivation of $\mathrm{C}^\omega(\mathsf{M})$. As we have seen, in the holomorphic case this does not generally establish a correspondence between vector fields and derivations, but it does for Stein manifolds. In the real analytic case, Grabowski [1981] shows that the map $X \mapsto \mathscr{L}_X$ is indeed an isomorphism of the $\mathbb{R}$-vector spaces of real analytic vector fields and derivations of real analytic functions. Thus the pursuit of a weak description of the C^ω-topology for vector fields does not seem to be out of line.

The definition of the weak-$\mathscr{L}$ topology proceeds much as in the smooth and holomorphic cases.

5.7 Definition: (Weak-$\mathscr{L}$ topology for space of real analytic vector fields) For a real analytic manifold M, the ***weak-$\mathscr{L}$ topology*** for $\Gamma^\omega(\mathsf{TM})$ is the weakest topology for which the map $X \mapsto \mathscr{L}_X f$ is continuous for every $f \in \mathrm{C}^\omega(\mathsf{M})$, if $\mathrm{C}^\omega(\mathsf{M})$ has the C^ω-topology. •

We now have the following result.

5.8 Theorem: (Weak-$\mathscr{L}$ characterisation of C^ω-topology for real analytic vector fields) *For a real analytic manifold* M*, the following topologies for* $\Gamma^\omega(\mathsf{TM})$ *agree:*

(i) the C^ω*-topology;*

(ii) the weak-$\mathscr{L}$ *topology.*

Proof: (i)⊆(ii) As we argued in the corresponding part of the proof of Theorem 3.5, it suffices to show that, for $K \subseteq \mathsf{M}$ compact and for $\boldsymbol{a} \in \mathrm{c}_{\downarrow 0}(\mathbb{Z}_{\geq 0}; \mathbb{R}_{>0})$, there exist compact sets $K_1, \dots, K_r \subseteq \mathsf{M}$, $\boldsymbol{a}_1, \dots, \boldsymbol{a}_r \in \mathrm{c}_{\downarrow 0}(\mathbb{Z}_{\geq 0}; \mathbb{R}_{>0})$, $f^1, \dots, f^r \in \mathrm{C}^\omega(\mathsf{M})$, and $C_1, \dots, C_r \in \mathbb{R}_{>0}$ such that

$$p^\omega_{K,\boldsymbol{a}}(X) \leq C_1 p^\omega_{K_1,\boldsymbol{a}_1}(\mathscr{L}_X f^1) + \dots + C_r p^\omega_{K_r,\boldsymbol{a}_r}(\mathscr{L}_X f^r), \qquad X \in \Gamma^\omega(\mathsf{TM}).$$

We begin with a simple technical lemma.

1 Lemma: *For each* $x \in \mathsf{M}$ *there exist* $f^1, \dots, f^n \in \mathrm{C}^\omega(\mathsf{M})$ *such that* $(\mathrm{d}f^1(x), \dots, \mathrm{d}f^n(x))$ *is a basis for* $\mathsf{T}^*_x\mathsf{M}$*.*

Proof: We are supposing, of course, that the connected component of M containing x has dimension n. There are many ways to prove this lemma, including applying Cartan's Theorem A to the sheaf of real analytic functions on M. We shall prove the lemma by embedding M in $\mathbb{R}^N$ by the embedding theorem of Grauert [1958]. Thus we have a proper real analytic embedding $\iota_{\mathsf{M}} \colon \mathsf{M} \to \mathbb{R}^N$. Let $g^1, \dots, g^N \in \mathrm{C}^\omega(\mathbb{R}^N)$ be the coordinate functions. Then we have a surjective linear map

$$\begin{aligned} \sigma_x \colon \mathbb{R}^N &\to \mathsf{T}^*_x\mathsf{M} \\ (c_1, \dots, c_N) &\mapsto \sum_{j=1}^N c_j \mathrm{d}(\iota^*_{\mathsf{M}} g^j)(x). \end{aligned}$$

Let $\boldsymbol{c}^1, \dots, \boldsymbol{c}^n \in \mathbb{R}^N$ be a basis for a complement of $\ker(\sigma_x)$. Then the functions

$$f^j = \sum_{k=1}^N c^j_k \iota^*_{\mathsf{M}} g^k$$

have the desired property. ▼

We assume that M has a well-defined dimension n. This assumption can easily be relaxed. We use the notation

$$p'^\omega_{K,\boldsymbol{a}}(f) = \sup\left\{ \frac{a_0 a_1 \cdots a_{|I|}}{I!} |\boldsymbol{D}^I f(\boldsymbol{x})| \;\middle|\; \boldsymbol{x} \in K,\ I \in \mathbb{Z}^n_{\geq 0} \right\}$$

for a function $f \in \mathrm{C}^\omega(\mathcal{U})$ defined on an open subset of $\mathbb{R}^n$ and with $K \subseteq \mathcal{U}$ compact. We shall also use this local coordinate notation for seminorms of local representatives of vector fields. Let $K \subseteq \mathsf{M}$ be compact and let $\boldsymbol{a} \in \mathrm{c}_{\downarrow 0}(\mathbb{Z}_{\geq 0}; \mathbb{R}_{>0})$. Let $x \in K$ and let $(\mathcal{U}_x, \phi_x)$ be a chart for M about x with the property that the coordinate functions x^j, $j \in \{1, \dots, n\}$, are restrictions to $\mathcal{U}_x$ of globally defined real analytic functions f^j_x, $j \in \{1, \dots, n\}$, on M. This is possible by the lemma above. Let $\boldsymbol{X} \colon \phi_x(\mathcal{U}_x) \to \mathbb{R}^n$ be the local representative of $X \in \Gamma^\omega(\mathsf{M})$. Then, in a neighbourhood of the closure of a relatively compact neighbourhood

$\mathcal{V}_x \subseteq \mathcal{U}_x$ of x, we have $\mathscr{L}_X f_x^j = X^j$, the jth component of X. By Lemma 2.4, there exist $C_x, \sigma_x \in \mathbb{R}_{>0}$ such that

$$\|j_m X(y)\|_{\overline{\mathbb{G}}_m} \leq C_x \sigma_x^{-m} \sup\left\{ \frac{1}{I!} |\boldsymbol{D}^I X^j(\phi_x(y))| \;\middle|\; |I| \leq m,\ j \in \{1, \dots, n\} \right\}$$

for $m \in \mathbb{Z}_{\geq 0}$ and $y \in \mathrm{cl}(\mathcal{V}_x)$. By equivalence of the ℓ^1 and ℓ^∞-norms for $\mathbb{R}^n$, there exists $C \in \mathbb{R}_{>0}$ such that

$$\sup\left\{ \frac{1}{I!} |\boldsymbol{D}^I X^j(\phi_x(y))| \;\middle|\; |I| \leq m,\ j \in \{1, \dots, n\} \right\} \\ \leq C \sum_{j=1}^n \sup\left\{ \frac{1}{I!} |\boldsymbol{D}^I (\mathscr{L}_X f_x^j)(\phi_x(y))| \;\middle|\; |I| \leq m \right\}$$

for $m \in \mathbb{Z}_{\geq 0}$ and $y \in \mathrm{cl}(\mathcal{V}_x)$. Another application of Lemma 2.4 gives $B_x, r_x \in \mathbb{R}_{>0}$ such that

$$\sup\left\{ \frac{1}{I!} |\boldsymbol{D}^I (\mathscr{L}_X f_x^j)(\phi_x(y))| \;\middle|\; |I| \leq m,\ j \in \{1, \dots, n\} \right\} \leq B_x r_x^{-m} \|j_m(\mathscr{L}_X f_x^j)(y)\|$$

for $m \in \mathbb{Z}_{\geq 0}$, $j \in \{1, \dots, n\}$, and $y \in \mathrm{cl}(\mathcal{V}_x)$. Combining the preceding three estimates and renaming constants gives

$$\|j_m X(y)\|_{\overline{\mathbb{G}}_m} \leq \sum_{j=1}^n C_x \sigma_x^{-m} \|j_m(\mathscr{L}_X f_x^j(\phi_x(y)))\|_{\overline{\mathbb{G}}_m}$$

for $m \in \mathbb{Z}_{\geq 0}$ and $y \in \mathrm{cl}(\mathcal{V}_x)$. Define

$$\boldsymbol{b}_x = (b_m)_{m \in \mathbb{Z}_{\geq 0}} \in \mathrm{c}_{\downarrow 0}(\mathbb{Z}_{\geq 0}; \mathbb{R}_{>0})$$

by $b_0 = C_x a_0$ and $b_m = \sigma_x^{-1} a_m$, $m \in \mathbb{Z}_{>0}$. Therefore,

$$a_0 a_1 \cdots a_m \|j_m X(y)\|_{\overline{\mathbb{G}}_m} \leq \sum_{j=1}^n b_0 b_1 \cdots b_m \|j_m(\mathscr{L}_X f_x^j(\phi_x(y)))\|_{\overline{\mathbb{G}}_m}$$

for $m \in \mathbb{Z}_{\geq 0}$ and $y \in \mathrm{cl}(\mathcal{V}_x)$. Supping over $y \in \mathrm{cl}(\mathcal{V}_x)$ and $m \in \mathbb{Z}_{\geq 0}$ on the right gives

$$a_0 a_1 \cdots a_m \|j_m X(y)\|_{\overline{\mathbb{G}}_m} \leq \sum_{j=1}^n p^\omega_{\mathrm{cl}(\mathcal{V}_x), \boldsymbol{b}_x}(\mathscr{L}_X f_x^j), \qquad m \in \mathbb{Z}_{\geq 0},\ y \in \mathrm{cl}(\mathcal{V}_x).$$

Let $x_1, \dots, x_k \in K$ be such that $K \subseteq \cup_{j=1}^k \mathcal{V}_{x_j}$, let $f^1, \dots, f^{kn}$ be the list of functions

$$f_{x_1}^1, \dots, f_{x_1}^n, \dots, f_{x_k}^1, \dots, f_{x_k}^n,$$

and let $\boldsymbol{b}_1, \dots, \boldsymbol{b}_{kn} \in \mathrm{c}_{\downarrow 0}(\mathbb{Z}_{\geq 0}; \mathbb{R}_{>0})$ be the list of sequences

$$\underbrace{\boldsymbol{b}_{x_1}, \dots, \boldsymbol{b}_{x_1}}_{n \text{ times}}, \dots, \underbrace{\boldsymbol{b}_{x_k}, \dots, \boldsymbol{b}_{x_k}}_{n \text{ times}}.$$

If $x \in K$, then $x \in \mathcal{V}_{x_j}$ for some $j \in \{1, \dots, k\}$ and so

$$a_0 a_1 \cdots a_m \|j_m X(x)\|_{\overline{\mathbb{G}}_m} \le \sum_{j=1}^{kn} p^{\omega}_{K,\boldsymbol{b}_j}(\mathscr{L}_X f^j),$$

and this part of the lemma follows upon taking the supremum over $x \in K$ and $m \in \mathbb{Z}_{\ge 0}$.

(ii)$\subseteq$(i) Here, as in the proof of the corresponding part of Theorem 3.5, it suffices to show that, for every $f \in \mathrm{C}^{\omega}(\mathsf{M})$, the map $\mathscr{L}_f \colon X \mapsto \mathscr{L}_X f$ is continuous from $\Gamma^{\omega}(\mathsf{TM})$ with the C^{ω}-topology to $\mathrm{C}^{\omega}(\mathsf{M})$ with the C^{ω}-topology.

We shall use the direct C^{ω}-topology to show this. Thus we work with an holomorphic manifold $\overline{\mathsf{M}}$ that is a complexification of M, as described in Section 5.1. We recall that $\mathscr{N}_{\mathsf{M}}$ denotes the directed set of neighbourhoods of M in $\overline{\mathsf{M}}$, and that the set $\mathscr{S}_{\mathsf{M}}$ of Stein neighbourhoods is cofinal in $\mathscr{N}_{\mathsf{M}}$. As we saw in Section 5.1, for $\overline{\mathcal{U}} \in \mathscr{N}_{\mathsf{M}}$, we have mappings

$$\begin{aligned} r_{\overline{\mathcal{U}},\mathsf{M}} \colon \Gamma^{\mathrm{hol},\mathbb{R}}(\mathsf{T}\overline{\mathcal{U}}) &\to \Gamma^{\omega}(\mathsf{TM}) \\ \overline{X} &\mapsto \overline{X}|\mathsf{M} \end{aligned}$$

and

$$\begin{aligned} r_{\overline{\mathcal{U}},\mathsf{M}} \colon \mathrm{C}^{\mathrm{hol},\mathbb{R}}(\overline{\mathcal{U}}) &\to \mathrm{C}^{\omega}(\mathsf{M}) \\ \overline{f} &\mapsto \overline{f}|\mathsf{M}, \end{aligned}$$

making an abuse of notation by using $r_{\overline{\mathcal{U}},\mathsf{M}}$ for two different things, noting that context will make it clear which we mean. For $K \subseteq \mathsf{M}$ compact, we also have the mapping

$$\begin{aligned} i_{\mathsf{M},K} \colon \mathrm{C}^{\omega}(\mathsf{M}) &\to \mathscr{C}^{\mathrm{hol},\mathbb{R}}_{K,\overline{\mathsf{M}}} \\ f &\mapsto [\overline{f}]_K, \end{aligned}$$

The C^{ω}-topology is the final topology induced by the mappings $r_{\overline{\mathcal{U}},\mathsf{M}}$. As such, by [Horváth 1966, Proposition 2.12.1], the map $\mathscr{L}_f$ is continuous if and only if $\mathscr{L}_f \circ r_{\overline{\mathcal{U}},\mathsf{M}}$ for every $\overline{\mathcal{U}} \in \mathscr{N}_{\mathsf{M}}$. Thus let $\overline{\mathcal{U}} \in \mathscr{N}_{\mathsf{M}}$. To show that $\mathscr{L}_f \circ r_{\overline{\mathcal{U}},\mathsf{M}}$ is continuous, it suffices by [Horváth 1966, §2.11] to show that $i_{\mathsf{M},K} \circ \mathscr{L}_f \circ r_{\overline{\mathcal{U}},\mathsf{M}}$ is continuous for every compact $K \subseteq \mathsf{M}$. Next, there is $\overline{\mathcal{U}} \supseteq \overline{\mathcal{S}} \in \mathscr{S}_{\mathsf{M}}$ so that f admits an holomorphic extension $\overline{f}$ to $\overline{\mathcal{S}}$. The following diagram shows how this all fits together.

$$\begin{tikzcd} \Gamma^{\mathrm{hol},\mathbb{R}}(\mathsf{T}\overline{\mathcal{U}}) \arrow[r, "r_{\overline{\mathcal{U}},\overline{\mathcal{S}}}"] \arrow[drr, dashed] & \Gamma^{\mathrm{hol},\mathbb{R}}(\mathsf{T}\overline{\mathcal{S}}) \arrow[r, "r_{\overline{\mathcal{S}},\mathsf{M}}"] \arrow[d, "\mathscr{L}_{\overline{f}}"'] & \Gamma^{\omega}(\mathsf{TM}) \arrow[d, dashed, "\mathscr{L}_f"] & \\ & \mathrm{C}^{\mathrm{hol},\mathbb{R}}(\overline{\mathcal{S}}) \arrow[r, "r_{\overline{\mathcal{S}},\mathsf{M}}"'] & \mathrm{C}^{\omega}(\mathsf{M}) \arrow[r, "i_{\mathsf{M},K}"'] & \mathscr{C}^{\mathrm{hol},\mathbb{R}}_{K,\overline{\mathsf{M}}} \end{tikzcd}$$

The dashed arrows signify maps whose continuity is *a priori* unknown to us. The diagonal dashed arrow is the one whose continuity we must verify to ascertain the continuity of the vertical dashed arrow. It is a simple matter of checking definitions to see that the diagram commutes. By Theorem 4.5, we have that $\mathscr{L}_{\overline{f}} \colon \Gamma^{\mathrm{hol},\mathbb{R}}(\mathsf{T}\overline{\mathcal{S}}) \to \mathrm{C}^{\mathrm{hol},\mathbb{R}}(\overline{\mathcal{S}})$ is continuous (keeping Remark 5.1 in mind). We deduce that, since

$$i_{\mathsf{M},K} \circ \mathscr{L}_f \circ r_{\overline{\mathcal{U}},\mathsf{M}} = i_{\mathsf{M},K} \circ r_{\overline{\mathcal{S}},\mathsf{M}} \circ \mathscr{L}_{\overline{f}} \circ r_{\overline{\mathcal{U}},\overline{\mathcal{S}}},$$

$i_{\mathsf{M},K} \circ \mathscr{L}_f \circ r_{\overline{\mathcal{U}},\mathsf{M}}$ is continuous for every $\overline{\mathcal{U}} \in \mathscr{N}_{\mathsf{M}}$ and for every compact $K \subseteq \mathsf{M}$, as desired. ■

As in the smooth and holomorphic cases, we can prove the equivalence of various topological notions between the weak-$\mathscr{L}$ and usual topologies.

5.9 Corollary: (Weak-$\mathscr{L}$ characterisations of boundedness, continuity, measurability, and integrability for the C^ω-topology) *Let* M *be a real analytic manifold, let* $(\mathcal{X},\mathscr{O})$ *be a topological space, let* $(\mathcal{T},\mathscr{M})$ *be a measurable space, and let* $\mu\colon\mathscr{M}\to\overline{\mathbb{R}}_{\ge 0}$ *be a finite measure. The following statements hold:*

(i) a subset $\mathcal{B}\subseteq\Gamma^\omega(\mathsf{TM})$ *is bounded in the von Neumann bornology if and only if it is weak-*$\mathscr{L}$ *bounded in the von Neumann bornology;*

(ii) a map $\Phi\colon\mathcal{X}\to\Gamma^\omega(\mathsf{TM})$ *is continuous if and only if it is weak-*$\mathscr{L}$ *continuous;*

(iii) a map $\Psi\colon\mathcal{T}\to\Gamma^\omega(\mathsf{TM})$ *is measurable if and only if it is weak-*$\mathscr{L}$ *measurable;*

(iv) a map $\Psi\colon\mathcal{T}\to\Gamma^\omega(\mathsf{TM})$ *is Bochner integrable if and only if it is weak-*$\mathscr{L}$ *Bochner integrable.*

Proof: The fact that $\{\mathscr{L}_f \mid f\in \mathrm{C}^\omega(\mathsf{M})\}$ contains a countable point separating subset follows from combining the lemma from the proof of Theorem 5.8 with the proof of the corresponding assertion in Corollary 3.6. Since $\Gamma^\omega(\mathsf{TM})$ is complete, separable, and Suslin, and since $\mathrm{C}^\omega(\mathsf{M})$ is Suslin by properties C^ω-2, C^ω-4, and C^ω-6 above, the corollary follows from Lemma 3.3, taking "$\mathsf{U}=\Gamma^\omega(\mathsf{TM})$," "$\mathsf{V}=\mathrm{C}^\omega(\mathsf{M})$," and "$\mathscr{A}=\{\mathscr{L}_f \mid f\in\mathrm{C}^\omega(\mathsf{M})\}$." ■

6. Time-varying vector fields

In this section we consider time-varying vector fields. The ideas in this section originate (for us) with the paper of Agrachev and Gamkrelidze [1978], and are nicely summarised in the more recent book of Agrachev and Sachkov [2004], at least in the smooth case. A geometric presentation of some of the constructions can be found in the paper of Sussmann [1998], again in the smooth case, and Sussmann also considers regularity less than smooth, e.g., finitely differentiable or Lipschitz. There is some consideration of the real analytic case in [Agrachev and Gamkrelidze 1978], but this consideration is restricted to real analytic vector fields admitting a bounded holomorphic extension to a fixed-width neighbourhood of $\mathbb{R}^n$ in $\mathbb{C}^n$. One of our results, the rather nontrivial Theorem 6.25, is that this framework of Agrachev and Gamkrelidze [1978] is sufficient for the purposes of local analysis. However, our treatment of the real analytic case is global, general, and comprehensive. To provide some context for our novel treatment of the real analytic case, we treat the smooth case in some detail, even though the results are probably mostly known. (However, we should say that, even in the smooth case, we could not find precise statements with proofs of some of the results we give.) We also treat the finitely differentiable and Lipschitz cases, so our theory also covers the "standard" Carathéodory existence and uniqueness theorem for time-varying ordinary differential equations, [e.g., Sontag 1998, Theorem 54]. We also consider holomorphic time-varying vector fields, as these have a relationship to real analytic time-varying vector fields that is sometimes useful to exploit.

One of the unique facets of our presentation is that we fully explain the rôle of the topologies developed in Sections 3, 4, and 5. Indeed, one way to understand the principal results of this section is that they show that the usual pointwise—in state and time—conditions placed on vector fields to regulate the character of their flows can be profitably phrased in terms of topologies for spaces of vector fields. While this idea is not entirely new—it is implicit in the approach of [Agrachev and Gamkrelidze 1978]—we do develop it comprehensively and in new directions.

While our principal interest is in vector fields, and also in functions, it is convenient to conduct much of the development for general vector bundles, subsequently specialising to vector fields and functions.

6.1. The smooth case. Throughout this section we will work with a smooth vector bundle $\pi\colon \mathsf{E} \to \mathsf{M}$ with a linear connection ∇^0 on E, an affine connection ∇ on M, a fibre metric $\mathbb{G}_0$ on E, and a Riemannian metric $\mathbb{G}$ on M. This defines the fibre norms $\|\cdot\|_{\overline{\mathbb{G}}_m}$ on $\mathsf{J}^m\mathsf{E}$ and seminorms $p^\infty_{K,m}$, $K \subseteq \mathsf{M}$ compact, $m \in \mathbb{Z}_{\geq 0}$, on $\Gamma^\infty(\mathsf{E})$ as in Section 3.1.

6.1 Definition: (Smooth Carathéodory section) Let $\pi\colon \mathsf{E} \to \mathsf{M}$ be a smooth vector bundle and let $\mathbb{T} \subseteq \mathbb{R}$ be an interval. A ***Carathéodory section of class*** C^∞ of E is a map $\xi\colon \mathbb{T} \times \mathsf{M} \to \mathsf{E}$ with the following properties:

(i) $\xi(t,x) \in \mathsf{E}_x$ for each $(t,x) \in \mathbb{T} \times \mathsf{M}$;

(ii) for each $t \in \mathbb{T}$, the map $\xi_t\colon \mathsf{M} \to \mathsf{E}$ defined by $\xi_t(x) = \xi(t,x)$ is of class C^∞;

(iii) for each $x \in \mathsf{M}$, the map $\xi^x\colon \mathbb{T} \to \mathsf{E}$ defined by $\xi^x(t) = \xi(t,x)$ is Lebesgue measurable.

We shall call $\mathbb{T}$ the ***time-domain*** for the section. By $\mathrm{CF}\Gamma^\infty(\mathbb{T};\mathsf{E})$ we denote the set of Carathéodory sections of class C^∞ of E. •

Note that the curve $t \mapsto \xi(t,x)$ is in the finite-dimensional vector space E_x, and so Lebesgue measurability of this is unambiguously defined, e.g., by choosing a basis and asking for Lebesgue measurability of the components with respect to this basis.

Now we put some conditions on the time dependence of the derivatives of the section.

6.2 Definition: (Locally integrally C^∞-bounded and locally essentially C^∞-bounded sections) Let $\pi\colon \mathsf{E} \to \mathsf{M}$ be a smooth vector bundle and let $\mathbb{T} \subseteq \mathbb{R}$ be an interval. A Carathéodory section $\xi\colon \mathbb{T} \times \mathsf{M} \to \mathsf{E}$ of class C^∞ is

(i) ***locally integrally*** C^∞***-bounded*** if, for every compact set $K \subseteq \mathsf{M}$ and every $m \in \mathbb{Z}_{\geq 0}$, there exists $g \in \mathrm{L}^1_{\mathrm{loc}}(\mathbb{T};\mathbb{R}_{\geq 0})$ such that

$$\|j_m\xi_t(x)\|_{\overline{\mathbb{G}}_m} \leq g(t), \qquad (t,x) \in \mathbb{T} \times K,$$

and is

(ii) ***locally essentially*** C^∞***-bounded*** if, for every compact set $K \subseteq \mathsf{M}$ and every $m \in \mathbb{Z}_{\geq 0}$, there exists $g \in \mathrm{L}^\infty_{\mathrm{loc}}(\mathbb{T};\mathbb{R}_{\geq 0})$ such that

$$\|j_m\xi_t(x)\|_{\overline{\mathbb{G}}_m} \leq g(t), \qquad (t,x) \in \mathbb{T} \times K.$$

The set of locally integrally C^∞-bounded sections of E with time-domain $\mathbb{T}$ is denoted by $\mathrm{LI}\Gamma^\infty(\mathbb{T},\mathsf{E})$ and the set of locally essentially C^∞-bounded sections of E with time-domain $\mathbb{T}$ is denoted by $\mathrm{LB}\Gamma^\infty(\mathbb{T};\mathsf{E})$. •

Note that $\mathrm{LB}\Gamma^\infty(\mathbb{T};\mathsf{M}) \subseteq \mathrm{LI}\Gamma^\infty(\mathbb{T};\mathsf{M})$, precisely because locally essentially bounded functions (in the usual sense) are locally integrable (in the usual sense).

We note that our definitions differ from those in [Agrachev and Gamkrelidze 1978, Agrachev and Sachkov 2004, Sussmann 1998]. The form of the difference is our use of connections and jet bundles, aided by Lemma 2.1. In [Agrachev and Gamkrelidze 1978] the presentation is developed on Euclidean spaces, and so the geometric treatment we give here is not necessary. (One way of understanding why it is not necessary is that Euclidean space has a canonical flat connection in which the decomposition of Lemma 2.1 becomes the usual decomposition of derivatives by their order.) In [Agrachev and Sachkov 2004] the treatment is on manifolds, and the seminorms are defined by an embedding of the manifold in Euclidean space by Whitney's Embedding Theorem [Whitney 1936]. Also, Agrachev and Sachkov [2004] use the weak-$\mathscr{L}$ topology in the case of vector fields, but we have seen that this is the same as the usual topology (Theorem 3.5). In [Sussmann 1998] the characterisation of Carathéodory functions uses Lie differentiation by smooth vector fields, and the locally convex topology for $\Gamma^\infty(\mathsf{TM})$ is not explicitly considered, although it is implicit in Sussmann's constructions. Sussmann also takes a weak-$\mathscr{L}$ approach to characterising properties of time-varying vector fields. In any case, all approaches can be tediously shown to be equivalent once the relationships are understood. An advantage of the approach we use here is that it does not require coordinate charts or embeddings to write the seminorms, and it makes the seminorms explicit, rather than implicitly present. The disadvantage of our approach is the added machinery and complication of connections and our jet bundle decomposition.

The following characterisation of Carathéodory sections and their relatives is also useful and insightful.

6.3 Theorem: (Topological characterisation of smooth Carathéodory sections) *Let $\pi\colon \mathsf{E} \to \mathsf{M}$ be a smooth vector bundle and let $\mathbb{T} \subseteq \mathbb{R}$ be an interval. For a map $\xi\colon \mathbb{T} \times \mathsf{M} \to \mathsf{E}$ satisfying $\xi(t,x) \in \mathsf{E}_x$ for each $(t,x) \in \mathbb{T} \times \mathsf{M}$, the following two statements are equivalent:*

(i) $\xi \in \mathrm{CF}\Gamma^\infty(\mathbb{T};\mathsf{E})$;

(ii) the map $\mathbb{T} \ni t \mapsto \xi_t \in \Gamma^\infty(\mathsf{E})$ *is measurable,*

the following two statements are equivalent:

(iii) $\xi \in \mathrm{LI}\Gamma^\infty(\mathbb{T};\mathsf{E})$;

(iv) the map $\mathbb{T} \ni t \mapsto \xi_t \in \Gamma^\infty(\mathsf{E})$ *is measurable and locally Bochner integrable,*

and the following two statements are equivalent:

(v) $\xi \in \mathrm{LB}\Gamma^\infty(\mathbb{T};\mathsf{E})$;

(vi) the map $\mathbb{T} \ni t \mapsto \xi_t \in \Gamma^\infty(\mathsf{E})$ *is measurable and locally essentially von Neumann bounded.*

Proof: It is illustrative, especially since we will refer to this proof at least three times subsequently, to understand the general framework of the proof. Much of the argument has already been carried out in a more general setting in Lemma 3.3.

So we let V be a locally convex topological vector space over $\mathbb{F} \in \{\mathbb{R}, \mathbb{C}\}$, let $(\mathcal{T}, \mathscr{M})$ be a measurable space, and let $\Psi\colon \mathcal{T} \to \mathsf{V}$. Let us first characterise measurability of Ψ. We use here the results of Thomas [1975] who studies integrability for functions taking values in locally convex Suslin spaces. Thus we assume that V is a Hausdorff Suslin space (as is the case for all spaces of interest to us in this paper). We let V' denote the topological dual of V. A subset $S \subseteq \mathsf{V}'$ is ***point separating*** if, for distinct $v_1, v_2 \in \mathsf{V}$, there exists $\alpha \in \mathsf{V}'$ such that $\alpha(v_1) \neq \alpha(v_2)$. Thomas [1975] proves the following result as his Theorem 1, and whose proof we provide, as it is straightforward and shows where the (not so straightforward) properties of Suslin spaces are used.

1 Lemma: *Let V be a Hausdorff, Suslin, locally convex topological vector space over $\mathbb{F} \in \{\mathbb{R}, \mathbb{C}\}$, let $(\mathcal{T}, \mathscr{M})$ be a measurable space, and let $\Psi\colon \mathcal{T} \to \mathsf{V}$. If $S \subseteq \mathsf{V}'$ is point separating, then Ψ is measurable if and only if $\alpha \circ \Psi$ is measurable for every $\alpha \in S$.*

Proof: If Ψ is measurable, then it is obvious that $\alpha \circ \Psi$ is measurable for every $\alpha \in \mathsf{V}'$ since such α are continuous.

Conversely, suppose that $\alpha \circ \Psi$ is measurable for every $\alpha \in S$. First of all, locally convex topological vector spaces are completely regular if they are Hausdorff [Schaefer and Wolff 1999, page 16]. Therefore, by [Bogachev 2007, Theorem 6.7.7], there is a countable subset of S that is point separating, so we may as well suppose that S is countable. We are now in the same framework as Lemma 3.3(iii), and the proof there applies by taking "$\mathsf{U} = \mathsf{V}$," "$\mathsf{V} = \mathbb{F}$," and "$\mathscr{A} = S$." ▼

The preceding lemma will allow us to characterise measurability. Let us now consider integrability.

2 Lemma: *Let V be a complete separable locally convex topological vector space over $\mathbb{F} \in \{\mathbb{R}, \mathbb{C}\}$ and let $(\mathcal{T}, \mathscr{M}, \mu)$ be a finite measure space. A measurable function $\Psi\colon \mathcal{T} \to \mathsf{V}$ is Bochner integrable if and only if $p \circ \Psi$ is integrable for every continuous seminorm p for V.*

Proof: It follows from [Beckmann and Deitmar 2011, Theorems 3.2, 3.3] that Ψ is integrable if $p \circ \Psi$ is integrable for every continuous seminorm p. Conversely, if Ψ is integrable, it is implied that Ψ is Bochner approximable, and so, by [Beckmann and Deitmar 2011, Theorem 3.2], we have that $p \circ \Psi$ is integrable for every continuous seminorm p. ▼

(i) $\iff$ (ii) For $x \in \mathsf{M}$ and $\alpha_x \in \mathsf{E}_x^*$, define $\mathrm{ev}_{\alpha_x}\colon \Gamma^\infty(\mathsf{E}) \to \mathbb{R}$ by $\mathrm{ev}_{\alpha_x}(\xi) = \langle \alpha_x; \xi(x)\rangle$. Clearly ev_{α_x} is $\mathbb{R}$-linear. We claim that ev_{α_x} is continuous. Indeed, for a directed set $(I, \preceq)$ and a net $(\xi)_{i\in I}$ converging to ξ,[10] we have

$$\lim_{i\in I} \mathrm{ev}_{\alpha_x}(\xi_i) = \lim_{i\in I} \alpha_x(\xi_i(x)) = \alpha_x\Big(\lim_{i\in I} \xi_i(x)\Big) = \alpha_x(\xi(x)) = \mathrm{ev}_{\alpha_x}(\xi),$$

using the fact that convergence in the CO^∞-topology implies pointwise convergence. It is obvious that the continuous linear functions ev_{α_x}, $\alpha_x \in \mathsf{E}^*$, are point separating. We now recall from property CO^∞-6 for the smooth CO^∞-topology that $\Gamma^\infty(\mathsf{E})$ is a Suslin space with the CO^∞-topology. Therefore, by the first lemma above, it follows that $t \mapsto \xi_t$ is measurable if and only if $t \mapsto \mathrm{ev}_{\alpha_x}(\xi_t) = \langle \alpha_x; \xi_t(x)\rangle$ is measurable for every $\alpha_x \in \mathsf{E}^*$. On the other hand, this is equivalent to $t \mapsto \xi_t(x)$ being measurable for every $x \in \mathsf{M}$ since $t \mapsto \xi_t(x)$ is a curve in the finite-dimensional vector space E_x. Finally, note that it is implicit in the statement of (ii) that ξ_t is smooth, and this part of the proposition follows easily from these observations.

(iii) $\iff$ (iv) Let $\mathbb{T}' \subseteq \mathbb{T}$ be compact.

First suppose that $\xi \in \mathrm{LI}\Gamma^\infty(\mathbb{T}; \mathsf{E})$. By definition of locally integrally C^∞-bounded, for each compact $K \subseteq \mathsf{M}$ and $m \in \mathbb{Z}_{\geq 0}$, there exists $g \in \mathrm{L}^1(\mathbb{T}'; \mathbb{R}_{\geq 0})$ such that

$$\|j_m \xi_t(x)\|_{\overline{\mathbb{G}}_m} \leq g(t), \qquad (t, x) \in \mathbb{T}' \times K \quad \implies \quad p^\infty_{K,m}(\xi_t) \leq g(t), \qquad t \in \mathbb{T}'.$$

[10] Since $\Gamma^\infty(\mathsf{E})$ is metrisable, it suffices to use sequences. However, we shall refer to this argument when we do not use metrisable spaces, so it is convenient to have the general argument here.

Note that continuity of $p^\infty_{K,m}$ implies that $t \mapsto p^\infty_{K,m}(\xi_t)$ is measurable. Therefore,

$$\int_{\mathbb{T}'} p^\infty_{K,m}(\xi_t)\,\mathrm{d}t < \infty, \qquad K \subseteq \mathsf{M} \text{ compact},\ m \in \mathbb{Z}_{\geq 0}.$$

Since $\Gamma^\infty(\mathsf{E})$ is complete and separable, it now follows from the second lemma above that $t \mapsto \xi_t$ is Bochner integrable on $\mathbb{T}'$. That is, since $\mathbb{T}'$ is arbitrary, $t \mapsto \xi_t$ is locally Bochner integrable.

Next suppose that $t \mapsto \xi_t$ is Bochner integrable on $\mathbb{T}$. By the second lemma above,

$$\int_{\mathbb{T}'} p^\infty_{K,m}(\xi_t)\,\mathrm{d}t < \infty, \qquad K \subseteq \mathsf{M} \text{ compact},\ m \in \mathbb{Z}_{\geq 0}.$$

Therefore, since

$$\|j_m\xi_t(x)\|_{\overline{\mathbb{G}}_m} \leq p^\infty_{K,m}(\xi_t), \qquad (t,x) \in \mathbb{T}' \times K,$$

we conclude that ξ is locally integrally C^∞-bounded since $\mathbb{T}'$ is arbitrary.

(v) $\iff$ (vi) We recall our discussion of von Neumann bounded sets in locally convex topological vector spaces preceding Lemma 3.1 above. With this in mind and using Lemma 3.1, this part of the theorem follows immediately. ■

Note that Theorem 6.3 applies, in particular, to vector fields and functions, giving the classes $\mathrm{CF}^\infty(\mathbb{T};\mathsf{M})$, $\mathrm{LIC}^\infty(\mathbb{T};\mathsf{M})$, and $\mathrm{LBC}^\infty(\mathbb{T};\mathsf{M})$ of functions, and the classes $\mathrm{CF}\Gamma^\infty(\mathbb{T};\mathsf{TM})$, $\mathrm{LI}\Gamma^\infty(\mathbb{T};\mathsf{TM})$, and $\mathrm{LB}\Gamma^\infty(\mathbb{T};\mathsf{TM})$ of vector fields. Noting that we have the alternative weak-$\mathscr{L}$ characterisation of the CO^∞-topology, we can summarise the various sorts of measurability, integrability, and boundedness for smooth time-varying vector fields as follows. In the statement of the result, ev_x is the "evaluate at x" map for both functions and vector fields.

6.4 Theorem: (Weak characterisations of measurability, integrability, and boundedness of smooth time-varying vector fields) *Let M be a smooth manifold, let $\mathbb{T} \subseteq \mathbb{R}$ be a time-domain, and let $X\colon \mathbb{T} \times \mathsf{M} \to \mathsf{TM}$ have the property that X_t is a smooth vector field for each $t \in \mathbb{T}$. Then the following four statements are equivalent:*

(i) $t \mapsto X_t$ is measurable;

(ii) $t \mapsto \mathscr{L}_{X_t}f$ is measurable for every $f \in \mathrm{C}^\infty(\mathsf{M})$;

(iii) $t \mapsto \mathrm{ev}_x \circ X_t$ is measurable for every $x \in \mathsf{M}$;

(iv) $t \mapsto \mathrm{ev}_x \circ \mathscr{L}_{X_t}f$ is measurable for every $f \in \mathrm{C}^\infty(\mathsf{M})$ and every $x \in \mathsf{M}$,

the following two statements are equivalent:

(v) $t \mapsto X_t$ is locally Bochner integrable;

(vi) $t \mapsto \mathscr{L}_{X_t}f$ is locally Bochner integrable for every $f \in \mathrm{C}^\infty(\mathsf{M})$,

and the following two statements are equivalent:

(vii) $t \mapsto X_t$ is locally essentially von Neumann bounded;

(viii) $t \mapsto \mathscr{L}_{X_t}f$ is locally essentially von Neumann bounded for every $f \in \mathrm{C}^\infty(\mathsf{M})$.

Proof: This follows from Theorem 6.3, along with Corollary 3.6. ■

Let us now discuss flows of vector fields from $\mathrm{LI}\Gamma^\infty(\mathbb{T};\mathsf{TM})$. To do so, let us provide the definition of the usual attribute of integral curves, but on manifolds.

6.5 Definition: (Locally absolutely continuous) Let M be a smooth manifold and let $\mathbb{T} \subseteq \mathbb{R}$ be an interval.

(i) A function $f\colon [a,b] \to \mathbb{R}$ is ***absolutely continuous*** if there exists $g \in \mathrm{L}^1([a,b];\mathbb{R})$ such that
$$f(t) = f(a) + \int_a^t g(\tau)\,\mathrm{d}\tau, \qquad t \in [a,b].$$

(ii) A function $f\colon \mathbb{T} \to \mathbb{R}$ is ***locally absolutely continuous*** if $f|\mathbb{T}'$ is absolutely continuous for every compact subinterval $\mathbb{T}' \subseteq \mathbb{T}$.

(iii) A curve $\gamma\colon \mathbb{T} \to \mathsf{M}$ is ***locally absolutely continuous*** if $\phi \circ \gamma$ is locally absolutely continuous for every $\phi \in \mathrm{C}^\infty(\mathsf{M})$. •

One easily verifies that a curve is locally absolutely continuous according to our definition if and only if its local representative is locally absolutely continuous in any coordinate chart.

We then have the following existence, uniqueness, and regularity result for locally integrally bounded vector fields. In the statement of the result, we use the notation
$$|a,b| = \begin{cases} [a,b], & a \leq b, \\ [b,a], & b < a. \end{cases}$$

In the following result, we do not provide the comprehensive list of properties of the flow, but only those required to make sense of its regularity with respect to initial conditions, as per our specification 3 for our theory in Section 1.2.

6.6 Theorem: (Flows of vector fields from $\mathrm{LI}\Gamma^\infty(\mathbb{T};\mathsf{TM})$) *Let M be a smooth manifold, let $\mathbb{T}$ be an interval, and let $X \in \mathrm{LI}\Gamma^\infty(\mathbb{T};\mathsf{TM})$. Then there exist a subset $D_X \subseteq \mathbb{T} \times \mathbb{T} \times \mathsf{M}$ and a map $\Phi^X\colon D_X \to \mathsf{M}$ with the following properties for each $(t_0,x_0) \in \mathbb{T} \times \mathsf{M}$:*

(i) the set
$$\mathbb{T}_X(t_0,x_0) = \{t \in \mathbb{T} \mid (t,t_0,x_0) \in D_X\}$$
is an interval;

(ii) there exists a locally absolutely continuous curve $t \mapsto \xi(t)$ satisfying
$$\xi'(t) = X(t,\xi(t)), \quad \xi(t_0) = x_0,$$
for almost all $t \in |t_0,t_1|$ if and only if $t_1 \in \mathbb{T}_X(t_0,x_0)$;

(iii) $\frac{\mathrm{d}}{\mathrm{d}t}\Phi^X(t,t_0,x_0) = X(t,\Phi^X(t,t_0,x_0))$ for almost all $t \in \mathbb{T}_X(t_0,x_0)$;

(iv) for each $t \in \mathbb{T}$ for which $(t,t_0,x_0) \in D_X$, there exists a neighbourhood $\mathcal{U}$ of x_0 such that the mapping $x \mapsto \Phi^X(t,t_0,x)$ is defined and of class C^∞ on $\mathcal{U}$.

Proof: We observe that the requirement that $X \in \mathrm{LI}\Gamma^\infty(\mathbb{T};\mathsf{TM})$ implies that, in any coordinate chart, the components of X and their derivatives are all bounded by a locally integrable function. This, in particular, implies that, in any coordinate chart for M, the ordinary differential equation associated to the vector field X satisfies the usual conditions for existence and uniqueness of solutions as per, for example, [Sontag 1998, Theorem 54]. Of course, the differential equation satisfies conditions much stronger than this, and we shall see how to use these in our argument below.

The first three assertions are now part of the standard existence theorem for solutions of ordinary differential equations, along with the usual Zorn's Lemma argument for the existence of a maximal interval on which integral curves is defined.

In the sequel we denote $\Phi^X_{t,t_0}(x) = \Phi^X(t, t_0, x_0)$.

For the fourth assertion we first make some constructions with vector fields on jet bundles, more or less following [Saunders 1989, §4.4]. We let $\mathsf{M}^2 = \mathsf{M} \times \mathsf{M}$ and we consider M^2 as a fibred manifold, indeed a trivial fibre bundle, over M by $\mathrm{pr}_1 \colon \mathsf{M}^2 \to \mathsf{M}$, i.e., by projection onto the first factor. A section of this fibred manifold is naturally identified with a smooth map $\Phi \colon \mathsf{M} \to \mathsf{M}$ by $x \mapsto (x, \Phi(x))$. We introduce the following notation:

1. $\mathsf{J}^m \mathrm{pr}_1$: the bundle of m-jets of sections of the fibred manifold $\mathrm{pr}_1 \colon \mathsf{M}^2 \to \mathsf{M}$;
2. $\mathsf{V}\,\mathrm{pr}_{1,m}$: the vertical bundle of the fibred manifold $\mathrm{pr}_{1,m} \colon \mathsf{J}^m \mathrm{pr}_1 \to \mathsf{M}$;
3. $\mathsf{V}\,\mathrm{pr}_1$: the vertical bundle of the fibred manifold $\mathrm{pr}_1 \colon \mathsf{M}^2 \to \mathsf{M}$;
4. ν: the projection $\mathrm{pr}_1 \circ (\pi_{\mathsf{TM}^2} | \mathsf{V}\,\mathrm{pr}_1)$;
5. $\mathsf{J}^m\nu$: the bundle of m-jets of sections of the fibred manifold $\nu \colon \mathsf{V}\,\mathrm{pr}_1 \to \mathsf{M}$.

With this notation, we have the following lemma.

1 Lemma: *There is a canonical diffeomorphism* $\alpha_m \colon \mathsf{J}^m\nu \to \mathsf{V}\,\mathrm{pr}_{1,m}$.

Proof: We describe the diffeomorphism, and then note that the verification that it is, in fact, a diffeomorphism is a fact easily checked in jet bundle coordinates.

Let $I \subseteq \mathbb{R}$ be an interval with $0 \in \mathrm{int}(I)$ and consider a smooth map $\phi \colon I \times \mathsf{M} \to \mathsf{M} \times \mathsf{M}$ of the form $\phi(t, x) = (x, \phi_1(t, x))$ for a smooth map ϕ_1. We let $\phi_t(x) = \phi^x(t) = \phi(t, x)$. We then have maps

$$\begin{aligned} j^x_m\phi \colon I &\to \mathsf{J}^m \mathrm{pr}_1 \\ t &\mapsto j_m\phi_t(x) \end{aligned}$$

and

$$\begin{aligned} \phi' \colon \mathsf{M} &\to \mathsf{V}\,\mathrm{pr}_1 \\ x &\mapsto \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} \phi^x(t). \end{aligned}$$

Note that the curve $j^x_m\phi$ is a curve in the fibre of $\mathrm{pr}_{1,m} \colon \mathsf{J}^m \mathrm{pr}_1 \to \mathsf{M}$. Thus we can sensibly define α_m by

$$\alpha_m(j_m\phi'(x)) = \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} j^x_m\phi(t).$$

In jet bundle coordinates, one can check that α_m has the local representative

$$((\boldsymbol{x}_1, (\boldsymbol{x}_2, \boldsymbol{A}_0)), (\boldsymbol{B}_1, \boldsymbol{A}_1, \dots, \boldsymbol{B}_m, \boldsymbol{A}_m)) \mapsto ((\boldsymbol{x}_1, (\boldsymbol{x}_2, \boldsymbol{B}_1, \dots, \boldsymbol{B}_m)), (\boldsymbol{A}_0, \boldsymbol{A}_1, \dots, \boldsymbol{A}_m)),$$

showing that α_m is indeed a diffeomorphism. ▼

Given a smooth vector field Y on M, we define a vector field $\tilde{Y}$ on M^2 by $\tilde{Y}(x_1, x_2) = (0_{x_1}, Y(x_2))$. Note that we have the following commutative diagram

$$\begin{tikzcd} \mathsf{M}^2 \arrow[r, "\tilde{Y}"] \arrow[d, "\mathrm{pr}_1"'] & \mathsf{V}\,\mathrm{pr}_1 \arrow[d, "\nu"] \\ \mathsf{M} \arrow[r, equal] & \mathsf{M} \end{tikzcd}$$

giving $\tilde{Y}$ as a morphism of fibred manifolds. It is thus a candidate to have its m-jet taken, giving a morphism of fibred manifolds $j_m\tilde{Y}\colon \mathsf{J}^m\,\mathrm{pr}_1 \to \mathsf{J}^m\nu$. By the lemma, $\alpha_m \circ j_m\tilde{Y}$ is a vertical vector field on $\mathsf{J}^m\,\mathrm{pr}_1$ that we denote by $\nu_m Y$, the ***mth vertical prolongation*** of Y. Let us verify that this is a vector field. First of all, for a section $\tilde{\Phi}$ of pr_1 given by $x \mapsto (x, \Phi(x))$, note that $\tilde{Y} \circ \tilde{\Phi}(x) = (0_x, Y(\Phi(x)))$, and so $j_m(\tilde{Y} \circ \tilde{\Phi})(x)$ is vertical. By the notation from the proof of the lemma, we can write $j_m(\tilde{Y} \circ \tilde{\Phi})(x) = j_m\phi'(x)$ for some suitable map ϕ as in the lemma. We then have

$$\alpha_m \circ j_m(\tilde{Y} \circ \tilde{\Phi})(x) = \alpha_m(j_m\phi'(x)) \in \mathsf{V}_{j_m\tilde{\Phi}(x)}\,\mathrm{pr}_{1,m}.$$

Therefore,

$$\pi_{\mathsf{T}\mathsf{J}^m\,\mathrm{pr}_1}(\alpha_m \circ j_m\tilde{Y}(j_m\tilde{\Phi}(x))) = \pi_{\mathsf{T}\mathsf{J}^m\,\mathrm{pr}_1}(\alpha_m \circ j_m(\tilde{Y} \circ \tilde{\Phi})(x)) = j_m\tilde{\Phi}(x).$$

Note that since $\mathsf{J}^m\,\mathrm{pr}_1$ is naturally identified with $\mathsf{J}^m(\mathsf{M};\mathsf{M})$ via the identification

$$j_m\tilde{\Phi}(x) \mapsto j_m\Phi(x)$$

if $\tilde{\Phi}(x) = (x, \Phi(x))$, we can as well think of $\nu_m Y$ as being a vector field on the latter space. Sorting through all the definitions gives the form of $\nu_m Y$ in coordinates as

$$((\boldsymbol{x}_1, \boldsymbol{x}_2), \boldsymbol{A}_1, \dots, \boldsymbol{A}_m) \mapsto (((\boldsymbol{x}_1, \boldsymbol{x}_2), \boldsymbol{A}_1, \dots, \boldsymbol{A}_m), \boldsymbol{0}, \boldsymbol{Y}, \boldsymbol{D}\boldsymbol{Y}, \dots, \boldsymbol{D}^m\boldsymbol{Y}). \tag{6.1}$$

We now apply the above constructions, for each fixed $t \in \mathbb{T}$, to get the vector field $\nu_m X_t$, and so the time-varying vector field $\nu_m X$ defined by $\nu_m X(t, j_m\Phi(x)) = \nu_m X_t(j_m\Phi(x))$ on $\mathsf{J}^m(\mathsf{M};\mathsf{M})$. The definition of $\mathrm{LI}\Gamma^1(\mathbb{T};\mathsf{TM})$, along with the coordinate formula (6.1), shows that $\nu_m X$ satisfies the standard conditions for existence and uniqueness of integral curves, and so its flow depends continuously on initial condition [Sontag 1998, Theorem 55].

The fourth part of the theorem, therefore, will follow if we can show that

1. for each $m \in \mathbb{Z}_{\geq 0}$, the flow of $\nu_m X$ depends on the initial condition in M in a C^m way,
2. $\Phi^{\nu_m X}_{t,t_0}(j_m\Phi^X_{t_0,t_0}(x_0)) = j_m\Phi^X_{t,t_0}(j_m\Phi^X_{t_0,t_0}(x_0))$, and
3. if $\{t\} \times \{t_0\} \times \mathcal{U} \subseteq D_X$, then $\{t\} \times \{t_0\} \times \mathrm{pr}_{1,m}^{-1}(\mathcal{U}) \subseteq D_{\nu_m X}$.

We ask for property 3 to ensure that the domain of differentiability does not get too small as the order of the derivatives gets large.

To prove these assertions, it suffices to work locally. According to (6.1), we have the time-dependent differential equation defined on

$$\mathcal{U} \times \mathrm{L}(\mathbb{R}^n;\mathbb{R}^n) \times \cdots \times \mathrm{L}^m_{\mathrm{sym}}(\mathbb{R}^n;\mathbb{R}^n),$$

where $\mathcal{U}$ is an open subset of $\mathbb{R}^n$, and given by

$$\begin{aligned}
\dot{\boldsymbol{\gamma}}(t) &= \boldsymbol{X}(t, \boldsymbol{\gamma}(t)),\\
\dot{\boldsymbol{A}}_1(t) &= \boldsymbol{D}\boldsymbol{X}(t, \boldsymbol{\gamma}(t)),\\
\dot{\boldsymbol{A}}_2(t) &= \boldsymbol{D}^2\boldsymbol{X}(t, \boldsymbol{\gamma}(t)),\\
&\ \,\vdots\\
\dot{\boldsymbol{A}}_m(t) &= \boldsymbol{D}^m\boldsymbol{X}(t, \boldsymbol{\gamma}(t)),
\end{aligned}$$

$(t, \boldsymbol{x}) \mapsto (\boldsymbol{x}, \boldsymbol{X}(t, \boldsymbol{x}))$ being the local representative of X. The initial conditions of interest for the vector field $\nu_m X$ are of the form $j_m\Phi^X_{t_0,t_0}(x)$. In coordinates, keeping in mind that $\Phi^X_{t_0,t_0} = \mathrm{id}_{\mathsf{M}}$, this gives

$$\boldsymbol{\gamma}(t_0) = \boldsymbol{x}_0,\ \ \boldsymbol{A}_1(t_0) = \boldsymbol{I}_n,\ \ \boldsymbol{A}_j(t_0) = \boldsymbol{0}, \qquad j \geq 2. \tag{6.2}$$

Let us denote by $t \mapsto \boldsymbol{\gamma}(t, t_0, \boldsymbol{x})$ and $t \mapsto \boldsymbol{A}_j(t, t_0, \boldsymbol{x})$, $j \in \{1, \dots, m\}$, the solutions of the differential equations above with these initial conditions.

We will show that assertions 1–3 hold by induction on m. In doing this, we will need to understand how differential equations depending differentiably on state also have solutions depending differentiably on initial condition. Such a result is not readily found in the textbook literature, as this latter is typically concerned with continuous dependence on initial conditions for cases with measurable time-dependence, and on differentiable dependence when the dependence on time is also differentiable. However, the general case (much more general than we need here) is worked out by Schuricht and von der Mosel [2000].

For $m = 0$, the assertions are simply the result of the usual continuous dependence on initial conditions [e.g., Sontag 1998, Theorem 55]. Let us consider the case $m = 1$. In this case, the properties of $\mathrm{LI}\Gamma^\infty(\mathbb{T}; \mathsf{TM})$ ensure that the hypotheses required to apply Theorem 2.1 of [Schuricht and von der Mosel 2000] hold for the differential equation

$$\begin{aligned} \dot{\boldsymbol{\gamma}}(t) &= \boldsymbol{X}(t, \boldsymbol{\gamma}(t)), \\ \dot{\boldsymbol{A}}_1(t) &= \boldsymbol{D}\boldsymbol{X}(t, \boldsymbol{\gamma}(t)). \end{aligned}$$

This allows us to conclude that $\boldsymbol{x} \mapsto \boldsymbol{\gamma}(t, t_0, \boldsymbol{x})$ is of class C^1. This establishes the assertion 1 in this case. Therefore, on a suitable domain, $j_1\Phi^X_{t,t_0}$ is well-defined. In coordinates the map $j_1\Phi^X_{t,t_0} \colon \mathsf{J}^1(\mathsf{M}; \mathsf{M}) \to \mathsf{J}^1(\mathsf{M}; \mathsf{M})$ is given by

$$(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{B}_1) \mapsto (\boldsymbol{x}, \boldsymbol{\gamma}(t, t_0, \boldsymbol{x}), \boldsymbol{D}_3\boldsymbol{\gamma}(t, t_0, \boldsymbol{x}) \circ \boldsymbol{B}_1), \tag{6.3}$$

this by the Chain Rule. We have

$$\frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{D}_3\boldsymbol{\gamma}(t, t_0, \boldsymbol{x}) = \boldsymbol{D}_3(\tfrac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{\gamma}(t, t_0, \boldsymbol{x})) = \boldsymbol{D}\boldsymbol{X}(t, \boldsymbol{\gamma}(t, t_0, \boldsymbol{x})),$$

the swapping of the time and spatial derivatives being valid by [Schuricht and von der Mosel 2000, Corollary 2.2]. Combining this with (6.3) and the initial conditions (6.2) shows that assertion 2 holds for $m = 1$. Moreover, since $\boldsymbol{A}_1(t, t_0, \boldsymbol{x})$ is obtained by merely integrating a continuous function of t from t_0 to t, we also conclude that assertion 3 holds.

Now suppose that assertions 1–3 hold for m. Again, the properties of $\mathrm{LI}\Gamma^\infty(\mathbb{T}; \mathsf{TM})$ imply that the hypotheses of Theorem 2.1 of [Schuricht and von der Mosel 2000] hold, and so solutions of the differential equation

$$\begin{aligned} \dot{\boldsymbol{\gamma}}(t) &= \boldsymbol{X}(t, \boldsymbol{\gamma}(t)), \\ \dot{\boldsymbol{A}}_1(t) &= \boldsymbol{D}\boldsymbol{X}(t, \boldsymbol{\gamma}(t)), \\ \dot{\boldsymbol{A}}_2(t) &= \boldsymbol{D}^2\boldsymbol{X}(t, \boldsymbol{\gamma}(t)), \\ &\ \ \vdots \\ \dot{\boldsymbol{A}}_m(t) &= \boldsymbol{D}^m\boldsymbol{X}(t, \boldsymbol{\gamma}(t)) \end{aligned}$$

depend continuously differentiably on initial condition. By the induction hypothesis applied to the assertion 2, this means that

$$(t,x) \mapsto \Phi^{\nu_m X}_{t,t_0}(j_m\Phi^X_{t_0,t_0}(x)) = j_m\Phi^X_{t,t_0}(x)$$

depends continuously differentiably on x, and so we conclude that $(t,x) \mapsto \Phi^X_{t,t_0}(x)$ depends on x in a C^{m+1} manner. This establishes assertion 1 for $m+1$. After an application of the Chain Rule for high-order derivatives (see [Abraham, Marsden, and Ratiu 1988, Supplement 2.4A]) we can, admittedly after just a few moments thought, see that the local representative of $j_{m+1}\Phi^X_{t,t_0}(j_{m+1}\Phi^X_{t_0,t_0}(x))$ is

$$(\boldsymbol{x}, \boldsymbol{\gamma}(t,t_0,\boldsymbol{x}), \boldsymbol{D}_3\boldsymbol{\gamma}(t,t_0,\boldsymbol{x}), \dots, \boldsymbol{D}_3^{m+1}\boldsymbol{\gamma}(t,t_0,\boldsymbol{x})),$$

keeping in mind the initial conditions (6.2) in coordinates.

By the induction hypothesis,

$$\frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{D}_3^j\boldsymbol{\gamma}(t) = \boldsymbol{D}^j\boldsymbol{X}(t,\boldsymbol{\gamma}(t,t_0,\boldsymbol{x})), \qquad j \in \{1,\dots,m\}.$$

Using Corollary 2.2 of [Schuricht and von der Mosel 2000] we compute

$$\frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{D}_3^{m+1}\boldsymbol{\gamma}(t,t_0,\boldsymbol{x}) = \boldsymbol{D}(\tfrac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{D}_3^m\boldsymbol{\gamma}(t,t_0,\boldsymbol{x})) = \boldsymbol{D}^{m+1}\boldsymbol{X}(t,\boldsymbol{\gamma}(t,t_0,\boldsymbol{x})),$$

giving assertion 2 for $m+1$. Finally, by the induction hypothesis and since $\boldsymbol{A}_{m+1}(t,t_0,\boldsymbol{x})$ is obtained by simple integration from t_0 to t, we conclude that assertion 3 holds for $m+1$. ■

6.2. The finitely differentiable or Lipschitz case. The requirement that the flow depends smoothly on initial conditions is not always essential, even when the vector field itself depends smoothly on the state. In such cases as this, one may want to consider classes of vector fields characterised by one of the weaker topologies described in Section 3.4. Let us see how to do this. In this section, so as to be consistent with our definition of Lipschitz norms in Section 3.5, we suppose that the affine connection ∇ on M is the Levi-Civita connection for the Riemannian metric $\mathbb{G}$ and that the vector bundle connection ∇^0 in E is $\mathbb{G}_0$-orthogonal.

6.7 Definition: (Finitely differentiable or Lipschitz Carathéodory section) Let $\pi\colon \mathsf{E} \to \mathsf{M}$ be a smooth vector bundle and let $\mathbb{T} \subseteq \mathbb{R}$ be an interval. Let $m \in \mathbb{Z}_{\geq 0}$ and let $m' \in \{0, \mathrm{lip}\}$. A ***Carathéodory section of class*** $\mathbf{C}^{m+m'}$ of E is a map $\xi\colon \mathbb{T} \times \mathsf{M} \to \mathsf{E}$ with the following properties:

(i) $\xi(t,x) \in \mathsf{E}_x$ for each $(t,x) \in \mathbb{T} \times \mathsf{M}$;

(ii) for each $t \in \mathbb{T}$, the map $\xi_t\colon \mathsf{M} \to \mathsf{E}$ defined by $\xi_t(x) = \xi(t,x)$ is of class $\mathrm{C}^{m+m'}$;

(iii) for each $x \in \mathsf{M}$, the map $\xi^x\colon \mathbb{T} \to \mathsf{E}$ defined by $\xi^x(t) = \xi(t,x)$ is Lebesgue measurable.

We shall call $\mathbb{T}$ the ***time-domain*** for the section. By $\mathrm{CF}\Gamma^{m+m'}(\mathbb{T};\mathsf{E})$ we denote the set of Carathéodory sections of class $\mathrm{C}^{m+m'}$ of E. •

Now we put some conditions on the time dependence of the derivatives of the section.

6.8 Definition: (Locally integrally $\mathrm{C}^{m+m'}$-bounded and locally essentially $\mathrm{C}^{m+m'}$-bounded sections) Let $\pi\colon \mathsf{E}\to\mathsf{M}$ be a smooth vector bundle and let $\mathbb{T}\subseteq\mathbb{R}$ be an interval. Let $m\in\mathbb{Z}_{\geq 0}$ and let $m'\in\{0,\mathrm{lip}\}$. A Carathéodory section $\xi\colon\mathbb{T}\times\mathsf{M}\to\mathsf{E}$ of class $\mathrm{C}^{m+m'}$ is

(i) ***locally integrally $\mathrm{C}^{m+m'}$-bounded*** if:

(a) $m'=0$: for every compact set $K\subseteq\mathsf{M}$, there exists $g\in\mathrm{L}^1_{\mathrm{loc}}(\mathbb{T};\mathbb{R}_{\geq 0})$ such that

$$\|j_m\xi_t(x)\|_{\overline{\mathbb{G}}_m}\leq g(t),\qquad (t,x)\in\mathbb{T}\times K;$$

(b) $m'=\mathrm{lip}$: for every compact set $K\subseteq\mathsf{M}$, there exists $g\in\mathrm{L}^1_{\mathrm{loc}}(\mathbb{T};\mathbb{R}_{\geq 0})$ such that

$$\operatorname{dil} j_m\xi_t(x),\|j_m\xi_t(x)\|_{\overline{\mathbb{G}}_m}\leq g(t),\qquad (t,x)\in\mathbb{T}\times K,$$

and is

(ii) ***locally essentially $\mathrm{C}^{m+m'}$-bounded*** if:

(a) $m'=0$: for every compact set $K\subseteq\mathsf{M}$, there exists $g\in\mathrm{L}^\infty_{\mathrm{loc}}(\mathbb{T};\mathbb{R}_{\geq 0})$ such that

$$\|j_m\xi_t(x)\|_{\overline{\mathbb{G}}_m}\leq g(t),\qquad (t,x)\in\mathbb{T}\times K;$$

(b) $m'=\mathrm{lip}$: for every compact set $K\subseteq\mathsf{M}$, there exists $g\in\mathrm{L}^\infty_{\mathrm{loc}}(\mathbb{T};\mathbb{R}_{\geq 0})$ such that

$$\operatorname{dil} j_m\xi_t(x),\|j_m\xi_t(x)\|_{\overline{\mathbb{G}}_m}\leq g(t),\qquad (t,x)\in\mathbb{T}\times K.$$

The set of locally integrally $\mathrm{C}^{m+m'}$-bounded sections of E with time-domain $\mathbb{T}$ is denoted by $\mathrm{LI}\Gamma^{m+m'}(\mathbb{T},\mathsf{E})$ and the set of locally essentially $\mathrm{C}^{m+m'}$-bounded sections of E with time-domain $\mathbb{T}$ is denoted by $\mathrm{LB}\Gamma^{m+m'}(\mathbb{T};\mathsf{E})$. •

6.9 Theorem: (Topological characterisation of finitely differentiable or Lipschitz Carathéodory sections) *Let $\pi\colon\mathsf{E}\to\mathsf{M}$ be a smooth vector bundle and let $\mathbb{T}\subseteq\mathbb{R}$ be an interval. Let $m\in\mathbb{Z}_{\geq 0}$ and let $m'\in\{0,\mathrm{lip}\}$. For a map $\xi\colon\mathbb{T}\times\mathsf{M}\to\mathsf{E}$ satisfying $\xi(t,x)\in\mathsf{E}_x$ for each $(t,x)\in\mathbb{T}\times\mathsf{M}$, the following two statements are equivalent:*

(i) $\xi\in\mathrm{CF}\Gamma^{m+m'}(\mathbb{T};\mathsf{E})$*;*

(ii) the map $\mathbb{T}\ni t\mapsto\xi_t\in\Gamma^{m+m'}(\mathsf{E})$ *is measurable,*

the following two statements are equivalent:

(iii) $\xi\in\mathrm{LI}\Gamma^{m+m'}(\mathbb{T};\mathsf{E})$*;*

(iv) the map $\mathbb{T}\ni t\mapsto\xi_t\in\Gamma^{m+m'}(\mathsf{E})$ *is measurable and locally Bochner integrable,*

and the following two statements are equivalent:

(v) $\xi\in\mathrm{LB}\Gamma^{m+m'}(\mathbb{T};\mathsf{E})$*;*

(vi) the map $\mathbb{T}\ni t\mapsto\xi_t\in\Gamma^{m+m'}(\mathsf{E})$ *is measurable and locally essentially von Neumann bounded.*

Proof: (i) $\iff$ (ii) For $x\in\mathsf{M}$ and $\alpha_x\in\mathsf{E}^*_x$, define $\mathrm{ev}_{\alpha_x}\colon\Gamma^{m+m'}(\mathsf{E})\to\mathbb{R}$ by $\mathrm{ev}_{\alpha_x}(\xi)=\langle\alpha_x;\xi(x)\rangle$. It is easy to show that ev_{α_x} is continuous and that the set of continuous functionals ev_{α_x}, $\alpha_x\in\mathsf{E}^*_x$, is point separating. Since $\Gamma^{m+m'}(\mathsf{E})$ is a Suslin space (properties

CO^m-6 and $\mathrm{CO}^{m+\mathrm{lip}}$-6), this part of the theorem follows in the same manner as the corresponding part of Theorem 6.3.

(iii) $\iff$ (iv) Since $\Gamma^{m+m'}(\mathsf{E})$ is complete and separable (by properties CO^m-2 and CO^m-4, and $\mathrm{CO}^{m+\mathrm{lip}}$-2 and $\mathrm{CO}^{m+\mathrm{lip}}$-4), the arguments from the corresponding part of Theorem 6.3 apply here, taking note of the definition of the seminorms $p_K^{\mathrm{lip}}(\xi)$ in case $m' = \mathrm{lip}$.

(v) $\iff$ (vi) We recall our discussion of von Neumann bounded sets in locally convex topological vector spaces preceding Lemma 3.1 above. With this in mind and using Lemma 4.3, this part of the proposition follows immediately. ■

Note that Theorem 6.9 applies, in particular, to vector fields and functions, giving the classes $\mathrm{CF}^{m+m'}(\mathbb{T};\mathsf{M})$, $\mathrm{LIC}^{m+m'}(\mathbb{T};\mathsf{M})$, and $\mathrm{LBC}^{m+m'}(\mathbb{T};\mathsf{M})$ of functions, and the classes $\mathrm{CF}\Gamma^{m+m'}(\mathbb{T};\mathsf{TM})$, $\mathrm{LI}\Gamma^{m+m'}(\mathbb{T};\mathsf{TM})$, and $\mathrm{LB}\Gamma^{m+m'}(\mathbb{T};\mathsf{TM})$ of vector fields. Noting that we have the alternative weak-$\mathscr{L}$ characterisation of the $\mathrm{CO}^{m+m'}$-topology, we can summarise the various sorts of measurability, integrability, and boundedness for smooth time-varying vector fields as follows. In the statement of the result, ev_x is the "evaluate at x" map for both functions and vector fields.

6.10 Theorem: (Weak characterisations of measurability, integrability, and boundedness of finitely differentiable or Lipschitz time-varying vector fields) *Let M be a smooth manifold, let $\mathbb{T} \subseteq \mathbb{R}$ be a time-domain, let $m \in \mathbb{Z}_{\geq 0}$, let $m' \in \{0, \mathrm{lip}\}$, and let $X : \mathbb{T} \times \mathsf{M} \to \mathsf{TM}$ have the property that X_t is a vector field of class $\mathrm{C}^{m+m'}$ for each $t \in \mathbb{T}$. Then the following four statements are equivalent:*

(i) $t \mapsto X_t$ is measurable;

(ii) $t \mapsto \mathscr{L}_{X_t} f$ is measurable for every $f \in \mathrm{C}^\infty(\mathsf{M})$;

(iii) $t \mapsto \mathrm{ev}_x \circ X_t$ is measurable for every $x \in \mathsf{M}$;

(iv) $t \mapsto \mathrm{ev}_x \circ \mathscr{L}_{X_t} f$ is measurable for every $f \in \mathrm{C}^\infty(\mathsf{M})$ and every $x \in \mathsf{M}$,

the following two statements are equivalent:

(v) $t \mapsto X_t$ is locally Bochner integrable;

(vi) $t \mapsto \mathscr{L}_{X_t} f$ is locally Bochner integrable for every $f \in \mathrm{C}^\infty(\mathsf{M})$,

and the following two statements are equivalent:

(vii) $t \mapsto X_t$ is locally essentially von Neumann bounded;

(viii) $t \mapsto \mathscr{L}_{X_t} f$ is locally essentially von Neumann bounded for every $f \in \mathrm{C}^\infty(\mathsf{M})$.

Proof: This follows from Theorem 6.9, along with Corollaries 3.9 and 3.15. ■

It is also possible to state an existence, uniqueness, and regularity theorem for flows of vector fields that depend on state in a finitely differentiable or Lipschitz manner.

6.11 Theorem: (Flows of vector fields from $\mathbf{LI}\boldsymbol{\Gamma}^{m+m'}(\mathbb{T};\mathsf{TM})$) *Let M be a smooth manifold, let $\mathbb{T}$ be an interval, let $m \in \mathbb{Z}_{\geq 0}$, and let $X \in \mathrm{LI}\Gamma^{m+\mathrm{lip}}(\mathbb{T};\mathsf{TM})$. Then there exist a subset $D_X \subseteq \mathbb{T} \times \mathbb{T} \times \mathsf{M}$ and a map $\Phi^X : D_X \to \mathsf{M}$ with the following properties for each $(t_0, x_0) \in \mathbb{T} \times \mathsf{M}$:*

(i) the set

$$\mathbb{T}_X(t_0, x_0) = \{t \in \mathbb{T} \mid (t, t_0, x_0) \in D_X\}$$

is an interval;

(ii) *there exists a locally absolutely continuous curve* $t \mapsto \xi(t)$ *satisfying*

$$\xi'(t) = X(t, \xi(t)), \quad \xi(t_0) = x_0,$$

for almost all $t \in |t_0, t_1|$ *if and only if* $t_1 \in \mathbb{T}_X(t_0, x_0)$*;*

(iii) $\frac{\mathrm{d}}{\mathrm{d}t}\Phi^X(t, t_0, x_0) = X(t, \Phi^X(t, t_0, x_0))$ *for almost all* $t \in \mathbb{T}_X(t_0, x_0)$*;*

(iv) *for each* $t \in \mathbb{T}$ *for which* $(t, t_0, x_0) \in D_X$*, there exists a neighbourhood* $\mathcal{U}$ *of* x_0 *such that the mapping* $x \mapsto \Phi^X(t, t_0, x)$ *is defined and of class* C^m *on* $\mathcal{U}$*.*

Proof: The proof here is by truncation of the proof of Theorem 6.6 from "∞" to "m." ■

6.3. The holomorphic case. While we are not *per se* interested in time-varying holomorphic vector fields, our understanding of time-varying real analytic vector fields—in which we are most definitely interested—is connected with an understanding of the holomorphic case, cf. Theorem 6.25.

We begin with definitions that are similar to the smooth case, but which rely on the holomorphic topologies introduced in Section 4.1. We will consider an holomorphic vector bundle $\pi \colon \mathsf{E} \to \mathsf{M}$ with an Hermitian fibre metric $\mathbb{G}$. This defines the seminorms p_K^{hol}, $K \subseteq \mathsf{M}$ compact, describing the $\mathrm{CO}^{\mathrm{hol}}$-topology for $\Gamma^{\mathrm{hol}}(\mathsf{E})$ as in Section 4.1.

Let us get started with the definitions.

6.12 Definition: (Holomorphic Carathéodory section) Let $\pi \colon \mathsf{E} \to \mathsf{M}$ be an holomorphic vector bundle and let $\mathbb{T} \subseteq \mathbb{R}$ be an interval. A ***Carathéodory section of class*** $\mathbf{C}^{\mathbf{hol}}$ of E is a map $\xi \colon \mathbb{T} \times \mathsf{M} \to \mathsf{E}$ with the following properties:

(i) $\xi(t, z) \in \mathsf{E}_z$ for each $(t, z) \in \mathbb{T} \times \mathsf{M}$;

(ii) for each $t \in \mathbb{T}$, the map $\xi_t \colon \mathsf{M} \to \mathsf{E}$ defined by $\xi_t(z) = \xi(t, z)$ is of class $\mathrm{C}^{\mathrm{hol}}$;

(iii) for each $z \in \mathsf{M}$, the map $\xi^z \colon \mathbb{T} \to \mathsf{E}$ defined by $\xi^z(t) = \xi(t, z)$ is Lebesgue measurable.

We shall call $\mathbb{T}$ the ***time-domain*** for the section. By $\mathrm{CF}\Gamma^{\mathrm{hol}}(\mathbb{T}; \mathsf{E})$ we denote the set of Carathéodory sections of class $\mathrm{C}^{\mathrm{hol}}$ of E. •

The associated notions for time-dependent sections compatible with the $\mathrm{CO}^{\mathrm{hol}}$-topology are as follows.

6.13 Definition: (Locally integrally $\mathbf{C}^{\mathbf{hol}}$-bounded and locally essentially $\mathbf{C}^{\mathbf{hol}}$-bounded sections) Let $\pi \colon \mathsf{E} \to \mathsf{M}$ be an holomorphic vector bundle and let $\mathbb{T} \subseteq \mathbb{R}$ be an interval. A Carathéodory section $\xi \colon \mathbb{T} \times \mathsf{M} \to \mathsf{E}$ of class $\mathrm{C}^{\mathrm{hol}}$ is

(i) ***locally integrally*** $\mathbf{C}^{\mathbf{hol}}$***-bounded*** if, for every compact set $K \subseteq \mathsf{M}$, there exists $g \in \mathrm{L}^1_{\mathrm{loc}}(\mathbb{T}; \mathbb{R}_{\geq 0})$ such that

$$\|\xi(t, z)\|_{\mathbb{G}} \leq g(t), \qquad (t, z) \in \mathbb{T} \times K$$

and is

(ii) ***locally essentially*** $\mathbf{C}^{\mathbf{hol}}$***-bounded*** if, for every compact set $K \subseteq \mathsf{M}$, there exists $g \in \mathrm{L}^\infty_{\mathrm{loc}}(\mathbb{T}; \mathbb{R}_{\geq 0})$ such that

$$\|\xi(t, z)\|_{\mathbb{G}} \leq g(t), \qquad (t, z) \in \mathbb{T} \times K.$$

The set of locally integrally $\mathrm{C}^{\mathrm{hol}}$-bounded sections of E with time-domain $\mathbb{T}$ is denoted by $\mathrm{LI}\Gamma^{\mathrm{hol}}(\mathbb{T},\mathsf{E})$ and the set of locally essentially $\mathrm{C}^{\mathrm{hol}}$-bounded sections of E with time-domain $\mathbb{T}$ is denoted by $\mathrm{LB}\Gamma^{\mathrm{hol}}(\mathbb{T};\mathsf{E})$. •

As with smooth sections, the preceding definitions admit topological characterisations, now using the $\mathrm{CO}^{\mathrm{hol}}$-topology for $\Gamma^{\mathrm{hol}}(\mathsf{E})$.

6.14 Theorem: (Topological characterisation of holomorphic Carathéodory sections) *Let $\pi\colon \mathsf{E}\to\mathsf{M}$ be an holomorphic vector bundle and let $\mathbb{T}\subseteq\mathbb{R}$ be an interval. For a map $\xi\colon\mathbb{T}\times\mathsf{M}\to\mathsf{E}$ satisfying $\xi(t,z)\in\mathsf{E}_z$ for each $(t,z)\in\mathbb{T}\times\mathsf{M}$, the following two statements are equivalent:*

(i) $\xi\in\mathrm{CF}\Gamma^{\mathrm{hol}}(\mathbb{T};\mathsf{E})$*;*

(ii) the map $\mathbb{T}\ni t\mapsto\xi_t\in\Gamma^{\mathrm{hol}}(\mathsf{E})$ *is measurable,*

the following two statements are equivalent:

(iii) $\xi\in\mathrm{LI}\Gamma^{\mathrm{hol}}(\mathbb{T};\mathsf{E})$*;*

(iv) the map $\mathbb{T}\ni t\mapsto\xi_t\in\Gamma^{\mathrm{hol}}(\mathsf{E})$ *is measurable and locally Bochner integrable,*

and the following two statements are equivalent:

(v) $\xi\in\mathrm{LB}\Gamma^{\mathrm{hol}}(\mathbb{T};\mathsf{E})$*;*

(vi) the map $\mathbb{T}\ni t\mapsto\xi_t\in\Gamma^{\mathrm{hol}}(\mathsf{E})$ *is measurable and locally essentially von Neumann bounded.*

Proof: (i) $\Longleftrightarrow$ (ii) For $z\in\mathsf{M}$ and $\alpha_z\in\mathsf{E}_z^*$, define $\mathrm{ev}_{\alpha_z}\colon\Gamma^{\mathrm{hol}}(\mathsf{E})\to\mathbb{C}$ by $\mathrm{ev}_{\alpha_z}(\xi)=\langle\alpha_z;\xi(z)\rangle$. It is easy to show that ev_{α_z} is continuous and that the set of continuous functionals ev_{α_z}, $\alpha_z\in\mathsf{E}_z^*$, is point separating. Since $\Gamma^{\mathrm{hol}}(\mathsf{E})$ is a Suslin space by $\mathrm{CO}^{\mathrm{hol}}$-6, this part of the theorem follows in the same manner as the corresponding part of Theorem 6.3.

(iii) $\Longleftrightarrow$ (iv) Since $\Gamma^{\mathrm{hol}}(\mathsf{E})$ is complete and separable (by properties $\mathrm{CO}^{\mathrm{hol}}$-2 and $\mathrm{CO}^{\mathrm{hol}}$-4), the arguments from the corresponding part of Theorem 6.3 apply here.

(v) $\Longleftrightarrow$ (vi) We recall our discussion of von Neumann bounded sets in locally convex topological vector spaces preceding Lemma 3.1 above. With this in mind and using Lemma 4.3, this part of the proposition follows immediately. ■

Since holomorphic vector bundles are smooth vector bundles (indeed, real analytic vector bundles), we have natural inclusions

$$\mathrm{LI}\Gamma^{\mathrm{hol}}(\mathbb{T};\mathsf{E})\subseteq\mathrm{CF}\Gamma^{\infty}(\mathbb{T};\mathsf{E}),\qquad \mathrm{LB}\Gamma^{\mathrm{hol}}(\mathbb{T};\mathsf{E})\subseteq\mathrm{CF}\Gamma^{\infty}(\mathbb{T};\mathsf{E}). \tag{6.4}$$

Moreover, by Proposition 4.2 we have the following.

6.15 Proposition: (Time-varying holomorphic sections as time-varying smooth sections) *For an holomorphic vector bundle $\pi\colon\mathsf{E}\to\mathsf{M}$ and an interval $\mathbb{T}$, the inclusions (6.4) actually induce inclusions*

$$\mathrm{LI}\Gamma^{\mathrm{hol}}(\mathbb{T};\mathsf{E})\subseteq\mathrm{LI}\Gamma^{\infty}(\mathbb{T};\mathsf{E}),\qquad \mathrm{LB}\Gamma^{\mathrm{hol}}(\mathbb{T};\mathsf{E})\subseteq\mathrm{LB}\Gamma^{\infty}(\mathbb{T};\mathsf{E}).$$

Note that Theorem 6.14 applies, in particular, to vector fields and functions, giving the classes $\mathrm{CF}^{\mathrm{hol}}(\mathbb{T};\mathsf{M})$, $\mathrm{LIC}^{\mathrm{hol}}(\mathbb{T};\mathsf{M})$, and $\mathrm{LBC}^{\mathrm{hol}}(\mathbb{T};\mathsf{M})$ of functions, and the classes $\mathrm{CF}\Gamma^{\mathrm{hol}}(\mathbb{T};\mathsf{TM})$, $\mathrm{LI}\Gamma^{\mathrm{hol}}(\mathbb{T};\mathsf{TM})$, and $\mathrm{LB}\Gamma^{\mathrm{hol}}(\mathbb{T};\mathsf{TM})$ of vector fields. Unlike in the smooth case preceding and the real analytic case following, there is, in general, not an equivalent weak-$\mathscr{L}$ version of the preceding definitions and results. This is because our Theorem 4.5 on the equivalence of the $\mathrm{CO}^{\mathrm{hol}}$-topology and the corresponding weak-$\mathscr{L}$ topology holds only on Stein manifolds. Let us understand the consequences of this with what we are doing here via an example.

6.16 Example: (Time-varying holomorphic vector fields on compact manifolds) Let M be a compact holomorphic manifold. By [Fritzsche and Grauert 2002, Corollary IV.1.3], the only holomorphic functions on M are the locally constant functions. Therefore, since $\partial f = 0$ for every $f \in \mathrm{C}^{\mathrm{hol}}(\mathsf{M})$, a literal application of the definition shows that, were we to make weak-$\mathscr{L}$ characterisations of vector fields, i.e., give their properties by ascribing those properties to the functions obtained after Lie differentiation, we would have $\mathrm{CF}\Gamma^{\mathrm{hol}}(\mathbb{T};\mathsf{TM})$, and, therefore, also $\mathrm{LI}\Gamma^{\mathrm{hol}}(\mathbb{T};\mathsf{TM})$ and $\mathrm{LB}\Gamma^{\mathrm{hol}}(\mathbb{T};\mathsf{TM})$, consisting of *all* maps $X\colon \mathbb{T}\times\mathsf{M}\to\mathsf{TM}$ satisfying $X(t,z)\in\mathsf{T}_z\mathsf{M}$ for all $z\in\mathsf{M}$. This is not a very useful class of vector fields. •

The following result summarises the various ways of verifying the measurability, integrability, and boundedness of holomorphic time-varying vector fields, taking into account that the preceding example necessitates that we restrict our consideration to Stein manifolds.

6.17 Theorem: (Weak characterisations of measurability, integrability, and boundedness of holomorphic time-varying vector fields) *Let M be a Stein manifold, let $\mathbb{T}\subseteq\mathbb{R}$ be a time-domain, and let $X\colon\mathbb{T}\times\mathsf{M}\to\mathsf{TM}$ have the property that X_t is an holomorphic vector field for each $t\in\mathbb{T}$. Then the following statements are equivalent:*

(i) $t\mapsto X_t$ *is measurable;*

(ii) $t\mapsto \mathscr{L}_{X_t}f$ *is measurable for every* $f\in\mathrm{C}^{\mathrm{hol}}(\mathsf{M})$*;*

(iii) $t\mapsto \mathrm{ev}_z\circ X_t$ *is measurable for every* $z\in\mathsf{M}$*;*

(iv) $t\mapsto \mathrm{ev}_z\circ\mathscr{L}_{X_t}f$ *is measurable for every* $f\in\mathrm{C}^{\mathrm{hol}}(\mathsf{M})$ *and every* $z\in\mathsf{M}$*,*

the following two statements are equivalent:

(v) $t\mapsto X_t$ *is locally Bochner integrable;*

(vi) $t\mapsto \mathscr{L}_{X_t}f$ *is locally Bochner integrable for every* $f\in\mathrm{C}^{\mathrm{hol}}(\mathsf{M})$*,*

and the following two statements are equivalent:

(vii) $t\mapsto X_t$ *is locally essentially von Neumann bounded;*

(viii) $t\mapsto \mathscr{L}_{X_t}f$ *is locally essentially von Neumann bounded for every* $f\in\mathrm{C}^{\mathrm{hol}}(\mathsf{M})$*.*

Proof: This follows from Theorem 6.14, along with Corollary 4.6. ■

Now we consider flows for the class of time-varying holomorphic vector fields defined above. Let $X\in\mathrm{LI}\Gamma^{\mathrm{hol}}(\mathbb{T};\mathsf{TM})$. According to Proposition 6.15, we can define the flow of X just as in the real case, and we shall continue to use the notation $D_X\subseteq\mathbb{T}\times\mathbb{T}\times\mathsf{M}$, Φ^X_{t,t_0}, and $\Phi^X\colon D_X\to\mathsf{M}$ as in the smooth case. The following result provides the attributes of the flow in the holomorphic case. This result follows easily from the constructions in the usual existence and uniqueness theorem for ordinary differential equations, but we could not find the result explicitly in the literature for measurable time-dependence. Thus we provide the details here.

6.18 Theorem: (Flows of vector fields from $\mathrm{LI}\Gamma^{\mathrm{hol}}(\mathbb{T};\mathsf{TM})$) *Let M be an holomorphic manifold, let $\mathbb{T}$ be an interval, and let $X \in \mathrm{LI}\Gamma^{\mathrm{hol}}(\mathbb{T};\mathsf{TM})$. Then there exist a subset $D_X \subseteq \mathbb{T}\times\mathbb{T}\times\mathsf{M}$ and a map $\Phi^X\colon D_X \to \mathsf{M}$ with the following properties for each $(t_0, z_0) \in \mathbb{T}\times\mathsf{M}$:*

(i) the set

$$\mathbb{T}_X(t_0, z_0) = \{t \in \mathbb{T} \mid \ (t, t_0, z_0) \in D_X\}$$

is an interval;

(ii) there exists a locally absolutely continuous curve $t \mapsto \xi(t)$ satisfying

$$\xi'(t) = X(t, \xi(t)), \quad \xi(t_0) = z_0,$$

for almost all $t \in |t_0, t_1|$ if and only if $t_1 \in \mathbb{T}_X(t_0, z_0)$;

(iii) $\frac{\mathrm{d}}{\mathrm{d}t}\Phi^X(t, t_0, z_0) = X(t, \Phi^X(t, t_0, z_0))$ for almost all $t \in \mathbb{T}_X(t_0, z_0)$;

(iv) for each $t \in \mathbb{T}$ for which $(t, t_0, z_0) \in D_X$, there exists a neighbourhood $\mathcal{U}$ of z_0 such that the mapping $z \mapsto \Phi^X(t, t_0, z)$ is defined and of class $\mathrm{C}^{\mathrm{hol}}$ on $\mathcal{U}$.

Proof: Given Proposition 6.15, the only part of the theorem that does not follow from Theorem 6.6 is the holomorphic dependence on initial conditions. This is a local assertion, so we let $(\mathcal{U}, \phi)$ be an holomorphic chart for M with coordinates denoted by $(z^1, \dots, z^n)$. We denote by $\boldsymbol{X}\colon \mathbb{T}\times\phi(\mathcal{U}) \to \mathbb{C}^n$ the local representative of X. By Proposition 6.15, this local representative is locally integrally C^∞-bounded. To prove holomorphicity of the flow, we recall the construction for the existence and uniqueness theorem for the solutions of the initial value problem

$$\dot{\boldsymbol{\gamma}}(t) = \boldsymbol{X}(t, \boldsymbol{\gamma}(t)), \qquad \boldsymbol{\gamma}(t_0) = \boldsymbol{z},$$

see [e.g., Schuricht and von der Mosel 2000, §1.2]. On some suitable product domain $\mathbb{T}'\times\mathsf{B}(r, \boldsymbol{z}_0)$ (the ball being contained in $\phi(\mathcal{U}) \subseteq \mathbb{C}^n$) we denote by $\mathrm{C}^0(\mathbb{T}'\times\mathsf{B}(r, \boldsymbol{z}_0);\mathbb{C}^n)$ the Banach space of continuous mappings with the ∞-norm [Hewitt and Stromberg 1975, Theorem 7.9]. We define an operator

$$\Phi\colon \mathrm{C}^0(\mathbb{T}'\times\mathsf{B}(r, \boldsymbol{z}_0);\mathbb{C}^n) \to \mathrm{C}^0(\mathbb{T}'\times\mathsf{B}(r, \boldsymbol{z}_0);\mathbb{C}^n)$$

by

$$\Phi(\boldsymbol{\gamma})(t, \boldsymbol{z}) = \boldsymbol{z} + \int_{t_0}^t \boldsymbol{X}(s, \boldsymbol{\gamma}(s, \boldsymbol{z}))\,\mathrm{d}s.$$

One shows that this mapping, with domains suitably defined, is a contraction mapping, and so, by iterating the mapping, one constructs a sequence in $\mathrm{C}^0(\mathbb{T}'\times\mathsf{B}(r, \boldsymbol{z}_0);\mathbb{C}^n)$ converging to a fixed point, and the fixed point, necessarily satisfying

$$\boldsymbol{\gamma}(t, \boldsymbol{z}) = \boldsymbol{z} + \int_{t_0}^t \boldsymbol{X}(s, \boldsymbol{\gamma}(s, \boldsymbol{z}))\,\mathrm{d}s$$

and $\boldsymbol{\gamma}(t_0, \boldsymbol{z}) = \boldsymbol{z}$, has the property that $\boldsymbol{\gamma}(t, \boldsymbol{z}) = \Phi^{\boldsymbol{X}}(t, t_0, \boldsymbol{z})$.

Let us consider the sequence one constructs in this procedure. We define $\boldsymbol{\gamma}_0 \in \mathrm{C}^0(\mathbb{T}'\times\mathsf{B}(r, \boldsymbol{z}_0);\mathbb{C}^n)$ by $\boldsymbol{\gamma}_0(t, \boldsymbol{z}) = \boldsymbol{z}$. Certainly $\boldsymbol{\gamma}_0$ is holomorphic in $\boldsymbol{z}$. Now define $\boldsymbol{\gamma}_1 \in \mathrm{C}^0(\mathbb{T}'\times\mathsf{B}(r, \boldsymbol{z}_0);\mathbb{C}^n)$ by

$$\boldsymbol{\gamma}_1(t, \boldsymbol{z}) = \Phi(\boldsymbol{\gamma}_0) = \boldsymbol{z} + \int_{t_0}^t \boldsymbol{X}(s, \boldsymbol{z})\,\mathrm{d}s.$$

Since $\boldsymbol{X} \in \mathrm{LI}\Gamma^{\mathrm{hol}}(\mathbb{T}'; \mathsf{TB}(r, \boldsymbol{z}_0))$, we have

$$\frac{\partial}{\partial \bar{z}^j}\boldsymbol{\gamma}_1(t, \boldsymbol{z}) = \frac{\partial}{\partial \bar{z}^j}\boldsymbol{z} + \int_{t_0}^{t} \frac{\partial}{\partial \bar{z}^j}\boldsymbol{X}(s, \boldsymbol{\gamma}_0(s, \boldsymbol{z}))\,\mathrm{d}s = 0, \qquad j \in \{1, \dots, n\},$$

swapping the derivative and the integral by the Dominated Convergence Theorem [Jost 2005, Theorem 16.11] (also noting by Proposition 6.15 that derivatives of $\boldsymbol{X}$ are bounded by an integrable function). Thus $\boldsymbol{\gamma}_1$ is holomorphic for each fixed $t \in \mathbb{T}'$. By iterating with t fixed, we have a sequence $(\boldsymbol{\gamma}_{j,t})_{j \in \mathbb{Z}_{\geq 0}}$ of holomorphic mappings from $\mathsf{B}(r, \boldsymbol{z}_0)$ converging uniformly to the function $\boldsymbol{\gamma}$ that describes how the solution at time t depends on the initial condition $\boldsymbol{z}$. The limit function is necessarily holomorphic [Gunning 1990a, page 5]. ■

6.4. The real analytic case. Let us now turn to describing real analytic time-varying sections. We thus will consider a real analytic vector bundle $\pi\colon \mathsf{E} \to \mathsf{M}$ with ∇^0 a real analytic linear connection on E, ∇ a real analytic affine connection on M, $\mathbb{G}_0$ a real analytic fibre metric on E, and $\mathbb{G}$ a real analytic Riemannian metric on M. This defines the seminorms $p^{\omega}_{K,\boldsymbol{a}}$, $K \subseteq \mathsf{M}$ compact, $\boldsymbol{a} \in \mathrm{c}_{\downarrow 0}(\mathbb{Z}_{\geq 0}; \mathbb{R}_{>0})$, describing the C^{ω}-topology as in Theorem 5.5.

6.19 Definition: (Real analytic Carathéodory section) Let $\pi\colon \mathsf{E} \to \mathsf{M}$ be a real analytic vector bundle and let $\mathbb{T} \subseteq \mathbb{R}$ be an interval. A ***Carathéodory section of class*** $\mathbf{C}^{\boldsymbol{\omega}}$ of E is a map $\xi\colon \mathbb{T} \times \mathsf{M} \to \mathsf{E}$ with the following properties:

(i) $\xi(t, x) \in \mathsf{E}_x$ for each $(t, x) \in \mathbb{T} \times \mathsf{M}$;

(ii) for each $t \in \mathbb{T}$, the map $\xi_t\colon \mathsf{M} \to \mathsf{E}$ defined by $\xi_t(x) = \xi_t(x)$ is of class C^{ω};

(iii) for each $x \in \mathsf{M}$, the map $\xi^x\colon \mathbb{T} \to \mathsf{E}$ defined by $\xi^x(t) = \xi(t, x)$ is Lebesgue measurable.

We shall call $\mathbb{T}$ the ***time-domain*** for the section. By $\mathrm{CF}\Gamma^{\omega}(\mathbb{T}; \mathsf{E})$ we denote the set of Carathéodory sections of class C^{ω} of E. •

Now we turn to placing restrictions on the time-dependence to allow us to do useful things.

6.20 Definition: (Locally integrally $\mathbf{C}^{\boldsymbol{\omega}}$-bounded and locally essentially $\mathbf{C}^{\boldsymbol{\omega}}$-bounded sections) Let $\pi\colon \mathsf{E} \to \mathsf{M}$ be a real analytic vector bundle and let $\mathbb{T} \subseteq \mathbb{R}$ be an interval. A Carathéodory section $\xi\colon \mathbb{T} \times \mathsf{M} \to \mathsf{E}$ of class C^{ω} is

(i) ***locally integrally*** $\mathbf{C}^{\boldsymbol{\omega}}$***-bounded*** if, for every compact set $K \subseteq \mathsf{M}$ and every $\boldsymbol{a} \in \mathrm{c}_{\downarrow 0}(\mathbb{Z}_{\geq 0}; \mathbb{R}_{>0})$, there exists $g \in \mathrm{L}^1_{\mathrm{loc}}(\mathbb{T}; \mathbb{R}_{\geq 0})$ such that

$$a_0 a_1 \cdots a_m \|j_m \xi_t(x)\|_{\overline{\mathbb{G}}_m} \leq g(t), \qquad (t, x) \in \mathbb{T} \times K,\ m \in \mathbb{Z}_{\geq 0},$$

and is

(ii) ***locally essentially*** $\mathbf{C}^{\boldsymbol{\omega}}$***-bounded*** if, for every compact set $K \subseteq \mathsf{M}$ and every $\boldsymbol{a} \in \mathrm{c}_{\downarrow 0}(\mathbb{Z}_{\geq 0}; \mathbb{R}_{>0})$, there exists $g \in \mathrm{L}^{\infty}_{\mathrm{loc}}(\mathbb{T}; \mathbb{R}_{\geq 0})$ such that

$$a_0 a_1 \cdots a_m \|j_m \xi_t(x)\|_{\overline{\mathbb{G}}_m} \leq g(t), \qquad (t, x) \in \mathbb{T} \times K,\ m \in \mathbb{Z}_{\geq 0}.$$

The set of locally integrally C^{ω}-bounded sections of E with time-domain $\mathbb{T}$ is denoted by $\mathrm{LI}\Gamma^{\omega}(\mathbb{T}, \mathsf{E})$ and the set of locally essentially C^{ω}-bounded sections of E with time-domain $\mathbb{T}$ is denoted by $\mathrm{LB}\Gamma^{\omega}(\mathbb{T}; \mathsf{E})$. •

As with smooth and holomorphic sections, the preceding definitions admit topological characterisations.

6.21 Theorem: (Topological characterisation of real analytic Carathéodory sections) *Let $\pi\colon \mathsf{E} \to \mathsf{M}$ be a real analytic manifold and let $\mathbb{T} \subseteq \mathbb{R}$ be an interval. For a map $\xi\colon \mathbb{T} \times \mathsf{M} \to \mathsf{E}$ satisfying $\xi(t,x) \in \mathsf{E}_x$ for each $(t,x) \in \mathbb{T} \times \mathsf{M}$, the following two statements are equivalent:*

(i) $\xi \in \mathrm{CF}\Gamma^\omega(\mathbb{T};\mathsf{E})$*;*

(ii) the map $\mathbb{T} \ni t \mapsto \xi_t \in \Gamma^\omega(\mathsf{E})$ *is measurable,*

the following two statements are equivalent:

(iii) $\xi \in \mathrm{LI}\Gamma^\omega(\mathbb{T};\mathsf{E})$*;*

(iv) the map $\mathbb{T} \ni t \mapsto \xi_t \in \Gamma^\omega(\mathsf{E})$ *is measurable and locally Bochner integrable,*

and the following two statements are equivalent:

(v) $\xi \in \mathrm{LB}\Gamma^\omega(\mathbb{T};\mathsf{E})$*;*

(vi) the map $\mathbb{T} \ni t \mapsto \xi_t \in \Gamma^\omega(\mathsf{E})$ *is measurable and locally essentially von Neumann bounded.*

Proof: Just as in the smooth case in Theorem 6.3, this is deduced from the following facts: (1) evaluation maps ev_{α_x}, $\alpha_x \in \mathsf{E}^*$, are continuous and point separating; (2) $\Gamma^\omega(\mathsf{E})$ is a Suslin space (property C^ω-6); (3) $\Gamma^\omega(\mathsf{E})$ is complete and separable (properties C^ω-2 and C^ω-4; (4) we understand von Neumann bounded subsets of $\Gamma^\omega(\mathsf{E})$ by Lemma 5.6. ■

Note that Theorem 6.21 applies, in particular, to vector fields and functions, giving the classes $\mathrm{CF}^\omega(\mathbb{T};\mathsf{M})$, $\mathrm{LIC}^\omega(\mathbb{T};\mathsf{M})$, and $\mathrm{LBC}^\omega(\mathbb{T};\mathsf{M})$ of functions, and the classes $\mathrm{CF}\Gamma^\omega(\mathbb{T};\mathsf{TM})$, $\mathrm{LI}\Gamma^\omega(\mathbb{T};\mathsf{TM})$, and $\mathrm{LB}\Gamma^\omega(\mathbb{T};\mathsf{TM})$ of vector fields. The following result then summarises the various ways of verifying the measurability, integrability, and boundedness of real analytic time-varying vector fields.

6.22 Theorem: (Weak characterisations of measurability, integrability, and boundedness of real analytic time-varying vector fields) *Let M be a real analytic manifold, let $\mathbb{T} \subseteq \mathbb{R}$ be a time-domain, and let $X\colon \mathbb{T} \times \mathsf{M} \to \mathsf{TM}$ have the property that X_t is a real analytic vector field for each $t \in \mathbb{T}$. Then the following statements are equivalent:*

(i) $t \mapsto X_t$ *is measurable;*

(ii) $t \mapsto \mathscr{L}_{X_t} f$ *is measurable for every* $f \in \mathrm{C}^\omega(\mathsf{M})$*;*

(iii) $t \mapsto \mathrm{ev}_x \circ X_t$ *is measurable for every* $x \in \mathsf{M}$*;*

(iv) $t \mapsto \mathrm{ev}_x \circ \mathscr{L}_{X_t} f$ *is measurable for every* $f \in \mathrm{C}^\omega(\mathsf{M})$ *and every* $x \in \mathsf{M}$*,*

the following two statements are equivalent:

(v) $t \mapsto X_t$ *is locally Bochner integrable;*

(vi) $t \mapsto \mathscr{L}_{X_t} f$ *is locally Bochner integrable for every* $f \in \mathrm{C}^\omega(\mathsf{M})$*,*

and the following two statements are equivalent:

(vii) $t \mapsto X_t$ *is locally essentially bounded;*

(viii) $t \mapsto \mathscr{L}_{X_t} f$ *is locally essentially bounded in the von Neumann bornology for every* $f \in \mathrm{C}^\omega(\mathsf{M})$*.*

Proof: This follows from Theorem 6.21, along with Corollary 5.9. ■

Let us verify that real analytic time-varying sections have the expected relationship to their smooth brethren.

6.23 Proposition: (Time-varying real analytic sections as time-varying smooth sections) *For a real analytic vector bundle $\pi\colon \mathsf{E} \to \mathsf{M}$ and an interval $\mathbb{T}$, we have*

$$\mathrm{LI}\Gamma^{\omega}(\mathbb{T}; \mathsf{E}) \subseteq \mathrm{LI}\Gamma^{\infty}(\mathbb{T}; \mathsf{E}), \qquad \mathrm{LB}\Gamma^{\omega}(\mathbb{T}; \mathsf{M}) \subseteq \mathrm{LB}\Gamma^{\infty}(\mathbb{T}; \mathsf{M}).$$

Proof: It is obvious that real analytic Carathéodory sections are smooth Carathéodory sections.

Let us verify only that $\mathrm{LI}\Gamma^{\omega}(\mathbb{T}; \mathsf{E}) \subseteq \mathrm{LI}\Gamma^{\infty}(\mathbb{T}; \mathsf{E})$, as the essentially bounded case follows in the same manner. We let $K \subseteq \mathsf{M}$ be compact and let $m \in \mathbb{Z}_{\geq 0}$. Choose (arbitrarily) $\boldsymbol{a} \in \mathrm{c}_{\downarrow 0}(\mathbb{Z}_{\geq 0}; \mathbb{R}_{>0})$. Then, if $\xi \in \mathrm{LI}\Gamma^{\omega}(\mathbb{T}; \mathsf{E})$, there exists $g \in \mathrm{L}^1_{\mathrm{loc}}(\mathbb{T}; \mathbb{R}_{\geq 0})$ such that

$$a_0 a_1 \cdots a_m \|j_m \xi_t(x)\|_{\overline{\mathbb{G}}_m} \leq g(t), \qquad x \in K,\ t \in \mathbb{T},\ m \in \mathbb{Z}_{\geq 0}.$$

Thus, taking $g_{\boldsymbol{a},m} \in \mathrm{L}^1_{\mathrm{loc}}(\mathbb{T}; \mathbb{R}_{\geq 0})$ defined by

$$g_{\boldsymbol{a},m}(t) = \frac{1}{a_0 a_1 \cdots a_m} g(t),$$

we have

$$\|j_m \xi_t(x)\|_{\overline{\mathbb{G}}_m} \leq g_{\boldsymbol{a},m}(t), \qquad x \in K,\ t \in \mathbb{T}$$

showing that $\xi \in \mathrm{LI}\Gamma^{\infty}(\mathbb{T}; \mathsf{E})$. ■

Having understood the comparatively simple relationship between real analytic and smooth time-varying sections, let us consider the correspondence between real analytic and holomorphic time-varying sections. First, note that if $\mathbb{T} \subseteq \mathbb{R}$ is an interval and if $\overline{\mathcal{U}} \in \mathscr{N}_{\mathsf{M}}$ is a neighbourhood of M in a complexification $\overline{\mathsf{M}}$, then we have an inclusion

$$\begin{aligned} \rho_{\overline{\mathcal{U}},\mathsf{M}}\colon \mathrm{CF}\Gamma^{\mathrm{hol},\mathbb{R}}(\mathbb{T}; \overline{\mathsf{E}}|\overline{\mathcal{U}}) &\to \mathrm{CF}\Gamma^{\omega}(\mathbb{T}; \mathsf{E}) \\ \overline{\xi} &\mapsto \overline{\xi}|\mathsf{M}. \end{aligned}$$

(Here the notation $\mathrm{CF}\Gamma^{\mathrm{hol},\mathbb{R}}(\mathbb{T}; \overline{\mathsf{E}}|\overline{\mathcal{U}})$ refers to those Carathéodory sections that are real when restricted to M, cf. the constructions of Section 5.1.) However, this inclusion does not characterise all real analytic Carathéodory sections, as the following example shows.

6.24 Example: (A real analytic Carathéodory function not extending to one that is holomorphic) Let $\mathbb{T}$ be any interval for which $0 \in \mathrm{int}(\mathbb{T})$. We consider the real analytic Carathéodory function on $\mathbb{R}$ with time-domain $\mathbb{T}$ defined by

$$f(t,x) = \begin{cases} \frac{t^2}{t^2+x^2}, & t \neq 0, \\ 0, & t = 0. \end{cases}$$

It is clear that $x \mapsto f(t,x)$ is real analytic for every $t \in \mathbb{T}$ and that $t \mapsto f(t,x)$ is measurable for every $x \in \mathbb{R}$. We claim that there is no neighbourhood $\overline{\mathcal{U}} \subseteq \mathbb{C}$ of $\mathbb{R} \subseteq \mathbb{C}$ such that f is the restriction to $\mathbb{R}$ of an holomorphic Carathéodory function on $\overline{\mathcal{U}}$. Indeed, let $\overline{\mathcal{U}} \subseteq \mathbb{C}$ be a neighbourhood of $\mathbb{R}$ and choose $t \in \mathbb{R}_{>0}$ sufficiently small that $\overline{\mathsf{D}}(t,0) \subseteq \overline{\mathcal{U}}$. Note that $f_t\colon x \mapsto \frac{1}{1+(x/t)^2}$ does not admit an holomorphic extension to any open set containing $\overline{\mathsf{D}}(t,0)$ since the radius of convergence of $z \mapsto \frac{1}{1+(z/t)^2}$ is t, cf. the discussion at the beginning of Section 5. Note that our construction actually shows that in no neighbourhood of $(0,0) \in \mathbb{R} \times \mathbb{R}$ is there an holomorphic extension of f. •

Fortunately, the example will not bother us, although it does serve to illustrate that the following result is not immediate.

6.25 Theorem: (Real analytic time-varying vector fields as restrictions of holomorphic time-varying vector fields) *Let $\pi\colon \mathsf{E} \to \mathsf{M}$ be a real analytic vector bundle with complexification $\overline{\pi}\colon \overline{\mathsf{E}} \to \overline{\mathsf{M}}$, and let $\mathbb{T}$ be a time-domain. For a map $\xi\colon \mathbb{T} \times \mathsf{M} \to \mathsf{E}$ satisfying $\xi(t,x) \in \mathsf{E}_x$ for every $(t,x) \in \mathbb{T} \times \mathsf{M}$, the following statements hold:*

(i) if $\xi \in \mathrm{LI}\Gamma^{\omega}(\mathbb{T};\mathsf{E})$, then, for each $(t_0,x_0) \in \mathbb{T} \times \mathsf{M}$ and each bounded subinterval $\mathbb{T}' \subseteq \mathbb{T}$ containing t_0, there exist a neighbourhood $\overline{\mathcal{U}}$ of x_0 in $\overline{\mathsf{M}}$ and $\overline{\xi} \in \Gamma^{\mathrm{hol}}(\mathbb{T}';\overline{\mathsf{E}}|\overline{\mathcal{U}})$ such that $\overline{\xi}(t,x) = \xi(t,x)$ for each $t \in \mathbb{T}'$ and $x \in \overline{\mathcal{U}} \cap \mathsf{M}$;

(ii) if, for each $x_0 \in \mathsf{M}$, there exist a neighbourhood $\overline{\mathcal{U}}$ of x_0 in $\overline{\mathsf{M}}$ and $\overline{\xi} \in \Gamma^{\mathrm{hol}}(\mathbb{T};\overline{\mathsf{E}}|\overline{\mathcal{U}})$ such that $\overline{\xi}(t,x) = \xi(t,x)$ for each $t \in \mathbb{T}$ and $x \in \overline{\mathcal{U}} \cap \mathsf{M}$, then $\xi \in \mathrm{LI}\Gamma^{\omega}(\mathbb{T};\mathsf{E})$.

Proof: (i) We let $\mathbb{T}' \subseteq \mathbb{T}$ be a bounded subinterval containing t_0 and let $\mathcal{U}$ be a relatively compact neighbourhood of x_0. Let $(\overline{\mathcal{U}}_j)_{j\in\mathbb{Z}_{>0}}$ be a sequence of neighbourhoods of $\mathrm{cl}(\mathcal{U})$ in $\overline{\mathsf{M}}$ with the properties that $\mathrm{cl}(\overline{\mathcal{U}}_{j+1}) \subseteq \overline{\mathcal{U}}_j$ and that $\cap_{j\in\mathbb{Z}_{>0}}\overline{\mathcal{U}}_j = \mathrm{cl}(\mathcal{U})$. We first note that

$$\mathrm{L}^1(\mathbb{T}';\Gamma^{\mathrm{hol},\mathbb{R}}(\overline{\mathsf{E}}|\overline{\mathcal{U}}_j)) \simeq \mathrm{L}^1(\mathbb{T}';\mathbb{R})\widehat{\otimes}_{\pi}\Gamma^{\mathrm{hol},\mathbb{R}}(\overline{\mathsf{E}}|\overline{\mathcal{U}}_j),$$

with $\widehat{\otimes}_{\pi}$ denoting the completed projective tensor product [Schaefer and Wolff 1999, Theorem III.6.5]. The theorem of Schaefer and Wolff is given for Banach spaces, and they also assert the validity of this for locally convex spaces; thus we also have

$$\mathrm{L}^1(\mathbb{T}';\mathscr{G}^{\mathrm{hol},\mathbb{R}}_{\mathrm{cl}(\mathcal{U}),\overline{\mathsf{E}}}) \simeq \mathrm{L}^1(\mathbb{T}';\mathbb{R})\widehat{\otimes}_{\pi}\mathscr{G}^{\mathrm{hol},\mathbb{R}}_{\mathrm{cl}(\mathcal{U}),\overline{\mathsf{E}}}.$$

In both cases, the isomorphisms are in the category of locally convex topological vector spaces. We now claim that

$$\mathrm{L}^1(\mathbb{T}';\mathbb{R}) \otimes_{\pi} \mathscr{G}^{\mathrm{hol},\mathbb{R}}_{\mathrm{cl}(\mathcal{U}),\overline{\mathsf{E}}}$$

is the direct limit of the directed system

$$(\mathrm{L}^1(\mathbb{T}';\mathbb{R}) \otimes_{\pi} \Gamma^{\mathrm{hol},\mathbb{R}}(\overline{\mathsf{E}}|\overline{\mathcal{U}}_j)))_{j\in\mathbb{Z}_{>0}}$$

with the associated mappings $\mathrm{id}\otimes_{\pi} r_{\mathrm{cl}(\mathcal{U}),j}$, $j \in \mathbb{Z}_{>0}$, where $r_{\mathrm{cl}(\mathcal{U}),j}$ is defined as in (5.1). (Here $\otimes_{\pi}$ is the uncompleted projective tensor product). We, moreover, claim that the direct limit topology is boundedly retractive, meaning that bounded sets in the direct limit are contained in and bounded in a single component of the directed system and, moreover, the topology on the bounded set induced by the component is the same as that induced by the direct limit.

Results of this sort have been the subject of research in the area of locally convex topologies, with the aim being to deduce conditions on the structure of the spaces comprising the directed system, and on the corresponding mappings (for us, the inclusion mappings and their tensor products with the identity on $\mathrm{L}^1(\mathbb{T}';\mathbb{R})$), that ensure that direct limits commute with tensor product, and that the associated direct limit topology is boundedly retractive. We shall make principal use of the results given by Mangino [1997]. To state the arguments with at least a little context, let us reproduce two conditions used by Mangino.

Condition (M) of Retakh [1970]: Let $(\mathsf{V}_j)_{j\in\mathbb{Z}_{>0}}$ be a directed system of locally convex spaces with strict direct limit V. The direct limit topology of V satisfies ***condition (M)*** if there exists a sequence $(\mathcal{O}_j)_{j\in\mathbb{Z}_{>0}}$ for which

(i) $\mathcal{O}_j$ is a balanced convex neighbourhood of $0 \in \mathsf{V}_j$,

(ii) $\mathcal{O}_j \subseteq \mathcal{O}_{j+1}$ for each $j \in \mathbb{Z}_{>0}$, and

(iii) for every $j \in \mathbb{Z}_{>0}$, there exists $k \geq j$ such that the topology induced on $\mathcal{O}_j$ by its inclusion in V_k and its inclusion in V agree. •

Condition (MO) of Mangino [1997]: Let $(\mathsf{V}_j)_{j\in\mathbb{Z}_{>0}}$ be a directed system of metrisable locally convex spaces with strict direct limit V. Let $i_{j,k}\colon \mathsf{V}_j \to \mathsf{V}_k$ be the inclusion for $k \geq j$ and let $i_j\colon \mathsf{V}_j \to \mathsf{V}$ be the induced map into the direct limit.

Suppose that, for each $j \in \mathbb{Z}_{>0}$, we have a sequence $(p_{j,l})_{l\in\mathbb{Z}_{>0}}$ of seminorms defining the topology of V_j such that $p_{j,l_1} \geq p_{j,l_2}$ if $l_1 \geq l_2$. Let

$$\mathsf{V}_{j,l} = \mathsf{V}_j/\{v \in \mathsf{V}_j \mid p_{j,l}(v) = 0\}$$

and denote by $\hat{p}_{j,l}$ the norm on $\mathsf{V}_{j,l}$ induced by $p_{j,l}$ [Schaefer and Wolff 1999, page 97]. Let $\pi_{j,l}\colon \mathsf{V}_j \to \mathsf{V}_{j,l}$ be the canonical projection. Let $\overline{\mathsf{V}}_{j,l}$ be the completion of $\mathsf{V}_{j,l}$. The family $(\overline{\mathsf{V}}_{j,l})_{j,l\in\mathbb{Z}_{>0}}$ is called a ***projective spectrum*** for V_j. Denote

$$\mathcal{O}_{j,l} = \{v \in \mathsf{V}_j \mid p_{j,l}(v) \leq 1\}.$$

The direct limit topology of V satisfies ***condition (MO)*** if there exists a sequence $(\mathcal{O}_j)_{j\in\mathbb{Z}_{>0}}$ and if, for every $j \in \mathbb{Z}_{>0}$, there exists a projective spectrum $(\overline{\mathsf{V}}_{j,l})_{j,l\in\mathbb{Z}_{>0}}$ for V_j for which

(i) $\mathcal{O}_j$ is a balanced convex neighbourhood of $0 \in \mathsf{V}_j$,

(ii) $\mathcal{O}_j \subseteq \mathcal{O}_{j+1}$ for each $j \in \mathbb{Z}_{>0}$, and

(iii) for every $j \in \mathbb{Z}_{>0}$, there exists $k \geq j$ such that, for every $l \in \mathbb{Z}_{>0}$, there exists $A \in \mathrm{L}(\mathsf{V};\overline{\mathsf{V}}_{k,l})$ satisfying

$$(\pi_{k,l} \circ i_{j,k} - A \circ i_j)(\mathcal{O}_j) \subseteq \mathrm{cl}(\pi_{k,l}(\mathcal{O}_{k,l})),$$

the closure on the right being taken in the norm topology of $\overline{\mathsf{V}}_{k,l}$. •

With these concepts, we have the following statements. We let $(\mathsf{V}_j)_{j\in\mathbb{Z}_{>0}}$ be a directed system of metrisable locally convex spaces with strict direct limit V.

1. If the direct limit topology on V satisfies condition (MO), then, for any Banach space U, $\mathsf{U} \otimes_\pi \mathsf{V}$ is the direct limit of the directed system $(\mathsf{U} \otimes_\pi \mathsf{V}_j)_{j\in\mathbb{Z}_{>0}}$, and the direct limit topology on $\mathsf{U} \otimes_\pi \mathsf{V}$ satisfies condition (M) [Mangino 1997, Theorem 1.3].
2. If the spaces V_j, $j \in \mathbb{Z}_{>0}$, are nuclear and if the direct limit topology on V is regular, then the direct limit topology on V satisfies condition (MO) [Mangino 1997, Theorem 1.3].
3. If the direct limit topology on V satisfies condition (M), then this direct limit topology is boundedly retractive [Wengenroth 1995].

Using these arguments we make the following conclusions.

4. The direct limit topology on $\mathscr{G}^{\mathrm{hol},\mathbb{R}}_{\mathrm{cl}(\mathcal{U}),\overline{\mathsf{E}}}$ satisfies condition (MO) (by virtue of assertion 2 above and by the properties of the direct limit topology enunciated in Section 5.3, specifically that the direct limit is a regular direct limit of nuclear Fréchet spaces).
5. The space $\mathrm{L}^1(\mathbb{T}';\mathbb{R}) \otimes_\pi \mathscr{G}^{\mathrm{hol},\mathbb{R}}_{\mathrm{cl}(\mathcal{U}),\overline{\mathsf{E}}}$ is the direct limit of the directed sequence $(\mathrm{L}^1(\mathbb{T}';\mathbb{R}) \otimes_\pi \Gamma^{\mathrm{hol},\mathbb{R}}(\overline{\mathsf{E}}|\overline{\mathcal{U}}_j))_{j\in\mathbb{Z}_{>0}}$ (by virtue of assertion 1 above).

6. The direct limit topology on $\mathrm{L}^1(\mathbb{T}';\mathbb{R}) \otimes_\pi \mathscr{G}^{\mathrm{hol},\mathbb{R}}_{\mathrm{cl}(\mathcal{U}),\overline{\mathsf{E}}}$ satisfies condition (M) (by virtue of assertion 1 above).
7. The direct limit topology on $\mathrm{L}^1(\mathbb{T}';\mathbb{R}) \otimes_\pi \mathscr{G}^{\mathrm{hol},\mathbb{R}}_{\mathrm{cl}(\mathcal{U}),\overline{\mathsf{E}}}$ is boundedly retractive (by virtue of assertion 3 above).

We shall also need the following lemma.

1 Lemma: *Let $K \subseteq \mathsf{M}$ be compact. If $[\overline{\xi}]_K \in \mathrm{L}^1(\mathbb{T}';\mathscr{G}^{\mathrm{hol},\mathbb{R}}_{K,\overline{\mathsf{E}}})$ then there exists a sequence $([\overline{\xi}_k]_K)_{k\in\mathbb{Z}_{>0}}$ in $\mathrm{L}^1(\mathbb{T}';\mathbb{R}) \otimes \mathscr{G}^{\mathrm{hol},\mathbb{R}}_{K,\overline{\mathsf{E}}}$ converging to $[\overline{\xi}]_K$ in the topology of $\mathrm{L}^1(\mathbb{T}';\mathscr{G}^{\mathrm{hol}}_{K,\overline{\mathsf{E}}})$.*

Proof: Since $\mathrm{L}^1(\mathbb{T}';\mathscr{G}^{\mathrm{hol},\mathbb{R}}_{K,\overline{\mathsf{E}}})$ is the completion of $\mathrm{L}^1(\mathbb{T}';\mathbb{R}) \otimes_\pi \mathscr{G}^{\mathrm{hol},\mathbb{R}}_{K,\overline{\mathsf{E}}}$, there exists a *net* $([\overline{\xi}_i]_K)_{i\in I}$ converging to $[\overline{\xi}]$, so the conclusion here is that we can actually find a converging *sequence*.

To prove this we argue as follows. Recall properties $\mathscr{G}^{\mathrm{hol},\mathbb{R}}$-5 and $\mathscr{G}^{\mathrm{hol},\mathbb{R}}$-6 of $\mathscr{G}^{\mathrm{hol},\mathbb{R}}_{K,\overline{\mathsf{E}}}$, indicating that it is reflexive and its dual is a nuclear Fréchet space. Thus $\mathscr{G}^{\mathrm{hol},\mathbb{R}}_{K,\overline{\mathsf{E}}}$ is the dual of a nuclear Fréchet space. Also recall from property $\mathscr{G}^{\mathrm{hol},\mathbb{R}}$-8 that $\mathscr{G}^{\mathrm{hol},\mathbb{R}}_{K,\overline{\mathsf{E}}}$ is a Suslin space. Now, by combining [Thomas 1975, Theorem 7] with remark (1) at the bottom of page 76 of [Thomas 1975] (and being aware that Bochner integrability as defined by Thomas is not *a priori* the same as Bochner integrability as we mean it), there exists a sequence $([\overline{\xi}_k]_K)_{k\in\mathbb{Z}_{>0}}$ of simple functions, i.e., elements of $\mathrm{L}^1(\mathbb{T}';\mathbb{R}) \otimes \mathscr{G}^{\mathrm{hol},\mathbb{R}}_{K,\overline{\mathsf{E}}}$, such that

$$\lim_{k\to\infty} [\overline{\xi}_k(t)]_K = [\overline{\xi}(t)]_K, \qquad \text{a.e. } t \in \mathbb{T}',$$

(this limit being in the topology of $\mathscr{G}^{\mathrm{hol},\mathbb{R}}_{K,\overline{\mathsf{E}}}$) and

$$\lim_{k\to\infty} \int_{\mathbb{T}'} ([\overline{\xi}(t)]_K - [\overline{\xi}_k(t)]_K)\,\mathrm{d}t = 0.$$

This implies, by the Dominated Convergence Theorem, that

$$\lim_{k\to\infty} \int_{\mathbb{T}'} p^{\omega}_{K,\boldsymbol{a}}([\overline{\xi}(t)]_K - [\overline{\xi}_k(t)]_K)\,\mathrm{d}t = 0$$

for every $\boldsymbol{a} \in \mathrm{c}_{\downarrow 0}(\mathbb{Z}_{\ge 0};\mathbb{R}_{>0})$, giving convergence in

$$\mathrm{L}^1(\mathbb{T}';\mathscr{G}^{\mathrm{hol},\mathbb{R}}_{K,\overline{\mathsf{E}}}) \simeq \mathrm{L}^1(\mathbb{T}';\mathbb{R}) \widehat{\otimes}_\pi \mathscr{G}^{\mathrm{hol},\mathbb{R}}_{K,\overline{\mathsf{E}}},$$

as desired. ▼

The remainder of the proof is straightforward. Since $\xi \in \mathrm{LI}\Gamma^{\omega}(\mathbb{T};\mathsf{E})$, the map

$$\mathbb{T}' \ni t \mapsto \xi_t \in \Gamma^{\omega}(\mathsf{E})$$

is an element of $\mathrm{L}^1(\mathbb{T}';\Gamma^{\omega}(\mathsf{E}))$ by Theorem 6.21. Therefore, if $[\overline{\xi}]_{\mathrm{cl}(\mathcal{U})}$ is the image of ξ under the natural mapping from $\Gamma^{\omega}(\mathsf{E})$ to $\mathscr{G}^{\mathrm{hol},\mathbb{R}}_{\mathrm{cl}(\mathcal{U}),\overline{\mathsf{E}}}$, the map

$$\mathbb{T}' \ni t \mapsto [\overline{\xi(t)}]_{\mathrm{cl}(\mathcal{U})} \in \mathscr{G}^{\mathrm{hol},\mathbb{R}}_{\mathrm{cl}(\mathcal{U}),\overline{\mathsf{E}}}$$

is an element of $\mathrm{L}^1(\mathbb{T}';\mathscr{G}^{\mathrm{hol},\mathbb{R}}_{\mathrm{cl}(\mathcal{U}),\overline{\mathsf{E}}})$, since continuous linear maps commute with integration [Beckmann and Deitmar 2011, Lemma 1.2]. Therefore, by the Lemma above, there exists a sequence $([\overline{\xi}_k]_{\mathrm{cl}(\mathcal{U})})_{k\in\mathbb{Z}_{>0}}$ in $\mathrm{L}^1(\mathbb{T}';\mathbb{R})\otimes\mathscr{G}^{\mathrm{hol},\mathbb{R}}_{\mathrm{cl}(\mathcal{U}),\overline{\mathsf{E}}}$ that converges to $[\overline{\xi}]_{\mathrm{cl}(\mathcal{U})}$. By our conclusion 5 above, the topology in which this convergence takes place is the completion of the direct limit topology associated to the directed system $(\mathrm{L}^1(\mathbb{T}';\mathbb{R})\otimes_\pi\Gamma^{\mathrm{hol},\mathbb{R}}(\overline{\mathsf{E}}|\overline{\mathcal{U}}_j))_{j\in\mathbb{Z}_{>0}}$. The direct limit topology on $\mathrm{L}^1(\mathbb{T}';\mathbb{R})\otimes_\pi\mathscr{G}^{\mathrm{hol},\mathbb{R}}_{\mathrm{cl}(\mathcal{U}),\overline{\mathsf{E}}}$ is boundedly retractive by our conclusion 7 above. This is easily seen to imply that the direct limit topology is sequentially retractive, i.e., that convergent sequences are contained in, and convergent in, a component of the direct limit [Fernández 1990]. This implies that there exists $j\in\mathbb{Z}_{>0}$ such that the sequence $(\overline{\xi}_k)_{k\in\mathbb{Z}_{>0}}$ converges in $\mathrm{L}^1(\mathbb{T}';\Gamma^{\mathrm{hol},\mathbb{R}}(\overline{\mathsf{E}}|\overline{\mathcal{U}}_j))$ and so converges to a limit $\overline{\eta}$ satisfying $[\overline{\eta}]_{\mathrm{cl}(\mathcal{U}_j)}=[\overline{\xi}]_{\mathrm{cl}(\mathcal{U}_j)}$. Thus $\overline{\xi}$ can be holomorphically extended to $\overline{\mathcal{U}}_j$. This completes this part of the proof.

(ii) Let $K\subseteq\mathsf{M}$ be compact and let $\boldsymbol{a}\in\mathrm{c}_{\downarrow0}(\mathbb{Z}_{\geq0};\mathbb{R}_{>0})$. Let $(\overline{\mathcal{U}}_j)_{j\in\mathbb{Z}_{>0}}$ be a sequence of neighbourhoods of K in $\overline{\mathsf{M}}$ such that $\mathrm{cl}(\overline{\mathcal{U}}_{j+1})\subseteq\overline{\mathcal{U}}_j$ and $K=\cap_{j\in\mathbb{Z}_{>0}}\overline{\mathcal{U}}_j$. By hypothesis, for $x\in K$, there is a relatively compact neighbourhood $\overline{\mathcal{U}}_x\subseteq\overline{\mathsf{M}}$ of x in $\overline{\mathsf{M}}$ such that there is an extension $\overline{\xi}_x\in\mathrm{LI}\Gamma^{\mathrm{hol},\mathbb{R}}(\mathbb{T};\overline{\mathsf{E}}|\overline{\mathcal{U}}_x)$ of $\xi|(\mathbb{T}\times(\overline{\mathcal{U}}_x\cap\mathsf{M}))$. Let $x_1,\dots,x_k\in K$ be such that $K\subseteq\cup_{j=1}^k\overline{\mathcal{U}}_{x_j}$ and let $l\in\mathbb{Z}_{>0}$ be sufficiently large that $\overline{\mathcal{U}}_l\subseteq\cup_{j=1}^k\overline{\mathcal{U}}_{x_j}$, so ξ admits an holomorphic extension $\overline{\xi}\in\mathrm{LI}\Gamma^{\mathrm{hol},\mathbb{R}}(\mathbb{T};\overline{\mathsf{E}}\overline{\mathcal{U}}_l)$.

Now we show that the above constructions imply that $\xi\in\mathrm{LI}\Gamma^\omega(\mathbb{T};\mathsf{TM})$. Let $\overline{g}\in\mathrm{L}^1_{\mathrm{loc}}(\mathbb{T};\mathbb{R}_{\geq0})$ be such that

$$\|\overline{\xi}(t,z)\|_{\overline{\mathbb{G}}}\leq\overline{g}(t),\qquad(t,z)\in\mathbb{T}\times\overline{\mathcal{U}}_l.$$

By Proposition 4.2, there exist $C,r\in\mathbb{R}_{>0}$ such that

$$\|j_m\xi(t,x)\|\leq Cr^{-m}\overline{g}(t)$$

for all $m\in\mathbb{Z}_{\geq0}$, $t\in\mathbb{T}$, and $x\in K$. Now let $N\in\mathbb{Z}_{\geq0}$ be such that $a_{N+1}<r$ and let $g\in\mathrm{L}^1_{\mathrm{loc}}(\mathbb{T};\mathbb{R}_{\geq0})$ be such that

$$Ca_0a_1\cdots a_mr^{-m}\overline{g}(t)\leq g(t)$$

for $m\in\{0,1,\dots,N\}$. Now, if $m\in\{0,1,\dots,N\}$, we have

$$a_0a_1\cdots a_m\|j_m\xi(t,x)\|_{\overline{\mathbb{G}}_m}\leq a_0a_1\cdots a_mCr^{-m}\overline{g}(t)\leq g(t)$$

for $(t,x)\in\mathbb{T}\times K$. If $m>N$ we also have

$$\begin{aligned}a_0a_1\cdots a_m\|j_m\xi(t,x)\|_{\overline{\mathbb{G}}_m}&\leq a_0a_1\cdots a_Nr^{-N}r^m\|j_m\xi(t,x)\|_{\overline{\mathbb{G}}_m}\\&\leq a_0a_1\cdots a_Nr^{-N}C\overline{g}(t)\leq g(t),\end{aligned}$$

for $(t,x)\in\mathbb{T}\times K$, as desired. ■

Finally, let us show that, according to our definitions, real analytic time-varying vector fields possess flows depending in a real analytic way on initial condition.

6.26 Theorem: (Flows of vector fields from $\mathrm{LI}\Gamma^{\omega}(\mathbb{T};\mathsf{TM})$) *Let M be a real analytic manifold, let $\mathbb{T}$ be an interval, and let $X \in \mathrm{LI}\Gamma^{\omega}(\mathbb{T};\mathsf{TM})$. Then there exist a subset $D_X \subseteq \mathbb{T}\times\mathbb{T}\times\mathsf{M}$ and a map $\Phi^X\colon D_X \to \mathsf{M}$ with the following properties for each $(t_0,x_0)\in\mathbb{T}\times\mathsf{M}$:*

(i) the set

$$\mathbb{T}_X(t_0,x_0) = \{t \in \mathbb{T} \mid (t,t_0,x_0)\in D_X\}$$

is an interval;

(ii) there exists a locally absolutely continuous curve $t\mapsto\xi(t)$ satisfying

$$\xi'(t) = X(t,\xi(t)), \quad \xi(t_0) = x_0,$$

for almost all $t \in |t_0,t_1|$ if and only if $t_1 \in \mathbb{T}_X(t_0,x_0)$;

(iii) $\frac{\mathrm{d}}{\mathrm{d}t}\Phi^X(t,t_0,x_0) = X(t,\Phi^X(t,t_0,x_0))$ for almost all $t\in\mathbb{T}_X(t_0,x_0)$;

(iv) for each $t\in\mathbb{T}$ for which $(t,t_0,x_0)\in D_X$, there exists a neighbourhood $\mathcal{U}$ of x_0 such that the mapping $x\mapsto\Phi^X(t,t_0,x)$ is defined and of class C^{ω} on $\mathcal{U}$.

Proof: The theorem follows from Theorems 6.18 and 6.25, noting that the flow of an holomorphic extension will leave invariant the real analytic manifold. ■

6.5. Mixing regularity hypotheses. It is possible to mix regularity conditions for vector fields. By this we mean that one can consider vector fields whose dependence on state is more regular than their joint state/time dependence. This can be done by considering $m\in\mathbb{Z}_{\geq 0}$, $m'\in\{0,\mathrm{lip}\}$, $r\in\mathbb{Z}_{\geq 0}\cup\{\infty,\omega\}$, and $r'\in\{0,\mathrm{lip}\}$ satisfying $m+m'<r+r'$, and considering vector fields in

$$\mathrm{CF}\Gamma^{r+r'}(\mathbb{T};\mathsf{TM})\cap\mathrm{LI}\Gamma^{m+m'}(\mathbb{T};\mathsf{TM}) \quad\text{or}\quad \mathrm{CF}\Gamma^{r+r'}(\mathbb{T};\mathsf{TM})\cap\mathrm{LB}\Gamma^{m+m'}(\mathbb{T};\mathsf{TM}),$$

using the obvious convention that $\infty+\mathrm{lip}=\infty$ and $\omega+\mathrm{lip}=\omega$. This does come across as quite unnatural in our framework, and perhaps it is right that it should. Moreover, because the $\mathrm{CO}^{m+m'}$-topology for $\Gamma^{r+r'}(\mathsf{TM})$ will be complete if and only if $m+m'=r+r'$, some of the results above will not translate to this mixed class of time-varying vector fields: particularly, the results on Bochner integrability require completeness. Nonetheless, this mixing of regularity assumptions is quite common in the literature. Indeed, this has *always* been done in the real analytic case, since the notions of "locally integrally C^{ω}-bounded" and "locally essentially C^{ω}-bounded" given in Definition 6.20 are being given for the first time in this paper.

7. Control systems

Now, having at hand a thorough accounting of time-varying vector fields, we turn to the characterisation of classes of control systems. These classes of systems will provide us with a precise point of comparison between our general development of Section 8 and the more common notion of a control system. Our system definitions are designed so that the act of "substituting in a control" leads to a time-varying vector field of the sort considered in Section 6. This essentially means that we need for our system vector fields to depend continuously on control in the appropriate topology. We note that, in practice, this is generally not a limitation, e.g., we show in Example 7.21 that control-affine systems satisfy our conditions. In cases where it is a limitation, the definitions and results here can be replaced with suitably modified versions with less smoothness, and we say a few words about this at the end of the section.

As we have been doing all along so far, we initially consider separately the finitely differentiable, Lipschitz, smooth, holomorphic, and real analytic cases. Also, the initial part of our discussion is carried out for parameterised sections of vector bundles (control systems are parameterised vector fields), as this allows us to handle vector fields and functions simultaneously, just as we did in Sections 3, 4, and 5.

When we turn to control systems starting in Section 7.2, we merge as much as possible the consideration of varying degrees of regularity to make clear the fact that, once the general framework is in place, much of the analysis proceeds along very similar lines, regardless of regularity.

We also include a brief discussion of differential inclusions since we shall use these, as well as usual control systems, in understanding the position of our "tautological control systems" from Section 8 in the existing order of things.

7.1. Parameterised vector fields.

One can think of a control system as a family of vector fields parameterised by control, as discussed in Section 1.1. It is the exact nature of this dependence on the parameter that we discuss in this section.

The smooth case. We begin by discussing parameter dependent smooth sections. Throughout this section we will work with a smooth vector bundle $\pi\colon \mathsf{E} \to \mathsf{M}$ with a linear connection ∇^0 on E, an affine connection ∇ on M, a fibre metric $\mathbb{G}_0$ on E, and a Riemannian metric $\mathbb{G}$ on M. These define the fibre metrics $\|\cdot\|_{\overline{\mathbb{G}}_m}$ and the seminorms $p^\infty_{K,m}$, $K \subseteq \mathsf{M}$ compact, $m \in \mathbb{Z}_{\geq 0}$, on $\Gamma^\infty(\mathsf{E})$ as in Section 3.1.

7.1 Definition: (Sections of parameterised class C^∞) Let $\pi\colon \mathsf{E} \to \mathsf{M}$ be a smooth vector bundle and let $\mathcal{P}$ be a topological space. A map $\xi\colon \mathsf{M} \times \mathcal{P} \to \mathsf{E}$ such that $\xi(x,p) \in \mathsf{E}_x$ for every $(x,p) \in \mathsf{M} \times \mathcal{P}$

(i) is a ***separately parameterised section of class C^∞*** if

  (a) for each $x \in \mathsf{M}$, the map $\xi_x\colon \mathcal{P} \to \mathsf{E}$ defined by $\xi_x(p) = \xi(x,p)$ is continuous and

  (b) for each $p \in \mathcal{P}$, the map $\xi^p\colon \mathsf{M} \to \mathsf{E}$ defined by $\xi^p(x) = \xi(x,p)$ is of class C^∞,

and

(ii) is a ***jointly parameterised section of class C^∞*** if it is a separately parameterised section of class C^∞ and if the map $(x,p) \mapsto j_m\xi^p(x)$ is continuous for every $m \in \mathbb{Z}_{\geq 0}$.

By $\mathrm{SP}\Gamma^\infty(\mathcal{P};\mathsf{E})$ we denote the set of separately parameterised sections of E of class C^∞ and by $\mathrm{JP}\Gamma^\infty(\mathcal{P};\mathsf{E})$ we denote the set of jointly parameterised sections of E of class C^∞. •

It is possible to give purely topological characterisations of this class of sections.

7.2 Proposition: (Characterisation of jointly parameterised sections of class C^∞) *Let $\pi\colon \mathsf{E} \to \mathsf{M}$ be a smooth vector bundle, let $\mathcal{P}$ be a topological space, and let $\xi\colon \mathsf{M} \times \mathcal{P} \to \mathsf{E}$ satisfy $\xi(x,p) \in \mathsf{E}_x$ for every $(x,p) \in \mathsf{M} \times \mathcal{P}$. Then $\xi \in \mathrm{JP}\Gamma^\infty(\mathcal{P};\mathsf{E})$ if and only if the map $p \mapsto \xi^p \in \Gamma^\infty(\mathsf{E})$ is continuous, where $\Gamma^\infty(\mathsf{E})$ has the CO^∞-topology.*

Proof: Given $\xi\colon \mathsf{M} \times \mathcal{P} \to \mathsf{E}$ we let $\xi_m\colon \mathsf{M} \times \mathcal{P} \to \mathsf{J}^m\mathsf{E}$ be the map $\xi_m(x,p) = j_m\xi^p(x)$. We also denote by $\sigma_\xi\colon \mathcal{P} \to \Gamma^\infty(\mathsf{E})$ the map given by $\sigma_\xi(p) = \xi^p$.

First suppose that ξ_m is continuous for every $m \in \mathbb{Z}_{\geq 0}$. Let $K \subseteq \mathsf{M}$ be compact, let $m \in \mathbb{Z}_{\geq 0}$, let $\epsilon \in \mathbb{R}_{>0}$, and let $p_0 \in \mathcal{P}$. Let $x \in K$ and let $\mathcal{W}_x$ be a neighbourhood of $\xi_m(x, p_0)$ in $\mathsf{J}^m\mathsf{E}$ for which

$$\mathcal{W}_x \subseteq \{j_m\eta(x') \in \mathsf{J}^m\mathsf{E} \mid \|j_m\eta(x') - \xi_m(x', p_0)\|_{\overline{\mathbb{G}}_m} < \epsilon\}.$$

By continuity of ξ_m, there exist a neighbourhood $\mathcal{U}_x \subseteq \mathsf{M}$ of x and a neighbourhood $\mathcal{O}_x \subseteq \mathcal{P}$ of p_0 such that $\xi_m(\mathcal{U}_x \times \mathcal{O}_x) \subseteq \mathcal{W}_x$. Now let $x_1, \dots, x_k \in K$ be such that $K \subseteq \cup_{j=1}^k \mathcal{U}_{x_j}$ and let $\mathcal{O} = \cap_{j=1}^k \mathcal{O}_{x_j}$. Then, if $p \in \mathcal{O}$ and $x \in K$, we have $x \in \mathcal{U}_{x_j}$ for some $j \in \{1, \dots, k\}$. Thus $\xi_m(x, p) \in \mathcal{W}_{x_j}$. Thus

$$\|\xi_m(x, p) - \xi_m(x, p_0)\|_{\overline{\mathbb{G}}_m} < \epsilon.$$

Therefore, taking supremums over $x \in K$, $p^\infty_{K,m}(\sigma_\xi(p) - \sigma_\xi(p_0)) \leq \epsilon$. As this can be done for every compact $K \subseteq \mathsf{M}$ and every $m \in \mathbb{Z}_{\geq 0}$, we conclude that σ_ξ is continuous.

Next suppose that σ_ξ is continuous and let $m \in \mathbb{Z}_{\geq 0}$. Let $(x_0, p_0) \in \mathsf{M} \times \mathcal{P}$ and let $\mathcal{W} \subseteq \mathsf{J}^m\mathsf{E}$ be a neighbourhood of $\xi_m(x_0, p_0)$. Let $\mathcal{U} \subseteq \mathsf{M}$ be a relatively compact neighbourhood of x_0 and let $\epsilon \in \mathbb{R}_{>0}$ be such that

$$\pi_m^{-1}(\mathcal{U}) \cap \{j_m\eta(x) \in \mathsf{J}^m\mathsf{E} \mid \|j_m\eta(x) - \xi_m(x, p_0)\|_{\overline{\mathbb{G}}_m} < \epsilon\} \subseteq \mathcal{W},$$

where $\pi_m \colon \mathsf{J}^m\mathsf{E} \to \mathsf{M}$ is the projection. By continuity of σ_ξ, let $\mathcal{O} \subseteq \mathcal{P}$ be a neighbourhood of p_0 such that $p^\infty_{\mathrm{cl}(\mathcal{U}),m}(\sigma_\xi(p) - \sigma_\xi(p_0)) < \epsilon$ for $p \in \mathcal{O}$. Therefore,

$$\|j_m\sigma_\xi(p)(x) - j_m\sigma_\xi(p_0)(x)\|_{\overline{\mathbb{G}}_m} < \epsilon, \qquad (x, p) \in \mathrm{cl}(\mathcal{U}) \times \mathcal{O}.$$

Therefore, if $(x, p) \in \mathcal{U} \times \mathcal{O}$, then $\pi_m(\xi_m(x, p)) = x \in \mathcal{U}$ and so $\xi_m(x, p) \in \mathcal{W}$, showing that ξ_m is continuous at (x_0, p_0). ■

Of course, the preceding discussion applies, in particular, to give vector fields of parameterised class C^∞ and functions of parameterised class C^∞. This gives the spaces $\mathrm{SPC}^\infty(\mathcal{P}; \mathsf{M})$ and $\mathrm{JPC}^\infty(\mathsf{M})$ of parameterised functions, and the spaces $\mathrm{SP}\Gamma^\infty(\mathcal{P}; \mathsf{TM})$ and $\mathrm{JP}\Gamma^\infty(\mathcal{P}; \mathsf{TM})$ of parameterised vector fields. Let us verify that we can as well use a weak-$\mathscr{L}$ version of this characterisation for jointly parameterised vector fields.

7.3 Proposition: (Weak-$\mathscr{L}$ characterisation of jointly parameterised vector fields of class C^∞) *Let M be a smooth manifold, let $\mathcal{P}$ be a topological space, and let $X \colon \mathsf{M} \times \mathcal{P} \to \mathsf{TM}$ satisfy $X(x, p) \in \mathsf{T}_x\mathsf{M}$ for every $(x, p) \in \mathsf{M} \times \mathcal{P}$. Then $X \in \mathrm{JP}\Gamma^\infty(\mathcal{P}; \mathsf{TM})$ if and only if $(x, p) \mapsto \mathscr{L}_{X^p} f$ is a jointly parameterised function of class C^∞ for every $f \in \mathrm{C}^\infty(\mathsf{M})$.*

Proof: This follows from Corollary 3.6(ii). ■

The finitely differentiable or Lipschitz case. The preceding development in the smooth case is easily extended to the finitely differentiable and Lipschitz cases, and we quickly give the results and definitions here. In this section, when considering the Lipschitz case, we assume that ∇ is the Levi-Civita connection associated to $\mathbb{G}$ and we assume that ∇^0 is $\mathbb{G}_0$-orthogonal.

7.4 Definition: (Sections of parameterised class $\mathrm{C}^{m+m'}$) Let $\pi\colon \mathsf{E} \to \mathsf{M}$ be a smooth vector bundle and let $\mathcal{P}$ be a topological space. A map $\xi\colon \mathsf{M} \times \mathcal{P} \to \mathsf{E}$ such that $\xi(x,p) \in \mathsf{E}_x$ for every $(x,p) \in \mathsf{M} \times \mathcal{P}$

(i) is a ***separately parameterised section of class $\mathrm{C}^{m+m'}$*** if

  (a) for each $x \in \mathsf{M}$, the map $\xi_x\colon \mathcal{P} \to \mathsf{E}$ defined by $\xi_x(p) = \xi(x,p)$ is continuous and

  (b) for each $p \in \mathcal{P}$, the map $\xi^p\colon \mathsf{M} \to \mathsf{E}$ defined by $\xi^p(x) = \xi(x,p)$ is of class $\mathrm{C}^{m+m'}$,

  and

(ii) is a ***jointly parameterised section of class $\mathrm{C}^{m+m'}$*** if it is a separately parameterised section of class $\mathrm{C}^{m+m'}$ and

  (a) $m' = 0$: the map $(x,p) \mapsto j_m\xi^p(x)$ is continuous;

  (b) $m' = \mathrm{lip}$: the map $(x,p) \mapsto j_m\xi^p(x)$ is continuous and, for each $(x_0,p_0) \in \mathsf{M} \times \mathcal{P}$ and each $\epsilon \in \mathbb{R}_{>0}$, there exist a neighbourhood $\mathcal{U} \subseteq \mathsf{M}$ of x_0 and a neighbourhood $\mathcal{O} \subseteq \mathcal{P}$ of p_0 such that

$$j_m\xi(\mathcal{U} \times \mathcal{O}) \subseteq \{j_m\eta(x) \in \mathsf{J}^m\mathsf{E} \mid \operatorname{dil}(j_m\eta - j_m\xi^{p_0})(x) < \epsilon\},$$

  where, of course, $j_m\xi(x,p) = j_m\xi^p(x)$.

By $\mathrm{SP}\Gamma^{m+m'}(\mathcal{P};\mathsf{E})$ we denote the set of separately parameterised sections of E of class $\mathrm{C}^{m+m'}$ and by $\mathrm{JP}\Gamma^{m+m'}(\mathcal{P};\mathsf{E})$ we denote the set of jointly parameterised sections of E of class $\mathrm{C}^{m+m'}$. •

Let us give the purely topological characterisation of this class of sections.

7.5 Proposition: (Characterisation of jointly parameterised sections of class $\mathrm{C}^{m+m'}$) *Let $\pi\colon \mathsf{E} \to \mathsf{M}$ be a smooth vector bundle, let $\mathcal{P}$ be a topological space, and let $\xi\colon \mathsf{M} \times \mathcal{P} \to \mathsf{E}$ satisfy $\xi(x,p) \in \mathsf{E}_x$ for every $(x,p) \in \mathsf{M} \times \mathcal{P}$. Then $\xi \in \mathrm{JP}\Gamma^{m+m'}(\mathcal{P};\mathsf{E})$ if and only if the map $p \mapsto \xi^p \in \Gamma^{m+m'}(\mathsf{E})$ is continuous, where $\Gamma^{m+m'}(\mathsf{E})$ has the $\mathrm{CO}^{m+m'}$-topology.*

Proof: We will prove the result only in the case that $m = 0$ and $m' = \mathrm{lip}$, as the general case follows by combining this case with the computations from the proof of Proposition 7.2. We denote $\sigma_\xi(p) = \xi(x,p)$.

Suppose that $(x,p) \mapsto \xi(x,p)$ is continuous and that, for every $(x_0,p_0) \in \mathsf{M} \times \mathcal{P}$ and for every $\epsilon \in \mathbb{R}_{>0}$, there exist a neighbourhood $\mathcal{U} \subseteq \mathsf{M}$ of x_0 and a neighbourhood $\mathcal{O} \subseteq \mathcal{P}$ of p_0 such that, if $(x,p) \in \mathcal{U} \times \mathcal{O}$, then $\operatorname{dil}(\xi^p - \xi^{p_0})(x) < \epsilon$. Let $K \subseteq \mathsf{M}$ be compact, let $\epsilon \in \mathbb{R}_{>0}$, and let $p_0 \in \mathcal{P}$. Let $x \in K$. By hypothesis, there exist a neighbourhood $\mathcal{U}_x \subseteq \mathsf{M}$ of x and a neighbourhood $\mathcal{O}_x \subseteq \mathcal{P}$ of p_0 such that

$$\xi(\mathcal{U}_x \times \mathcal{O}_x) \subseteq \{\eta(x') \in \mathsf{J}^m\mathsf{E} \mid \operatorname{dil}(\eta - \xi^{p_0})(x') < \epsilon\}.$$

Now let $x_1,\dots,x_k \in K$ be such that $K \subseteq \cup_{j=1}^k \mathcal{U}_{x_j}$ and let $\mathcal{O} = \cap_{j=1}^k \mathcal{O}_{x_j}$. Then, if $p \in \mathcal{O}$ and $x \in K$, we have $x \in \mathcal{U}_{x_j}$ for some $j \in \{1,\dots,k\}$. Thus

$$\operatorname{dil}(\xi(x,p) - \xi(x,p_0))_{\overline{\mathbb{G}}_m} < \epsilon.$$

Therefore, taking supremums over $x \in K$, we have $\lambda_K(\sigma_\xi(p) - \sigma_\xi(p_0)) \le \epsilon$. By choosing $\mathcal{O}$ to be possibly smaller, the argument of Proposition 7.2 ensures that $p_K^0(\sigma_\xi(p) - \sigma_\xi(p_0)) \le \epsilon$, and so $p_K^{\mathrm{lip}}(\sigma_\xi(p) - \sigma_\xi(p_0)) < \epsilon$ for $p \in \mathcal{O}$. As this can be done for every compact $K \subseteq \mathsf{M}$, we conclude that σ_ξ is continuous.

Next suppose that σ_ξ is continuous, let $(x_0, p_0) \in \mathsf{M} \times \mathcal{P}$, and let $\epsilon \in \mathbb{R}_{>0}$. Let $\mathcal{U}$ be a relatively compact neighbourhood of x_0. Since σ_ξ is continuous, let $\mathcal{O}$ be a neighbourhood of p_0 such that

$$p_{\mathrm{cl}(\mathcal{U})}^{\mathrm{lip}}(\sigma_\xi(p) - \sigma_\xi(p_0)) < \epsilon, \qquad p \in \mathcal{O}.$$

Thus, for every $(x, p) \in \mathcal{U} \times \mathcal{O}$, $\mathrm{dil}\,(\xi^p - \xi^{p_0})(x) < \epsilon$. Following the argument of Proposition 7.2 one also shows that ξ is continuous at (x_0, p_0), which shows that $\xi \in \mathrm{JP}\Gamma^{\mathrm{lip}}(\mathcal{P}; \mathsf{E})$. ■

Of course, the preceding discussion applies, in particular, to give vector fields of jointly parameterised class $\mathrm{C}^{m+m'}$ and functions of jointly parameterised class $\mathrm{C}^{m+m'}$. This gives the spaces $\mathrm{SPC}^{m+m'}(\mathcal{P}; \mathsf{M})$ and $\mathrm{JPC}^{m+m'}(\mathsf{M})$ of parameterised functions, and the spaces $\mathrm{SP}\Gamma^{m+m'}(\mathcal{P}; \mathsf{TM})$ and $\mathrm{JP}\Gamma^{m+m'}(\mathcal{P}; \mathsf{TM})$ of parameterised vector fields. Let us verify that we can as well use a weak-$\mathscr{L}$ version of this characterisation for jointly parameterised vector fields.

7.6 Proposition: (Weak-$\mathscr{L}$ characterisation of jointly parameterised vector fields of class $\mathrm{C}^{m+m'}$) *Let M be a smooth manifold, let $\mathcal{P}$ be a topological space, and let $X\colon \mathsf{M} \times \mathcal{P} \to \mathsf{TM}$ satisfy $X(x, p) \in \mathsf{T}_x\mathsf{M}$ for every $(x, p) \in \mathsf{M} \times \mathcal{P}$. Then $X \in \mathrm{JP}\Gamma^{m+m'}(\mathcal{P}; \mathsf{TM})$ if and only if $(x, p) \mapsto \mathscr{L}_{X^p} f$ is a jointly parameterised function of class $\mathrm{C}^{m+m'}$ for every $f \in \mathrm{C}^\infty(\mathsf{M})$.*

Proof: This follows from Corollary 3.15(ii). ■

The holomorphic case. As with time-varying vector fields, we are not really interested, *per se*, in holomorphic control systems, and in fact we will not even define the notion. However, it is possible, and possibly sometimes easier, to verify that a control system satisfies our rather technical criterion of being a "real analytic control system" by verifying that it possesses an holomorphic extension. Thus, in this section, we present the required holomorphic definitions. We will consider an holomorphic vector bundle $\pi\colon \mathsf{E} \to \mathsf{M}$ with an Hermitian fibre metric $\mathbb{G}$. This defines the seminorms p_K^{hol}, $K \subseteq \mathsf{M}$ compact, describing the $\mathrm{CO}^{\mathrm{hol}}$-topology for $\Gamma^{\mathrm{hol}}(\mathsf{E})$ as in Section 4.1.

7.7 Definition: (Sections of parameterised class $\mathrm{C}^{\mathrm{hol}}$) Let $\pi\colon \mathsf{E} \to \mathsf{M}$ be an holomorphic vector bundle and let $\mathcal{P}$ be a topological space. A map $\xi\colon \mathsf{M} \times \mathcal{P} \to \mathsf{E}$ such that $\xi(z, p) \in \mathsf{E}_z$ for every $(z, p) \in \mathsf{M} \times \mathcal{P}$

(i) is a ***separately parameterised section of class $\mathrm{C}^{\mathrm{hol}}$*** if

 (a) for each $z \in \mathsf{M}$, the map $\xi_z\colon \mathcal{P} \to \mathsf{E}$ defined by $\xi_z(p) = \xi(z, p)$ is continuous and

 (b) for each $p \in \mathcal{P}$, the map $\xi^p\colon \mathsf{M} \to \mathsf{E}$ defined by $\xi^p(z) = \xi(z, p)$ is of class $\mathrm{C}^{\mathrm{hol}}$,

 and

(ii) is a ***jointly parameterised section of class $\mathrm{C}^{\mathrm{hol}}$*** if it is a separately parameterised section of class $\mathrm{C}^{\mathrm{hol}}$ and if the map $(z, p) \mapsto \xi^p(z)$ is continuous.

By $\mathrm{SP}\Gamma^{\mathrm{hol}}(\mathcal{P}; \mathsf{E})$ we denote the set of separately parameterised sections of E of class $\mathrm{C}^{\mathrm{hol}}$ and by $\mathrm{JP}\Gamma^{\mathrm{hol}}(\mathcal{P}; \mathsf{E})$ we denote the set of jointly parameterised sections of E of class $\mathrm{C}^{\mathrm{hol}}$. •

As in the smooth case, it is possible to give purely topological characterisations of these classes of sections.

7.8 Proposition: (Characterisation of jointly parameterised sections of class $\mathrm{C}^{\mathrm{hol}}$) *Let $\pi\colon \mathsf{E} \to \mathsf{M}$ be an holomorphic vector bundle, let $\mathcal{P}$ be a topological space, and let $\xi\colon \mathsf{M} \times \mathcal{P} \to \mathsf{E}$ satisfy $\xi(z,p) \in \mathsf{E}_z$ for every $(z,p) \in \mathsf{M} \times \mathcal{P}$. Then $\xi \in \mathrm{JP}\Gamma^{\mathrm{hol}}(\mathcal{P};\mathsf{E})$ if and only if the map $p \mapsto \xi^p \in \Gamma^{\mathrm{hol}}(\mathsf{E})$ is continuous, where $\Gamma^{\mathrm{hol}}(\mathsf{E})$ has the $\mathrm{CO}^{\mathrm{hol}}$-topology.*

Proof: We define $\sigma_\xi\colon \mathcal{P} \to \Gamma^{\mathrm{hol}}(\mathsf{E})$ by $\sigma_\xi(p) = \xi^p$.

First suppose that ξ is continuous. Let $K \subseteq \mathsf{M}$ be compact, let $\epsilon \in \mathbb{R}_{>0}$, and let $p_0 \in \mathcal{P}$. Let $z \in K$ and let $\mathcal{W}_z \subseteq \mathsf{E}$ be a neighbourhood of $\xi(z,p_0)$ for which

$$\mathcal{W}_z \subseteq \{\eta(z') \in \mathsf{E} \mid \ \|\eta(z') - \xi(z',p_0)\|_{\overline{\mathbb{G}}} < \epsilon\}.$$

By continuity of ξ, there exist a neighbourhood $\mathcal{U}_z \subseteq \mathsf{M}$ of z and a neighbourhood $\mathcal{O}_z \subseteq \mathcal{P}$ of p_0 such that $\xi(\mathcal{U}_z \times \mathcal{O}_z) \subseteq \mathcal{W}_z$. Now let $z_1,\dots,z_k \in K$ be such that $K \subseteq \cup_{j=1}^k \mathcal{U}_{z_j}$ and let $\mathcal{O} = \cap_{j=1}^k \mathcal{O}_{z_j}$. Then, if $p \in \mathcal{O}$ and $z \in K$, we have $z \in \mathcal{U}_{z_j}$ for some $j \in \{1,\dots,k\}$. Thus $\xi(z,p) \in \mathcal{W}_{z_j}$. Thus $\|\xi(z,p) - \xi(z,p_0)\|_{\overline{\mathbb{G}}} < \epsilon$. Therefore, taking supremums over $z \in K$, $p_K^{\mathrm{hol}}(\sigma_\xi(p) - \sigma_\xi(p_0)) \le \epsilon$. As this can be done for every compact $K \subseteq \mathsf{M}$, we conclude that σ_ξ is continuous.

Next suppose that σ_ξ is continuous. Let $(z_0,p_0) \in \mathsf{M} \times \mathcal{P}$ and let $\mathcal{W} \subseteq \mathsf{E}$ be a neighbourhood of $\xi(z_0,p_0)$. Let $\mathcal{U} \subseteq \mathsf{M}$ be a relatively compact neighbourhood of z_0 and let $\epsilon \in \mathbb{R}_{>0}$ be such that

$$\pi^{-1}(\mathcal{U}) \cap \{\eta(z) \in \mathsf{E} \mid \ \|\eta(z) - \xi(z,p_0)\|_{\overline{\mathbb{G}}} < \epsilon\} \subseteq \mathcal{W}.$$

By continuity of σ_ξ, let $\mathcal{O} \subseteq \mathcal{P}$ be a neighbourhood of p_0 such that $p^{\mathrm{hol}}_{\mathrm{cl}(\mathcal{U})}(\sigma_\xi(p) - \sigma_\xi(p_0)) < \epsilon$ for $p \in \mathcal{O}$. Therefore,

$$\|\sigma_\xi(p)(z) - \sigma_\xi(p_0)(z)\|_{\overline{\mathbb{G}}} < \epsilon, \qquad (z,p) \in \mathrm{cl}(\mathcal{U}) \times \mathcal{O}.$$

Therefore, if $(z,p) \in \mathcal{U} \times \mathcal{O}$, we have $\xi(z,p) \in \mathcal{W}$, showing that ξ is continuous at (z_0,p_0). ■

The specialisation of the preceding constructions to vector fields and functions is immediate. This gives the spaces $\mathrm{SPC}^{\mathrm{hol}}(\mathcal{P};\mathsf{M})$ and $\mathrm{JPC}^{\mathrm{hol}}(\mathsf{M})$ of parameterised functions, and the spaces $\mathrm{SP}\Gamma^{\mathrm{hol}}(\mathcal{P};\mathsf{TM})$ and $\mathrm{JP}\Gamma^{\mathrm{hol}}(\mathcal{P};\mathsf{TM})$ of parameterised vector fields. Let us verify that we can as well use a weak-$\mathscr{L}$ version of the preceding definitions for vector fields in the case when the base manifold is Stein.

7.9 Proposition: (Weak-$\mathscr{L}$ characterisation of jointly parameterised vector fields of class $\mathrm{C}^{\mathrm{hol}}$ on Stein manifolds) *Let M be a Stein manifold, let $\mathcal{P}$ be a topological space, and let $X\colon \mathsf{M} \times \mathcal{P} \to \mathsf{TM}$ satisfy $X(z,p) \in \mathsf{T}_z\mathsf{M}$ for every $(z,p) \in \mathsf{M} \times \mathcal{P}$. Then $X \in \mathrm{JP}\Gamma^{\mathrm{hol}}(\mathcal{P};\mathsf{TM})$ if and only if $(x,p) \mapsto \mathscr{L}_{X^p} f$ is a jointly parameterised function of class $\mathrm{C}^{\mathrm{hol}}$ for every $f \in \mathrm{C}^\infty(\mathsf{M})$.*

Proof: This follows from Corollary 4.6(ii). ■

The real analytic case. Now we repeat the procedure above for real analytic sections. We thus will consider a real analytic vector bundle $\pi\colon \mathsf{E} \to \mathsf{M}$ with ∇^0 a real analytic linear connection on E, ∇ a real analytic affine connection on M, $\mathbb{G}_0$ a real analytic fibre metric on E, and $\mathbb{G}$ a real analytic Riemannian metric on M. This defines the seminorms $p^\omega_{K,\boldsymbol{a}}$, $K \subseteq \mathsf{M}$ compact, $\boldsymbol{a} \in \mathrm{c}_{\downarrow 0}(\mathbb{Z}_{\ge 0};\mathbb{R}_{>0})$, describing the C^ω-topology as in Theorem 5.5.

7.10 Definition: (Sections of parameterised class C^ω) Let $\pi\colon \mathsf{E}\to\mathsf{M}$ be a real analytic vector bundle and let $\mathcal{P}$ be a topological space. A map $\xi\colon \mathsf{M}\times\mathcal{P}\to\mathsf{E}$ such that $\xi(x,p)\in\mathsf{E}_x$ for every $(x,p)\in\mathsf{M}\times\mathcal{P}$

(i) is a ***separately parameterised section of class C^ω*** if

  (a) for each $x\in\mathsf{M}$, the map $\xi_x\colon\mathcal{P}\to\mathsf{E}$ defined by $\xi_x(p)=\xi(x,p)$ is continuous and

  (b) for each $p\in\mathcal{P}$, the map $\xi^p\colon\mathsf{M}\to\mathsf{E}$ defined by $\xi^p(x)=\xi(x,p)$ is of class C^ω,

 and

(ii) is a ***jointly parameterised section of class C^ω*** if it is a separately parameterised section of class C^∞ and if, for each $(x_0,p_0)\in\mathsf{M}\times\mathcal{P}$, for each $\boldsymbol{a}\in c_{\downarrow 0}(\mathbb{Z}_{\geq 0},\mathbb{R}_{>0})$, and for each $\epsilon\in\mathbb{R}_{>0}$, there exist a neighbourhood $\mathcal{U}\subseteq\mathsf{M}$ of x_0 and a neighbourhood $\mathcal{O}\subseteq\mathcal{P}$ of p_0 such that

$$j_m\xi(\mathcal{U}\times\mathcal{O})\subseteq\{j_m\eta(x)\in\mathsf{J}^m\mathsf{E}\mid\ a_0a_1\cdots a_m\|j_m\eta(x)-j_m\xi^{p_0}(x)\|_{\overline{\mathbb{G}}_m}<\epsilon\}$$

 for every $m\in\mathbb{Z}_{\geq 0}$, where, of course, $j_m\xi(x,p)=j_m\xi^p(x)$.

By $\mathrm{SP}\Gamma^\omega(\mathcal{P};\mathsf{E})$ we denote the set of separately parameterised sections of E of class C^ω and by $\mathrm{JP}\Gamma^\omega(\mathcal{P};\mathsf{E})$ we denote the set of jointly parameterised sections of E of class C^ω. •

7.11 Remark: (Jointly parameterised sections of class C^ω) The condition that $\xi\in\mathrm{JP}\Gamma^\infty(\mathcal{P};\mathsf{E})$ can be restated like this: for each $(x_0,p_0)\in\mathsf{M}\times\mathcal{P}$, for each $m\in\mathbb{Z}_{\geq 0}$, and for each $\epsilon\in\mathbb{R}_{>0}$, there exist a neighbourhood $\mathcal{U}\subseteq\mathsf{M}$ of x_0 and a neighbourhood $\mathcal{O}\subseteq\mathcal{P}$ of p_0 such that

$$j_m\xi(\mathcal{U}\times\mathcal{O})\subseteq\{j_m\eta(x)\in\mathsf{J}^m\mathsf{E}\mid\ \|j_m\eta(x)-j_m\xi^{p_0}(x)\|_{\overline{\mathbb{G}}_m}<\epsilon\};$$

that this is so is, more or less, the idea of the proof of Proposition 7.2. Phrased this way, one sees clearly the grammatical similarity between the smooth and real analytic definitions. Indeed, the grammatical transformation from the smooth to the real analytic definition is, *put a factor of $a_0a_1\cdots a_m$ before the norm, precede the condition with "for every $\boldsymbol{a}\in c_{\downarrow 0}(\mathbb{Z}_{\geq 0};\mathbb{R}_{>0})$", and move the "for every $m\in\mathbb{Z}_{\geq 0}$" from before the condition to after.* This was also seen in the definitions of locally integrally bounded and locally essentially bounded sections in Section 6. Indeed, the grammatical similarity will be encountered many times in the sequel, and we shall refer to this to keep ourselves from repeating arguments in the real analytic case that mirror their smooth counterparts. •

The following result records topological characterisations of jointly parameterised sections in the real analytic case.

7.12 Proposition: (Characterisation of jointly parameterised sections of class C^ω) *Let $\pi\colon\mathsf{E}\to\mathsf{M}$ be a real analytic vector bundle, let $\mathcal{P}$ be a topological space, and let $\xi\colon\mathsf{M}\times\mathcal{P}\to\mathsf{E}$ satisfy $\xi(x,p)\in\mathsf{E}_x$ for every $(x,p)\in\mathsf{M}\times\mathcal{P}$. Then $\xi\in\mathrm{JP}\Gamma^\omega(\mathcal{P};\mathsf{E})$ if and only if the map $p\mapsto\xi^p\in\Gamma^\omega(\mathsf{E})$ is continuous, where $\Gamma^\omega(\mathsf{E})$ has the C^ω-topology.*

Proof: For $\boldsymbol{a}\in c_{\downarrow 0}(\mathbb{Z}_{\geq 0};\mathbb{R}_{>0})$ and $m\in\mathbb{Z}_{\geq 0}$, given $\xi\colon\mathsf{M}\times\mathcal{P}\to\mathsf{E}$ satisfying $\xi^p\in\Gamma^\omega(\mathsf{E})$, we let $\xi_{\boldsymbol{a},m}\colon\mathsf{M}\times\mathcal{P}\to\mathsf{J}^m\mathsf{E}$ be the map

$$\xi_{\boldsymbol{a},m}(x,p)=a_0a_1\cdots a_mj_m\xi^p(x).$$

We also denote by $\sigma_\xi\colon\mathcal{P}\to\Gamma^\omega(\mathsf{E})$ the map given by $\sigma_\xi(p)=\xi^p$.

Suppose that, for every $(x_0, p_0) \in \mathsf{M} \times \mathcal{P}$, for every $\boldsymbol{a} \in \mathrm{c}_{\downarrow 0}(\mathbb{Z}_{\geq 0}; \mathbb{R}_{>0})$, and for every $\epsilon \in \mathbb{R}_{>0}$, there exist a neighbourhood $\mathcal{U} \subseteq \mathsf{M}$ of x_0 and a neighbourhood $\mathcal{O} \subseteq \mathcal{P}$ of p_0 such that, if $(x, p) \in \mathcal{U} \times \mathcal{O}$, then

$$\|\xi_{\boldsymbol{a},m}(x, p) - \xi_{\boldsymbol{a},m}(x, p_0)\|_{\overline{\mathbb{G}}_m} < \epsilon, \qquad m \in \mathbb{Z}_{\geq 0}.$$

Let $K \subseteq \mathsf{M}$ be compact, let $\boldsymbol{a} \in \mathrm{c}_{\downarrow 0}(\mathbb{Z}_{\geq 0}; \mathbb{R}_{>0})$, let $\epsilon \in \mathbb{R}_{>0}$, and let $p_0 \in \mathcal{P}$. Let $x \in K$. By hypothesis, there exist a neighbourhood $\mathcal{U}_x \subseteq \mathsf{M}$ of x and a neighbourhood $\mathcal{O}_x \subseteq \mathcal{P}$ of p_0 such that

$$\xi_{\boldsymbol{a},m}(\mathcal{U}_x \times \mathcal{O}_x) \subseteq \{j_m\eta(x') \in \mathsf{J}^m\mathsf{E} \mid \ \|a_0a_1 \cdots a_m j_m\eta(x') - \xi_{\boldsymbol{a},m}(x', p_0)\|_{\overline{\mathbb{G}}_m} < \epsilon\},$$

for each $m \in \mathbb{Z}_{\geq 0}$. Now let $x_1, \dots, x_k \in K$ be such that $K \subseteq \cup_{j=1}^k \mathcal{U}_{x_j}$ and let $\mathcal{O} = \cap_{j=1}^k \mathcal{O}_{x_j}$. Then, if $p \in \mathcal{O}$ and $x \in K$, we have $x \in \mathcal{U}_{x_j}$ for some $j \in \{1, \dots, k\}$. Thus

$$\|\xi_{\boldsymbol{a},m}(x, p) - \xi_{\boldsymbol{a},m}(x, p_0)\|_{\overline{\mathbb{G}}_m} < \epsilon, \qquad m \in \mathbb{Z}_{\geq 0}.$$

Therefore, taking supremums over $x \in K$ and $m \in \mathbb{Z}_{\geq 0}$, we have $p^\omega_{K,\boldsymbol{a}}(\sigma_\xi(p) - \sigma_\xi(p_0)) \leq \epsilon$. As this can be done for every compact $K \subseteq \mathsf{M}$ and every $\boldsymbol{a} \in \mathrm{c}_{\downarrow 0}(\mathbb{Z}_{\geq 0}; \mathbb{R}_{>0})$, we conclude that σ_ξ is continuous.

Next suppose that σ_ξ is continuous, let $(x_0, p_0) \in \mathsf{M} \times \mathcal{P}$, let $\boldsymbol{a} \in \mathrm{c}_{\downarrow 0}(\mathbb{Z}_{\geq 0}; \mathbb{R}_{>0})$, and let $\epsilon \in \mathbb{R}_{>0}$. Let $\mathcal{U}$ be a relatively compact neighbourhood of x_0. Since σ_ξ is continuous, let $\mathcal{O}$ be a neighbourhood of p_0 such that

$$p^\omega_{\mathrm{cl}(\mathcal{U}),\boldsymbol{a}}(\sigma_\xi(p) - \sigma_\xi(p_0)) < \epsilon, \qquad p \in \mathcal{O}.$$

Thus, for every $(x, p) \in \mathcal{U} \times \mathcal{O}$,

$$a_0a_1 \cdots a_m \|j_m\xi(x, p) - j_m\xi(x, p_0)\|_{\overline{\mathbb{G}}_m} < \epsilon, \qquad m \in \mathbb{Z}_{\geq 0},$$

which shows that $\xi \in \mathrm{JP}\Gamma^\omega(\mathcal{P}; \mathsf{E})$. ■

As we have done in the smooth and holomorphic cases above, we can specialise the preceding discussion from sections to vector fields and functions, giving the spaces $\mathrm{SPC}^\omega(\mathcal{P}; \mathsf{M})$ and $\mathrm{JPC}^\omega(\mathsf{M})$ of parameterised functions, and the spaces $\mathrm{SP}\Gamma^\omega(\mathcal{P}; \mathsf{TM})$ and $\mathrm{JP}\Gamma^\omega(\mathcal{P}; \mathsf{TM})$ of parameterised vector fields. We then have the following weak-$\mathscr{L}$ characterisation for jointly parameterised vector fields.

7.13 Proposition: (Weak-$\mathscr{L}$ characterisation of jointly parameterised vector fields of class C^ω) *Let M be a real analytic manifold, let $\mathcal{P}$ be a topological space, and let $X \colon \mathsf{M} \times \mathcal{P} \to \mathsf{TM}$ satisfy $X(x, p) \in \mathsf{T}_x\mathsf{M}$ for every $(x, p) \in \mathsf{M} \times \mathcal{P}$. Then $X \in \mathrm{JP}\Gamma^\omega(\mathcal{P}; \mathsf{TM})$ if and only if $(x, p) \mapsto \mathscr{L}_{X^p} f(x)$ is a jointly parameterised function of class C^ω for every $f \in \mathrm{C}^\omega(\mathsf{M})$.*

Proof: This follows from Corollary 5.9(ii). ■

One can wonder about the relationship between sections of jointly parameterised class C^ω and sections that are real restrictions of sections of jointly parameterised class $\mathrm{C}^{\mathrm{hol}}$. We address this with a result and an example. First the result.

7.14 Theorem: (Jointly parameterised real analytic sections as restrictions of jointly parameterised holomorphic sections) *Let $\pi\colon \mathsf{E} \to \mathsf{M}$ be a real analytic vector bundle with holomorphic extension $\overline{\pi}\colon \overline{\mathsf{E}} \to \overline{\mathsf{M}}$ and let $\mathcal{P}$ be a topological space. For a map $\xi\colon \mathsf{M} \times \mathcal{P} \to \mathsf{E}$ satisfying $\xi(x,p) \in \mathsf{E}_x$ for all $(x,p) \in \mathsf{M} \times \mathcal{P}$, the following statements hold:*

(i) if $\xi \in \mathrm{JP}\Gamma^\omega(\mathcal{P};\mathsf{E})$ and if $\mathcal{P}$ is locally compact and Hausdorff, then, for each $(x_0,p_0) \in \mathsf{M} \times \mathcal{P}$, there exist a neighbourhood $\overline{\mathcal{U}} \subseteq \overline{\mathsf{M}}$ of x_0, a neighbourhood $\mathcal{O} \subseteq \mathcal{P}$ of p_0, and $\overline{\xi} \in \mathrm{JP}\Gamma^{\mathrm{hol}}(\mathcal{O};\overline{\mathsf{E}}|\overline{\mathcal{U}})$ such that $\xi(x,p) = \overline{\xi}(x,p)$ for all $(x,p) \in (\mathsf{M} \cap \overline{\mathcal{U}}) \times \mathcal{O}$;

(ii) if there exists a section $\overline{\xi} \in \mathrm{JP}\Gamma^{\mathrm{hol}}(\mathcal{P};\overline{\mathsf{E}})$ such that $\xi(x,p) = \overline{\xi}(x,p)$ for every $(x,p) \in \mathsf{M} \times \mathcal{P}$, then $\xi \in \mathrm{JP}\Gamma^\omega(\mathcal{P};\mathsf{E})$.

Proof: (i) Let $p_0 \in \mathcal{P}$ and let $\mathcal{O}$ be a relatively compact neighbourhood of p_0, this being possible since $\mathcal{P}$ is locally compact. Let $x_0 \in \mathsf{M}$, let $\mathcal{U}$ be a relatively compact neighbourhood of x_0, and let $(\overline{\mathcal{U}}_j)_{j\in\mathbb{Z}_{>0}}$ be a sequence of neighbourhoods of $\mathrm{cl}(\mathcal{U})$ in $\overline{\mathsf{M}}$ with the properties that $\mathrm{cl}(\overline{\mathcal{U}}_{j+1}) \subseteq \overline{\mathcal{U}}_j$ and that $\cap_{j\in\mathbb{Z}_{>0}}\overline{\mathcal{U}}_j = \mathrm{cl}(\mathcal{U})$. We first note that

$$\mathrm{C}^0(\mathrm{cl}(\mathcal{O});\mathscr{G}^{\mathrm{hol},\mathbb{R}}_{\mathrm{cl}(\mathcal{U}),\overline{\mathsf{E}}}) \simeq \mathrm{C}^0(\mathrm{cl}(\mathcal{O}))\check{\otimes}_e\mathscr{G}^{\mathrm{hol},\mathbb{R}}_{\mathrm{cl}(\mathcal{U}),\overline{\mathsf{E}}}$$

and

$$\mathrm{C}^0(\mathrm{cl}(\mathcal{O});\Gamma^{\mathrm{hol},\mathbb{R}}(\overline{\mathsf{E}}|\overline{\mathcal{U}}_j)) \simeq \mathrm{C}^0(\mathrm{cl}(\mathcal{O}))\check{\otimes}_e\Gamma^{\mathrm{hol},\mathbb{R}}(\overline{\mathsf{E}}|\overline{\mathcal{U}}_j),$$

with $\check{\otimes}_e$ denoting the completed injective tensor product; see [Jarchow 1981, Chapter 16] for the injective tensor product for locally convex spaces and [Diestel, Fourie, and Swart 2008, Theorem 1.1.10] for the preceding isomorphisms for Banach spaces (the constructions apply more or less verbatim to locally convex spaces [Bierstedt 2007, Proposition 5.4]). One can also prove, using the argument from the proof of [Diestel, Fourie, and Swart 2008, Theorem 1.1.10] (see top of page 15 of that reference), that, if $[\overline{\xi}]_K \in \mathrm{C}^0(\mathrm{cl}(\mathcal{O});\mathscr{G}^{\mathrm{hol},\mathbb{R}}_{\mathrm{cl}(\mathcal{U}),\overline{\mathsf{E}}})$, then there is a *sequence* (we know there is a *net*) $([\overline{\xi}_k]_{\mathrm{cl}(\mathcal{U})})_{k\in\mathbb{Z}_{>0}}$ in $\mathrm{C}^0(\mathrm{cl}(\mathcal{O}))\otimes\mathscr{G}^{\mathrm{hol},\mathbb{R}}_{\mathrm{cl}(\mathcal{U}),\overline{\mathsf{E}}}$ converging to $[\overline{\xi}]_K$ in the completed injective tensor product topology. Note that since $\mathscr{G}^{\mathrm{hol},\mathbb{R}}_{\mathrm{cl}(\mathcal{U}),\overline{\mathsf{E}}}$ and $\Gamma^{\mathrm{hol},\mathbb{R}}(\overline{\mathsf{E}}|\overline{\mathcal{U}}_j)$, $j \in \mathbb{Z}_{>0}$, are nuclear, the injective tensor product can be swapped with the projective tensor product in the above constructions [Pietsch 1969, Proposition 5.4.2]. One can now reproduce the argument from the proof of Theorem 6.25, swapping $\mathrm{L}^1(\mathbb{T}';\mathbb{R})$ with $\mathrm{C}^0(\mathrm{cl}(\mathcal{O}))$ and using the results of Mangino [1997], to complete the proof in this case.

(ii) Let $(x_0,p_0) \in \mathsf{M} \times \mathcal{P}$, let $\boldsymbol{a} \in \mathrm{c}_{\downarrow 0}(\mathbb{Z}_{\geq 0};\mathbb{R}_{>0})$, and let $\epsilon \in \mathbb{R}_{>0}$. Let $\mathcal{U} \subseteq \mathsf{M}$ be a relatively compact neighbourhood of x_0 and let $\overline{\mathcal{U}}$ be a relatively compact neighbourhood of $\mathrm{cl}(\mathcal{U})$. By Proposition 4.2, there exist $C, r \in \mathbb{R}_{>0}$ such that

$$p^\infty_{\mathrm{cl}(\mathcal{U}),m}(\sigma_\xi(p) - \sigma_\xi(p_0)) \leq Cr^{-m}\sup\{\|\overline{\xi}(z,p) - \overline{\xi}(z,p_0)\|_{\overline{\mathbb{G}}} \mid z \in \overline{\mathcal{U}}\}$$

for all $m \in \mathbb{Z}_{\geq 0}$ and $p \in \mathcal{P}$. Now let $N \in \mathbb{Z}_{\geq 0}$ be such that $a_{N+1} < r$ and let $\mathcal{O}$ be a neighbourhood of p_0 such that

$$\|\overline{\xi}(z,p) - \overline{\xi}(z,p_0)\|_{\overline{\mathbb{G}}} < \frac{\epsilon r^m}{Ca_0a_1\cdots a_m}, \qquad m \in \{0,1,\dots,N\},$$

for $(z,p) \in \overline{\mathcal{U}} \times \mathcal{O}$. Then, if $m \in \{0,1,\dots,N\}$, we have

$$\begin{aligned} a_0a_1\cdots a_m&\|j_m\xi^p(x) - j_m\xi^{p_0}(x)\|_{\overline{\mathbb{G}}_m} \\ &\leq a_0a_1\cdots a_mCr^{-m}\sup\{\|\overline{\xi}(z,p) - \overline{\xi}(z,p_0)\|_{\overline{\mathbb{G}}_m} \mid z \in \overline{\mathcal{U}}\} < \epsilon, \end{aligned}$$

for $(x,p) \in \mathcal{U} \times \mathcal{O}$. If $m > N$ we also have

$$\begin{aligned} a_0 a_1 \cdots a_m \| j_m \xi^p(x) - j_m \xi^{p_0}(x) \|_{\overline{\mathbb{G}}_m} & \\ &\leq a_0 a_1 \cdots a_N r^{-N} r^m \| j_m \xi^p(x) - j_m \xi^{p_0}(x) \|_{\overline{\mathbb{G}}_m} \\ &\leq a_0 a_1 \cdots a_N r^{-N} r^m C r^{-m} \sup\{ \| \overline{\xi}(z,p) - \overline{\xi}(z,p_0) \|_{\overline{\mathbb{G}}_m} \mid z \in \overline{\mathcal{U}} \} < \epsilon, \end{aligned}$$

for $(x,p) \in \mathcal{U} \times \mathcal{O}$, as desired. ■

The next example shows that the assumption of local compactness cannot be generally relaxed.

7.15 Example: (Jointly parameterised real analytic sections are not always restrictions of jointly parameterised holomorphic sections) Let $\mathsf{M} = \mathbb{R}$, let $\mathcal{P} = \mathrm{C}^\omega(\mathbb{R})$, and define $f\colon \mathbb{R} \times \mathcal{P} \to \mathbb{R}$ by $f(x,g) = g(x)$. Since $g \mapsto f^g$ is the identity map, we conclude from Proposition 7.12 that $f \in \mathrm{JPC}^\omega(\mathcal{P};\mathsf{M})$. Let $x_0 \in \mathbb{R}$. We claim that, for any neighbourhood $\overline{\mathcal{U}}$ of x_0 in $\mathbb{C}$ and any neighbourhood $\mathcal{O}$ of $0 \in \mathcal{P}$, there exists $g \in \mathcal{O}$ such that g, and therefore f^g, does not have an holomorphic extension to $\overline{\mathcal{U}}$. To see this, let $\sigma \in \mathbb{R}_{>0}$ be such that the disk $\overline{\mathsf{D}}(\sigma, x_0)$ in $\mathbb{C}$ is contained in $\overline{\mathcal{U}}$. Let $K_1, \dots, K_r \subseteq \mathbb{R}$ be compact, let $\boldsymbol{a}_1, \dots, \boldsymbol{a}_r \in \mathrm{c}_{\downarrow 0}(\mathbb{Z}_{\geq 0}; \mathbb{R}_{>0})$, and let $\epsilon_1, \dots, \epsilon_r \in \mathbb{R}_{>0}$ be such that

$$\cap_{j=1}^r \{ g \in \mathcal{P} \mid\ p_{K_j, \boldsymbol{a}_j}(g) \leq \epsilon_j \} \subseteq \mathcal{O}.$$

Now define

$$g(x) = \frac{\alpha}{1 + ((x - x_0)/\sigma)^2}, \qquad x \in \mathbb{R},$$

with $\alpha \in \mathbb{R}_{>0}$ chosen sufficiently small that $p_{K_j, \boldsymbol{a}_j}(g) < \epsilon_j$, $j \in \{1, \dots, r\}$, and note that $g \in \mathcal{O}$ does not have an holomorphic extension to $\overline{\mathcal{U}}$, cf. the discussion at the beginning of Section 5. •

Mixing regularity hypotheses. Just as we discussed with time-varying vector fields in Section 6.5, it is possible to consider parameterised sections with mixed regularity hypotheses. Indeed, the conditions of Definitions 7.1, 7.4, and 7.10 are joint on state and parameter. Thus we may consider the following situation. Let $m \in \mathbb{Z}_{\geq 0}$, $m' \in \{0, \mathrm{lip}\}$, $r \in \mathbb{Z}_{\geq 0} \cup \{\infty, \omega\}$, and $r' \in \{0, \mathrm{lip}\}$. If $r + r' \geq m + m'$ (with the obvious convention that $\infty + \mathrm{lip} = \infty$ and $\omega + \mathrm{lip} = \omega$), we may then consider a parameterised section in

$$\mathrm{SP}\Gamma^{r+r'}(\mathcal{P};\mathsf{E}) \cap \mathrm{JP}\Gamma^{m+m'}(\mathcal{P};\mathsf{E})$$

As with time-varying vector fields, there is nothing wrong with this—indeed this is often done—as long as one remembers what is true and what is not in the case when $r + r' > m + m'$.

7.2. Control systems with locally essentially bounded controls. Let us first establish some terminology we will use throughout the remainder of the paper.

7.16 Notation: (Regularity hypotheses and proofs with regularity hypotheses) Starting in this section, and continuing throughout the remainder of the paper, we will simultaneously be considering finitely differentiable, Lipschitz, smooth, and real analytic hypotheses. To do this, we will let $m \in \mathbb{Z}_{\geq 0}$ and $m' \in \{0, \mathrm{lip}\}$, and consider the regularity classes $\nu \in \{m + m', \infty, \omega\}$. In such cases we shall require that the underlying manifold be of class "C^r, $r \in \{\infty, \omega\}$, as required." This has the obvious meaning, namely that we consider class C^ω if $\nu = \omega$ and class C^∞ otherwise.

Proofs will typically break into the four cases $\nu = \infty$, $\nu = m$, $\nu = m + \mathrm{lip}$, and $\nu = \omega$. In most cases there is a structural similarity in the way arguments are carried out, so we will oftentimes do all cases at once. In doing this, we will, for $K \subseteq \mathsf{M}$ be compact, for $k \in \mathbb{Z}_{\geq 0}$, and for $\boldsymbol{a} \in \mathrm{c}_{\downarrow 0}(\mathbb{Z}_{\geq 0}; \mathbb{R}_{>0})$, denote

$$p_K = \begin{cases} p^\infty_{K,k}, & \nu = \infty, \\ p^m_K, & \nu = m, \\ p^{m+\mathrm{lip}}_K, & \nu = m + \mathrm{lip}, \\ p^\omega_{K,\boldsymbol{a}}, & \nu = \omega. \end{cases}$$

Then, using the fact that $\xi \in \mathrm{LI}\Gamma^\nu(\mathbb{T}; \mathsf{E})$ if and only if there exists $g \in \mathrm{L}^1_{\mathrm{loc}}(\mathbb{T}; \mathbb{R}_{\geq 0})$ such that $p_K(\xi_t) \leq g(t)$ (with a similar sort of assertion for parameterised section), we argue all cases simultaneously. The convenience and brevity more than make up for the slight loss of preciseness in this approach. •

With the notions of parameterised sections from the preceding section, we readily define what we mean by a control system.

7.17 Definition: (Control system) Let $m \in \mathbb{Z}_{\geq 0}$ and $m' \in \{0, \mathrm{lip}\}$, let $\nu \in \{m + m', \infty, \omega\}$, and let $r \in \{\infty, \omega\}$, as required. A ***$\mathbf{C}^\nu$-control system*** is a triple $\Sigma = (\mathsf{M}, F, \mathcal{C})$, where

(i) M is a C^r-manifold whose elements are called ***states***,

(ii) $\mathcal{C}$ is a topological space called the ***control set***, and

(iii) $F \in \mathrm{JP}\Gamma^\nu(\mathcal{C}; \mathsf{TM})$. •

The governing equations for a control system are

$$\xi'(t) = F(\xi(t), \mu(t)),$$

for suitable functions $t \mapsto \mu(t) \in \mathcal{C}$ and $t \mapsto \xi(t) \in \mathsf{M}$. To ensure that these equations make sense, the differential equation should be shown to have the properties needed for existence and uniqueness of solutions, as well as appropriate dependence on initial conditions. We do this by allowing the controls for the system to be as general as reasonable.

7.18 Proposition: (Property of control system when the control is specified) *Let $m \in \mathbb{Z}_{\geq 0}$ and $m' \in \{0, \mathrm{lip}\}$, let $\nu \in \{m + m', \infty, \omega\}$, and let $r \in \{\infty, \omega\}$, as required. Let $\Sigma = (\mathsf{M}, F, \mathcal{C})$ be a C^ν-control system. If $\mu \in \mathrm{L}^\infty_{\mathrm{loc}}(\mathbb{T}; \mathcal{C})$ (boundedness here being taking with respect to the compact bornology) then $F^\mu \in \mathrm{LB}\Gamma^\nu(\mathbb{T}, \mathsf{TM})$, where $F^\mu \colon \mathbb{T} \times \mathsf{M} \to \mathsf{TM}$ is defined by $F^\mu(t, x) = F(x, \mu(t))$.*

Proof: Let us define $\hat{F}^\mu \colon \mathbb{T} \to \Gamma^\nu(\mathsf{TM})$ by $\hat{F}^\mu(t) = F^\mu_t$. By Propositions 7.2, 7.5, and 7.12, the mapping $u \mapsto F^u$ is continuous. Since $\hat{F}^\mu$ is thus the composition of the measurable function μ and the continuous mapping $u \mapsto F^u$, it follows that $\hat{F}^\mu$ is measurable. It follows from Theorems 6.3, 6.9, and 6.21 that F^μ is a Carathéodory vector field of class C^ν.

Let $\mathbb{T}' \subseteq \mathbb{T}$ be compact. Since μ is locally essentially bounded, there exists a compact set $K \subseteq \mathcal{C}$ such that

$$\lambda(\{t \in \mathbb{T}' \mid \mu(t) \notin K\}) = 0.$$

Since the mapping $u \mapsto F^u$ is continuous,

$$\{F^\mu_t \mid t \in \mathbb{T}'\}$$

is contained in a compact subset of $\Gamma^\nu(\mathsf{TM})$, i.e., F^μ is locally essentially bounded. ■

The notion of a trajectory is, of course, well known. However, we make the definitions clear for future reference.

7.19 Definition: (Trajectory for control system) Let $m \in \mathbb{Z}_{\geq 0}$ and $m' \in \{0, \mathrm{lip}\}$, let $\nu \in \{m + m', \infty, \omega\}$, and let $r \in \{\infty, \omega\}$, as required. Let $\Sigma = (\mathsf{M}, F, \mathcal{C})$ be a C^ν-control system. For an interval $\mathbb{T} \subseteq \mathbb{R}$, a ***$\mathbb{T}$-trajectory*** is a locally absolutely continuous curve $\xi\colon \mathbb{T} \to \mathsf{M}$ for which there exists $\mu \in \mathrm{L}^\infty_{\mathrm{loc}}(\mathbb{T}; \mathcal{C})$ such that

$$\xi'(t) = F(\xi(t), \mu(t)), \qquad \text{a.e. } t \in \mathbb{T}.$$

The set of $\mathbb{T}$-trajectories we denote by $\mathrm{Traj}(\mathbb{T}, \Sigma)$. If $\mathcal{U}$ is open, we denote by $\mathrm{Traj}(\mathbb{T}, \mathcal{U}, \Sigma)$ those trajectories taking values in $\mathcal{U}$.[11] •

One may also wish to restrict the class of controls one uses. Thus we can consider, for each time-domain $\mathbb{T}$, a subset $\mathscr{C}(\mathbb{T}) \subseteq \mathrm{L}^\infty_{\mathrm{loc}}(\mathbb{T}; \mathcal{C})$. Generally, one will ask for some compatibility conditions for these subsets, like, for example, that, if $\mathbb{T}' \subseteq \mathbb{T}$, then $\mu|\mathbb{T}' \in \mathscr{C}(\mathbb{T}')$ for every $\mu \in \mathscr{C}(\mathbb{T})$. For example, one may consider things like piecewise continuous or piecewise constant controls. In this case, we denote by $\mathrm{Traj}(\mathbb{T}, \mathscr{C})$ the set of trajectories arising from using controls from $\mathscr{C}(\mathbb{T})$. Similarly, by $\mathrm{Traj}(\mathbb{T}, \mathcal{U}, \mathscr{C})$ we denote the trajectories from this set taking values in an open set $\mathcal{U}$. We shall see in Section 8 that our tautological control systems provide a natural means of capturing issues such as this.

7.3. Control systems with locally integrable controls. In this section we specialise the discussion from the preceding section in one direction, while generalising it in another. To be precise, we now consider the case where our control set $\mathcal{C}$ is a subset of a locally convex topological vector space, and the system structure is such that the notion of integrability is preserved (in a way that will be made clear in Proposition 7.22 below).

7.20 Definition: (Sublinear control system) Let $m \in \mathbb{Z}_{\geq 0}$ and $m' \in \{0, \mathrm{lip}\}$, let $\nu \in \{m + m', \infty, \omega\}$, and let $r \in \{\infty, \omega\}$, as required. A ***C^ν-sublinear control system*** is a triple $\Sigma = (\mathsf{M}, F, \mathcal{C})$, where

(i) M is a C^r-manifold whose elements are called ***states***,

(ii) $\mathcal{C}$ is a subset of a locally convex topological vector space V, $\mathcal{C}$ being called the ***control set***, and

[11] This is not a common notion in this context, and our introduction of this is for the convenience of making comparisons in the next section; see Theorems 8.35 and 8.37.

(iii) $F\colon \mathsf{M} \times \mathcal{C} \to \mathsf{TM}$ has the following property: for every continuous seminorm p for $\Gamma^\nu(\mathsf{TM})$, there exists a continuous seminorm q for V such that

$$p(F^{u_1} - F^{u_2}) \leq q(u_1 - u_2), \qquad u_1, u_2 \in \mathcal{C}. \qquad \bullet$$

Note that, by Propositions 7.2, 7.5, and 7.12, the sublinearity condition (iii) implies that a C^ν-sublinear control system is a C^ν-control system.

Let us demonstrate a class of sublinear control systems in which we will be particularly interested.

7.21 Example: (Control-linear systems and control-affine systems) The class of sublinear control systems we consider seems quite particular, but will turn out to be extremely general in our framework. We let $m \in \mathbb{Z}_{\geq 0}$ and $m' \in \{0, \mathrm{lip}\}$, let $\nu \in \{m + m', \infty, \omega\}$, and let $r \in \{\infty, \omega\}$, as required. Let V be a locally convex topological vector space, and let $\mathcal{C} \subseteq \mathsf{V}$. We suppose that we have a continuous linear map $\Lambda \in \mathrm{L}(\mathsf{V}; \Gamma^\nu(\mathsf{TM}))$ and we correspondingly define $F_\Lambda\colon \mathsf{M} \times \mathcal{C} \to \mathsf{TM}$ by $F_\Lambda(x, u) = \Lambda(u)(x)$. Continuity of Λ immediately gives that the control system $(\mathsf{M}, F_\Lambda, \mathcal{C})$ is sublinear, and we shall call a system such as this a ***$\mathbf{C}^\nu$-control-linear system***.

Note that we can regard a control-affine system as a control-linear system as follows. For a control-affine system with $\mathcal{C} \subseteq \mathbb{R}^k$ and with

$$F(x, \boldsymbol{u}) = f_0(x) + \sum_{a=1}^{k} u^a f_a(x),$$

we let $\mathsf{V} = \mathbb{R}^{k+1} \simeq \mathbb{R} \oplus \mathbb{R}^k$ and take

$$\mathcal{C}' = \{(u^0, \boldsymbol{u}) \in \mathbb{R} \oplus \mathbb{R}^k \mid u^0 = 1, \ \boldsymbol{u} \in \mathcal{C}\}, \quad \Lambda(u^0, \boldsymbol{u}) = \sum_{a=0}^{k} u^a f_a.$$

Clearly we have $F(x, \boldsymbol{u}) = F_\Lambda(x, (1, \boldsymbol{u}))$ for every $\boldsymbol{u} \in \mathcal{C}$. Since linear maps from finite-dimensional locally convex spaces are continuous [Horváth 1966, Proposition 2.10.2], we conclude that control-affine systems are control-linear systems. Thus they are also control systems as per Definition 7.17. $\bullet$

One may want to regard the generalisation from the case where the control set is a subset of $\mathbb{R}^k$ to being a subset of a locally convex topological vector space to be mere fancy generalisation, but this is, actually, far from being the case as we shall see in Section 8.

We also have a version of Proposition 7.18 for sublinear control systems.

7.22 Proposition: (Property of sublinear control system when the control is specified) *Let $m \in \mathbb{Z}_{\geq 0}$ and $m' \in \{0, \mathrm{lip}\}$, let $\nu \in \{m + m', \infty, \omega\}$, and let $r \in \{\infty, \omega\}$, as required. Let $\Sigma = (\mathsf{M}, F, \mathcal{C})$ be a C^ν-sublinear control system for which $\mathcal{C}$ is a subset of a locally convex topological vector space V. If $\mu \in \mathrm{L}^1_{\mathrm{loc}}(\mathbb{T}; \mathcal{C})$, then $F^\mu \in \mathrm{LI}\Gamma^\nu(\mathbb{T}; \mathsf{TM})$, where $F^\mu\colon \mathbb{T} \times \mathsf{M} \to \mathsf{TM}$ is defined by $F^\mu(t, x) = F(x, \mu(t))$.*

Proof: The proof that F^μ is a Carathéodory vector field of class C^ν goes exactly as in Proposition 7.18.

To prove that $F^\mu \in \mathrm{LI}\Gamma^\nu(\mathbb{T};\mathsf{TM})$, let $K \subseteq \mathsf{M}$ be compact, let $k \in \mathbb{Z}_{\geq 0}$, let $\boldsymbol{a} \in \mathrm{c}_{\downarrow 0}(\mathbb{Z}_{\geq 0};\mathbb{R}_{>0})$, and denote

$$p_K = \begin{cases} p^\infty_{K,k}, & \nu = \infty, \\ p^m_K, & \nu = m, \\ p^{m+\mathrm{lip}}_K, & \nu = m+\mathrm{lip}, \\ p^\omega_{K,\boldsymbol{a}}, & \nu = \omega. \end{cases}$$

Define $g\colon \mathbb{T} \to \mathbb{R}_{\geq 0}$ by $g(t) = p_K(F^\mu_t)$. We claim that $g \in \mathrm{L}^\infty_{\mathrm{loc}}(\mathbb{T};\mathbb{R}_{\geq 0})$. From the first part of the proof of Proposition 7.18, $t \mapsto F^\mu_t(x)$ is measurable for every $x \in \mathsf{M}$. By Theorems 6.3, 6.9, and 6.21, it follows that $t \mapsto F^\mu_t$ is measurable. Since p_K is a continuous function on $\Gamma^\nu(\mathsf{TM})$, it follows that $t \mapsto p_K(F^\mu_t)$ is measurable, as claimed. We claim that $g \in \mathrm{L}^1_{\mathrm{loc}}(\mathbb{T};\mathbb{R}_{\geq 0})$. Note that $X \mapsto p_K(X)$ is a continuous seminorm on $\Gamma^\infty(\mathsf{TM})$. By hypothesis, there exists a continuous seminorm q for the locally convex topology for V such that

$$p_K(F^{u_1} - F^{u_2}) \leq q(u_1 - u_2)$$

for every $u_1, u_2 \in \mathcal{C}$. Therefore, if $\mathbb{T}' \subseteq \mathbb{T}$ is compact and if $u_0 \in \mathcal{C}$, we also have

$$\begin{aligned} \int_{\mathbb{T}'} g(t)\,\mathrm{d}t &= \int_{\mathbb{T}'} p_K(F^\mu_t) \\ &\leq \int_{\mathbb{T}'} p_K(F^\mu_t - F^{u_0})\,\mathrm{d}t + \int_{\mathbb{T}'} p_K(F^{u_0})\,\mathrm{d}t \\ &\leq \int_{\mathbb{T}'} q(\mu(t))\,\mathrm{d}t + (q(u_0) + p_K(F^{u_0}))\lambda(\mathbb{T}') < \infty, \end{aligned}$$

the last inequality by the characterisation of Bochner integrability from [Beckmann and Deitmar 2011, Theorems 3.2 and 3.3]. Thus g is locally integrable. It follows from Theorems 6.3, 6.9, and 6.21 that $F^\mu \in \mathrm{LI}\Gamma^\nu(\mathbb{T};\mathsf{TM})$, as desired. ■

There is also a version of the notion of trajectory that is applicable to the case when the control set is a subset of a locally convex topological space.

7.23 Definition: (Trajectory for sublinear control system) Let $m \in \mathbb{Z}_{\geq 0}$ and $m' \in \{0, \mathrm{lip}\}$, let $\nu \in \{m+m', \infty, \omega\}$, and let $r \in \{\infty, \omega\}$, as required. Let $\Sigma = (\mathsf{M}, F, \mathcal{C})$ be a C^ν-control system. For an interval $\mathbb{T} \subseteq \mathbb{R}$, a ***$\mathbb{T}$-trajectory*** is a locally absolutely continuous curve $\xi\colon \mathbb{T} \to \mathsf{M}$ for which there exists $\mu \in \mathrm{L}^1_{\mathrm{loc}}(\mathbb{T};\mathcal{C})$ such that

$$\xi'(t) = F(\xi(t), \mu(t)), \qquad \text{a.e. } t \in \mathbb{T}.$$

The set of $\mathbb{T}$-trajectories we denote by $\mathrm{Traj}(\mathbb{T}, \Sigma)$. If $\mathcal{U}$ is open, we denote by $\mathrm{Traj}(\mathbb{T}, \mathcal{U}, \Sigma)$ those trajectories taking values in $\mathcal{U}$. •

7.4. Differential inclusions. We briefly mentioned differential inclusions in Section 1.1, but now let us define them properly and give a few attributes of, and constructions for, differential inclusions of which we shall subsequently make use.

First the definition.

7.24 Definition: (Differential inclusion, trajectory) For a smooth manifold M, a ***differential inclusion*** on M is a set-valued map $\mathscr{X}: \mathsf{M} \twoheadrightarrow \mathsf{TM}$ with nonempty values for which $\mathscr{X}(x) \subseteq \mathsf{T}_x\mathsf{M}$. A ***trajectory*** for a differential inclusion $\mathscr{X}$ is a locally absolutely continuous curve $\xi: \mathbb{T} \to \mathsf{M}$ defined on an interval $\mathbb{T} \subseteq \mathbb{R}$ for which $\xi'(t) \in \mathscr{X}(\xi(t))$ for almost every $t \in \mathbb{T}$. If $\mathbb{T} \subseteq \mathbb{R}$ is an interval and if $\mathcal{U} \subseteq \mathsf{M}$ is open, by $\mathrm{Traj}(\mathbb{T}, \mathcal{U}, \mathscr{X})$ we denote the trajectories of $\mathscr{X}$ defined on $\mathbb{T}$ and taking values in $\mathcal{U}$. •

Of course, differential inclusions will generally not have trajectories, and to ensure that they do various hypotheses can be made. Two common attributes of differential inclusions in this vein are the following.

7.25 Definition: (Lower and upper semicontinuity of differential inclusions) A differential inclusion $\mathscr{X}$ on a smooth manifold M is:

(i) ***lower semicontinuous*** at $x_0 \in \mathsf{M}$ if, for any $v_0 \in \mathscr{X}(x_0)$ and any neighbourhood $\mathcal{V} \subseteq \mathsf{TM}$ of v_0, there exists a neighbourhood $\mathcal{U} \subseteq \mathsf{M}$ of x_0 such that $\mathscr{X}(x) \cap \mathcal{V} \neq \varnothing$ for every $x \in \mathcal{U}$;

(ii) ***lower semicontinuous*** if it is lower semicontinuous at every $x \in \mathsf{M}$;

(iii) ***upper semicontinuous*** at $x_0 \in \mathsf{M}$ if, for every open set $\mathsf{TM} \supseteq \mathcal{V} \supseteq \mathscr{X}(x_0)$, there exists a neighbourhood $\mathcal{U} \subseteq \mathsf{M}$ of x_0 such that $\mathscr{X}(\mathcal{U}) \subseteq \mathcal{V}$;

(iv) ***upper semicontinuous*** if it is upper semicontinuous at each $x \in \mathsf{M}$;

(v) ***continuous*** at $x_0 \in \mathsf{M}$ if it is both lower and upper semicontinuous at x_0;

(vi) ***continuous*** if it is both lower and upper semicontinuous. •

Other useful properties of differential inclusions are the following.

7.26 Definition: (Closed-valued, compact-valued, convex-valued differential inclusions) A differential inclusion $\mathscr{X}$ on a smooth manifold M is:

(i) ***closed-valued*** (resp. ***compact-valued***, ***convex-valued***) at $x \in \mathsf{M}$ if $\mathscr{X}(x)$ is closed (resp., compact, convex);

(ii) ***closed-valued*** (resp. ***compact-valued***, ***convex-valued***) if $\mathscr{X}(x)$ is closed (resp., compact, convex) for every $x \in \mathsf{M}$. •

Some standard hypotheses for existence of trajectories are then:

1. $\mathscr{X}$ is lower semicontinuous with closed and convex values [Aubin and Cellina 1984, Theorem 2.1.1];
2. $\mathscr{X}$ is upper semicontinuous with compact and convex values [Aubin and Cellina 1984, Theorem 2.1.4];
3. $\mathscr{X}$ is continuous with compact values [Aubin and Cellina 1984, Theorem 2.2.1].

These are not matters with which we shall be especially concerned.

A standard operation is to take "hulls" of differential inclusions in the following manner.

7.27 Definition: (Convex hull, closure of a differential inclusion) Let $r \in \{\infty, \omega\}$, let M be a C^r-manifold, and let $\mathscr{X}: \mathsf{M} \twoheadrightarrow \mathsf{TM}$ be a differential inclusion.

(i) The ***convex hull*** of $\mathscr{X}$ is the differential inclusion $\mathrm{conv}(\mathscr{X})$ defined by

$$\mathrm{conv}(\mathscr{X})(x) = \mathrm{conv}(\mathscr{X}(x)), \qquad x \in \mathsf{M}.$$

(ii) The ***closure*** of $\mathscr{X}$ is the differential inclusion $\mathrm{cl}(\mathscr{X})$ defined by

$$\mathrm{cl}(\mathscr{X})(x) = \mathrm{cl}(\mathscr{X}(x)), \qquad x \in \mathsf{M}.$$ •

To close this section, let us make an observation regarding the connection between control systems and differential inclusions. Let $m \in \mathbb{Z}_{\geq 0}$ and $m' \in \{0, \mathrm{lip}\}$, let $\nu \in \{m + m', \infty, \omega\}$, and let $r \in \{\infty, \omega\}$, as required. Let $\Sigma = (\mathsf{M}, F, \mathcal{C})$ be a C^ν-control system. To this system we associate the differential inclusion $\mathscr{X}_\Sigma$ by

$$\mathscr{X}_\Sigma(x) = \{F^u(x) \mid u \in \mathcal{C}\}.$$

Since the differential inclusion $\mathscr{X}_\Sigma$ is defined by a family of vector fields, one might try to recover the vector fields F^u, $u \in \mathcal{C}$, from $\mathscr{X}_\Sigma$. The obvious way to do this is to consider

$$\Gamma^\nu(\mathscr{X}_\Sigma) \triangleq \{X \in \Gamma^\nu(\mathsf{TM}) \mid X(x) \in \mathscr{X}_\Sigma(x),\ x \in \mathsf{M}\}.$$

Clearly we have $F^u \in \Gamma^\nu(\mathscr{X}_\Sigma)$ for every $u \in \mathcal{C}$. However, $\mathscr{X}_\Sigma$ will generally contain vector fields not of the form F^u for some $u \in \mathcal{C}$. Let us give an illustration of this. Let us consider a smooth control system $(\mathsf{M}, F, \mathcal{C})$ with the following properties:

1. $\mathcal{C}$ is a disjoint union of sets $\mathcal{C}_1$ and $\mathcal{C}_2$;
2. there exist disjoint open sets $\mathcal{U}_1$ and $\mathcal{U}_2$ such that $\mathrm{supp}(F^u) \subseteq \mathcal{U}_1$ for $u \in \mathcal{C}_1$ and $\mathrm{supp}(F^u) \subseteq \mathcal{U}_2$ for $u \in \mathcal{C}_2$.

One then has that

$$\{c_1 F^{u_1} + c_1 F^{u_2} \mid u_1 \in \mathcal{C}_1,\ u_2 \in \mathcal{C}_2,\ c_1, c_2 \in \{0,1\},\ c_1^2 + c_2^2 \neq 0\} \subseteq \Gamma^\nu(\mathscr{X}_\Sigma),$$

showing that there are more sections of $\mathscr{X}_\Sigma$ than there are control vector fields. This is very much related to presheaves and sheaves, to which we shall now turn our attention.

8. Tautological control systems: Definitions and fundamental properties

In this section we introduce the class of control systems we propose as being useful mathematical models for the investigation of geometric system structure. The reader would do well to remember that this definition makes no pretences of being simple or user-friendly. However, we can do some interesting things with these models, and to illustrate this we present in Section 8.8 an elegant formulation of sub-Riemannian geometry in the framework of tautological control systems.

8.1. Presheaves and sheaves of sets of vector fields. We choose to phrase our notion of control systems in the language of sheaf theory. This will seem completely pointless to a reader not used to thinking in this sort of language. However, we do believe there are benefits to the sheaf approach including (1) sheaves are the proper framework for constructing germs of control systems which are often important in the study of local system structure and (2) sheaf theory provides us with a natural class of mappings between systems that we use to advantage in Section 8.7.

We do not even come close to discussing sheaves in any generality; we merely give the definitions we require, a few of the most elementary consequences of these definitions, and some representative (for us) examples.

8.1 Definition: (Presheaf of sets of vector fields) Let $m \in \mathbb{Z}_{\geq 0}$ and $m' \in \{0, \mathrm{lip}\}$, let $\nu \in \{m + m', \infty, \omega\}$, and let $r \in \{\infty, \omega\}$, as required. Let M be a manifold of class C^r. A ***presheaf of sets of*** $\mathbf{C}^\nu$***-vector fields*** is an assignment to each open set $\mathcal{U} \subseteq \mathsf{M}$ a subset $\mathscr{F}(\mathcal{U})$ of $\Gamma^\nu(\mathsf{T}\mathcal{U})$ with the property that, for open sets $\mathcal{U}, \mathcal{V} \subseteq \mathsf{M}$ with $\mathcal{V} \subseteq \mathcal{U}$, the map

$$\begin{aligned} r_{\mathcal{U},\mathcal{V}} \colon \mathscr{F}(\mathcal{U}) &\to \Gamma^\nu(\mathsf{T}\mathcal{V}) \\ X &\mapsto X|\mathcal{V} \end{aligned}$$

takes values in $\mathscr{F}(\mathcal{V})$. Elements of $\mathscr{F}(\mathcal{U})$ are called ***local sections*** over $\mathcal{U}$. •

Let us give some notation to the presheaf of sets of vector fields of which every other such presheaf is a subset.

8.2 Example: (Presheaf of all vector fields) Let $m \in \mathbb{Z}_{\geq 0}$ and $m' \in \{0, \mathrm{lip}\}$, let $\nu \in \{m + m', \infty, \omega\}$, and let $r \in \{\infty, \omega\}$, as required. Let M be a manifold of class C^r. The presheaf of *all* vector fields of class C^ν is denoted by $\mathscr{G}^\nu_{\mathsf{TM}}$. Thus $\mathscr{G}^\nu_{\mathsf{TM}}(\mathcal{U}) = \Gamma^\nu(\mathsf{T}\mathcal{U})$ for every open set $\mathcal{U}$. Presheaves such as this are extremely important in the "normal" applications of sheaf theory. For those with some background in these more standard applications of sheaf theory, we mention that our reasons for using the theory are not quite the usual ones. Such readers will be advised to be careful not to overlay too much of their past experience on what we do with sheaf theory here. •

The preceding notion of a presheaf is intuitively clear, but it does have some defects. One of these defects is that one can describe local data that does not patch together to give global data. Let us illustrate this with a few examples.

8.3 Examples: (Local definitions not globally consistent)

1. Let $m \in \mathbb{Z}_{\geq 0}$ and $m' \in \{0, \mathrm{lip}\}$, let $\nu \in \{m+m', \infty, \omega\}$, and let $r \in \{\infty, \omega\}$, as required. Let us take a manifold M of class C^r with a Riemannian metric $\mathbb{G}$. Let us define a presheaf $\mathscr{F}_{\mathrm{bdd}}$ by asking that

$$\mathscr{F}_{\mathrm{bdd}}(\mathcal{U}) = \{X \in \Gamma^\nu(\mathsf{TM}) \mid \ \sup\{\|X(x)\|_{\mathbb{G}} \mid x \in \mathcal{U}\} < \infty\}.$$

   Thus $\mathscr{F}_{\mathrm{bdd}}$ is comprised of vector fields that are "bounded." This is a perfectly sensible requirement. However, the following phenomenon can happen if M is not compact. There can exist an open cover $(\mathcal{U}_a)_{a\in A}$ for M and local sections $X_a \in \mathscr{F}_{\mathrm{bdd}}(\mathcal{U}_a)$ that are "compatible" in the sense that $X_a|\mathcal{U}_a \cap \mathcal{U}_b = X_b|\mathcal{U}_a \cap \mathcal{U}_b$, for each $a, b \in A$, but such that there is no globally defined section $X \in \mathscr{F}_{\mathrm{bdd}}(\mathsf{M})$ such that $X|\mathcal{U}_a = X_a$ for every $a \in A$. We leave to the reader the easy job of coming up with a concrete instance of this.

2. Let $m \in \mathbb{Z}_{\geq 0}$ and $m' \in \{0, \mathrm{lip}\}$, let $\nu \in \{m+m', \infty, \omega\}$, and let $r \in \{\infty, \omega\}$, as required. Let M be a manifold of class C^r. If $\mathscr{X} \subseteq \Gamma^\nu(\mathsf{TM})$ is any family of vector fields on M, then we can define an associated presheaf $\mathscr{F}_{\mathscr{X}}$ of sets of vector fields by

$$\mathscr{F}_{\mathscr{X}}(\mathcal{U}) = \{X|\mathcal{U} \mid X \in \mathscr{X}\}.$$

   Note that $\mathscr{F}(\mathsf{M})$ is necessarily equal to $\mathscr{X}$, and so we shall typically use $\mathscr{F}(\mathsf{M})$ to denote the set of globally defined vector fields giving rise to this presheaf. A presheaf of this sort will be called ***globally generated***.

This sort of presheaf will almost never have nice "local to global" properties. Let us illustrate why this is so. Let M be a connected Hausdorff manifold. Suppose that the set of globally defined vector fields $\mathscr{F}(\mathsf{M})$ has cardinality strictly larger than 1 and has the following property: there exists a disconnected open set $\mathcal{U} \subseteq \mathsf{M}$ such that the mapping from $\mathscr{F}(\mathcal{U})$ to $\mathscr{F}(\mathsf{M})$ given by $X|\mathcal{U} \mapsto X$ is injective. This property will hold for real analytic families of vector fields, because we can take as $\mathcal{U}$ the union of a pair of disconnected open sets. However, the property will also hold for many reasonable families of smooth vector fields.

We write $\mathcal{U} = \mathcal{U}_1 \cup \mathcal{U}_2$ for disjoint open sets $\mathcal{U}_1$ and $\mathcal{U}_2$. By hypothesis, there exist vector fields $X_1, X_2 \in \mathscr{F}(\mathsf{M})$ such that $X_1|\mathcal{U} \neq X_2|\mathcal{U}$. Define local sections $X'_a \in \mathscr{F}(\mathcal{U}_a)$ by $X'_a = X_a|\mathcal{U}_a$, $a \in \{1, 2\}$. The condition

$$X'_1|\mathcal{U}_1 \cap \mathcal{U}_2 = X'_2|\mathcal{U}_1 \cap \mathcal{U}_2$$

is vacuously satisfied. But there can be no $X \in \mathscr{F}(\mathsf{M})$ such that, if $X' \in \mathscr{F}(\mathcal{U})$ is given by $X' = X|\mathcal{U}$, then $X'|\mathcal{U}_1 = X'_1$ and $X'|\mathcal{U}_2 = X'_2$.

While a globally generated presheaf is unlikely to allow patching from local to global, this can be easily redressed by undergoing a process known as "sheafification" that we will describe below. •

The preceding examples suggest that if one wishes to make compatible local constructions that give rise to a global construction, additional properties need to be ascribed to a presheaf of sets of vector fields. This we do as follows.

8.4 Definition: (Sheaf of sets of vector fields) Let $m \in \mathbb{Z}_{\geq 0}$ and $m' \in \{0, \mathrm{lip}\}$, let $\nu \in \{m + m', \infty, \omega\}$, and let $r \in \{\infty, \omega\}$, as required. Let M be a manifold of class C^r. A presheaf $\mathscr{F}$ of sets of C^ν-vector fields is a ***sheaf of sets of C^ν-vector fields*** if, for every open set $\mathcal{U} \subseteq \mathsf{M}$, for every open cover $(\mathcal{U}_a)_{a\in A}$ of $\mathcal{U}$, and for every choice of local sections $X_a \in \mathscr{F}(\mathcal{U}_a)$ satisfying $X_a|\mathcal{U}_a \cap \mathcal{U}_b = X_b|\mathcal{U}_a \cap \mathcal{U}_b$, there exists $X \in \mathscr{F}(\mathcal{U})$ such that $X|\mathcal{U}_a = X_a$ for every $a \in A$. •

The condition in the definition is called the ***gluing condition***. Readers familiar with sheaf theory will note the absence of the other condition, sometimes called the separation condition, normally placed on a presheaf in order for it to be a sheaf: it is automatically satisfied for presheaves of sets of vector fields.

Many of the presheaves that we encounter will not be sheaves, as they will be globally generated. Thus let us give some examples of sheaves, just as a point of reference.

8.5 Examples: (Sheaves of sets of vector fields)

1. Let $m \in \mathbb{Z}_{\geq 0}$ and $m' \in \{0, \mathrm{lip}\}$, let $\nu \in \{m + m', \infty, \omega\}$, and let $r \in \{\infty, \omega\}$, as required. Let M be a C^r-manifold. The presheaf $\mathscr{G}^\nu_{\mathsf{TM}}$ of all C^ν-vector fields is a sheaf. We leave the simple and standard working out of this to the reader; it will provide some facility in working with sheaf concepts for those not already having this.
2. If instead of considering bounded vector fields as in part Example 8.3–1, we consider the presheaf of vector fields satisfying a *fixed* bound, then the resulting presheaf is a sheaf. Let us be clear. Let $m \in \mathbb{Z}_{\geq 0}$ and $m' \in \{0, \mathrm{lip}\}$, let $\nu \in \{m + m', \infty, \omega\}$, and let $r \in \{\infty, \omega\}$, as required. We let M be a C^r-manifold with Riemannian metric $\mathbb{G}$ and, for $B \in \mathbb{R}_{>0}$, define a presheaf $\mathscr{F}_{\leq B}$ by

$$\mathscr{F}_{\leq B}(\mathcal{U}) = \{X \in \Gamma^\nu(\mathsf{T}\mathcal{U}) \mid \sup\{\|X(x)\|_{\mathbb{G}} \mid x \in \mathcal{U}\} \leq B\}.$$

The presheaf $\mathscr{F}_{\leq B}$ is a sheaf, as is easily verified. In this case, the local constraints for membership are compatible with a global one.

3. Let $m \in \mathbb{Z}_{\geq 0}$ and $m' \in \{0, \mathrm{lip}\}$, let $\nu \in \{m+m', \infty, \omega\}$, and let $r \in \{\infty, \omega\}$, as required. Let M be a C^r-manifold. Let $A \subseteq \mathsf{M}$ and define a presheaf $\mathscr{I}_A$ of sets of vector fields by

$$\mathscr{I}_A(\mathcal{U}) = \{X \in \Gamma^\nu(\mathsf{T}\mathcal{U}) \mid \ X(x) = 0, \ x \in A\}.$$

This is a sheaf (again, we leave the verification to the reader) called the ***ideal sheaf*** of A. •

Let us now turn to localising sheaves of sets of vector fields. Let $m \in \mathbb{Z}_{\geq 0}$ and $m' \in \{0, \mathrm{lip}\}$, let $\nu \in \{m+m', \infty, \omega\}$, and let $r \in \{\infty, \omega\}$, as required. Let M be a C^r-manifold, let $A \subseteq \mathsf{M}$, and let $\mathscr{N}_A$ be the set of neighbourhoods of A in M, i.e., the open subsets of M containing A. This is a directed set in the usual way by inclusion, i.e., $\mathcal{U} \preceq \mathcal{V}$ if $\mathcal{V} \subseteq \mathcal{U}$. Let $\mathscr{F}$ be a sheaf of sets of C^ν-vector fields. The ***stalk*** of $\mathscr{F}$ over A is the direct limit $\operatorname{dir\,lim}_{\mathcal{U} \in \mathscr{N}_A} \mathscr{F}(\mathcal{U})$. Let us be less cryptic about this. Let $\mathcal{U}, \mathcal{V} \in \mathscr{N}_A$, and let $X \in \mathscr{F}(\mathcal{U})$ and $Y \in \mathscr{F}(\mathcal{V})$. We say X and Y are ***equivalent*** if there exists $\mathcal{W} \subseteq \mathcal{U} \cap \mathcal{V}$ such that $X|\mathcal{W} = Y|\mathcal{W}$. The ***germ*** of $X \in \mathscr{F}(\mathcal{U})$ for $\mathcal{U} \in \mathscr{N}_A$ is the equivalence class of X under this equivalence relation. If $\mathcal{U} \in \mathscr{N}_A$ and if $X \in \mathscr{F}(\mathcal{U})$, then we denote by $[X]_A$ the equivalence class of X in $\mathscr{F}_A$. The ***stalk*** of $\mathscr{F}$ over A is the set of all equivalence classes. The stalk of $\mathscr{F}$ over A is denoted by $\mathscr{F}_A$, and we write $\mathscr{F}_{\{x\}}$ as $\mathscr{F}_x$.

Let us now describe how a presheaf can be converted in a natural way into a sheaf. The description of how to do this for general presheaves is a little complicated. However, in the case we are dealing with here, we can be explicit about this.

8.6 Lemma: (A sheaf associated to every presheaf of sets of vector fields) *Let $m \in \mathbb{Z}_{\geq 0}$ and $m' \in \{0, \mathrm{lip}\}$, let $\nu \in \{m+m', \infty, \omega\}$, and let $r \in \{\infty, \omega\}$, as required. Let M be a C^r-manifold and let $\mathscr{F}$ be a presheaf of sets of C^ν-vector fields. For an open set $\mathcal{U} \subseteq \mathsf{M}$, define*

$$\mathrm{Sh}(\mathscr{F})(\mathcal{U}) = \{X \in \Gamma^\nu(\mathsf{T}\mathcal{U}) \mid \ [X]_x \in \mathscr{F}_x \textit{ for every } x \in \mathcal{U}\}.$$

Then $\mathrm{Sh}(\mathscr{F})$ *is a sheaf.*

Proof: Let $\mathcal{U} \subseteq \mathsf{M}$ be open and let $(\mathcal{U}_a)_{a \in A}$ be an open cover of $\mathcal{U}$. Suppose that local sections $X_a \in \mathrm{Sh}(\mathscr{F})(\mathcal{U}_a)$, $a \in A$, satisfy $X_a|\mathcal{U}_a \cap \mathcal{U}_b = X_b|\mathcal{U}_a \cap \mathcal{U}_b$ for each $a, b \in A$. Since $\mathscr{G}^\nu_{\mathsf{TM}}$ is a sheaf, there exists $X \in \Gamma^\nu(\mathsf{T}\mathcal{U})$ such that $X|\mathcal{U}_a = X_a$, $a \in A$. It remains to show that $X \in \mathrm{Sh}(\mathscr{F})(\mathcal{U})$. Let $x \in \mathcal{U}$ and let $a \in A$ be such that $x \in \mathcal{U}_a$. Then we have $[X]_x = [X_a]_x \in \mathscr{F}_x$, as desired. ■

With the lemma in mind we have the following definition.

8.7 Definition: (Sheafification of a presheaf of sets of vector fields) Let $m \in \mathbb{Z}_{\geq 0}$ and $m' \in \{0, \mathrm{lip}\}$, let $\nu \in \{m+m', \infty, \omega\}$, and let $r \in \{\infty, \omega\}$, as required. Let M be a C^r-manifold and let $\mathscr{F}$ be a presheaf of sets of C^ν-vector fields. The ***sheafification*** of $\mathscr{F}$ is the sheaf $\mathrm{Sh}(\mathscr{F})$ of sets of vector fields defined by

$$\mathrm{Sh}(\mathscr{F})(\mathcal{U}) = \{X \in \Gamma^\nu(\mathsf{T}\mathcal{U}) \mid \ [X]_x \in \mathscr{F}_x \text{ for all } x \in \mathcal{U}\}. \quad \bullet$$

Let us consider some examples of sheafification.

8.8 Examples: (Sheafification)

1. Let us consider the presheaf of bounded vector fields from Example 8.3–1. Let $m \in \mathbb{Z}_{\geq 0}$ and $m' \in \{0, \mathrm{lip}\}$, let $\nu \in \{m + m', \infty, \omega\}$, and let $r \in \{\infty, \omega\}$, as required. Let M be a C^r-manifold and consider the presheaf $\mathscr{F}_{\mathrm{bdd}}$ of bounded vector fields. One easily sees that the stalk of this presheaf at $x \in \mathsf{M}$ is given by
$$\mathscr{F}_{\mathrm{bdd},x} = \{[X]_x \mid X \in \Gamma^\nu(\mathsf{TM})\},$$
i.e., there are no restrictions on the stalks coming from the boundedness restriction on vector fields. Therefore, $\mathrm{Sh}(\mathscr{F}_{\mathrm{bdd}}) = \mathscr{G}^\nu_{\mathsf{TM}}$.
2. Let us now examine the sheafification of a globally generated presheaf of sets of vector fields as in Example 8.3–2. Let $m \in \mathbb{Z}_{\geq 0}$ and $m' \in \{0, \mathrm{lip}\}$, let $\nu \in \{m + m', \infty, \omega\}$, and let $r \in \{\infty, \omega\}$, as required. Let M be a C^r-manifold and let $\mathscr{F}$ be a globally generated presheaf of sets of C^ν-vector fields, with $\mathscr{F}(\mathsf{M})$ the global generators. We will contrast $\mathscr{F}(\mathcal{U})$ with $\mathrm{Sh}(\mathscr{F})(\mathcal{U})$ to get an understanding of what the sheaf $\mathrm{Sh}(\mathscr{F})$ "looks like."

   To do so, for $\mathcal{U} \subseteq \mathsf{M}$ open and for $X \in \Gamma^\nu(\mathsf{T}\mathcal{U})$, let us define a set-valued map $\kappa_{X,\mathcal{U}} \colon \mathcal{U} \rightarrowtail \mathscr{F}(\mathsf{M})$ by
$$\kappa_{X,\mathcal{U}}(x) = \{X' \in \mathscr{F}(\mathsf{M}) \mid X'(x) = X(x)\}.$$
   Generally, since we have asked nothing of the vector field X, we might have $\kappa_{X,\mathcal{U}}(x) = \varnothing$ for a chosen x, or for some x, or for every x. If, however, we take $X \in \mathscr{F}(\mathcal{U})$, then $X = X'|\mathcal{U}$ for some $X' \in \mathscr{F}(\mathsf{M})$. Therefore, there exists a constant selection of $\kappa_{X,\mathcal{U}}$, i.e., a constant function $s \colon \mathcal{U} \to \mathscr{F}(\mathsf{M})$ such that $s(x) \in \kappa_{X,\mathcal{U}}(x)$ for every $x \in \mathcal{U}$. Note that if, for example, M is connected and $\nu = \omega$, then there will be a *unique* such constant selection since a real analytic vector field known on an open subset uniquely determines the vector field on the connected component containing this open set; this is the Identity Theorem, cf. [Gunning 1990a, Theorem A.3] in the holomorphic case and the same proof applies in the real analytic case. Moreover, this constant selection in this case will completely characterise $\kappa_{X,\mathcal{U}}$ in the sense that $\kappa_{X,\mathcal{U}}(x) = \{s(x)\}$.

   Let us now contrast this with the character of the map $\kappa_{X,\mathcal{U}}$ for a local section $X \in \mathrm{Sh}(\mathscr{F})(\mathcal{U})$. In this case, for each $x \in \mathcal{U}$, we have $[X]_x = [X_x]_x$ for some $X_x \in \mathscr{F}(\mathsf{M})$. Thus there exists a neighbourhood $\mathcal{V}_x \subseteq \mathcal{U}$ such that $X|\mathcal{V}_x = X_x|\mathcal{V}_x$. What this shows is that there is a locally constant selection of $\kappa_{X,\mathcal{U}}$, i.e., a locally constant map $s \colon \mathcal{U} \to \mathscr{F}(\mathsf{M})$ such that $s(x) \in \kappa_{X,\mathcal{U}}(x)$ for each $x \in \mathcal{U}$. As above, in the real analytic case when M is connected, this locally constant selection is uniquely determined, and determines $\kappa_{X,\mathcal{U}}$ in the sense that $\kappa_{X,\mathcal{U}}(x) = \{s(x)\}$.

   Note that locally constant functions are those that are constant on connected components. Thus, by passing to the sheafification, we have gained flexibility by allowing local sections to differ on connected components of an open set. While this does not completely characterise the difference between local sections of the globally generated sheaf $\mathscr{F}$ and its sheafification $\mathrm{Sh}(\mathscr{F})$, it captures the essence of the matter, and *does* completely characterise the difference when $\nu = \omega$ and M is connected. •

8.2. Tautological control systems. Our definition of a tautological control system is relatively straightforward, given the definitions of the preceding section.

8.9 Definition: (Tautological control system and related notions) Let $m \in \mathbb{Z}_{\geq 0}$ and $m' \in \{0, \mathrm{lip}\}$, let $\nu \in \{m + m', \infty, \omega\}$, and let $r \in \{\infty, \omega\}$, as required.

(i) A $\mathbf{C}^\nu$***-tautological control system*** is a pair $\mathfrak{G} = (\mathsf{M}, \mathscr{F})$, where M is a manifold of class C^r whose elements are called ***states*** and where $\mathscr{F}$ is a presheaf of sets of C^ν-vector fields on M.

(ii) A tautological control system $\mathfrak{G} = (\mathsf{M}, \mathscr{F})$ is ***complete*** if $\mathscr{F}$ is a sheaf and is ***globally generated*** if $\mathscr{F}$ is globally generated.

(iii) The ***completion*** of $\mathfrak{G} = (\mathsf{M}, \mathscr{F})$ is the tautological control system $\mathrm{Sh}(\mathfrak{G}) = (\mathsf{M}, \mathrm{Sh}(\mathscr{F}))$. •

This is a pretty featureless definition, sorely in need of some connection to control theory. Let us begin to build this connection by pointing out the manner in which more common constructions give rise to tautological control systems, and vice versa.

8.10 Examples: (Correspondences between tautological control systems and other sorts of control systems) One of the topics of interest to us will be the relationship between our notion of tautological control systems and the more common notions of control systems (as in Sections 7.2 and 7.3) and differential inclusions (as in Section 7.4). We begin here by making some more or less obvious associations.

1. Let $m \in \mathbb{Z}_{\geq 0}$ and $m' \in \{0, \mathrm{lip}\}$, let $\nu \in \{m+m', \infty, \omega\}$, and let $r \in \{\infty, \omega\}$, as required. Let $\Sigma = (\mathsf{M}, F, \mathcal{C})$ be a C^ν-control system. To this control system we associate the C^ν-tautological control system $\mathfrak{G}_\Sigma = (\mathsf{M}, \mathscr{F}_\Sigma)$ by
$$\mathscr{F}_\Sigma(\mathcal{U}) = \{F^u|\mathcal{U} \in \Gamma^\nu(\mathsf{T}\mathcal{U}) \mid \ u \in \mathcal{C}\}.$$
The presheaf of sets of vector fields in this case is of the globally generated variety, as in Example 8.3–2. According to Example 8.3–2 we should generally not expect tautological control systems such as this to be *a priori* complete. We can, however, sheafify so that the tautological control system $\mathrm{Sh}(\mathfrak{G}_\Sigma)$ is complete.

2. Let us consider a means of going from a large class of tautological control systems to a control system. Let $m \in \mathbb{Z}_{\geq 0}$ and $m' \in \{0, \mathrm{lip}\}$, let $\nu \in \{m + m', \infty, \omega\}$, and let $r \in \{\infty, \omega\}$, as required. We suppose that we have a C^ν-tautological control system $\mathfrak{G} = (\mathsf{M}, \mathscr{F})$ where the presheaf $\mathscr{F}$ is globally generated. We define a C^ν-control system $\Sigma_{\mathfrak{G}} = (\mathsf{M}, F_{\mathscr{F}}, \mathcal{C}_{\mathscr{F}})$ as follows. We take $\mathcal{C}_{\mathscr{F}} = \mathscr{F}(\mathsf{M})$, i.e., the control set is our family of globally defined vector fields and the topology is that induced from $\Gamma^\nu(\mathsf{T}\mathsf{M})$. We define
$$\begin{aligned} F_{\mathscr{F}} \colon \mathsf{M} \times \mathcal{C}_{\mathscr{F}} &\to \mathsf{T}\mathsf{M} \\ (x, X) &\mapsto X(x). \end{aligned}$$
(Note that one has to make an awkward choice between writing a vector field as u or a control as X, since vector fields are controls. We have gone with the latter awkward choice, since it more readily mandates thinking about what the symbols mean.) Note that $F^X_{\mathscr{F}} = X$, and so this is somehow the identity map in disguise. In order for this construction to provide a *bona fide* control system, we should check that $F_{\mathscr{F}}$ is a parameterised vector field of class C^ν according to our Definitions 7.1, 7.4, and 7.10. According to Propositions 7.2, 7.5, and 7.12, it is sufficient to check that the map $X \mapsto F^X_{\mathscr{F}}$ is continuous. But this is the identity map, which is obviously continuous! Note that $\Sigma_{\mathfrak{G}}$ is a control-linear system, according to Example 7.21.

3. Let $m \in \mathbb{Z}_{\geq 0}$ and $m' \in \{0, \mathrm{lip}\}$, let $\nu \in \{m+m', \infty, \omega\}$, and let $r \in \{\infty, \omega\}$, as required. Let $\mathscr{X} \colon \mathsf{M} \twoheadrightarrow \mathsf{TM}$ be a differential inclusion. If $\mathcal{U} \subseteq \mathsf{M}$ is open, we denote

$$\Gamma^{\nu}(\mathscr{X}|\mathcal{U}) = \{X \in \Gamma^{\nu}(\mathsf{T}\mathcal{U}) \mid X(x) \in \mathscr{X}(x),\ x \in \mathcal{U}\}.$$

One should understand, of course, that we may very well have $\Gamma^{\nu}(\mathscr{X}|\mathcal{U}) = \varnothing$. This might happen for two reasons.

   (a) First, the differential inclusion may lack sufficient regularity to permit even local sections of the prescribed regularity.

   (b) Second, even if it permits local sections, there may be be problems finding sections defined on "large" open sets, because there may be global obstructions. One might anticipate this to be especially problematic in the real analytic case, where the specification of a vector field locally determines its behaviour globally by the Identity Theorem, cf. [Gunning 1990a, Theorem A.3].

This caveat notwithstanding, we can go ahead and define a tautological control system $\mathfrak{G}_{\mathscr{X}} = (\mathsf{M}, \mathscr{F}_{\mathscr{X}})$ with $\mathscr{F}_{\mathscr{X}}(\mathcal{U}) = \Gamma^{\nu}(\mathscr{X}|\mathcal{U})$.

We claim that $\mathfrak{G}_{\mathscr{X}}$ is complete. To see this, let $\mathcal{U} \subseteq \mathsf{M}$ be open and let $(\mathcal{U}_a)_{a \in A}$ be an open cover for $\mathcal{U}$. For each $a \in A$, let $X_a \in \mathscr{F}_{\mathscr{X}}(\mathcal{U}_a)$ and suppose that, for $a, b \in A$,

$$X_a|\mathcal{U}_a \cap \mathcal{U}_b = X_b|\mathcal{U}_a \cap \mathcal{U}_b.$$

Since $\mathscr{G}^{\nu}_{\mathsf{TM}}$ is a sheaf, let $X \in \Gamma^{\nu}(\mathsf{T}\mathcal{U})$ be such that $X|\mathcal{U}_a = X_a$ for each $a \in A$. We claim that $X \in \mathscr{F}_{\mathscr{X}}(\mathcal{U})$. Indeed, for $x \in \mathcal{U}$ we have $X(x) = X_a(x) \in \mathscr{X}(x)$ if we take $a \in A$ such that $x \in \mathcal{U}_a$.

The sheaf $\mathscr{F}_{\mathscr{X}}$ is not necessarily globally generated. Here is a stupid counterexample. Let us define $\mathscr{X}(x) = \mathsf{T}_x\mathsf{M}$, $x \in \mathsf{M}$, so that $\mathscr{F}_{\mathscr{X}} = \mathscr{G}^{\nu}_{\mathsf{TM}}$. For an open set $\mathcal{U}$, there will generally be local sections $X \in \Gamma^{\nu}(\mathsf{T}\mathcal{U})$ that are not restrictions to $\mathcal{U}$ of globally defined vector fields; vector fields that "blow up" at some point in the boundary of $\mathcal{U}$ are what one should have in mind.

4. Let $m \in \mathbb{Z}_{\geq 0}$ and $m' \in \{0, \mathrm{lip}\}$, let $\nu \in \{m+m', \infty, \omega\}$, and let $r \in \{\infty, \omega\}$, as required. Note that there is also associated to any C^{ν}-tautological control system $\mathfrak{G} = (\mathsf{M}, \mathscr{F})$ a differential inclusion $\mathscr{X}_{\mathfrak{G}}$ by

$$\mathscr{X}_{\mathfrak{G}}(x) = \{X(x) \mid [X]_x \in \mathscr{F}_x\},$$

recalling that $\mathscr{F}_x$ is the stalk of $\mathscr{F}$ at x. •

Now note that we can iterate the four constructions and ask to what extent we end up back where we started. More precisely, we have the following result.

8.11 Proposition: (Going back and forth between classes of systems) *Let $m \in \mathbb{Z}_{\geq 0}$ and $m' \in \{0, \mathrm{lip}\}$, let $\nu \in \{m+m', \infty, \omega\}$, and let $r \in \{\infty, \omega\}$, as required. Let $\mathfrak{G} = (\mathsf{M}, \mathscr{F})$ be a C^{ν}-tautological control system, let $\Sigma = (\mathsf{M}, F, \mathcal{C})$ be a C^{ν}-control system, and let $\mathscr{X}$ be a differential inclusion. Then the following statements hold:*

(i) if $\mathfrak{G}$ is globally generated, then $\mathfrak{G}_{\Sigma_{\mathfrak{G}}} = \mathfrak{G}$;

(ii) if the map $u \mapsto F^u$ from $\mathcal{C}$ to $\Gamma^{\nu}(\mathsf{TM})$ is injective and open onto its image, then $\Sigma_{\mathfrak{G}_{\Sigma}} = \Sigma$;

(iii) $\mathscr{F}(\mathcal{U}) \subseteq \mathscr{F}_{\mathscr{X}_{\mathfrak{G}}}(\mathcal{U})$ for every open $\mathcal{U} \subseteq \mathsf{M}$;

(iv) $\mathscr{X}_{\mathfrak{G}_{\mathscr{X}}} \subseteq \mathscr{X}$.

Proof: (i) Let $\mathcal{U} \subseteq \mathsf{M}$ be open and let $X \in \mathscr{F}(\mathcal{U})$. Then $X = X'|\mathcal{U}$ for $X' \in \mathscr{F}(\mathsf{M})$. Thus $X' \in \mathcal{C}_{\mathscr{F}}$ and $X'(x) = F(x, X')$ and so $X \in \mathscr{F}_{\Sigma_{\mathfrak{G}}}(\mathcal{U})$. Conversely, let $X \in \mathscr{F}_{\Sigma_{\mathfrak{G}}}(\mathcal{U})$. Then $X(x) = F(x, X')$, $x \in \mathcal{U}$, for some $X' \in \mathcal{C}_{\mathscr{F}}$. But this means that $X(x) = X'(x)$ for $X' \in \mathscr{F}(\mathcal{U})$ and for all $x \in \mathcal{U}$. In other words, $X \in \mathscr{F}(\mathcal{U})$.

(ii) Note that $\mathfrak{G}_\Sigma$ is globally generated. Thus we have

$$\mathcal{C}_{\mathscr{F}_\Sigma} = \mathscr{F}_\Sigma(\mathsf{M}) = \{F^u \mid u \in \mathcal{C}\}.$$

Since the map $u \mapsto F^u$ is continuous (by Propositions 7.2, 7.5, and 7.12), and injective and open onto its image (by hypothesis), it is an homeomorphism onto its image. Thus $\mathcal{C}_{\mathscr{F}_\Sigma}$ is homeomorphic to $\mathcal{C}$. Since $u \mapsto F^u$ is injective we can unambiguously write

$$F_{\mathscr{F}_\Sigma}(x, F^u) = F^u(x) = F(x, u).$$

(iii) Let $\mathcal{U} \subseteq \mathsf{M}$ be open. If $X \in \mathscr{F}(\mathcal{U})$, then clearly we have $X(x) \in \mathscr{X}_{\mathfrak{G}}(x)$ for every $x \in \mathcal{U}$ and so $\mathscr{F}(\mathcal{U}) \subseteq \mathscr{F}_{\mathscr{X}_{\mathfrak{G}}}(\mathcal{U})$, giving the assertion.

(iv) This is obvious. ■

8.12 Remark: (Correspondence between control systems and control-linear systems) The result establishes the rather surprising correspondence between control systems $\Sigma = (\mathsf{M}, F, \mathcal{C})$ for which the map $u \mapsto F^u$ is injective and open onto its image, and the associated control-linear system $\Sigma_{\mathfrak{G}_\Sigma} = (\mathsf{M}, \mathscr{F}_{\Sigma_{\mathfrak{G}}}, \mathcal{C}_{\mathscr{F}_\Sigma})$. That is to say, at least at the system level, in our treatment every system corresponds in a natural way to a control-linear system, albeit with a rather complicated control set. This correspondence carries over to trajectories as well, but one can also weaken these conditions to obtain trajectory correspondence in more general situations. These matters we discuss in detail in Section 8.6. •

Let us make some comments on the hypotheses present in the preceding result.

8.13 Remarks: (Going back and forth between classes of systems)

1. Since $\mathfrak{G}_\Sigma$ is necessarily globally generated for any control system Σ, the requirement that $\mathfrak{G}$ be globally generated cannot be dropped in part (i).
2. The requirement that the map $u \mapsto F^u$ be injective in part (ii) cannot be relaxed. Without this assumption, there is no way to recover F from $\{F^u \mid u \in \mathcal{C}\}$. Similarly, if this map is not open onto its image, while there may be a bijection between $\mathcal{C}$ and $\mathcal{C}_{\mathscr{F}_\Sigma}$, it will not be an homeomorphism which one needs for the control systems to be the same.
3. The converse assertion in part (iii) does not generally hold, as many counterexamples show. Here are two, each of a different character.
   (a) We take $\mathsf{M} = \mathbb{R}$ and consider the C^ω-tautological control system $\mathfrak{G} = (\mathsf{M}, \mathscr{F})$ where $\mathscr{F}$ is the globally generated presheaf defined by the single vector field $x^2\frac{\partial}{\partial x}$. Note that

$$\mathscr{X}_{\mathfrak{G}}(x) = \begin{cases} \{0\}, & x = 0, \\ \mathsf{T}_x\mathbb{R}, & x \neq 0. \end{cases}$$

Therefore,

$$\mathscr{F}_{\mathscr{X}}(\mathcal{U}) = \begin{cases} \{X \in \Gamma^\omega(\mathsf{T}\mathcal{U}) \mid X(0) = 0\}, & 0 \in \mathcal{U}, \\ \Gamma^\omega(\mathsf{T}\mathcal{U}), & 0 \notin \mathcal{U}. \end{cases}$$

It holds, therefore, that the vector field $x\frac{\partial}{\partial x}$ is a global section of $\mathscr{F}_{\mathscr{X}}$, but is not a global section of $\mathscr{F}$.

(b) Let us again take $\mathsf{M} = \mathbb{R}$ and now define a smooth tautological control system $\mathfrak{G} = (\mathsf{M}, \mathscr{F})$ by asking that $\mathscr{F}$ be the globally generated presheaf defined by the vector fields $X_1, X_2 \in \Gamma^\infty(\mathbb{R})$, where

$$X_1(x) = \begin{cases} \mathrm{e}^{-1/x}\frac{\partial}{\partial x}, & x > 0, \\ 0, & x \le 0, \end{cases}$$

and

$$X_2(x) = \begin{cases} \mathrm{e}^{-1/x}\frac{\partial}{\partial x}, & x < 0, \\ 0, & x \ge 0. \end{cases}$$

In this case,

$$\mathscr{X}_{\mathfrak{G}}(x) = \begin{cases} \{0\}, & x = 0, \\ \{0\} \cup \{\mathrm{e}^{-1/x}\frac{\partial}{\partial x}\}, & x \neq 0. \end{cases}$$

Therefore, $\mathscr{F}_{\mathscr{X}}$ is the sheafification of the globally generated presheaf defined by the vector fields X_1, X_2, X_3, and X_4, where

$$X_3(x) = \begin{cases} \mathrm{e}^{-1/x}\frac{\partial}{\partial x}, & x \neq 0, \\ 0, & x = 0, \end{cases}$$

and X_4 is the zero vector field.

4. Given the discussion in Example 8.10–3, one cannot reasonably expect that we will generally have equality in part (iv) of the preceding result. Indeed, one might even be inclined to say that it is only differential inclusions satisfying $\mathscr{X} = \mathscr{X}_{\mathfrak{G}_{\mathscr{X}}}$ that are useful in geometric control theory... •

While we are not yet finished with the task of formulating our theory—trajectories have yet to appear—it is worthwhile to make a pause at this point to reflect upon what we have done and have not done. After a moments thought, one realises that the difference between a control system $\Sigma = (\mathsf{M}, F, \mathcal{C})$ and its associated tautological control system $\mathfrak{G}_\Sigma = (\mathsf{M}, \mathscr{F}_\Sigma)$ is that, in the former case, the control vector fields are from the *indexed family* $(F^u)_{u\in\mathcal{C}}$, while for the tautological control system we have the *set* $\{F^u \mid u \in \mathcal{C}\}$. In going from the former to the latter we have "forgotten" the index u which we are explicitly keeping track of for control systems. If the map $u \mapsto F^u$ is injective, as in Proposition 8.11(ii), then there is no information lost as one goes from the indexed family to the set. If $u \mapsto F^u$ is not injective, then this is a signal that the control set is too large, and perhaps one should collapse it in some way. In other words, one can probably suppose injectivity of $u \mapsto F^u$ without loss of generality. (Openness of this map is another matter. As we shall see in Section 8.6 below, openness (and a little more) is crucial for there to be trajectory correspondence between systems and tautological control systems.) This then leaves us with the mathematical semantics of distinguishing between the indexed family $(F^u)_{u\in\mathcal{C}}$ and the subset $\{F^u \mid u \in \mathcal{C}\}$. About this, let us make two observations.

1. The entire edifice of nonlinear control theory seems, in some sense, to be built upon the preference of the indexed family over the set. As we discuss in the introduction, in applications there are very good reasons for doing this. But from the point of view of the general theory, the idea that one should carefully maintain the labelling of the vector fields from the set $\{F^u \mid u \in \mathcal{C}\}$ seems to be a really unnecessary distraction. And, moreover, it is a distraction upon which is built the whole notion of "feedback transformation," plus entire methodologies in control theory that are not feedback-invariant, e.g., linearisation, cf. Example 1.1. So, semantics? Possibly, but sometimes semantic choices are important.
2. Many readers will probably not be convinced by our attempts to magnify the distinction between the indexed family $(F^u)_{u\in\mathcal{C}}$ and the set $\{F^u \mid u \in \mathcal{C}\}$. As we shall see, however, this distinction becomes more apparent if one is really dedicated to using sets rather than indexed families. Indeed, this deprives one of the notion of "control," and one is forced to be more thoughtful about what one means by "trajectory." It is to this more thoughtful undertaking that we now turn, slowly.

8.3. Open-loop systems. Trajectories are associated to "open-loop systems," so we first discuss these. We first introduce some notation. Let $m \in \mathbb{Z}_{\geq 0}$ and $m' \in \{0, \mathrm{lip}\}$, let $\nu \in \{m + m', \infty, \omega\}$, and let $r \in \{\infty, \omega\}$, as required. For a C^ν-tautological control system $\mathfrak{G} = (\mathsf{M}, \mathscr{F})$, we then denote

$$\mathrm{LI}\Gamma^\nu(\mathbb{T}; \mathscr{F}(\mathcal{U})) = \{X : \mathbb{T} \to \mathscr{F}(\mathcal{U}) \mid X \in \mathrm{LI}\Gamma^\nu(\mathbb{T}; \mathsf{T}\mathcal{U})\},$$

for $\mathbb{T} \subseteq \mathbb{R}$ an interval and $\mathcal{U} \subseteq \mathsf{M}$ open.

8.14 Definition: (Open-loop system) Let $m \in \mathbb{Z}_{\geq 0}$ and $m' \in \{0, \mathrm{lip}\}$, let $\nu \in \{m + m', \infty, \omega\}$, and let $r \in \{\infty, \omega\}$, as required. Let $\mathfrak{G} = (\mathsf{M}, \mathscr{F})$ be a C^ν-tautological control system. An ***open-loop system*** for $\mathfrak{G}$ is a triple $\mathfrak{G}_{\mathrm{ol}} = (X, \mathbb{T}, \mathcal{U})$ where

(i) $\mathbb{T} \subseteq \mathbb{R}$ is an interval called the ***time-domain***;

(ii) $\mathcal{U} \subseteq \mathsf{M}$ is open;

(iii) $X \in \mathrm{LI}\Gamma^\nu(\mathbb{T}; \mathscr{F}(\mathcal{U}))$. •

Note that an open-loop system for $\mathfrak{G} = (\mathsf{M}, \mathscr{F})$ is also an open-loop system for the completion $\mathrm{Sh}(\mathfrak{G})$, just because $\mathscr{F}(\mathcal{U}) \subseteq \mathrm{Sh}(\mathscr{F})(\mathcal{U})$. However, of course, there may be open-loop systems for $\mathrm{Sh}(\mathfrak{G})$ that are not open-loop systems for $\mathfrak{G}$. This is as it should be, and has no significant ramifications for the theory, as we shall see as we go along.

In order to see how we should think about an open-loop system, let us consider this notion in the special case of control systems.

8.15 Example: (Open-loop systems associated to control systems) Let $m \in \mathbb{Z}_{\geq 0}$ and $m' \in \{0, \mathrm{lip}\}$, let $\nu \in \{m + m', \infty, \omega\}$, and let $r \in \{\infty, \omega\}$, as required. Let $\Sigma = (\mathsf{M}, F, \mathcal{C})$ be a C^ν-control system with $\mathfrak{G}_\Sigma$ the associated C^ν-tautological control system. If we let $\mu \in \mathrm{L}^\infty_{\mathrm{loc}}(\mathbb{T}; \mathcal{C})$, then we have the associated open-loop system $\mathfrak{G}_{\Sigma,\mu} = (F^\mu, \mathbb{T}, \mathsf{M})$ defined by

$$F^\mu(t)(x) = F(x, \mu(t)), \qquad t \in \mathbb{T},\ x \in \mathsf{M}.$$

Proposition 7.18 ensures that this is an open-loop system for the tautological control system $\mathfrak{G}_\Sigma$.

A similar assertion holds if $\mathcal{C}$ is a subset of a locally convex topological vector space and F defines a sublinear control system, and if $\mu \in \mathrm{L}^1_{\mathrm{loc}}(\mathbb{T}; \mathcal{C})$, cf. Proposition 7.22. •

8.16 Notation: (Open-loop systems) For an open-loop system $\mathfrak{G}_{\mathrm{ol}}(X, \mathbb{T}, \mathcal{U})$, the notation $X(t)(x)$, while accurate, is unnecessarily cumbersome, and we will often instead write $X(t, x)$ or $X_t(x)$, with no loss of clarity and a gain in aesthetics. •

Generally one might wish to place a restriction on the set of open-loop systems one will use. This is tantamount to, for usual control systems, placing restrictions on the controls one might use; one may wish to use piecewise continuous controls or piecewise constant controls, for example. For tautological control systems we do this as follows.

8.17 Definition: (Open-loop subfamily) Let $m \in \mathbb{Z}_{\geq 0}$ and $m' \in \{0, \mathrm{lip}\}$, let $\nu \in \{m + m', \infty, \omega\}$, and let $r \in \{\infty, \omega\}$, as required. Let $\mathfrak{G} = (\mathsf{M}, \mathscr{F})$ be a C^ν-tautological control system. An ***open-loop subfamily*** for $\mathfrak{G}$ is an assignment, to each interval $\mathbb{T} \subseteq \mathbb{R}$ and each open set $\mathcal{U} \subseteq \mathsf{M}$, a subset $\mathscr{O}_{\mathfrak{G}}(\mathbb{T}, \mathcal{U}) \subseteq \mathrm{LI}\Gamma^\nu(\mathbb{T}; \mathscr{F}(\mathcal{U}))$ with the property that, if $(\mathbb{T}_1, \mathcal{U}_1)$ and $(\mathbb{T}_2, \mathcal{U}_2)$ are such that $\mathbb{T}_1 \subseteq \mathbb{T}_2$ and $\mathcal{U}_1 \subseteq \mathcal{U}_2$, then

$$\{X|\mathbb{T}_1 \times \mathcal{U}_1 \mid X \in \mathscr{O}_{\mathfrak{G}}(\mathbb{T}_2, \mathcal{U}_2)\} \subseteq \mathscr{O}_{\mathfrak{G}}(\mathbb{T}_1, \mathcal{U}_1). \quad \bullet$$

Here are a few common examples of open-loop subfamilies.

8.18 Examples: (Open-loop subfamilies) Let $m \in \mathbb{Z}_{\geq 0}$ and $m' \in \{0, \mathrm{lip}\}$, let $\nu \in \{m + m', \infty, \omega\}$, and let $r \in \{\infty, \omega\}$, as required. Let $\mathfrak{G} = (\mathsf{M}, \mathscr{F})$ be a C^ν-tautological control system.

1. The ***full subfamily*** for $\mathfrak{G}$ is the open-loop subfamily $\mathscr{O}_{\mathfrak{G},\mathrm{full}}$ defined by

$$\mathscr{O}_{\mathfrak{G},\mathrm{full}}(\mathbb{T}, \mathcal{U}) = \mathrm{LI}\Gamma^\nu(\mathbb{T}; \mathscr{F}(\mathcal{U})).$$

   Thus the full subfamily contains all possible open-loop systems. Of course, every open-loop subfamily will be contained in this one.

2. The ***locally essentially bounded subfamily*** for $\mathfrak{G}$ is the open-loop subfamily $\mathscr{O}_{\mathfrak{G},\infty}$ defined by asking that

$$\mathscr{O}_{\mathfrak{G},\infty}(\mathbb{T}, \mathcal{U}) = \{X \in \mathscr{O}_{\mathfrak{G},\mathrm{full}}(\mathbb{T}, \mathcal{U}) \mid X \in \mathrm{LB}\Gamma^\nu(\mathbb{T}; \mathsf{T}\mathcal{U})\}.$$

   Thus, for the locally essentially bounded subfamily, we require that the condition of being locally integrally C^ν-bounded be replaced with the stronger condition of being locally essentially C^ν-bounded.

3. The ***locally essentially compact subfamily*** for $\mathfrak{G}$ is the open-loop subfamily $\mathscr{O}_{\mathfrak{G},\mathrm{cpt}}$ defined by asking that

$$\begin{aligned} \mathscr{O}_{\mathfrak{G},\mathrm{cpt}}(\mathbb{T}, \mathcal{U}) = \{X \in \mathscr{O}_{\mathfrak{G},\mathrm{full}}(\mathbb{T}, \mathcal{U}) | \ & \text{for every compact subinterval } \mathbb{T}' \subseteq \mathbb{T} \\ & \text{there exists a compact } K \subseteq \Gamma^\nu(\mathbb{T}; \mathsf{T}\mathcal{U}) \\ & \text{such that } X(t) \subseteq K \text{ for almost every } t \in \mathbb{T}'\}. \end{aligned}$$

   Thus, for the locally essentially compact subfamily, we require that the condition of being locally essentially bounded in the von Neumann bornology (that defines the locally essentially bounded subfamily) be replaced with being locally essentially bounded in the compact bornology.

We comment that in cases when the compact and von Neumann bornologies agree, then of course we have $\mathscr{O}_{\mathfrak{G},\infty} = \mathscr{O}_{\mathfrak{G},\mathrm{cpt}}$. As we have seen in CO$^\infty$-5 and C$^\omega$-5, this is the case when $\nu \in \{\infty, \omega\}$.

4. The ***piecewise constant subfamily*** for $\mathfrak{G}$ is the open-loop subfamily $\mathscr{O}_{\mathfrak{G},\mathrm{pwc}}$ defined by asking that

$$\mathscr{O}_{\mathfrak{G},\mathrm{pwc}}(\mathbb{T};\mathcal{U}) = \{X \in \mathscr{O}_{\mathfrak{G},\mathrm{full}}(\mathbb{T},\mathcal{U}) \mid\ t \mapsto X(t) \text{ is piecewise constant}\}.$$

Let us be clear what we mean by piecewise constant. We mean that there is a partition $(\mathbb{T}_j)_{j\in J}$ of $\mathbb{T}$ into pairwise disjoint intervals such that

(a) for any compact interval $\mathbb{T}' \subseteq \mathbb{T}$, the set

$$\{j \in J \mid\ \mathbb{T}' \cap \mathbb{T}_j \neq \varnothing\}$$

is finite and such that

(b) $X|\mathbb{T}_j$ is constant for each $j \in J$.

One might imagine that the piecewise constant open-loop subfamily will be useful for studying orbits and controllability of tautological control systems.

5. We can associate an open-loop subfamily to an open-loop system as follows. Let $m \in \mathbb{Z}_{\geq 0}$ and $m' \in \{0, \mathrm{lip}\}$, let $\nu \in \{m+m', \infty, \omega\}$, and let $r \in \{\infty, \omega\}$, as required. Let $\mathfrak{G} = (\mathsf{M}, \mathscr{F})$ be a C$^\nu$-tautological control system, let $\mathscr{O}_{\mathfrak{G}}$ be an open-loop subfamily for $\mathfrak{G}$, let $\mathbb{T}$ be a time-domain, let $\mathcal{U} \subseteq \mathsf{M}$ be open, and let $X \in \mathscr{O}_{\mathfrak{G}}(\mathbb{T},\mathcal{U})$. We denote by $\mathscr{O}_{\mathfrak{G},X}$ the open-loop subfamily defined as follows. If $\mathbb{T}' \subseteq \mathbb{T}$ and $\mathcal{U}' \subseteq \mathcal{U}$, then we let

$$\mathscr{O}_{\mathfrak{G},X}(\mathbb{T}',\mathcal{U}') = \{X' \in \mathscr{O}_{\mathfrak{G}}(\mathbb{T}',\mathcal{U}') \mid\ X' = X|\mathbb{T}' \times \mathcal{U}'\}.$$

If $\mathbb{T}' \not\subseteq \mathbb{T}$ and/or $\mathcal{U}' \not\subseteq \mathcal{U}$, then we take $\mathscr{O}_{\mathfrak{G},X} = \varnothing$. Thus $\mathscr{O}_{\mathfrak{G},X}$ is comprised of those vector fields from $\mathscr{O}_{\mathfrak{G}}$ that are merely restrictions of X to smaller domains. Just why this might be interesting we will only see when we discuss linearisation about a reference flow in Section 9.4.

6. Let $m \in \mathbb{Z}_{\geq 0}$ and $m' \in \{0, \mathrm{lip}\}$, let $\nu \in \{m+m', \infty, \omega\}$, and let $r \in \{\infty, \omega\}$, as required. In Proposition 8.11 we saw that there was a pretty robust correspondence between C$^\nu$-control systems and C$^\nu$-tautological control systems, *at the system level*. As we make our way towards trajectories, as we are now doing, this robustness breaks down a little. To frame this, we can define an open-loop subfamily for the tautological control system associated to a C$^\nu$-control system $\Sigma = (\mathsf{M}, F, \mathcal{C})$ as follows. For a time-domain $\mathbb{T}$ and an open $\mathcal{U} \subseteq \mathsf{M}$, we define

$$\mathscr{O}_\Sigma(\mathbb{T},\mathcal{U}) = \{F^\mu|\mathcal{U} \mid\ \mu \in \mathrm{L}^\infty_{\mathrm{loc}}(\mathbb{T};\mathcal{C})\},$$

recalling that $F^\mu(t,x) = F(x,\mu(t))$. We clearly have $\mathscr{O}_\Sigma(\mathbb{T};\mathcal{U}) \subseteq \mathscr{O}_{\mathfrak{G}_\Sigma,\mathrm{cpt}}(\mathbb{T};\mathcal{U})$ for every time-domain $\mathbb{T}$ and every open $\mathcal{U} \subseteq \mathsf{M}$; this was proved in the course of proving Proposition 7.18. Of course, by virtue of Proposition 7.22, we have a corresponding construction if the control set $\mathcal{C}$ is a subset of a locally convex topological vector space, if F is sublinear, and if $\mu \in \mathrm{L}^1_{\mathrm{loc}}(\mathbb{T};\mathcal{C})$. However, we do not generally expect to have equality of these two open-loop subfamilies. This, in turn, will have repercussions on the nature of the trajectories for these subfamilies, and, therefore, on the relationship of trajectories of a control system and the corresponding tautological control system. We will consider these matters in Section 8.6, and we will see that, for many interesting classes of control systems, there is, in fact, a natural trajectory correspondence between the system and its associated tautological control system. •

Our notion of an open-loop subfamily is very general, and working with the full generality will typically lead to annoying problems. There are many attributes that one may wish for open-loop subfamilies to satisfy in order to relax some the annoyance. To illustrate, let us define a typical attribute that one may require, that of translation-invariance. Let us define some notation so that we can easily make the definition. For a time-domain $\mathbb{T}$ and for $s \in \mathbb{R}$, we denote

$$s + \mathbb{T} = \{s + t \mid t \in \mathbb{T}\}$$

and we denote by $\tau_s\colon s + \mathbb{T} \to \mathbb{T}$ the translation map $\tau_s(t) = t - s$.

8.19 Definition: (Translation-invariant open-loop subfamily) Let $m \in \mathbb{Z}_{\geq 0}$ and $m' \in \{0, \mathrm{lip}\}$, let $\nu \in \{m + m', \infty, \omega\}$, and let $r \in \{\infty, \omega\}$, as required. Let $\mathfrak{G} = (\mathsf{M}, \mathscr{F})$ be a C^ν-tautological control system. An open-loop subfamily $\mathscr{O}_{\mathfrak{G}}$ for $\mathfrak{G}$ is ***translation-invariant*** if, for every $s \in \mathbb{R}$, every time-domain $\mathbb{T}$, and every open set $\mathcal{U} \subseteq \mathsf{M}$, the map

$$\begin{aligned}(\tau_s \times \mathrm{id}_{\mathcal{U}})^*\colon \mathscr{O}_{\mathfrak{G}}(s + \mathbb{T}, \mathcal{U}) &\to \mathscr{O}_{\mathfrak{G}}(\mathbb{T}, \mathcal{U}) \\ X &\mapsto X \circ (\tau_s \times \mathrm{id}_{\mathcal{U}})\end{aligned}$$

is a bijection. •

An immediate consequence of the definition is, of course, that if $t \mapsto \xi(t)$ is a trajectory (we will formally define the notion of "trajectory" in the next section), then so is $t \mapsto \xi(s+t)$ for every $s \in \mathbb{R}$.

Let us now think about how open-loop subfamilies interact with completion. In order for the definition we are about to make make sense, we should verify the following lemma.

8.20 Lemma: (Time-varying vector fields characterised by their germs) *Let $m \in \mathbb{Z}_{\geq 0}$ and $m' \in \{0, \mathrm{lip}\}$, let $\nu \in \{m + m', \infty, \omega\}$, and let $r \in \{\infty, \omega\}$, as required. Let M be a C^r-manifold, let $\mathbb{T} \subseteq \mathbb{R}$ be an interval, and let $X\colon \mathbb{T} \times \mathsf{M} \to \mathsf{TM}$ have the property that $X(t, x) \in \mathsf{T}_x\mathsf{M}$ for each $(t, x) \in \mathbb{T} \times \mathsf{M}$. Then the following statements hold:*

(i) if, for each $x \in \mathsf{M}$, there exist a neighbourhood $\mathcal{U}$ of x and $X' \in \mathrm{CF}\Gamma^\nu(\mathbb{T}; \mathsf{T}\mathcal{U})$ such that $[X_t]_x = [X'_t]_x$ for every $t \in \mathbb{T}$, then $X \in \mathrm{CF}\Gamma^\nu(\mathbb{T}; \mathsf{TM})$;

(ii) if, for each $x \in \mathsf{M}$, there exist a neighbourhood $\mathcal{U}$ of x and $X' \in \mathrm{LI}\Gamma^\nu(\mathbb{T}; \mathsf{T}\mathcal{U})$ such that $[X_t]_x = [X'_t]_x$ for every $t \in \mathbb{T}$, then $X \in \mathrm{LI}\Gamma^\nu(\mathbb{T}; \mathsf{TM})$;

(iii) if, for each $x \in \mathsf{M}$, there exist a neighbourhood $\mathcal{U}$ of x and $X' \in \mathrm{LB}\Gamma^\nu(\mathbb{T}; \mathsf{T}\mathcal{U})$ such that $[X_t]_x = [X'_t]_x$ for every $t \in \mathbb{T}$, then $X \in \mathrm{LB}\Gamma^\nu(\mathbb{T}; \mathsf{TM})$.

Proof: (i) Let $x \in \mathsf{M}$. Since X agrees in some neighbourhood of x with a Carathéodory vector field X', it follows that $t \mapsto X_t(x) = X'_t(x)$ is measurable. In like manner, let $t \in \mathbb{T}$ and let $x_0 \in \mathsf{M}$. Then $x \mapsto X_t(x) = X'_t(x)$ is of class C^ν in a neighbourhood of x_0, and so $x \mapsto X_t(x)$ is of class C^ν.

(ii) For $K \subseteq \mathsf{M}$ compact, for $k \in \mathbb{Z}_{\geq 0}$, and for $\boldsymbol{a} \in \mathrm{c}_{\downarrow 0}(\mathbb{Z}_{\geq 0}; \mathbb{R}_{>0})$, denote

$$p_K = \begin{cases} p^{\infty}_{K,k}, & \nu = \infty, \\ p^m_K, & \nu = m, \\ p^{m+\mathrm{lip}}_K, & \nu = m + \mathrm{lip}, \\ p^{\omega}_{K,\boldsymbol{a}}, & \nu = \omega. \end{cases}$$

Let $K \subseteq \mathsf{M}$ be compact, let $x \in K$, let $\mathcal{U}_x$ be a relatively compact neighbourhood of x, and let $X_x \in \mathrm{LI}\Gamma^{\nu}(\mathbb{T}; \mathcal{U}_x)$ be such that $[X_t]_x = [X_{x,t}]_x$ for every $t \in \mathbb{T}$. Then there exists $g_x \in \mathrm{L}^1_{\mathrm{loc}}(\mathbb{T}; \mathbb{R}_{\geq 0})$ such that

$$p_{\mathrm{cl}(\mathcal{U}_x)}(X_{x,t}) \leq g_x(t), \qquad t \in \mathbb{T}.$$

Now let $x_1, \dots, x_k \in K$ be such that $K \subseteq \cup_{j=1}^k \mathcal{U}_{x_j}$. Let $g(t) = \max\{g_{x_1}(t), \dots, g_{x_k}(t)\}$, noting that the associated function g is measurable by [Cohn 2013, Proposition 2.1.4] and is locally integrable by the triangle inequality, along with the fact that

$$g(t) \leq C(g_{x_1}(t) + \dots + g_{x_k}(t))$$

for some suitable $C \in \mathbb{R}_{>0}$ (this is simply the statement of the equivalence of the ℓ^1 and ℓ^∞ norms for $\mathbb{R}^n$). We then have

$$p_K(X_t) \leq g(t), \qquad t \in \mathbb{T},$$

showing that $X \in \mathrm{LI}\Gamma^{\nu}(\mathbb{T}; \mathsf{TM})$.

(iii) This is proved in exactly the same manner, *mutatis mutandis*, as the preceding part of the lemma. ■

The following definition can now be made.

8.21 Definition: (Completion of an open-loop subfamily) Let $m \in \mathbb{Z}_{\geq 0}$ and $m' \in \{0, \mathrm{lip}\}$, let $\nu \in \{m + m', \infty, \omega\}$, and let $r \in \{\infty, \omega\}$, as required. Let $\mathfrak{G} = (\mathsf{M}, \mathscr{F})$ be a C^ν-tautological control system and let $\mathscr{O}_{\mathfrak{G}}$ be an open-loop subfamily for $\mathfrak{G}$. The ***completion*** of $\mathscr{O}_{\mathfrak{G}}$ is the open-loop subfamily $\mathrm{Sh}(\mathscr{O}_{\mathfrak{G}})$ for $\mathrm{Sh}(\mathfrak{G})$ defined by specifying that $(X, \mathbb{T}, \mathcal{U}) \in \mathrm{Sh}(\mathscr{O}_{\mathfrak{G}})$ if, for each $x \in \mathcal{U}$, there exist a neighbourhood $\mathcal{U}' \subseteq \mathcal{U}$ of x and $(X', \mathbb{T}, \mathcal{U}') \in \mathscr{O}_{\mathfrak{G}}(\mathbb{T}, \mathcal{U}')$ such that $[X_t]_x = [X'_t]_x$ for each $t \in \mathbb{T}$. •

Clearly the completion of an open-loop subfamily is an open-loop subfamily for the completion. Moreover, if $(X, \mathbb{T}, \mathcal{U}) \in \mathscr{O}_{\mathfrak{G}}(\mathbb{T}, \mathcal{U})$, then $(X, \mathbb{T}, \mathcal{U}) \in \mathrm{Sh}(\mathscr{O}_{\mathfrak{G}}(\mathbb{T}, \mathcal{U}))$, but one cannot expect the converse assertion to generally hold.

8.4. Trajectories. With the concept of open-loop system just developed, it is relatively easy to provide a notion of a trajectory for a tautological control system.

8.22 Definition: (Trajectory for tautological control system) Let $m \in \mathbb{Z}_{\geq 0}$ and $m' \in \{0, \mathrm{lip}\}$, let $\nu \in \{m + m', \infty, \omega\}$, and let $r \in \{\infty, \omega\}$, as required. Let $\mathfrak{G} = (\mathsf{M}, \mathscr{F})$ be a C^ν-tautological control system and let $\mathscr{O}_{\mathfrak{G}}$ be an open-loop subfamily for $\mathfrak{G}$.

(i) For a time-domain $\mathbb{T}$, an open set $\mathcal{U} \subseteq \mathsf{M}$, and for $X \in \mathscr{O}_{\mathfrak{G}}(\mathbb{T}, \mathcal{U})$, an ***$(X, \mathbb{T}, \mathcal{U})$-trajectory*** for $\mathscr{O}_{\mathfrak{G}}$ is a curve $\xi \colon \mathbb{T} \to \mathcal{U}$ such that $\xi'(t) = X(t, \xi(t))$ for almost every $t \in \mathbb{T}$.

(ii) For a time-domain $\mathbb{T}$ and an open set $\mathcal{U} \subseteq \mathsf{M}$, a ***$(\mathbb{T}, \mathcal{U})$-trajectory*** for $\mathscr{O}_{\mathfrak{G}}$ is a curve $\xi \colon \mathbb{T} \to \mathcal{U}$ such that $\xi'(t) = X(t, \xi(t))$ for almost every $t \in \mathbb{T}$ for some $X \in \mathscr{O}_{\mathfrak{G}}(\mathbb{T}, \mathcal{U})$.

(iii) A ***trajectory*** for $\mathscr{O}_{\mathfrak{G}}$ is a curve that is a $(\mathbb{T}, \mathcal{U})$-trajectory for $\mathscr{O}_{\mathfrak{G}}$ for some time-domain $\mathbb{T}$ and some open set $\mathcal{U} \subseteq \mathsf{M}$.

We denote by:

(iv) $\mathrm{Traj}(X, \mathbb{T}; \mathcal{U})$ the set of $(X, \mathbb{T}, \mathcal{U})$-trajectories for $\mathscr{O}_{\mathfrak{G}}$;

(v) $\mathrm{Traj}(\mathbb{T}, \mathcal{U}, \mathscr{O}_{\mathfrak{G}})$ the set of $(\mathbb{T}, \mathcal{U})$-trajectories for $\mathscr{O}_{\mathfrak{G}}$;

(vi) $\mathrm{Traj}(\mathscr{O}_{\mathfrak{G}})$ the set of trajectories for $\mathscr{O}_{\mathfrak{G}}$.

We shall abbreviate $\mathrm{Traj}(\mathbb{T}, \mathcal{U}, \mathfrak{G}) = \mathrm{Traj}(\mathbb{T}, \mathcal{U}, \mathscr{O}_{\mathfrak{G},\mathrm{full}})$ and $\mathrm{Traj}(\mathfrak{G}) = \mathrm{Traj}(\mathscr{O}_{\mathfrak{G},\mathrm{full}})$. •

Sometimes one wishes to keep track of the fact that, associated with a trajectory is an open-loop system. The following notion is designed to capture this.

8.23 Definition: (Referenced trajectory) Let $m \in \mathbb{Z}_{\geq 0}$ and $m' \in \{0, \mathrm{lip}\}$, let $\nu \in \{m + m', \infty, \omega\}$, and let $r \in \{\infty, \omega\}$, as required. Let $\mathfrak{G} = (\mathsf{M}, \mathscr{F})$ be a C^ν-tautological control system and let $\mathscr{O}_{\mathfrak{G}}$ be an open-loop subfamily for $\mathfrak{G}$. A ***referenced $\mathscr{O}_{\mathfrak{G}}$-trajectory*** is a pair (X, ξ) where $X \in \mathscr{O}_{\mathfrak{G}}(\mathbb{T}; \mathcal{U})$ and $\xi \in \mathrm{Traj}(X, \mathbb{T}, \mathcal{U})$. By $\mathrm{Rtraj}(\mathbb{T}, \mathcal{U}, \mathscr{O}_{\mathfrak{G}})$ we denote the set of referenced $\mathscr{O}_{\mathfrak{G}}$-trajectories for which $X \in \mathscr{O}_{\mathfrak{G}}(\mathbb{T}; \mathcal{U})$.

In Section 8.6 below, we shall explore trajectory correspondences between tautological control systems, control systems, and differential inclusions.

The notion of a trajectory immediately gives rise to a certain open-loop subfamily. At present it may not be clear why this construction is interesting, but it will come up in Section 9.4 when we talk about linearisations about trajectories.

8.24 Example: (The open-loop subfamily defined by a trajectory) Let $m \in \mathbb{Z}_{\geq 0}$ and $m' \in \{0, \mathrm{lip}\}$, let $\nu \in \{m + m', \infty, \omega\}$, and let $r \in \{\infty, \omega\}$, as required. Let $\mathfrak{G} = (\mathsf{M}, \mathscr{F})$ be a C^ν-tautological control system, let $\mathscr{O}_{\mathfrak{G}}$ be an open-loop subfamily for $\mathfrak{G}$, and let $\xi \in \mathrm{Traj}(\mathbb{T}, \mathcal{U}, \mathscr{O}_{\mathfrak{G}})$. We denote by $\mathscr{O}_{\mathfrak{G},\xi}$ the open-loop subfamily defined as follows. If $\mathbb{T}' \subseteq \mathbb{T}$ and $\mathcal{U}' \subseteq \mathcal{U}$ are such that $\xi(\mathbb{T}') \subseteq \mathcal{U}'$, then we let

$$\mathscr{O}_{\mathfrak{G},\xi}(\mathbb{T}', \mathcal{U}') = \{X \in \mathscr{O}_{\mathfrak{G}}(\mathbb{T}', \mathcal{U}') \mid \ \xi'(t) = X(t, \xi(t)), \ \text{a.e. } t \in \mathbb{T}'\}.$$

If $\mathbb{T}' \not\subseteq \mathbb{T}$ or $\mathcal{U}' \not\subseteq \mathcal{U}$, or if $\mathbb{T}' \subseteq \mathbb{T}$ and $\mathcal{U}' \subseteq \mathcal{U}$ but $\xi(\mathbb{T}') \not\subseteq \mathcal{U}'$, then we take $\mathscr{O}_{\mathfrak{G},\xi} = \varnothing$. Thus $\mathscr{O}_{\mathfrak{G},\xi}$ is comprised of those vector fields from $\mathscr{O}_{\mathfrak{G}}$ possessing ξ (restricted to the appropriate subinterval) as an integral curve. •

In control theory, trajectories are of paramount importance, often far more important, say, than systems *per se*. For this reason, one might ask that completion of a tautological control system preserve trajectories. However, this will generally not be the case, as the following counterexample illustrates.

8.25 Example: (Sheafification does not preserve trajectories) We will chat our way through a general example; the reader can very easily create a specific concrete instance from the general discussion.

Let $m \in \mathbb{Z}_{\geq 0}$ and $m' \in \{0, \mathrm{lip}\}$, let $\nu \in \{m+m', \infty, \omega\}$, and let $r \in \{\infty, \omega\}$, as required. We let M be a C^r-manifold with Riemannian metric $\mathbb{G}$. We consider the presheaf $\mathscr{F}_{\mathrm{bdd}}$ of bounded C^ν-vector fields on M, initially discussed in Example 8.3–1. We let $\mathfrak{G}_{\mathrm{bdd}} = (\mathsf{M}, \mathscr{F}_{\mathrm{bdd}})$ so that, as we saw in Example 8.8–1, $\mathrm{Sh}(\mathscr{F}_{\mathrm{bdd}}) = \mathscr{G}^\nu_{\mathsf{TM}}$. Let X be a vector field possessing an integral curve $\xi\colon \mathbb{T} \to \mathsf{M}$ for which

$$\limsup_{t \to \sup \mathbb{T}} \|\xi'(t)\|_{\mathbb{G}} = \infty$$

(this requires that $\mathbb{T}$ be noncompact, of course).

Now let us see how this gives rise to a trajectory for $\mathrm{Sh}(\mathfrak{G}_{\mathrm{bdd}})$ that is not a trajectory for $\mathfrak{G}_{\mathrm{bdd}}$. We let $\mathbb{T}$ be the interval of definition of the integral curve ξ described above. We consider the open subset $\mathsf{M} \subseteq \mathsf{M}$. We then have the open-loop system $(X, \mathbb{T}, \mathsf{M})$ specified by letting $X(t) = X$ (abusing notation), i.e., we consider a time-independent open-loop system. It is clear, then, that $\xi \in \mathrm{Traj}(\mathbb{T}, \mathsf{M}, \mathrm{Sh}(\mathfrak{G}_{\mathrm{bdd}}))$ (since $\mathrm{Sh}(\mathfrak{G}_{\mathrm{bdd}}) = (\mathsf{M}, \mathscr{G}^\nu_{\mathsf{TM}})$ as we showed in Example 8.8–1), but that ξ cannot be a trajectory for $\mathfrak{G}_{\mathrm{bdd}}$ since any vector field possessing ξ as an integral curve cannot be bounded. •

Thus we cannot expect sheafification to generally preserve trajectories. This should be neither a surprise nor a disappointment to us. It is gratifying, however, that sheafification *does* preserve trajectories in at least one important case.

8.26 Proposition: (Trajectories are preserved by sheafification of globally generated systems) *Let $m \in \mathbb{Z}_{\geq 0}$ and $m' \in \{0, \mathrm{lip}\}$, let $\nu \in \{m+m', \infty, \omega\}$, and let $r \in \{\infty, \omega\}$, as required. Let $\mathfrak{G} = (\mathsf{M}, \mathscr{F})$ be a globally generated C^ν-tautological control system, let $\mathbb{T}$ be a time-domain, and let $\mathscr{O}_{\mathfrak{G}}$ be an open-loop subfamily for $\mathfrak{G}$. For a locally absolutely continuous curve $\xi\colon \mathbb{T} \to \mathsf{M}$ the following statements are equivalent:*

(i) $\xi \in \mathrm{Traj}(\mathbb{T}, \mathcal{U}, \mathscr{O}_{\mathfrak{G}})$ for some open set $\mathcal{U} \subseteq \mathsf{M}$;

(ii) $\xi \in \mathrm{Traj}(\mathbb{T}, \mathcal{U}', \mathrm{Sh}(\mathscr{O}_{\mathfrak{G}}))$ for some open set $\mathcal{U}' \subseteq \mathsf{M}$.

Proof: Since $\mathscr{O}_{\mathfrak{G}}(\mathbb{T}, \mathcal{U}) \subseteq \mathrm{Sh}(\mathscr{O}_{\mathfrak{G}})(\mathbb{T}, \mathcal{U})$, the first assertion clearly implies the second. So it is the opposite implication we need to prove.

Thus let $\mathcal{U}' \subseteq \mathsf{M}$ be open and suppose that $\xi \in \mathrm{Traj}(\mathbb{T}, \mathcal{U}', \mathrm{Sh}(\mathscr{O}_{\mathfrak{G}}))$. Let $X \in \mathrm{LI}\Gamma^\nu(\mathbb{T}; \mathsf{T}\mathcal{U}')$ be such that ξ is an integral curve for X and such that $X_t \in \mathrm{Sh}(\mathscr{F})(\mathcal{U}')$ for every $t \in \mathbb{T}$. For each *fixed* $\tau \in \mathbb{T}$, there exists $X_\tau \in \mathrm{LI}\Gamma^\nu(\mathbb{T}; \mathscr{F}(\mathsf{M}))$ such that $[X_{\tau,t}]_{\xi(\tau)} = [X_t]_{\xi(\tau)}$ for every $t \in \mathbb{T}$. (This is the definition of $\mathrm{Sh}(\mathscr{O}_{\mathfrak{G}})$, noting that $\mathscr{F}$ is globally generated.) This means that around τ we have a bounded open interval $\mathbb{T}_\tau \subseteq \mathbb{T}$ and a neighbourhood $\mathcal{U}_\tau$ of $\xi(\tau)$ so that $\xi(\mathbb{T}_\tau) \subseteq \mathcal{U}_\tau$ and so that $\xi'(t) = X_\tau(t, \xi(t))$ for almost every $t \in \mathbb{T}_\tau$. By paracompactness, we can choose a locally finite refinement of these intervals that also covers $\mathbb{T}$. By repartitioning, we arrive at a locally finite pairwise disjoint covering $(\mathbb{T}_j)_{j \in J}$ of $\mathbb{T}$ by subintervals with the following property: the index set J is a finite or countable subset of $\mathbb{Z}$ chosen so that $t_1 < t_2$ whenever $t_1 \in \mathbb{T}_{j_1}$ and $t_2 \in \mathbb{T}_{j_2}$ with $j_1 < j_2$. That is, we order the labels for the elements of the partition in the natural way, this making sense since the cover is locally finite. By construction, we have $X_j \in \mathrm{LI}\Gamma^\nu(\mathbb{T}_j; \mathscr{F}(\mathsf{M}))$ with the property that $\xi|\mathbb{T}_j$ is an integral curve for X_j. We then define $\overline{X}\colon \mathbb{T} \to \mathscr{F}(\mathsf{M})$ by asking that $\overline{X}|\mathbb{T}_j = X_j$. It remains to show that $\overline{X} \in \mathrm{LI}\Gamma^\nu(\mathbb{T}; \mathscr{F}(\mathsf{M}))$.

Because each of the vector fields X_j, $j \in J$, is a Carathéodory vector field, we easily conclude that $\overline{X}$ is also a Carathéodory vector field.

Let $K \subseteq \mathsf{M}$ be compact, $k \in \mathbb{Z}_{\geq 0}$, and $\boldsymbol{a} \in \mathrm{c}_{\downarrow 0}(\mathbb{Z}_{\geq 0}; \mathbb{R}_{>0})$, and denote

$$p_K = \begin{cases} p^\infty_{K,k}, & \nu = \infty, \\ p^m_K, & \nu = m, \\ p^{m+\mathrm{lip}}_K, & \nu = m + \mathrm{lip}, \\ p^\omega_{K,\boldsymbol{a}}, & \nu = \omega. \end{cases}$$

For each $j \in J$, there then exists $g_j \in \mathrm{L}^1_{\mathrm{loc}}(\mathbb{T}_j; \mathbb{R}_{\geq 0})$ such that

$$p_K(X_{j,t}) \leq g_j(t), \qquad t \in \mathbb{T}_j.$$

Define $g\colon \mathbb{T} \to \mathbb{R}_{\geq 0}$ by asking that $g|\mathbb{T}_j = g_j$. We claim that $g \in \mathrm{L}^1_{\mathrm{loc}}(\mathbb{T}; \mathbb{R}_{\geq 0})$. Let $\mathbb{T}' \subseteq \mathbb{T}$ be a compact subinterval. The set

$$J_{\mathbb{T}'} = \{j \in J \mid \mathbb{T}' \cap \mathbb{T}_j \neq \varnothing\}.$$

is finite by local finiteness of the cover $(\mathbb{T}_j)_{j \in J}$. Now we have

$$\int_{\mathbb{T}'} g(t)\,\mathrm{d}t \leq \sum_{j \in J_{\mathbb{T}'}} \int_{\mathbb{T}_j} g_j(t)\,\mathrm{d}t < \infty.$$

Since

$$p_K(\overline{X}_t) \leq g(t), \qquad t \in \mathbb{T},$$

we conclude that $\overline{X} \in \mathrm{LI}\Gamma^\nu(\mathbb{T}; \mathsf{TM})$, as desired. ■

8.5. Attributes that can be given to tautological control systems. In this section we show that some typical assumptions that are made for control systems also can be made for tautological control systems. None of this is particularly earth-shattering, but it does serves as a plausibility check for our framework, letting us know that it has some common ground with familiar constructions from control theory.

A construction that often occurs in control theory is to determine a trajectory as the limit of a sequence of trajectories in some manner. To ensure the existence of such limits, the following property for tautological control systems is useful.

8.27 Definition: (Closed tautological control system) Let $m \in \mathbb{Z}_{\geq 0}$ and $m' \in \{0, \mathrm{lip}\}$, let $\nu \in \{m + m', \infty, \omega\}$, and let $r \in \{\infty, \omega\}$, as required. A C^ν-tautological control system $\mathfrak{G} = (\mathsf{M}, \mathscr{F})$ is ***closed*** if $\mathscr{F}(\mathcal{U})$ is closed in the topology of $\Gamma^\nu(\mathsf{T}\mathcal{U})$ for every open set $\mathcal{U} \subseteq \mathsf{M}$. •

Here are some examples of control systems that give rise to closed tautological control systems.

8.28 Proposition: (Control systems with closed tautological control systems) *Let $m \in \mathbb{Z}_{\geq 0}$ and $m' \in \{0, \mathrm{lip}\}$, let $\nu \in \{m+m', \infty, \omega\}$, and let $r \in \{\infty, \omega\}$, as required. Let $\Sigma = (\mathsf{M}, F, \mathcal{C})$ be a C^ν-control system with $\mathfrak{G}_\Sigma$ the associated C^ν-tautological control system as in Example 8.10–1. Then $\mathfrak{G}_\Sigma$ is closed if Σ has either of the following two attributes:*

(i) $\mathcal{C}$ is compact;

(ii) $\mathcal{C}$ is a closed subset of $\mathbb{R}^k$ and the system is control-affine, i.e.,

$$F(x, \boldsymbol{u}) = f_0(x) + \sum_{a=1}^{k} u^a f_a(x),$$

for $f_0, f_1, \dots, f_k \in \Gamma^\nu(\mathsf{TM})$.

Proof: (i) Let $\mathcal{U} \subseteq \mathsf{M}$ be open. By Propositions 7.2, 7.5, and 7.12, the map

$$\mathcal{C} \ni u \mapsto F^u \in \Gamma^\nu(\mathsf{T}\mathcal{U})$$

is continuous. Now let $\mathcal{U} \subseteq \mathsf{M}$ be open and note that $\mathscr{F}_\Sigma(\mathcal{U})$ is the image of $\mathcal{C}$ under the mapping

$$\mathcal{C} \ni u \mapsto F^u|\mathcal{U} \in \Gamma^\nu(\mathsf{T}\mathcal{U}).$$

Thus $\mathscr{F}_\Sigma(\mathcal{U})$ is compact, and so closed, being the image of a compact set under a continuous mapping [Willard 1970, Theorem 17.7].

(ii) Let $\mathcal{U} \subseteq \mathsf{M}$ be open. Just as in the preceding part of the proof, we consider the mapping $\boldsymbol{u} \mapsto F^{\boldsymbol{u}}|\mathcal{U}$. Note that the image of the mapping

$$\boldsymbol{u} \mapsto F^{\boldsymbol{u}} = f_0 + \sum_{a=1}^{k} u^a f_a$$

is a finite-dimensional affine subspace of the $\mathbb{R}$-vector space $\Gamma^\nu(\mathsf{T}\mathcal{U})$. Therefore, this image is closed since (1) locally convex topologies are translation invariant (by construction) and since (2) finite-dimensional subspaces of locally convex spaces are closed [Horváth 1966, Proposition 2.10.1]. Moreover, the map $\boldsymbol{u} \mapsto F^{\boldsymbol{u}}|\mathcal{U}$ is closed onto its image since any surjective linear map between finite-dimensional locally convex spaces is closed. We conclude, therefore, that if we restrict this map from all of $\mathbb{R}^k$ to $\mathcal{C}$, then the image is closed. ■

Let us next turn to attributes of tautological control systems arising from the fact, shown in Example 8.10–4, that tautological control systems give rise to differential inclusions in a natural way.

8.29 Proposition: (Continuity of differential inclusions arising from tautological control systems) *Let $m \in \mathbb{Z}_{\geq 0}$ and $m' \in \{0, \mathrm{lip}\}$, let $\nu \in \{m+m', \infty, \omega\}$, and let $r \in \{\infty, \omega\}$, as required. If $\mathfrak{G} = (\mathsf{M}, \mathscr{F})$ is a C^ν-tautological control system, then*

(i) $\mathscr{X}_\mathfrak{G}$ is lower semicontinuous and

(ii) $\mathscr{X}_\mathfrak{G}$ is upper semicontinuous if $\mathfrak{G}$ is globally generated and $\mathscr{F}(\mathsf{M})$ is compact.

Proof: (i) Let $x_0 \in \mathsf{M}$ and let $v_{x_0} \in \mathscr{X}_{\mathfrak{G}}(x_0)$. Then there exists a neighbourhood $\mathcal{W}$ of x_0 and $X \in \mathscr{F}(\mathcal{W})$ such that $X(x_0) = v_{x_0}$. Let $\mathcal{V} \subseteq \mathsf{TM}$ be a neighbourhood of v_{x_0}. By continuity of X, there exists a neighbourhood $\mathcal{U} \subseteq \mathcal{W}$ of x_0 such that $X(\mathcal{U}) \subseteq \mathcal{V}$. This implies that $X(x) \in \mathscr{X}_{\mathfrak{G}}(x)$ for every $x \in \mathcal{U}$, giving lower semicontinuity of $\mathscr{X}_{\mathfrak{G}}$.

(ii) Let $x_0 \in \mathsf{M}$ and let $\mathcal{V} \subseteq \mathsf{TM}$ be a neighbourhood of $\mathscr{X}_{\mathfrak{G}}(x_0)$. For each $X \in \mathscr{F}(\mathsf{M})$, $\mathcal{V}$ is a neighbourhood of $X(x_0)$ and so there exist neighbourhoods $\mathcal{M}_X \subseteq \mathsf{M}$ of x_0 and $\mathcal{C}_X \subseteq \mathscr{F}(\mathsf{M})$ of X such that

$$\{X'(x) \mid x \in \mathcal{M}_X,\ X' \in \mathcal{C}_X\} \subseteq \mathcal{V}.$$

Since $\mathscr{F}(\mathsf{M})$ is compact, let $X_1, \dots, X_k \in \mathscr{F}(\mathsf{M})$ be such that $\mathscr{F}(\mathsf{M}) = \cup_{j=1}^{k} \mathcal{C}_{X_j}$. Then the neighbourhood $\mathcal{U} = \cap_{j=1}^{k} \mathcal{M}_{X_j}$ of x_0 has the property that $\mathscr{X}_{\mathfrak{G}}(\mathcal{U}) \subseteq \mathcal{V}$. ■

There are many easy examples to illustrate that compactness of $\mathscr{F}(\mathsf{M})$ is generally required in part (ii) of the preceding result. Here is one.

8.30 Example: (A tautological control system with non-upper semicontinuous differential inclusion) Let $m \in \mathbb{Z}_{\geq 0}$ and $m' \in \{0, \mathrm{lip}\}$, let $\nu \in \{m + m', \infty, \omega\}$, and let $r \in \{\infty, \omega\}$, as required. Let M be a C^r-manifold and let $x_0 \in \mathsf{M}$. Let $\mathscr{F}(x_0)$ be the globally generated sheaf of sets of C^ν-vector fields defined by

$$\mathscr{F}(x_0)(\mathsf{M}) = \{X \in \Gamma^\nu(\mathsf{TM}) \mid X(x_0) = 0\}.$$

We claim that, if we take $\mathfrak{G} = (\mathsf{M}, \mathscr{F}(x_0))$, then we have

$$\mathscr{X}_{\mathfrak{G}}(x) = \begin{cases} \{0_{x_0}\}, & x = x_0, \\ \mathsf{T}_x\mathsf{M}, & x \neq x_0. \end{cases} \tag{8.1}$$

In the case $\nu = \infty$ or $\nu = m$, this is straightforward. Let $\mathcal{U}$ be a neighbourhood of $x \neq x_0$ such that $x_0 \notin \mathrm{cl}(\mathcal{U})$. By the smooth Tietze Extension Theorem [Abraham, Marsden, and Ratiu 1988, Proposition 5.5.8], if $X \in \Gamma^\infty(\mathsf{TM})$, then there exists $X' \in \Gamma^\infty(\mathsf{TM})$ such that $X'|\mathcal{U} = X|\mathcal{U}$ and such that $X'(x_0) = 0_{x_0}$. Thus $[X]_x = [X'_x]$ and so we have $\mathscr{F}(x_0)_x = \mathscr{G}^\nu_{x,\mathsf{M}}$ in this case. From this, (8.1) follows.

The case of $\nu = m + \mathrm{lip}$ follows as does the case $\nu = m$, noting that a locally Lipschitz vector field multiplied by a smooth function is still a locally Lipschitz vector field [Weaver 1999, Proposition 1.5.3].

The case of $\nu = \omega$ is a little more difficult, and relies on Cartan's Theorem A for coherent sheaves on real analytic manifolds [Cartan 1957]. Here is the argument for those who know a little about sheaves. First, define a sheaf of sets (in fact, submodules) of real analytic vector fields by

$$\mathscr{I}_{x_0}(\mathcal{U}) = \begin{cases} \{X \in \Gamma^\omega(\mathsf{T}\mathcal{U}) \mid X(x_0) = 0_{x_0}\}, & x_0 \in \mathcal{U}, \\ \Gamma^\omega(\mathsf{T}\mathcal{U}), & x_0 \notin \mathcal{U}. \end{cases}$$

We note that $\mathscr{I}_{x_0}$ is a coherent sheaf since it is a finitely generated subsheaf of the coherent sheaf $\mathscr{G}^\omega_{\mathsf{TM}}$ [Demailly 2012, Theorem 3.16].[12] Let $x \neq x_0$ and let $v_x \in \mathsf{T}_x\mathsf{M}$. By Cartan's

[12] This relies on the fact that Oka's Theorem, in the version of "the sheaf of sections of a vector bundle is coherent," holds in the real analytic case. It does, and the proof is the same as for the holomorphic case [Demailly 2012, Theorem 3.19] since the essential ingredient is the Weierstrass Preparation Theorem, which holds in the real analytic case [Krantz and Parks 2002, Theorem 6.1.3].

Theorem A, there exist $X_1, \dots, X_k \in \mathscr{I}_{x_0}(\mathsf{M}) = \mathscr{F}(x_0)(\mathsf{M})$ such that $[X_1]_x, \dots, [X_k]_x$ generate $(\mathscr{I}_{x_0})_x = \mathscr{G}^{\omega}_{x,\mathsf{TM}}$ as a module over the ring $\mathscr{C}^{\omega}_{x,\mathsf{M}}$ of germs of functions at x. Let $[X]_x \in \mathscr{G}^{\omega}_{x,\mathsf{TM}}$ be such that $X(x) = v_x$. There then exist $[f^1]_x, \dots, [f^k]_x \in \mathscr{C}^{\omega}_{x,\mathsf{M}}$ such that

$$[f^1]_x[X_1]_x + \dots + [f^k]_x[X_k]_x = [X]_x.$$

Therefore,

$$v_x = X(x) = f^1(x)X_1(x) + \dots + f^k(x)X_k(x),$$

and so, taking

$$X = f^1X_1 + \dots + f^kX_k \in \mathscr{I}_{x_0}(\mathsf{M}) = \mathscr{F}(x_0)(\mathsf{M}),$$

we see that $v_x = X(x) \in \mathscr{X}_{\mathfrak{G}}(x)$, which establishes (8.1) in this case.

In any event, (8.1) holds, and it is easy to see that this differential inclusion is not upper semicontinuous. •

We can make the following definitions, rather analogous to those of Definition 7.27 for differential inclusions.

8.31 Definition: (Attributes of tautological control systems coming from the associated differential inclusion) Let $m \in \mathbb{Z}_{\geq 0}$ and $m' \in \{0, \mathrm{lip}\}$, let $\nu \in \{m + m', \infty, \omega\}$, and let $r \in \{\infty, \omega\}$, as required. The C^{ν}-tautological control system $\mathfrak{G} = (\mathsf{M}, \mathscr{F})$ is:

(i) ***closed-valued*** (resp. ***compact-valued***, ***convex-valued***) at $x \in \mathsf{M}$ if $\mathscr{X}_{\mathfrak{G}}(x)$ is closed (resp., compact, convex);

(ii) ***closed-valued*** (resp. ***compact-valued***, ***convex-valued***) if $\mathscr{X}_{\mathfrak{G}}(x)$ is closed (resp., compact, convex) for every $x \in \mathsf{M}$. •

One can now talk about taking "hulls" under various properties. Let us discuss this for the properties of closedness and convexity. First we need the definitions we will use.

8.32 Definition: (Convex hull, closure of a tautological control system) Let $m \in \mathbb{Z}_{\geq 0}$ and $m' \in \{0, \mathrm{lip}\}$, let $\nu \in \{m + m', \infty, \omega\}$, and let $r \in \{\infty, \omega\}$, as required. Let $\mathfrak{G} = (\mathsf{M}, \mathscr{F})$ be a C^{ν}-tautological control system.

(i) The ***convex hull*** of $\mathfrak{G}$ is the C^{ν}-tautological control system $\mathrm{conv}(\mathfrak{G}) = (\mathsf{M}, \mathrm{conv}(\mathscr{F}))$, where $\mathrm{conv}(\mathscr{F})$ is the presheaf of subsets of C^{ν}-vector fields given by

$$\mathrm{conv}(\mathscr{F})(\mathcal{U}) = \mathrm{conv}(\mathscr{F}(\mathcal{U})),$$

the convex hull on the right being that in the $\mathbb{R}$-vector space $\Gamma^{\nu}(\mathsf{T}\mathcal{U})$.

(ii) The ***closure*** of $\mathfrak{G}$ is the C^{ν}-tautological control system

$$\mathrm{cl}(\mathfrak{G}) = (\mathsf{M}, \mathrm{cl}(\mathscr{F})),$$

where $\mathrm{cl}(\mathscr{F})$ is the presheaf of subsets of C^{ν}-vector fields given by $\mathrm{cl}(\mathscr{F})(\mathcal{U}) = \mathrm{cl}(\mathscr{F}(\mathcal{U}))$, the closure on the right being that in the $\mathbb{R}$-topological vector space $\Gamma^{\nu}(\mathsf{T}\mathcal{U})$. The reader should verify that $\mathrm{cl}(\mathscr{F})$ is indeed a presheaf. •

Let us now relate the two different sorts of "hulls" we have.

8.33 Proposition: (Convex hull and closure commute with taking differential inclusions) *Let $m \in \mathbb{Z}_{\geq 0}$ and $m' \in \{0, \mathrm{lip}\}$, let $\nu \in \{m+m', \infty, \omega\}$, and let $r \in \{\infty, \omega\}$, as required. Let $\mathfrak{G} = (\mathsf{M}, \mathscr{F})$ be a C^ν-tautological control system with $\mathscr{X}_{\mathfrak{G}}$ the associated differential inclusion. Then the following statements hold:*

(i) $\mathrm{conv}(\mathscr{X}_{\mathfrak{G}}) = \mathscr{X}_{\mathrm{conv}(\mathfrak{G})}$;

(ii) $\mathscr{X}_{\mathrm{cl}(\mathfrak{G})} \subseteq \mathrm{cl}(\mathscr{X}_{\mathfrak{G}})$ *and* $\mathscr{X}_{\mathrm{cl}(\mathfrak{G})} = \mathrm{cl}(\mathscr{X}_{\mathfrak{G}})$ *if* $\mathfrak{G}$ *is globally generated and* $\mathscr{F}(\mathsf{M})$ *is bounded in the compact bornology (or, equivalently, the von Neumann bornology if* $\nu \in \{\infty, \omega\}$*).*

Proof: (i) Let $x \in \mathsf{M}$. If $v \in \mathrm{conv}(\mathscr{X}_{\mathfrak{G}}(x))$, then there exist $v_1, \dots, v_k \in \mathscr{X}_{\mathfrak{G}}(x)$ and $c_1, \dots, c_k \in [0,1]$ satisfying $\sum_{j=1}^k c_j = 1$ such that

$$v = c_1 v_1 + \dots + c_k v_k.$$

Let $\mathcal{U}_1, \dots, \mathcal{U}_k$ be neighbourhoods of x and let $X_j \in \mathscr{F}(\mathcal{U}_j)$ be such that $X_j(x) = v_j$, $j \in \{1, \dots, k\}$. Then, taking $\mathcal{U} = \cap_{j=1}^k \mathcal{U}_j$,

$$c_1 X_1|\mathcal{U} + \dots + c_k X_k|\mathcal{U} \in \mathrm{conv}(\mathscr{F}(\mathcal{U})),$$

showing that $\mathrm{conv}(\mathscr{X}_{\mathfrak{G}}(x)) \subseteq \mathscr{X}_{\mathrm{conv}(\mathfrak{G})}(x)$.

Conversely, let $v \in \mathscr{X}_{\mathrm{conv}(\mathfrak{G})}$, let $\mathcal{U}$ be a neighbourhood of x, and let $X \in \mathrm{conv}(\mathscr{F}(\mathcal{U}))$ be such that $X(x) = v$. Then

$$X = c_1 X_1 + \dots + c_k X_k$$

for $X_1, \dots, X_k \in \mathscr{F}(\mathcal{U})$ and for $c_1, \dots, c_k \in [0,1]$ satisfying $\sum_{j=1}^k c_j = 1$. We then have

$$v = c_1 X_1(x) + \dots + c_k X_k(x) \in \mathrm{conv}(\mathscr{X}_{\mathfrak{G}})(x),$$

completing the proof of the proposition as concerns convex hulls.

(ii) Let $x \in \mathsf{M}$, let $v \in \mathscr{X}_{\mathrm{cl}(\mathfrak{G})}(x)$, let $\mathcal{U}$ be a neighbourhood of x, and let $X \in \mathrm{cl}(\mathscr{F}(\mathcal{U}))$ be such that $X(x) = v$. Let $(I, \preceq)$ be a directed set and let $(X_i)_{i \in I}$ be a net in $\mathscr{F}(\mathcal{U})$ converging to X in the appropriate topology. Then we have $\lim_{i \in I} X_i(x) = X(x)$ since the net $(X_i)_{i \in I}$ converges uniformly in some neighbourhood of x (this is true for all cases of ν). Thus $v \in \mathrm{cl}(\mathscr{X}_{\mathfrak{G}}(x))$, as desired.

Suppose that $\mathscr{F}$ is globally generated with $\mathscr{F}(\mathsf{M})$ bounded, let $x \in \mathsf{M}$, and let $v \in \mathrm{cl}(\mathscr{X}_{\mathfrak{G}})(x)$. Thus there exists a sequence $(v_j)_{j \in \mathbb{Z}_{>0}}$ in $\mathscr{X}_{\mathfrak{G}}(x)$ converging to v. Let $X_j \in \mathscr{F}(\mathsf{M})$ be such that $X_j(x) = v_j$, $j \in \mathbb{Z}_{>0}$. Since $\mathrm{cl}(\mathscr{F}(\mathsf{M}))$ is compact, there is a subsequence $(X_{j_k})_{j_k}$ in $\mathscr{F}(\mathsf{M})$ converging to $X \in \mathrm{cl}(\mathscr{F}(\mathsf{M}))$. Moreover,

$$X(x) = \lim_{k \to \infty} X_{j_k}(x) = \lim_{j \to \infty} v_j = v$$

since $(X_{j_k})_{k \in \mathbb{Z}_{>0}}$ converges to X uniformly in some neighbourhood of x (again, this is true for all ν). Thus $v \in \mathscr{X}_{\mathrm{cl}(\mathfrak{G})}(x)$.

The parenthetical comment in the final assertion of the proof follows since the compact and von Neumann bornologies agree for nuclear spaces [Pietsch 1969, Proposition 4.47]. ■

The following example shows that the opposite inclusion stated in the proposition for closures does not generally hold.

8.34 Example: (A tautological control system for which the closure of the differential inclusion is not the differential inclusion of the closure) We will talk our way through a general sort of example, leaving to the reader the job of instantiating this to give a concrete example.

Let $m \in \mathbb{Z}_{>0}$ and $m' \in \{0, \mathrm{lip}\}$, let $\nu \in \{m+m', \infty, \omega\}$, and let $r \in \{\infty, \omega\}$, as required. Let M be a C^r-manifold. Let $x \in \mathsf{M}$ and let $(X_j)_{j\in\mathbb{Z}_{>0}}$ be a sequence of C^ν-vector fields with the following properties:

1. $(X_j(x))_{j\in\mathbb{Z}_{>0}}$ converges to 0_x;
2. $X_j(x) \neq 0_x$ for all $j \in \mathbb{Z}_{>0}$;
3. there exists a neighbourhood $\mathcal{O}$ of zero in $\Gamma^\nu(\mathsf{TM})$ such that, for each $j \in \mathbb{Z}_{>0}$,

$$\{k \in \mathbb{Z}_{>0} \setminus \{j\} \mid X_k - X_j \in \mathcal{O}\} = \varnothing.$$

Let $\mathscr{F}$ be the globally generated presheaf of sets of C^ν-vector fields given by $\mathscr{F}(\mathsf{M}) = \{X_j \mid j \in \mathbb{Z}_{>0}\}$. Then $0_x \in \mathrm{cl}(\mathscr{X}_{\mathfrak{G}}(x))$. We claim that $0_x \notin \mathscr{X}_{\mathrm{cl}(\mathfrak{G})}(x)$. To see this, suppose that $0_x \in \mathscr{X}_{\mathrm{cl}(\mathfrak{G})}(x)$. Since $\mathscr{F}(\mathsf{M})$ is countable, this implies that there is a subsequence $(X_{j_k})_{k\in\mathbb{Z}_{>0}}$ that converges in $\Gamma^\nu(\mathsf{TM})$. But this is prohibited by the construction of the sequence $(X_j)_{j\in\mathbb{Z}_{>0}}$. •

8.6. Trajectory correspondence between tautological control systems and other sorts of control systems. In Example 8.10 and Proposition 8.11 we made precise the connections between various models for control systems: control systems, differential inclusions, and tautological control systems. In order to flesh out these connections more deeply, in this section we investigate the possible correspondences between the trajectories for the various models.

We first consider correspondences between trajectories of control systems and their associated tautological control systems. Thus we let $m \in \mathbb{Z}_{\geq 0}$ and $m' \in \{0, \mathrm{lip}\}$, let $\nu \in \{m+m', \infty, \omega\}$, and let $r \in \{\infty, \omega\}$, as required. Let $\Sigma = (\mathsf{M}, F, \mathcal{C})$ be a C^ν-control system with $\mathfrak{G}_\Sigma$ the associated C^ν-tautological control system, as in Example 8.10–1. As we saw in Proposition 8.11(ii), the correspondence between Σ and $\mathfrak{G}_\Sigma$ is perfect, at the system level, when the map $u \mapsto F^u$ is injective and open onto its image. Part (ii) of the following result shows that this perfect correspondence almost carries over at the level of trajectories as well. Included with this statement we include a few other related ideas concerning trajectory correspondences.

8.35 Theorem: (Correspondence between trajectories of a control system and its associated tautological control system) *Let $m \in \mathbb{Z}_{\geq 0}$ and $m' \in \{0, \mathrm{lip}\}$, let $\nu \in \{m+m', \infty, \omega\}$, and let $r \in \{\infty, \omega\}$, as required. Let $\Sigma = (\mathsf{M}, F, \mathcal{C})$ be a C^ν-control system with $\mathfrak{G}_\Sigma$ the associated C^ν-tautological control system, as in Example 8.10–1. Then the following statements hold:*

(i) $\mathrm{Traj}(\mathbb{T}, \mathcal{U}, \Sigma) \subseteq \mathrm{Traj}(\mathbb{T}, \mathcal{U}, \mathscr{O}_{\mathfrak{G}_\Sigma,\mathrm{cpt}})$;

(ii) if the map $u \mapsto F^u$ is injective and proper, then $\mathrm{Traj}(\mathbb{T}, \mathcal{U}, \mathscr{O}_{\mathfrak{G}_\Sigma,\mathrm{cpt}}) \subseteq \mathrm{Traj}(\mathbb{T}, \mathcal{U}, \Sigma)$;

(iii) if $\mathcal{C}$ is a Suslin topological space[13] *and if F is proper, then* $\mathrm{Traj}(\mathbb{T}, \mathcal{U}, \mathscr{O}_{\mathfrak{G}_\Sigma,\infty}) \subseteq \mathrm{Traj}(\mathbb{T}, \mathcal{U}, \Sigma)$.

[13] Recall that this means that $\mathcal{C}$ is the continuous image of a complete, separable, metric space.

(iv) if, in addition, $\nu \in \{\infty, \omega\}$, then we may replace $\mathrm{Traj}(\mathbb{T}, \mathcal{U}, \mathscr{O}_{\mathfrak{G}_\Sigma,\mathrm{cpt}})$ *with* $\mathrm{Traj}(\mathbb{T}, \mathcal{U}, \mathscr{O}_{\mathfrak{G}_\Sigma,\infty})$ *in statements (i), (ii), and (iii).*

Proof: (i) Let $\xi \in \mathrm{Traj}(\mathbb{T}, \mathcal{U}, \Sigma)$ and let $\mu \in \mathrm{L}^\infty_{\mathrm{loc}}(\mathbb{T}; \mathcal{C})$ be such that

$$\xi'(t) = F(\xi(t), \mu(t)), \qquad \text{a.e. } t \in \mathbb{T}.$$

Note that, as we saw in Example 8.15, $F^\mu|\mathcal{U} \in \mathscr{O}_{\mathfrak{G}_\Sigma,\infty}(\mathbb{T}, \mathcal{U})$, making sure to note that the conclusions of Proposition 7.18 imply that $F^\mu \in \mathrm{LB}\Gamma^\nu(\mathbb{T}; \mathsf{TM})$. Thus $\xi \in \mathrm{Traj}(\mathbb{T}, \mathcal{U}, \mathscr{O}_{\mathfrak{G}_\Sigma,\infty})$. To show that, in fact, $\xi \in \mathrm{Traj}(\mathbb{T}, \mathcal{U}, \mathscr{O}_{\mathfrak{G}_\Sigma,\mathrm{cpt}})$, let $\mathbb{T}' \subseteq \mathbb{T}$ be a compact subinterval and let $K \subseteq \mathcal{C}$ be a compact set such that $\mu(t) \in K$ for almost every $t \in \mathbb{T}'$. Denote

$$\begin{aligned} \hat{F} \colon \mathcal{C} &\to \Gamma^\nu(\mathsf{TM}) \\ u &\mapsto F^u. \end{aligned}$$

Since $\hat{F}$ is continuous, $\hat{F}(K)$ is compact [Willard 1970, Theorem 17.7]. Since $F^\mu_t \in \hat{F}(K)$ for almost every $t \in \mathbb{T}'$, we conclude that $\xi \in \mathrm{Traj}(\mathbb{T}, \mathcal{U}, \mathscr{O}_{\mathfrak{G}_\Sigma,\mathrm{cpt}})$, as claimed.

(ii) Recall from [Bourbaki 1989b, Proposition I.10.2] that, if $\hat{F}$ (as defined above) is proper, then it has a closed image, and is a homeomorphism onto its image. If $\xi \in \mathrm{Traj}(\mathbb{T}, \mathcal{U}, \mathscr{O}_{\mathfrak{G}_\Sigma,\mathrm{cpt}})$, then there exists $X \in \mathscr{O}_{\mathfrak{G}_\Sigma,\mathrm{cpt}}$ such that $\xi'(t) = X(t, \xi(t))$ for almost every $t \in \mathbb{T}$. Note that, since $X \in \mathscr{O}_{\mathfrak{G}_\Sigma,\mathrm{cpt}}$, we have $X(t) \in \mathscr{F}_\Sigma(\mathsf{M}) = \mathrm{image}(\hat{F})$. Thus, by hypothesis, there exists a unique $\mu \colon \mathbb{T} \to \mathcal{C}$ such that $\hat{F} \circ \mu = X$. To show that μ is measurable, let $\mathcal{O} \subseteq \mathcal{C}$ be open so that $\hat{F}(\mathcal{O})$ is an open subset of $\mathrm{image}(\hat{F})$. Thus there exists an open set $\mathcal{O}' \subseteq \Gamma^\nu(\mathsf{TM})$ such that $\hat{F}(\mathcal{O}) = \mathrm{image}(\hat{F}) \cap \mathcal{O}'$. Then we have

$$\mu^{-1}(\mathcal{O}) = X^{-1}(\hat{F}(\mathcal{O})) = X^{-1}(\mathcal{O}'),$$

giving the desired measurability. To show that $\mu \in \mathrm{L}^\infty_{\mathrm{loc}}(\mathbb{T}; \mathcal{C})$, let $\mathbb{T}' \subseteq \mathbb{T}$ be a compact subinterval and let $K \subseteq \Gamma^\nu(\mathsf{TM})$ be such that $X(t) \in K$ for almost every $t \in \mathbb{T}'$. Then, since $\hat{F}$ is proper, $\hat{F}^{-1}(K)$ is a compact subset of $\mathcal{C}$. Since $\mu(t) \in \hat{F}^{-1}(K)$ for almost every $t \in \mathbb{T}'$ we conclude that $\mu \in \mathrm{L}^\infty_{\mathrm{loc}}(\mathbb{T}; \mathcal{C})$.

(iii) Let $\xi \in \mathrm{Traj}(\mathbb{T}, \mathcal{U}, \mathscr{O}_{\mathfrak{G}_\Sigma,\infty})$ and let $X \in \mathscr{O}_{\mathfrak{G}_\Sigma,\infty}(\mathbb{T}, \mathcal{U})$ be such that $\xi'(t) = X(t, \xi(t))$ for almost every $t \in \mathbb{T}$. We wish to construct $\mu \in \mathrm{L}^\infty_{\mathrm{loc}}(\mathbb{T}, \mathcal{C})$ such that

$$\xi' = F(\xi(t), \mu(t)), \qquad \text{a.e. } t \in \mathbb{T}.$$

We fix an arbitrary element $\bar{u} \in \mathcal{C}$ (it matters not which) and then define a set-valued map $U \colon \mathbb{T} \twoheadrightarrow \mathcal{C}$ by

$$U(t) = \begin{cases} \{u \in \mathcal{C} \mid \ \xi'(t) = F(\xi(t), u)\}, & \xi'(t) \text{ exists}, \\ \{\bar{u}\}, & \text{otherwise}. \end{cases}$$

Since $X(t) \in \mathscr{F}_\Sigma(\mathsf{M})$, we conclude that $X(t) \in \mathrm{image}(\hat{F})$ for every $t \in \mathbb{T}$, i.e., $X(t) = F^u$ for some $u \in \mathcal{C}$, and so $U(t) \neq \varnothing$ for every $t \in \mathbb{T}$.

Properness of F ensures that $U(t)$ is compact for every $t \in \mathbb{T}$. The following lemma shows that *any* selection μ of U is locally essentially bounded in the compact bornology.

1 Lemma: *If $\mathbb{T}' \subseteq \mathbb{T}$ is a compact subinterval, then the set $\cup\{U(t) \mid t \in \mathbb{T}'\}$ is contained in a compact subset of $\mathcal{C}$.*

Proof: Let us define $F_\xi \colon \mathbb{T} \times \mathcal{C} \to \mathsf{TM}$ by $F_\xi(t, u) = F(\xi(t), u)$. We claim that, if $\mathbb{T}' \subseteq \mathbb{T}$ is compact, then $F_\xi|\mathbb{T}' \times \mathcal{C}$ is proper. To see this, first define

$$\begin{aligned} G_\xi \colon \mathbb{T}' \times \mathcal{C} &\to \mathsf{M} \times \mathcal{C} \\ (t, u) &\mapsto (\xi(t), u), \end{aligned}$$

i.e., $G_\xi = \xi \times \mathrm{id}_\mathcal{C}$. With this notation, we have $F_\xi = F \circ G_\xi$. Since $F_\xi^{-1}(K) = G_\xi^{-1}(F^{-1}(K))$ and since F is proper, to show that F_ξ is proper it suffices to show that G_ξ is proper. Let $K \subseteq \mathsf{M} \times \mathcal{C}$ be compact. We let $\mathrm{pr}_1 \colon \mathsf{M} \times \mathcal{C} \to \mathsf{M}$ and $\mathrm{pr}_2 \colon \mathsf{M} \times \mathcal{C} \to \mathcal{C}$ be the projections. Note that

$$G_\xi^{-1}(K) = (\xi \times \mathrm{id}_\mathcal{C})^{-1}(K) \subseteq \xi^{-1}(\mathrm{pr}_1(K)) \times \mathrm{id}_\mathcal{C}^{-1}(\mathrm{pr}_2(K)).$$

Since the projections are continuous, $\mathrm{pr}_1(K)$ and $\mathrm{pr}_2(K)$ are compact [Willard 1970, Theorem 17.7]. Since ξ is a continuous function whose domain (for our present purposes) is the compact set $\mathbb{T}'$, $\xi^{-1}(\mathrm{pr}_1(K))$ is compact. Since the identity map is proper, $\mathrm{id}_\mathcal{C}^{-1}(\mathrm{pr}_2(K))$ is compact. Thus $G_\xi^{-1}(K)$ is contained in a product of compact sets. Since a product of compact sets is compact [Willard 1970, Theorem 17.8] and $G_\xi^{-1}(K)$ is closed by continuity of G_ξ, it follows that $G_\xi^{-1}(K)$ is compact, as claimed. Thus $F_\xi|\mathbb{T}' \times \mathcal{C}$ is proper.

Now, since ξ is a trajectory for the $\mathscr{O}_{\mathfrak{G}_\Sigma,\infty}$ open-loop subfamily, there exists a compact set $K' \subseteq \mathsf{TM}$ such that

$$\{\xi'(t) \mid t \in \mathbb{T}'\} \subseteq K',$$

adopting the convention that $\xi'(t)$ is taken to satisfy $\xi'(t) = F(\xi(t), \bar{u})$ when $\xi'(t)$ does not exist; this is an arbitrary and inconsequential choice. By our argument above, $K'' \triangleq (F_\xi|\mathbb{T}' \times \mathcal{C})^{-1}(K')$ is compact. Therefore, for each $t \in \mathbb{T}'$,

$$\begin{aligned} \{(t, u) \in \mathbb{T}' \times \mathcal{C} \mid u \in U(t)\} &= \{(t, u) \in \mathbb{T}' \times \mathcal{C} \mid F(\xi(t), u) = \xi'(t)\} \\ &\subseteq \{(t, u) \in \mathbb{T}' \times \mathcal{C} \mid F(\xi(t), u) \in K'\} \subseteq K''. \end{aligned}$$

Defining the compact set (compact by [Willard 1970, Theorem 17.7]) $K = \mathrm{pr}_2(K'')$, with $\mathrm{pr}_2 \colon \mathbb{T}' \times \mathcal{C} \to \mathcal{C}$ the projection, we then have

$$\cup\{U(t) \mid t \in \mathbb{T}'\} \subseteq K.$$ ▼

We shall now make a series of observations about the set-valued map U, using results of Himmelberg [1975] on measurable set-valued mappings, particularly with values in Suslin spaces.

2 Lemma: *The set-valued map U is measurable, i.e., if $\mathcal{O} \subseteq \mathcal{C}$ is open, then*

$$U^{-1}(\mathcal{O}) = \{t \in \mathbb{T} \mid U(t) \cap \mathcal{O} \neq \varnothing\}$$

is measurable.

Proof: Define

$$\begin{aligned} F_\xi \colon \mathbb{T} \times \mathcal{C} &\to \mathsf{TM} \\ (t, u) &\mapsto F(\xi(t), u), \end{aligned}$$

noting that $t \mapsto F_\xi(t, u)$ is measurable for each $u \in \mathcal{C}$ and that $u \mapsto F_\xi(t, u)$ is continuous for every $t \in \mathbb{T}$. It follows from [Himmelberg 1975, Theorem 6.4] that U is measurable as stated. ▼

3 Lemma: *There exists a measurable function $\mu\colon \mathbb{T} \to \mathcal{C}$ such that $\mu(t) \in U(t)$ for almost every $t \in \mathbb{T}$.*

Proof: First note that $U(t)$ is a closed subset of $\mathcal{C}$ since it is either the singleton $\{\bar{u}\}$ or the preimage of the closed set $\{\xi'(t)\}$ under the continuous map $u \mapsto F(\xi(t), u)$. It follows from [Himmelberg 1975, Theorem 3.5] that

$$\operatorname{graph}(U) = \{(t,u) \in \mathbb{T} \times \mathcal{C} \mid \ u \in U(t)\}$$

is measurable with respect to the product σ-algebra of the Lebesgue measurable sets in $\mathbb{T}$ and the Borel sets in $\mathcal{C}$. The lemma now follows from [Himmelberg 1975, Theorem 5.7]. ▼

Now, for $t \in \mathbb{T}$ having the property that $\xi'(t)$ exists and that $\mu(t) \in U(t)$ (with μ from the preceding lemma), we have $\xi'(t) = F(\xi(t), \mu(t))$.

(iv) This follows by our observation of Example 8.18–3. ■

Let us make some comments on the hypotheses of the preceding theorem.

8.36 Remarks: (Trajectory correspondence between control systems and tautological control systems)

1. Part (ii) of the result has assumptions that the map $u \mapsto F^u$ be injective and proper. An investigation of the proof shows that injectivity and openness onto the image of this map are enough to give trajectories for Σ that correspond to measurable controls. The additional assumption of properness, which gives the further consequence of the image of the map $u \mapsto F^u$ being closed, allows us to conclude boundedness of the controls. Let us look at these assumptions.
   (a) By the map $u \mapsto F^u$ being injective, we definitely do not mean that the map $u \mapsto F(x,u)$ is injective for each $x \in \mathsf{M}$; this is a very strong assumption whose adoption eliminates a large number of interesting control systems. For example, if we take $\mathsf{M} = \mathbb{R}$, $\mathcal{C} = \mathbb{R}$, and $F(x,u) = ux\frac{\partial}{\partial x}$ to define a C^ν-control system for any $\nu \in \{m+m', \infty, \omega\}$ with $m \in \mathbb{Z}_{\geq 0}$ and $m' \in \{0, \mathrm{lip}\}$, then the map $u \mapsto F^u$ is injective, but the map $u \mapsto F(0,u)$ is not.
   (b) Let us take $\mathsf{M} = \mathbb{R}^2$, $\mathcal{C} = \mathbb{R}$, and

$$F((x_1,x_2),u) = f_1(u)\frac{\partial}{\partial x_1} + f_2(u)\frac{\partial}{\partial x_2},$$

   where $f_1, f_2\colon \mathbb{R} \to \mathbb{R}$ are such that the map $u \mapsto (f_1(u), f_2(u))$ is injective and continuous, but not a homeomorphism onto its image. Such a system may be verified to be a C^ν-control system for any $\nu \in \{m+m', \infty, \omega\}$ with $m \in \mathbb{Z}_{\geq 0}$ and $m' \in \{0, \mathrm{lip}\}$ (using Propositions 7.2, 7.5, and 7.12). In this case, we claim that the map $\hat{F}\colon u \mapsto F^u$ is injective and continuous, but not a homeomorphism onto its image. Injectivity of the map is clear and continuity follows since F is a jointly parameterised vector field of class C^ν. Define a linear map

$$\begin{aligned} \kappa\colon \mathbb{R}^2 &\to \Gamma^\nu(\mathsf{TM}) \\ (v_1, v_2) &\mapsto v_1\frac{\partial}{\partial x_1} + v_2\frac{\partial}{\partial x_2}, \end{aligned}$$

i.e., $\kappa(\boldsymbol{v})$ is the constant vector field with components (v_1, v_2). Using the seminorms for our locally convex topologies and the standard seminorm characterisations of continuous linear maps (as in [Schaefer and Wolff 1999, §III.1.1]), we can easily see that κ is a continuous linear map, and so is a homeomorphism onto its closed image (arguing as in the proof of Proposition 8.28(ii)). Then $\hat{F} = \kappa \circ (f_1 \times f_2)$, and so we conclude that $\hat{F}$ is a homeomorphism onto its image if and only if $f_1 \times f_2$ is a homeomorphism onto its image, and this gives our claim.

(c) Let us take $\mathsf{M} = \mathbb{R}$, $\mathcal{C} = \mathbb{R}$, and $F(x, u) = \tan^{-1}(u)\frac{\partial}{\partial x}$. As with the examples above, we regard this as a control system of class C^ν for any $\nu \in \{m + m', \infty, \omega\}$, for $m \in \mathbb{Z}_{\geq 0}$ and $m' \in \{0, \mathrm{lip}\}$. We claim that $\hat{F} \colon u \mapsto F^u$ is a homeomorphism onto its image, but is not proper. This is verified in exactly the same manner as in the preceding example.

(d) If $\mathcal{C}$ is compact, then $\hat{F}$ is proper because, if $K \subseteq \Gamma^\nu(\mathsf{TM})$ is compact, then $\hat{F}^{-1}(K)$ is closed, and so compact [Willard 1970, Theorem 17.5]. This gives trajectory correspondence between a C^ν-control system and its corresponding tautological control system for compact control sets when the map $\hat{F}$ is injective.

2. Part (iii) of the result has two assumptions, that $\mathcal{C}$ is a Suslin space and that F is proper. Let us consider some cases where these hypotheses hold.

   (a) Complete separable metric spaces are Suslin spaces.

   (b) If $\mathcal{C}$ is an open or a closed subspace of Suslin space, it is a Suslin space [Bogachev 2007, Lemma 6.6.5(ii)].

   (c) For $m \in \mathbb{Z}_{\geq 0}$, $m' \in \{0, \mathrm{lip}\}$, and $\nu \in \{m + m', \infty, \omega\}$, $\Gamma^\nu(\mathsf{TM})$ is a Suslin space. In all except the case of $\nu = \omega$, this follows since $\Gamma^\nu(\mathsf{TM})$ is a separable, complete, metrisable space. However, $\Gamma^\omega(\mathsf{TM})$ is not metrisable. Nonetheless, it is Suslin, as argued in Section 5.3.

   (d) If $\mathcal{C}$ is compact, then F is proper. Indeed, if $K \subseteq \mathsf{TM}$ is compact, then $\pi_{\mathsf{TM}}(K)$ is compact, and
$$F^{-1}(K) \subseteq \pi_{\mathsf{TM}}(K) \times \mathcal{C},$$
and so the set on the left is compact, being a closed subset of a compact set [Willard 1970, Theorem 17.5]. •

We also have a version of the preceding theorem in the case that the control set $\mathcal{C}$ is a subset of a locally convex topological vector space, cf. Proposition 7.22. Here we also specialise for one of the implications to control-linear systems introduced in Example 7.21.

8.37 Theorem: (Correspondence between trajectories of a control-linear system and its associated tautological control system) *Let $m \in \mathbb{Z}_{\geq 0}$ and $m' \in \{0, \mathrm{lip}\}$, let $\nu \in \{m + m', \infty, \omega\}$, and let $r \in \{\infty, \omega\}$, as required. Let $\Sigma = (\mathsf{M}, F, \mathcal{C})$ be a C^ν-sublinear control system for which $\mathcal{C}$ is a subset of a locally convex topological vector space V, and let $\mathfrak{G}_\Sigma$ be the associated C^ν-tautological control system, as in Example 8.10–1. If $\mathbb{T}$ is a time-domain and if $\mathcal{U}$ is open, then $\mathrm{Traj}(\mathbb{T}, \mathcal{U}, \Sigma) \subseteq \mathrm{Traj}(\mathbb{T}, \mathcal{U}, \mathscr{O}_{\mathfrak{G}_\Sigma,\mathrm{full}})$.*

Conversely, if

(i) Σ is a C^ν-control-linear system, i.e., there exists $\Lambda \in \mathrm{L}(\mathsf{V}; \Gamma^\nu(\mathsf{TM}))$ such that $F(x, u) = \Lambda(u)(x)$,

(ii) Λ is injective, and

(iii) Λ is an open mapping onto its image,

then it is also the case that $\mathrm{Traj}(\mathbb{T},\mathcal{U},\mathscr{O}_{\mathfrak{G}_\Sigma,\mathrm{full}}) \subseteq \mathrm{Traj}(\mathbb{T},\mathcal{U},\Sigma)$.

Proof: We first show that $\mathrm{Traj}(\mathbb{T},\mathcal{U},\Sigma) \subseteq \mathrm{Traj}(\mathbb{T},\mathcal{U},\mathscr{O}_{\mathfrak{G}_\Sigma,\mathrm{full}})$. Suppose that $\xi \in \mathrm{Traj}(\mathbb{T},\mathcal{U},\Sigma)$. Thus there exists $\mu \in \mathrm{L}^1_{\mathrm{loc}}(\mathbb{T};\mathcal{C})$ such that

$$\xi'(t) = F(\xi(t),\mu(t)), \qquad \text{a.e. } t \in \mathbb{T}.$$

By Proposition 7.22 and Example 8.15, $F^\mu|\mathcal{U} \in \mathscr{O}_{\mathfrak{G}_\Sigma,\mathrm{full}}(\mathbb{T},\mathcal{U})$ and so $\xi \in \mathrm{Traj}(\mathbb{T},\mathcal{U},\mathscr{O}_{\mathfrak{G}_\Sigma,\mathrm{full}})$.

Now let us prove the "conversely" assertion of the theorem. Thus we let $\xi \in \mathrm{Traj}(\mathbb{T},\mathcal{U},\mathscr{O}_{\mathfrak{G}_\Sigma,\mathrm{full}})$ so that there exists $X \in \mathscr{O}_{\mathfrak{G}_\Sigma,\mathrm{full}}(\mathbb{T},\mathcal{U})$ for which $\xi'(t) = X(t,\xi(t))$ for almost every $t \in \mathbb{T}$. Since Λ is injective and since $X_t \in \Lambda(\mathcal{C})$ for each $t \in \mathbb{T}$ (this is the definition of $\mathfrak{G}_\Sigma$), we uniquely define $\mu(t) \in \mathcal{C}$ by $\Lambda(\mu(t)) = X_t$. We need only show that μ is locally Bochner integrable. Let Λ^{-1} denote the inverse of Λ, thought of as a map from image(Λ) to V. As Λ is open, Λ^{-1} is continuous. From this, measurability of μ follows immediately. To show that μ is locally Bochner integrable, let q be a continuous seminorm for the locally convex topology of V and, as per [Schaefer and Wolff 1999, §III.1.1], let p be a continuous seminorm for the locally convex topology of $\Gamma^\nu(\mathsf{TM})$ such that $q(\Lambda^{-1}(Y)) \le p(Y)$ for every $Y \in \Gamma^\nu(\mathsf{TM})$. Then we have, for any compact subinterval $\mathbb{T}' \subseteq \mathbb{T}$,

$$\int_{\mathbb{T}'} q(\mu(t))\,\mathrm{d}t \le \int_{\mathbb{T}'} p(X_t)\,\mathrm{d}t < \infty,$$

giving Bochner integrability of μ by [Beckmann and Deitmar 2011, Theorems 3.2 and 3.3]. ■

Let us make some observations about the preceding theorem.

8.38 Remarks: (Trajectory correspondence between control systems and tautological control systems) The converse part of Theorem 8.37 has three hypotheses: that the system is control-linear; that the map from controls to vector fields is injective; that the map from controls to vector fields is open onto its image. The first hypothesis, linearity of the system, cannot be weakened except in sort of artificial ways. As can be seen from the proof, linearity allows us to talk about the integrability of the associated control. Injectivity can be assumed without loss of generality by quotienting out the kernel if it is not. Let us consider some cases where the third hypothesis holds. Let $m \in \mathbb{Z}_{\ge 0}$ and $m' \in \{0, \mathrm{lip}\}$, let $\nu \in \{m+m', \infty, \omega\}$, and let $r \in \{\infty, \omega\}$, as required.

1. Let $\mathcal{C} \subseteq \mathbb{R}^k$ and suppose that our system is C^ν-control-affine, i.e.,

$$F(x,\boldsymbol{u}) = f_0(x) + \sum_{a=1}^k u^a f_a(x)$$

for C^ν-vector fields $f_0, f_1, \dots, f_m$. As we pointed out in Example 7.21, this can be regarded as a control-linear system by taking $\mathsf{V} = \mathbb{R} \oplus \mathbb{R}^k$

$$\mathcal{C}' = \{(u^0,\boldsymbol{u}) \in \mathsf{V} \mid\ u^0 = 1,\ \boldsymbol{u} \in \mathcal{C}\},$$

and

$$\Lambda(u^0,\boldsymbol{u}) = \sum_{a=0}^k u^a f_a.$$

We can assume Λ is injective, as mentioned above. In this case, the map Λ is a homeomorphism onto its image since any map from a finite-dimensional locally convex space is continuous [Horváth 1966, Proposition 2.10.2]. Thus Theorem 8.37 applies to control-affine systems, and gives trajectory equivalence in this case.

2. The other case of interest to us is that when $\mathsf{V} = \Gamma^\nu(\mathsf{TM})$ and when $\mathcal{C} \subseteq \mathsf{V}$ is then a family of globally defined vector fields of class C^ν on M. In this case, Λ is the identity map on $\Gamma^\nu(\mathsf{TM})$, so the hypotheses of Theorem 8.37 are easily satisfied. The trajectory equivalence one gets in this case is that between a globally generated tautological control system and its corresponding control system as in Example 8.10–2. •

One of the conclusions enunciated above is sufficiently interesting to justify its own theorem.

8.39 Theorem: (Correspondence between trajectories of a tautological control system and its associated control system) *Let $m \in \mathbb{Z}_{\geq 0}$ and $m' \in \{0, \mathrm{lip}\}$, let $\nu \in \{m + m', \infty, \omega\}$, and let $r \in \{\infty, \omega\}$, as required. Let $\mathfrak{G} = (\mathsf{M}, \mathscr{F})$ be a globally generated C^ν-tautological control system. As in Example 8.10–2, let $\Sigma_{\mathfrak{G}} = (\mathsf{M}, \Sigma_{\mathfrak{G}}, \mathcal{C}_{\mathscr{F}})$ be the corresponding C^ν-control system. Then, for each time-domain $\mathbb{T}$ and each open set $\mathcal{U} \subseteq \mathsf{M}$, $\mathrm{Traj}(\mathbb{T}, \mathcal{U}, \mathscr{O}_{\mathfrak{G},\mathrm{full}}) = \mathrm{Traj}(\mathbb{T}, \mathcal{U}, \Sigma_{\mathfrak{G}})$.*

Proof: This is the observation made in Remark 8.38–2. ■

Now we turn to relationships between trajectories for tautological control systems and differential inclusions. In Example 8.10–3 we showed how a tautological control system can be built from a differential inclusion. However, as we mentioned in that example, we cannot expect any sort of general correspondence between trajectories of the differential inclusion and the tautological control system constructed from it; differential inclusions are just too irregular. We can, however, consider the correspondence in the other direction, as the following theorem indicates.

8.40 Theorem: (Correspondence between trajectories of a tautological control system and its associated differential inclusion) *Let $m \in \mathbb{Z}_{\geq 0}$ and $m' \in \{0, \mathrm{lip}\}$, let $\nu \in \{m + m', \infty, \omega\}$, and let $r \in \{\infty, \omega\}$, as required. Let $\mathfrak{G} = (\mathsf{M}, \mathscr{F})$ be a C^ν-tautological control system and let $\mathscr{X}_{\mathfrak{G}}$ be the associated differential inclusion, as in Example 8.10–4. For $\mathbb{T}$ a time-domain and $\mathcal{U} \subseteq \mathsf{M}$ an open set, $\mathrm{Traj}(\mathbb{T}, \mathcal{U}, \mathfrak{G}) \subseteq \mathrm{Traj}(\mathbb{T}, \mathcal{U}, \mathscr{X}_{\mathfrak{G}})$.*

Conversely, if $\mathscr{F}$ is globally generated and if $\mathscr{F}(\mathsf{M})$ is a compact subset of $\Gamma^\nu(\mathsf{TM})$, then $\mathrm{Traj}(\mathbb{T}, \mathcal{U}, \mathscr{X}_{\mathfrak{G}}) \subseteq \mathrm{Traj}(\mathbb{T}, \mathcal{U}, \mathfrak{G})$.

Proof: Since, for an open-loop system $(X, \mathbb{T}, \mathcal{U})$, $X(t) \in \mathscr{F}(\mathcal{U})$ for every $t \in \mathbb{T}$, we have $X(t, x) \in \mathscr{X}_{\mathfrak{G}}(x)$ for every $(t, x) \in \mathbb{T} \times \mathcal{U}$. Thus, if $\xi \in \mathrm{Traj}(\mathbb{T}, \mathcal{U}, \mathfrak{G})$, then we have $\xi'(t) \in \mathscr{X}_{\mathfrak{G}}(\xi(t))$ for almost every $t \in \mathbb{T}$.

For the "conversely" part of the theorem, if ξ is a trajectory for the differential inclusion $\mathscr{X}_{\mathfrak{G}}$ then, for almost every $t \in \mathbb{T}$, $\xi'(t) = X(\xi(t))$ for some $X \in \mathscr{F}(\mathsf{M})$. Therefore, let us fix an arbitrary $\overline{X} \in \mathscr{F}(\mathsf{M})$ and let us define $U \colon \mathbb{T} \twoheadrightarrow \mathscr{F}(\mathsf{M})$ by

$$U(t) = \begin{cases} \{X \in \mathscr{F}(\mathsf{M}) \mid \xi'(t) = X(\xi(t))\}, & \xi'(t) \text{ exists}, \\ \{\overline{X}\}, & \text{otherwise}. \end{cases}$$

Now we note that

1. $\mathcal{C}_{\mathscr{F}} = \mathscr{F}(\mathsf{M})$ is a Suslin space, being a closed subset of a Suslin space, and
2. the map $F_{\mathscr{F}}$ is proper by Remark 8.36–2(d).

Thus we are in exactly the right framework to use the proof of Theorem 8.35(iii) to show that there exists a locally essentially bounded measurable control $t \mapsto X(t)$ for which

$$\xi'(t) = F_{\mathscr{F}}(\xi(t), X(t)), \qquad \text{a.e. } t \in \mathbb{T},$$

and so $\xi \in \mathrm{Traj}(\mathbb{T}, \mathcal{U}, \Sigma_{\mathfrak{G}})$, as desired. ■

Let us comment on the hypotheses of this theorem.

8.41 Remark: (Trajectory correspondence between tautological control systems and differential inclusions) The assumption that $\mathscr{F}(\mathsf{M})$ be compact in the “conversely” part of the preceding theorem is indispensable. The connection going from differential inclusion to tautological control system is too “loose” to get any sort of useful trajectory correspondence, without restricting the class of vector fields giving rise to the differential inclusion. Roughly speaking, this is because a differential inclusion only prescribes the values of vector fields, and the topologies have to do with derivatives as well. •

8.7. The category of tautological control systems. In our discussion of feedback equivalence in Section 1.1 we indicated that the notion of equivalence in our framework is not interesting to us. In this section, we illustrate why it not interesting by defining a natural notion of equivalence, and then seeing that it degenerates to something trivial under natural hypotheses. We do this in a general way by considering first how one might define a “category” of tautological control systems with objects and morphisms. The problem of equivalence is then the problem of understanding isomorphisms in this category. By imposing a naturality condition on morphisms via trajectories, we prove that isomorphisms are uniquely determined by diffeomorphisms of the underlying manifolds for the two tautological control systems. The notion of “direct image” we use here is common in sheaf theory, and we refer to [e.g., Kashiwara and Schapira 1990, Definition 2.3.1] for some discussion. However, by far the best presentation that we could find of direct images of presheaves such as we use here is in the online documentation (Stacks 2013).

Let us first describe how to build maps between tautological control systems. This is done first by making the following definition.

8.42 Definition: (Direct image of tautological control systems) Let $m \in \mathbb{Z}_{\geq 0}$ and $m' \in \{0, \mathrm{lip}\}$, let $\nu \in \{m + m', \infty, \omega\}$, and let $r \in \{\infty, \omega\}$, as required. Let $\mathfrak{G} = (\mathsf{M}, \mathscr{F})$ be a C^{ν}-tautological control system, let N be C^r-manifold, and let $\Phi \in \mathrm{C}^r(\mathsf{M}; \mathsf{N})$. The ***direct image*** of $\mathfrak{G}$ by Φ is the tautological control system $\Phi_*\mathfrak{G} = (\mathsf{N}, \Phi_*\mathscr{F})$ defined by $\Phi_*\mathscr{F}(\mathcal{V}) = \mathscr{F}(\Phi^{-1}(\mathcal{V}))$ for $\mathcal{V} \subseteq \mathsf{N}$ open. •

One easily verifies that if $\mathscr{F}$ is a sheaf, then so too is $\Phi_*\mathscr{F}$.

With the preceding sheaf construction, we can define what we mean by a morphism of tautological control systems.

8.43 Definition: (Morphism of tautological control systems) Let $m \in \mathbb{Z}_{\geq 0}$ and $m' \in \{0, \mathrm{lip}\}$, let $\nu \in \{m + m', \infty, \omega\}$, and let $r \in \{\infty, \omega\}$, as required. Let $\mathfrak{G} = (\mathsf{M}, \mathscr{F})$ and $\mathfrak{H} = (\mathsf{N}, \mathscr{G})$ be C^{ν}-tautological control systems. A ***morphism*** from $\mathfrak{G}$ to $\mathfrak{H}$ is a pair $(\Phi, \Phi^{\sharp})$ such that

(i) $\Phi \in \mathrm{C}^r(\mathsf{M};\mathsf{N})$ and

(ii) $\Phi^\sharp = (\Phi^\sharp_{\mathcal{V}})_{\mathcal{V}\,\mathrm{open}}$ is a family of mappings $\Phi^\sharp_{\mathcal{V}}\colon \mathscr{G}(\mathcal{V}) \to \Phi_*\mathscr{F}(\mathcal{V})$, $\mathcal{V} \subseteq \mathsf{N}$ defined as follows:

  (a) there exists a family $L_{\mathcal{V}} \in \mathrm{L}(\Gamma^\nu(\mathsf{T}\mathcal{V});\Gamma^\nu(\mathsf{T}(\Phi^{-1}(\mathcal{V}))))$ of continuous linear mappings satisfying $L_{\mathcal{V}'} = L_{\mathcal{V}}|\Gamma^\nu(\mathsf{T}\mathcal{V}')$ if $\mathcal{V},\mathcal{V}' \subseteq \mathsf{N}$ are open with $\mathcal{V}' \subseteq \mathcal{V}$;

  (b) $\Phi^\sharp_{\mathcal{V}} = L_{\mathcal{V}}|\mathscr{G}(\mathcal{V})$. •

By the preceding definition, we arrive at the "category of C^ν-tautological control systems" whose objects are tautological control systems and whose morphisms are as just defined. From the point of view of control theory, one wishes to restrict these definitions further to account for the fact that morphisms ought to preserve trajectories. Therefore, let us see how trajectories come into the picture. First we consider open-loop systems. Thus let $\mathbb{T}$ be a time-domain and let $\mathcal{V} \subseteq \mathsf{N}$ be open. If $Y\colon \mathbb{T} \to \mathscr{G}(\mathcal{U})$ then we have $\Phi^\sharp(Y)_t \triangleq \Phi^\sharp_{\mathcal{V}}(Y_t) \in \mathscr{F}(\Phi^{-1}(\mathcal{V}))$ for each $t \in \mathbb{T}$. That is, an open-loop system $(Y,\mathbb{T},\mathcal{V})$ for $\mathfrak{H}$ gives rise to an open-loop system $(\Phi^\sharp(Y),\mathbb{T},\Phi^{-1}(\mathcal{V}))$ for $\mathfrak{G}$. For such a correspondence to have significance, it must do the more or less obvious thing to trajectories.

8.44 Definition: (Natural morphisms of tautological control systems) Let $m \in \mathbb{Z}_{\geq 0}$ and $m' \in \{0,\mathrm{lip}\}$, let $\nu \in \{m+m',\infty,\omega\}$, and let $r \in \{\infty,\omega\}$, as required. Let $\mathfrak{G} = (\mathsf{M},\mathscr{F})$ and $\mathfrak{H} = (\mathsf{N},\mathscr{G})$ be C^ν-tautological control systems. A morphism $(\Phi,\Phi^\sharp)$ from $\mathfrak{G}$ to $\mathfrak{H}$ is ***natural*** if, for each time-domain $\mathbb{T}$, each open $\mathcal{V} \subseteq \mathsf{N}$, and each $Y \in \mathrm{LI}\Gamma^\nu(\mathbb{T};\mathscr{G}(\mathcal{V}))$, any integral curve $\xi\colon \mathbb{T}' \to \Phi^{-1}(\mathcal{V})$ for the time-varying vector field $t \mapsto \Phi^\sharp(Y_t)$ defined on $\mathbb{T}' \subseteq \mathbb{T}$ has the property that $\Phi\circ\xi$ is an integral curve for Y. •

Note that the time-varying vector field $t \mapsto \Phi^\sharp(Y_t)$ from the definition is locally integrally bounded by [Beckmann and Deitmar 2011, Lemma 1.2].

We can now characterise these natural morphisms.

8.45 Proposition: (Characterisation of natural morphisms) *Let $m \in \mathbb{Z}_{\geq 0}$ and $m' \in \{0,\mathrm{lip}\}$, let $\nu \in \{m+m',\infty,\omega\}$, and let $r \in \{\infty,\omega\}$, as required. Let $\mathfrak{G} = (\mathsf{M},\mathscr{F})$ and $\mathfrak{H} = (\mathsf{M},\mathscr{G})$ be C^ν-tautological control systems. A morphism $(\Phi,\Phi^\sharp)$ from $\mathfrak{G}$ to $\mathfrak{H}$ is natural if and only if, for each open $\mathcal{V} \subseteq \mathsf{N}$, each $Y \in \mathscr{G}(\mathcal{V})$, each $y \in \mathcal{V}$, and each $x \in \Phi^{-1}(y)$, we have $T_x\Phi(\Phi^\sharp(Y)(x)) = Y(y)$.*

Proof: First suppose that $(\Phi,\Phi^\sharp)$ is natural, and let $\mathcal{V} \subseteq \mathsf{N}$ be open, let $Y \in \mathscr{G}(\mathcal{V})$, let $y \in \mathcal{V}$, and let $x \in \Phi^{-1}(\mathcal{V})$. Let $\mathbb{T} \subseteq \mathbb{R}$ be a time-domain for which $0 \in \mathrm{int}(\mathbb{T})$ and for which the integral curve η for Y through y is defined on $\mathbb{T}$. We consider $Y \in \mathrm{LI}\Gamma^\nu(\mathbb{T};\mathscr{G}(\mathcal{V}))$ by taking $Y_t = Y$, i.e., Y is a time-independent time-varying vector field. Note that integral curves of Y can, therefore, be chosen to be differentiable [Coddington and Levinson 1955, Theorem 1.3], and *will be* differentiable if $\nu > 0$. Let $\mathbb{T}' \subseteq \mathbb{T}$ be such that the differentiable integral curve ξ for $\Phi^\sharp(Y)$ through x is defined on $\mathbb{T}'$. Since $(\Phi,\Phi^\sharp)$ is natural, we have $\eta = \Phi\circ\xi$ on $\mathbb{T}'$. Therefore,

$$Y(y) = \eta'(0) = T_x\Phi(\xi'(0)) = T_x\Phi(\Phi^\sharp(Y)(x)).$$

Next suppose that, for each open $\mathcal{V} \subseteq \mathsf{N}$, each $Y \in \mathscr{G}(\mathcal{V})$, each $y \in \mathcal{V}$, and each $x \in \Phi^{-1}(y)$, we have $T_x\Phi(\Phi^\sharp(Y)(x)) = Y(y)$. Let $\mathbb{T}$ be a time-domain, let $\mathcal{V} \subseteq \mathsf{N}$ be open, let $Y \in \mathrm{LI}\Gamma^\nu(\mathbb{T};\mathscr{G}(\mathcal{V}))$, and let $\xi\colon \mathbb{T}' \to \Phi^{-1}(\mathcal{V})$ be an integral curve for the time-varying vector field $t \mapsto \Phi^\sharp(Y_t)$ defined on $\mathbb{T}' \subseteq \mathbb{T}$. Let $\eta = \Phi \circ \xi$. Then we have

$$\eta'(t) = T_{\xi(t)}\Phi(\Phi^\sharp(Y_t)(\xi(t))) = Y_t(\eta(t))$$

for almost every $t \in \mathbb{T}'$, showing that η is an integral curve for Y. ■

Note that the condition $T_x\Phi(\Phi^\sharp(Y)(x)) = Y(y)$ is consistent with the regularity conditions for Y. In the cases $\nu \in \{m, \infty, \omega\}$, this is a consequence of the Chain Rule (see [Krantz and Parks 2002, Proposition 2.2.8] for the real analytic case). In the Lipschitz case this is a consequence of [Gromov 2007, Example 1.4(c)] combined with [Weaver 1999, Proposition 1.2.2].

To make a connection with more common notions of mappings between control systems, let us do the following. Let $m \in \mathbb{Z}_{\geq 0}$ and $m' \in \{0, \mathrm{lip}\}$, let $\nu \in \{m + m', \infty, \omega\}$, and let $r \in \{\infty, \omega\}$, as required. Suppose that we have two C^ν-control systems $\Sigma_1 = (\mathsf{M}_1, F_1, \mathcal{C}_1)$ and $\Sigma_2 = (\mathsf{M}_2, F_2, \mathcal{C}_2)$. As tautological control systems, these are globally generated, so let us not fuss with general open sets for the purpose of this illustrative discussion. We then suppose that we have a mapping $\Phi \in \mathrm{C}^r(\mathsf{M}_1; \mathsf{M}_2)$ and a mapping $\kappa\colon \mathsf{M}_1 \times \mathcal{C}_2 \to \mathcal{C}_1$, which gives rise to a correspondence between the system vector fields by

$$\Phi^\sharp(F_2^{u_2})(x_1) = F_1^{\kappa(x_1,u_2)}(x_1).$$

The condition of naturality means that a trajectory ξ_1 for Σ_1 satisfying

$$\xi_1'(t) = F_1(\xi_1(t), \kappa(\xi_1(t), \mu_2(t)))$$

gives rise to a trajectory $\xi_2 = \Phi \circ \xi_1$ for Σ_2, implying that

$$\xi_2' = T_{\xi_1(t)}\Phi(\xi_1'(t)) = T_{\xi_1(t)}\Phi \circ F_1(\xi_1(t), \kappa(\xi_1(t), \mu_2(t))).$$

Thus

$$F_2(x_2, u_2) = T_{x_1}\Phi \circ F_1(x_1, \kappa(x_1, u_2))$$

for every $x_1 \in \Phi^{-1}(x_2)$.

There may well be some interest in studying general natural morphisms, but we will not pursue this right at the moment. Instead, let us simply think about isomorphisms in the category of tautological control systems.

8.46 Definition: (Isomorphisms of tautological control systems) Let $m \in \mathbb{Z}_{\geq 0}$ and $m' \in \{0, \mathrm{lip}\}$, let $\nu \in \{m + m', \infty, \omega\}$, and let $r \in \{\infty, \omega\}$, as required. Let $\mathfrak{G} = (\mathsf{M}, \mathscr{F})$ and $\mathfrak{H} = (\mathsf{N}, \mathscr{G})$ be C^ν-tautological control systems. An ***isomorphism*** from $\mathfrak{G}$ to $\mathfrak{H}$ is a morphism $(\Phi, \Phi^\sharp)$ such that Φ is a diffeomorphism and $L_{\mathcal{V}}$ is an isomorphism (in the category of locally convex topological vector spaces) for every open $\mathcal{V} \subseteq \mathsf{N}$, where $L_{\mathcal{V}}$ is such that $\Phi^\sharp_{\mathcal{V}} = L_{\mathcal{V}}|\mathscr{G}(\mathcal{V})$ as in Definition 8.43. •

It is now easy to describe the natural isomorphisms.

8.47 Proposition: (Characterisation of natural isomorphisms) *Let $m \in \mathbb{Z}_{\geq 0}$ and $m' \in \{0, \mathrm{lip}\}$, let $\nu \in \{m+m', \infty, \omega\}$, and let $r \in \{\infty, \omega\}$, as required. Let $\mathfrak{G} = (\mathsf{M}, \mathscr{F})$ and $\mathfrak{H} = (\mathsf{N}, \mathscr{G})$ be C^ν-tautological control systems. A morphism $(\Phi, \Phi^\sharp)$ from $\mathfrak{G}$ to $\mathfrak{H}$ is a natural isomorphism if and only if Φ is a diffeomorphism and*

$$\mathscr{G}(\Phi(\mathcal{U})) = \{(\Phi|\mathcal{U})_* X \mid X \in \mathscr{F}(\mathcal{U})\}$$

for every open set $\mathcal{U} \subseteq \mathsf{M}$.

Proof: According to Proposition 8.45, if $\mathcal{V} \subseteq \mathsf{N}$ is open and if $Y \in \mathscr{G}(\mathcal{V})$, we have $(\Phi|\Phi^{-1}(\mathcal{V}))_*(\Phi^\sharp(Y)) = Y$ or $\Phi^\sharp(Y) = (\Phi|\Phi^{-1}(\mathcal{V}))^* Y$. Since $\Phi^\sharp$ is a bijection from $\mathscr{G}(\mathcal{V})$ to $\mathscr{F}(\Phi^{-1}(\mathcal{V}))$, we conclude that

$$\mathscr{F}(\Phi^{-1}(\mathcal{V})) = \{(\Phi|\Phi^{-1}(\mathcal{V}))^* Y \mid Y \in \mathscr{G}(\mathcal{V})\}.$$

This is clearly equivalent to the assertion of the theorem since Φ must be a diffeomorphism. ■

In words, natural isomorphisms simply amount to the natural correspondence of vector fields under the push-forward Φ_*. (One should verify that push-forward is continuous as a mapping between locally convex spaces. This amounts to proving continuity of composition, and for this we point to places in the literature from which this can be deduced. In the smooth and finitely differentiable cases this can be shown using an argument fashioned after that from [Mather 1969, Proposition 1]. In the Lipschitz case, this follows because the Lipschitz constant of a composition is bounded by the product of the Lipschitz constants [Weaver 1999, Proposition 1.2.2]. In the real analytic case, this follows from Sublemma 6 from the proof of Lemma 2.4.) In particular, if one wishes to consider only the identity diffeomorphism, i.e., only consider the "feedback part" of a feedback transformation, we see that the only natural isomorphism is simply the identity morphism. In this way we see that the notion of equivalence for tautological control systems is either very trivial (it is easy to understand when systems are equivalent) or very difficult (the study of equivalence classes contains as a special case the classification of vector fields up to diffeomorphism), depending on your tastes. It is our view that the triviality (or impossibility) of equivalence is a virtue of the formulation since all structure except that of the manifold and the vector fields has been removed; there is no extraneous structure. We refer to Section 1.1 for further discussion.

8.8. A tautological control system formulation of sub-Riemannian geometry. In our preceding discussion of tautological control systems, we strove to make connections between tautological control systems and standard control models. We do not wish to give the impression, however, that tautological control systems are mere fancy reformulations of standard control systems. In this section we give an application, sub-Riemannian geometry, that illustrates the *per se* value of tautological control systems.

Let us define the basic structure of sub-Riemannian geometry.

8.48 Definition: (Sub-Riemannian manifold) Let $m \in \mathbb{Z}_{\geq 0}$ and $m' \in \{0, \mathrm{lip}\}$, let $\nu \in \{m+m', \infty, \omega\}$, and let $r \in \{\infty, \omega\}$, as required. A ***C^ν-sub-Riemannian manifold*** is a pair $(\mathsf{M}, \mathbb{G})$ where M is a C^r-manifold and $\mathbb{G}$ is a C^ν-tensor field of type $(2,0)$ such that $\mathbb{G}(x)$ is positive-semidefinite as a quadratic function on $\mathsf{T}_x^*\mathsf{M}$. •

Associated with a sub-Riemannian structure $\mathbb{G}$ on M is a distribution that we now describe. First of all, we have a map $\mathbb{G}^\sharp\colon \mathsf{T}^*\mathsf{M} \to \mathsf{T}\mathsf{M}$ defined by

$$\langle \beta_x; \mathbb{G}^\sharp(\alpha_x)\rangle = \mathbb{G}(\beta_x, \alpha_x).$$

We then denote by $\mathsf{D}_{\mathbb{G}} = \mathrm{image}(\mathbb{G}^\sharp)$ the associated distribution. Note that $\mathsf{D}_{\mathbb{G}}$ is a distribution of class C^ν since, for each $x \in \mathsf{M}$, there exist a neighbourhood $\mathcal{U}$ of x and a family of C^ν-vector fields $(X_a)_{a\in A}$ on $\mathcal{U}$ (namely the images under $\mathbb{G}^\sharp$ of the coordinate basis vector fields, if we choose $\mathcal{U}$ to be a coordinate chart domain) such that

$$\mathsf{D}_{\mathbb{G},y} = \mathsf{D}_{\mathbb{G}} \cap \mathsf{T}_y\mathsf{M} = \mathrm{span}_{\mathbb{R}}(X_a(y)|\ a \in A)$$

for every $y \in \mathcal{U}$. There is also an associated ***sub-Riemannian metric*** for $\mathsf{D}_{\mathbb{G}}$, i.e., an assignment to each $x \in \mathsf{M}$ an inner product $\mathbb{G}(x)$ on $\mathsf{D}_{\mathbb{G},x}$. This is denoted also by $\mathbb{G}$ and defined by

$$\mathbb{G}(u_x, v_x) = \mathbb{G}(\alpha_x, \beta_x),$$

where $u_x = \mathbb{G}^\sharp(\alpha_x)$ and $v_x = \mathbb{G}^\sharp(\beta_x)$, and where we joyously abuse notation.

An absolutely continuous curve $\gamma\colon [a,b] \to \mathsf{M}$ is ***$\mathsf{D}_{\mathbb{G}}$-admissible*** if $\gamma'(t) \in \mathsf{D}_{\mathbb{G},\gamma(t)}$ for almost every $t \in [a,b]$. The ***length*** of a $\mathsf{D}_{\mathbb{G}}$-admissible curve $\gamma\colon [a,b] \to \mathsf{M}$ is

$$\ell_{\mathbb{G}}(\gamma) = \int_a^b \sqrt{\mathbb{G}(\gamma'(t), \gamma'(t))}\,\mathrm{d}t.$$

As in Riemannian geometry, the length of a $\mathsf{D}_{\mathbb{G}}$-admissible curve is independent of parameterisation, and so curves can be considered to be defined on $[0,1]$. We can then define the ***sub-Riemannian distance*** between $x_1, x_2 \in \mathsf{M}$ by

$$\begin{aligned}\mathrm{d}_{\mathbb{G}}(x_1, x_2) = \inf\{\ell_{\mathbb{G}}(\gamma)|\ \gamma\colon [0,1] \to \mathsf{M} \text{ is an absolutely}\\ \text{continuous curve for which } \gamma(0) = x_1 \text{ and } \gamma(1) = x_2\}.\end{aligned}$$

One of the problems of sub-Riemannian geometry is to determine length minimising curves, i.e., sub-Riemannian geodesics.

A common means of converting sub-Riemannian geometry into a standard control problem is to choose a $\mathbb{G}$-orthonormal basis $(X_1, \dots, X_k)$ for $\mathsf{D}_{\mathbb{G}}$ and so consider the control-affine system with dynamics prescribed by

$$F(x, \boldsymbol{u}) = \sum_{a=1}^k u^a X_a(x), \qquad x \in \mathsf{M},\ \boldsymbol{u} \in \mathbb{R}^k.$$

Upon doing this, $\mathsf{D}_{\mathbb{G}}$-admissible curves are evidently trajectories for this control-affine system. Moreover, for a trajectory $\xi\colon [0,1] \to \mathsf{M}$ satisfying

$$\xi'(t) = \sum_{a=1}^k u^a(t) X_a(\xi(t)),$$

we have

$$\ell_{\mathbb{G}}(\xi) = \int_0^1 \|\boldsymbol{u}(t)\|\,\mathrm{d}t.$$

The difficulty, of course, with the preceding approach to sub-Riemannian geometry is that there may be no $\mathbb{G}$-orthonormal basis for $\mathsf{D}_{\mathbb{G}}$. This can be the case for at least two reasons: (1) the distribution $\mathsf{D}_{\mathbb{G}}$ may not have locally constant rank; (2) when the distribution $\mathsf{D}_{\mathbb{G}}$ has locally constant rank, the global topology of M may prohibit the existence of a global basis, e.g., on even-dimensional spheres there is no global basis for vector fields, orthonormal or otherwise. However, one can formulate sub-Riemannian geometry in terms of a tautological control system in a natural way. Indeed, associated to $\mathsf{D}_{\mathbb{G}}$ is the tautological control system $\mathfrak{G}_{\mathbb{G}} = (\mathsf{M}, \mathscr{F}_{\mathbb{G}})$, where, for an open subset $\mathcal{U} \subseteq \mathsf{M}$,

$$\mathscr{F}_{\mathbb{G}}(\mathcal{U}) = \{X \in \Gamma^{\nu}(\mathsf{T}\mathcal{U}) \mid \; X(x) \in \mathsf{D}_{\mathbb{G},x}, \; x \in \mathcal{U}\}.$$

One readily verifies that $\mathscr{F}_{\mathbb{G}}$ is a sheaf.

Let us see how we can regard our tautological control system formulation as that for an "ordinary" control system, with a suitable control set, as per Example 8.10–2. First of all, note that the sheaf $\mathscr{F}_{\mathbb{G}}$ is not globally generated; this is because it is a sheaf, cf. Example 8.3–2. However, it can be regarded as the sheafification of the globally generated sheaf with global generators $\mathscr{F}_{\mathbb{G}}(\mathsf{M})$.

8.49 Lemma: (The sheaf of vector fields for the sub-Riemannian tautological control problem) *The sheaf $\mathscr{F}_{\mathbb{G}}$ is the sheafification of the globally generated presheaf with generators $\mathscr{F}_{\mathbb{G}}(\mathsf{M})$.*

Proof: This is a result about sheaf cohomology, and we will not give all details here. Instead we will simply point to the main facts from which the conclusion follows. First of all, to prove the assertion, it suffices by Lemma 8.6 to show that $\mathscr{F}_{\mathbb{G},x}$ is generated, as a module over the ring $\mathscr{C}^{\nu}_{x,\mathsf{M}}$, by germs of global sections. In the cases $\nu \in \{m, m+\mathrm{lip}, \infty\}$, the fact that the sheaf of rings of smooth functions admits partitions of unity implies that the sheaf $\mathscr{C}^{\nu}_{\mathsf{M}}$ is a fine sheaf of rings [Wells Jr. 2008, Example 3.4(d)]. It then follows from [Wells Jr. 2008, Example 3.4(e)] that the sheaf $\mathscr{F}_{\mathbb{G}}$ is also fine and so soft [Wells Jr. 2008, Proposition 3.5]. Because of this, the cohomology groups of positive degree for this sheaf vanish [Wells Jr. 2008, Proposition 3.11], and this ensures that germs of global sections generate all stalks (more or less by definition of cohomology in degree 1). In the case $\nu = \omega$, the result is quite nontrivial. First of all, by a real analytic adaptation of [Gunning 1990b, Corollary H9], one can show that $\mathscr{F}_{\mathbb{G}}$ is locally finitely generated. Then, $\mathscr{F}_{\mathbb{G}}$ being a finitely generated subsheaf of the coherent sheaf $\mathscr{G}^{\omega}_{\mathsf{TM}}$, it is itself coherent [Demailly 2012, Theorem 3.16]. Then, by Cartan's Theorem A [Cartan 1957], we conclude that $\mathscr{F}_{\mathbb{G},x}$ is generated by germs of global sections. ■

By the preceding lemma and Proposition 8.26, we can as well consider the globally generated presheaf with global generators $\mathscr{F}_{\mathbb{G}}(\mathsf{M})$, and so trajectories are those of the associated "ordinary" control system $\Sigma_{\mathbb{G}} = (\mathsf{M}, F_{\mathbb{G}}, \mathcal{C}_{\mathbb{G}})$, where $\mathcal{C}_{\mathbb{G}} = \mathscr{F}_{\mathbb{G}}(\mathsf{M})$ and $F_{\mathbb{G}}(x, X) = X(x)$.

Let us next formulate the sub-Riemannian geodesic problem in the framework of tautological control systems. First of all, it is convenient when performing computations to work with energy rather than length as the quantity we are minimising. To this end, for an absolutely continuous $\mathsf{D}_{\mathbb{G}}$-admissible curve $\gamma\colon [a,b] \to \mathsf{M}$, we define the ***energy*** of this curve to be

$$E_{\mathbb{G}}(\gamma) = \frac{1}{2}\int_a^b \mathbb{G}(\gamma'(t), \gamma'(t))\,\mathrm{d}t.$$

A standard argument shows that curves that minimise energy are in 1–1 correspondence with curves that minimise length and are parameterised to have an appropriate constant speed [Montgomery 2002, Proposition 1.4.3]. We can and do, therefore, consider the energy minimisation problem. We let $x_1, x_2 \in \mathsf{M}$ and let $\mathscr{O}_{x_1,x_2}$ be the open-loop subfamily for which the members of $\mathscr{O}_{x_1,x_2}(\mathbb{T}, \mathcal{U})$ are those vector fields $X \in \mathrm{LI}\Gamma^\nu(\mathbb{T}; \mathscr{F}_{\mathbb{G}}(\mathcal{U}))$ having the property that there exist $t_1, t_2 \in \mathbb{T}$ with $t_1 < t_2$, $\mathcal{U}' \subseteq \mathcal{U}$, and $\xi \in \mathrm{Traj}([t_1, t_2], \mathcal{U}', \mathscr{O}_{\mathfrak{G}_{\mathbb{G}},X})$ (see Example 8.18–5 for notation) such that $\xi(t_1) = x_1$ and $\xi(t_2) = x_2$. If $X \in \mathscr{O}_{q_1,q_2}(\mathbb{T}, \mathcal{U})$, let us denote by $\mathrm{Traj}(X, x_1, x_2)$ those integral curves $\xi\colon [t_1, t_2] \to \mathsf{M}$ for X with the property that $\xi(t_1) = x_1$ and $\xi(t_2) = x_2$. We can then define

$$\mathscr{C}_{\mathbb{G}}(X) = \inf\{E_{\mathbb{G}}(\xi) \mid \xi \in \mathrm{Traj}(X, x_1, x_2)\}.$$

The goal, then, is to find an interval $\mathbb{T}_* \subseteq \mathbb{R}$, an open set $\mathcal{U}_*$, and $X_* \in \mathscr{O}_{x_1,x_2}(\mathbb{T}_*, \mathcal{U}_*)$ such that

$$\mathscr{C}_{\mathbb{G}}(X_*) \le \mathscr{C}_{\mathbb{G}}(X), \qquad X \in \mathscr{O}_{x_1,x_2}(\mathbb{T}, \mathcal{U}),\ \mathbb{T} \text{ an interval}, \mathcal{U} \subseteq \mathsf{M} \text{ open}.$$

Let us apply the classical Maximum Principle of Pontryagin, Boltyanskiĭ, Gamkrelidze, and Mishchenko [1961], leaving aside the technicalities caused by the complicated topology of the control set. The dealing with of these technicalities will be the subject of future work. We thus suppose that we have a length minimising trajectory $\xi_* \in \mathrm{Traj}(X_*, x_1, x_2)$ for $X_* \in \mathscr{O}_{x_1,x_2}(\mathbb{T}_*, \mathcal{U}_*)$. The ***Hamiltonian*** for the system has the form

$$\begin{aligned} H_{\mathbb{G}}\colon \mathsf{T}^*\mathcal{U}_* \times \mathscr{F}_{\mathbb{G}}(\mathcal{U}_*) &\to \mathbb{R} \\ (\alpha_x, X) &\mapsto \langle \alpha_x; X(x)\rangle + \lambda_0 \tfrac{1}{2}\mathbb{G}(X(x), X(x)), \end{aligned}$$

where $\lambda_0 \in \{0, -1\}$. If we consider only normal extremals, i.e., supposing that $\lambda_0 = -1$, then the Maximum Principle prescribes that $X_*\colon \mathsf{T}^*\mathcal{U}_* \to \mathsf{T}\mathcal{U}_*$ should be a bundle map over $\mathrm{id}_{\mathcal{U}_*}$ chosen so that $X_*(\alpha_x)$ maximises the function

$$v_x \mapsto \langle \alpha_x; v_x\rangle - \tfrac{1}{2}\mathbb{G}(v_x, v_x).$$

Standard finite-dimensional optimisation gives $X_*(x) = \mathbb{G}^\sharp(\alpha_x)$. The ***maximum Hamiltonian*** is then obtained by substituting this value of the "control" into the Hamiltonian:

$$\begin{aligned} H_{\mathbb{G}}^{\max}\colon \mathsf{T}^*\mathsf{M} &\to \mathbb{R} \\ \alpha_x &\mapsto \tfrac{1}{2}\mathbb{G}(\alpha_x, \alpha_x). \end{aligned}$$

The normal extremals are then integral curves of the Hamiltonian vector field associated with the Hamiltonian $H_{\mathbb{G}}^{\max}$.

The preceding computations, having banished the usual parameterisation by control, are quite elegant when compared to manner in which one applies the Maximum Principle to the "usual" control formulation of sub-Riemannian geometry. The calculations are also more general and global. However, to make sense of them, one has to prove an appropriate version of the Maximum Principle, something which will be forthcoming. For the moment, we mention that a significant rôle in this will be played by appropriate needle variations constructed by dragging variations along a trajectory to the final endpoint. The manner in which one drags these variations has to do with linearisation, to which we now turn our attention.

9. Linearisation of tautological control systems

As an illustration of the fact that it is possible to do non-elementary things in the framework of tautological control systems, we present a fully developed theory for the linearisation of these systems. This theory is both satisfying and revealing. It is satisfying because it is very simple (if one knows a little tangent bundle geometry) and it is revealing because, for example, it clarifies and rectifies the hiccup with classical linearisation theory that was revealed in Example 1.1.

Before we begin, it is worth pointing out that, apart from the problem revealed in Example 1.1, there are other difficulties with the very idea of classical Jacobian linearisation to which blind eyes seem to be routinely turned in practice. First of all, for models of the form "$F(x,u)$," one must assume that differentiation with respect to u can be done. For models of this sort, there is no reason to assume the control set to be a subset of $\mathbb{R}^m$, and so one runs into a problem right away. Even so, if one restricts to control-affine systems, where the notion of differentiation with respect to u seems not to be problematic, one must ignore the fact that the control set is generally not an open set, and so these derivatives are not so easily made sense of. Therefore, even for the typical models one studies in control theory, there are good reasons to revisit the notion of linearisation.

We point out that geometric linearisation of control-affine systems, and a Linear Quadratic Regulator theory in this framework, has been carried out by Lewis and Tyner [2010]. But even the geometric approach in that work is refined and clarified by what we present here.

In this section we work with systems of general regularity, only requiring that they be at least once differentiable so that we can easily define their linearisation. For dealing with Lipschitz systems, we will use the following result.

9.1 Lemma: ($\mathbf{C}^{(m+\mathrm{lip})-1} = \mathbf{C}^{(m-1)+\mathrm{lip}}$) *For a smooth vector bundle $\pi\colon \mathsf{E} \to \mathsf{M}$ and for $m \in \mathbb{Z}_{>0}$, if $\xi \in \Gamma^{m+\mathrm{lip}}(\mathsf{E})$, then $j_1\xi \in \Gamma^{(m-1)+\mathrm{lip}}(\mathsf{J}^1\mathsf{E})$. Moreover, $\operatorname{dil} j_{m-1}(j_1\xi)(x) = \operatorname{dil} j_m\xi(x)$ for every $x \in \mathsf{M}$.*

Proof: We need to show that $j_{m-1}(j_1\xi)$ is locally Lipschitz. This, however, is clear since $j_{m-1}j_1\xi$ is the image of $j_m\xi$ under the injection of $\mathsf{J}^m\mathsf{E}$ in $\mathsf{J}^{m-1}\mathsf{J}^1\mathsf{E}$ [Saunders 1989, Definition 6.2.25], and since $j_m\xi$ is Lipschitz by hypothesis.

The last formula in the statement of the lemma requires us to make sense of $\operatorname{dil} j_{m-1}(j_1\xi)$. This is made sense of using the fact that, by Lemma 2.1, one has $\mathsf{J}^1\mathsf{E} \simeq \mathsf{T}^*\mathsf{M} \otimes \mathsf{E}$, and so the Riemannian metric $\mathbb{G}$ on M, the fibre metric $\mathbb{G}_0$, the Levi-Civita connection ∇ on M, and the $\mathbb{G}_0$-orthogonal linear connection ∇^0 induce a fibre metric and linear connection in the vector bundle $\mathsf{J}^1\mathsf{E}$ as in Sections 2.1 and 2.2. Now let us examine the inclusion of $\mathsf{J}^m\mathsf{E}$ in $\mathsf{J}^{m-1}\mathsf{J}^1\mathsf{E}$ to verify the final assertion of the lemma. We use Lemma 2.1 to write

$$\mathsf{J}^m\mathsf{E} \simeq \oplus_{j=0}^m \mathsf{S}^j(\mathsf{T}^*\mathsf{M}) \otimes \mathsf{E}.$$

In this case, the inclusion of $\mathsf{J}^m\mathsf{E}$ in $\mathsf{J}^1\mathsf{J}^{m-1}\mathsf{E}$ becomes identified with the natural inclusions

$$\mathsf{S}^j(\mathsf{T}^*\mathsf{M}) \otimes \mathsf{E} \to \mathsf{S}^{j-1}(\mathsf{T}^*\mathsf{M}) \otimes \mathsf{T}^*\mathsf{M} \otimes \mathsf{E}, \qquad j \in \{0,1,\dots,m-1\},$$

given by

$$\alpha^1 \odot \cdots \odot \alpha^j \otimes e \mapsto \sum_{k=1}^{j} \alpha^1 \odot \cdots \odot \alpha^{k-1} \odot \alpha^{k+1} \odot \cdots \odot \alpha^j \otimes \alpha^k \otimes e.$$

The fibre metric on $\mathrm{S}^j(\mathsf{T}^*\mathsf{M})$ is the restriction of that on $\mathrm{T}^j(\mathsf{T}^*\mathsf{M})$. Thus the preceding inclusion preserves the fibre metrics since these are defined componentwise on the tensor product. Similarly, since the connection in the symmetric and tensor products is defined so as to satisfy the Leibniz rule for the tensor product, the injection above commutes with parallel translation. It now follows from the definition of dilatation that the final formula in the statement of the lemma holds. ■

9.1. Tangent bundle geometry. To make the constructions in this section, we recall a little tangent bundle geometry. Throughout this section, we let $m \in \mathbb{Z}_{>0}$, $m' \in \{0, \mathrm{lip}\}$, and let $\nu \in \{m + m', \infty, \omega\}$. We take $r \in \{\infty, \omega\}$, as required. The meaning of "$\nu - 1$" is obvious for all ν. But, to be clear, $\infty - 1 = \infty$, $\omega - 1 = \omega$, and, given Lemma 9.1, $(m + \mathrm{lip}) - 1 = (m - 1) + \mathrm{lip}$.

Let $X \in \Gamma^\nu(\mathsf{TM})$. We will lift X to a vector field on TM in two ways. The first is the vertical lift, and is described first by a vector bundle map $\mathrm{vlft}\colon \pi^*_{\mathsf{TM}}\mathsf{TM} \to \mathsf{TTM}$ as follows. Let $x \in \mathsf{M}$ and let $v_x, w_x \in \mathsf{T}_x\mathsf{M}$. The **vertical lift** of u_x to v_x is given by

$$\mathrm{vlft}(v_x, u_x) = \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} (v_x + t u_x).$$

Now, if $X \in \Gamma^\nu(\mathsf{TM})$, we define $X^V \in \Gamma^\nu(\mathsf{TTM})$ by $X^V(v_x) = \mathrm{vlft}(v_x, X(x))$. In coordinates $(x^1, \dots, x^n)$ for M with $((x^1, \dots, x^n), (v^1, \dots, v^n))$ the associated natural coordinates for TM, if $X = X^j \frac{\partial}{\partial x^j}$, then $X^V = X^j \frac{\partial}{\partial v^j}$. The vertical lift is a very simple vector field. It is tangent to the fibres of TM, and is in fact constant on each fibre.

The other lift of $X \in \Gamma^\nu(\mathsf{TM})$ that we shall use is the ***tangent lift***[14] which is the vector field X^T on TM of class $\mathrm{C}^{\nu-1}$ whose flow is given by $\Phi_t^{X^T}(v_x) = T_x\Phi_t^X(v_x)$. Therefore, explicitly,

$$X^T(v_x) = \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} T_x\Phi_t^X(v_x).$$

In coordinates as above, if $X = X^j \frac{\partial}{\partial x^j}$, then

$$X^T = X^j \frac{\partial}{\partial x^j} + \frac{\partial X^j}{\partial x^k} v^k \frac{\partial}{\partial v^j}. \tag{9.1}$$

One recognises the "linearisation" of X in this expression, but one should understand that the second term in this coordinate expression typically has no meaning by itself. The flow for X^T is related to that for X according to the following commutative diagram:

$$\begin{tikzcd} \mathsf{TM} \arrow[r, "\Phi_t^{X^T}"] \arrow[d, "\pi_{\mathsf{TM}}"'] & \mathsf{TM} \arrow[d, "\pi_{\mathsf{TM}}"] \\ \mathsf{M} \arrow[r, "\Phi_t^X"'] & \mathsf{M} \end{tikzcd} \tag{9.2}$$

[14] This is also frequently called the ***complete lift***. However, "tangent lift" so much better captures the essence of the construction, that we prefer our terminology. Also, the dual of the tangent lift is used in the Maximum Principle, and this is then conveniently called the "cotangent lift."

Thus X^T projects to X in the sense that $T_{v_x}\pi_{\mathsf{TM}}(X^T(v_x)) = X(x)$. Moreover, X^T is a "linear" vector field (as befits its appearance in "linearisation" below), which means that the diagram

$$\begin{tikzcd} \mathsf{TM} \arrow[r, "X^T"] \arrow[d, "\pi_{\mathsf{TM}}"'] & \mathsf{TTM} \arrow[d, "T\pi_{\mathsf{TM}}"] \\ \mathsf{M} \arrow[r, "X"'] & \mathsf{TM} \end{tikzcd} \tag{9.3}$$

defines X^T as a vector bundle map over X.

We will be interested in the flow of the tangent lift in the time-varying case, and the next lemma indicates how this works.

9.2 Lemma: (Tangent lift for time-varying vector fields) *Let $m \in \mathbb{Z}_{>0}$ and $m' \in \{0, \mathrm{lip}\}$, let $\nu \in \{m + m', \infty, \omega\}$, and let $r \in \{\infty, \omega\}$, as required. Let M be a C^r-manifold and let $\mathbb{T} \subseteq \mathbb{R}$ be a time-domain. For $X \in \mathrm{LI}\Gamma^\nu(\mathbb{T}; \mathsf{TM})$ define $X^T : \mathbb{T} \times \mathsf{TM} \to \mathsf{TTM}$ by $X^T(t, v_x) = (X(t))^T(v_x)$. Then*

(i) $X^T \in \mathrm{LI}\Gamma^{\nu-1}(\mathbb{T}; \mathsf{TTM})$,

(ii) if $(t, t_0, x_0) \in D_X$, then $(t, t_0, v_{x_0}) \in D_{X^T}$ for every $v_{x_0} \in \mathsf{T}_{x_0}\mathsf{M}$, and

(iii) $X^T(t, v_x) = \frac{\mathrm{d}}{\mathrm{d}\tau}\big|_{\tau=0} T_x\Phi^X_{t+\tau,t}(v_x)$.

Proof: (i) Since differentiation with respect to x preserves measurability in t,[15] and since the coordinate expression for X^T involves differentiating the coordinate expression for X, we conclude that X^T is a Carathéodory vector field. To show that $X^T \in \mathrm{LI}\Gamma^{\nu-1}(\mathbb{T}; \mathsf{TTM})$ requires, according to our definitions of Section 6, an affine connection on TM and a Riemannian metric on TM. We suppose, of course, that we have an affine connection ∇ and a Riemannian metric $\mathbb{G}$ on M. For simplicity of some of the computations below, and without loss of generality, we shall suppose that ∇ is torsion-free. In case $r = \omega$, we suppose these are real analytic, according to Lemma 2.3. In case $\nu = m + \mathrm{lip}$ for some $m \in \mathbb{Z}_{>0}$, we assume that ∇ is the Levi-Civita connection associated with $\mathbb{G}$.

Let us first describe the Riemannian metric on TM we shall use. The affine connection ∇ gives a splitting $\mathsf{TTM} \simeq \pi^*_{\mathsf{TM}}\mathsf{TM} \oplus \pi^*_{\mathsf{TM}}\mathsf{TM}$ [Kolář, Michor, and Slovák 1993, §11.11]. We adopt the convention that the second component of this decomposition is the vertical component so $T_{v_x}\pi_{\mathsf{TM}}$ restricted to the first component is an isomorphism onto $\mathsf{T}_x\mathsf{M}$, i.e., the first component is "horizontal." If $X \in \Gamma^\nu(\mathsf{TM})$ we denote by $X^H \in \Gamma^\nu(\mathsf{TTM})$ the unique horizontal vector field for which $T_{v_x}\pi_{\mathsf{TM}}(X^H(v_x)) = X(x)$ for every $v_x \in \mathsf{TM}$, i.e., X^H is the "horizontal lift" of X. Let us denote by $\mathrm{hor}, \mathrm{ver} : \mathsf{TTM} \to \pi^*_{\mathsf{TM}}\mathsf{TM}$ the projections onto the first and second components of the direct sum decomposition. This then immediately gives a Riemannian metric $\mathbb{G}^T$ on TM by

$$\mathbb{G}^T(X_{v_x}, Y_{v_x}) = \mathbb{G}(\mathrm{hor}(X_{v_x}), \mathrm{hor}(Y_{v_x})) + \mathbb{G}(\mathrm{ver}(X_{v_x}), \mathrm{ver}(Y_{v_x})).$$

This is called the ***Sasaki metric*** [Sasaki 1958] in the case that ∇ is the Levi-Civita connection associated with $\mathbb{G}$.

Now let us determine how an affine connection on TM can be constructed from ∇. There are a number of ways to lift an affine connection from M to one on TM, many of

[15] Derivatives are limits of sequences of difference quotients, each of which is measurable, and limits of sequences of measurable functions are measurable [Cohn 2013, Proposition 2.1.5].

these being described by Yano and Ishihara [1973]. We shall use the so-called "tangent lift" of ∇, which is the unique affine connection ∇^T on TM satisfying $\nabla^T_{X^T} Y^T = (\nabla_X Y)^T$ for $X, Y \in \Gamma^\nu(\mathsf{TM})$ [Yano and Kobayashi 1966, §7], [Yano and Ishihara 1973, page 30].

We have the following sublemma.

1 Sublemma: *If $X \in \Gamma^\nu(\mathsf{TM})$, if $v_x \in \mathsf{TM}$, if $k \in \mathbb{Z}_{\geq 0}$ satisfies $k \leq \nu$, if $X_1, \dots, X_k \in \Gamma^\infty(\mathsf{TM})$, and if $Z_a \in \{X_a^T, X_a^V\}$, $a \in \{1, \dots, k\}$, then the following formula holds:*

$$(\nabla^T)^{(k)} X^T(Z_1, \dots, Z_k) = \begin{cases} (\nabla^{(k)} X(X_1, \dots, X_k))^V, & Z_a \text{ vertical for some } a \in \{1, \dots, k\}, \\ (\nabla^{(k)} X(X_1, \dots, X_k))^T, & \text{otherwise.} \end{cases}$$

Proof: By [Yano and Kobayashi 1966, Proposition 7.2], we have

$$\nabla^T X^T(X_1^T) = (\nabla X(X_1))^T, \quad \nabla^T X^T(X_1^V) = (\nabla X(X_1))^V,$$

giving the result when $k = 1$. Suppose the result is true for $k \in \mathbb{Z}_{>0}$, and let $Z_a \in \{X_a^T, X_a^V\}$, $a \in \{1, \dots, m+1\}$. First suppose that $Z_{k+1} = X_{k+1}^T(v_x)$. We then compute, using the fact that covariant differentiation commutes with contraction [Dodson and Poston 1991, Theorem 7.03(F)],

$$\begin{aligned} (\nabla^T)^{(k+1)} X^T(Z_1, \dots, Z_m, Z_{k+1}) = \nabla^T_{X^T_{k+1}}&((\nabla^T)^{(k)} X^T)(Z_1, \dots, Z_k) \\ &- \sum_{j=1}^k (\nabla^T)^{(k)} X^T(Z_1, \dots, \nabla^T_{X^T_{k+1}} Z_j, \dots, Z_k). \end{aligned} \tag{9.4}$$

We now consider two cases.

1. *None of $Z_1, \dots, Z_k$ are vertical:* In this case, by the induction hypothesis,

$$((\nabla^T)^{(k)} X^T)(Z_1, \dots, Z_k) = (\nabla^{(k)} X)(U_1, \dots, U_k))^T,$$

and [Yano and Kobayashi 1966, Proposition 7.2] gives

$$\nabla^T_{X^T_{k+1}}((\nabla^T)^{(k)} X^T)(Z_1, \dots, Z_k) = (\nabla_{X_{k+1}}(\nabla^{(k)} X)(U_1, \dots, U_k))^T.$$

Again using [Yano and Kobayashi 1966, Proposition 7.2] and also using the induction hypothesis, we have, for $j \in \{1, \dots, k\}$,

$$(\nabla^T)^{(k)} X^T(Z_1, \dots, \nabla^T_{X^T_{k+1}} Z_j, \dots, Z_k) = (\nabla^{(k)} X(U_1, \dots, \nabla_{X_{k+1}} U_j, \dots, U_k))^T.$$

Combining the preceding two formulae with (9.4) gives the desired conclusion for $k+1$ in this case.

2. *At least one of $Z_1, \dots, Z_k$ is vertical:* In this case, we have

$$((\nabla^T)^{(k)} X^T)(Z_1, \dots, Z_k) = (\nabla^{(k)} X)(U_1, \dots, U_k))^V$$

by the induction hypothesis. Applications of [Yano and Kobayashi 1966, Proposition 7.2] and the induction hypothesis give the formulae

$$\nabla^T_{X^T_{k+1}}((\nabla^T)^{(k)} X^T)(Z_1, \dots, Z_k) = (\nabla_{X_{k+1}}(\nabla^{(k)} X)(U_1, \dots, U_k))^V.$$

and, for $j \in \{1, \dots, k\}$,

$$(\nabla^T)^{(k)} X^T(Z_1, \dots, \nabla^T_{X^T_{k+1}} Z_j, \dots, Z_k) = (\nabla^{(k)} X(U_1, \dots, \nabla_{X_{k+1}} U_j, \dots, U_k))^V.$$

Combining the preceding two formulae with (9.4) again gives the desired conclusion for $k+1$ in this case.

If we take $Z_{k+1} = X^V_{k+1}$, an entirely similar argument gives the result for this case for $k+1$, and so completes the proof of the sublemma. ▼

To complete the proof of the lemma, let us for the moment simply regard X as a vector field of class C^ν, not depending on time. We will make use of the fact that, for every $v_x \in \mathsf{TM}$, $\mathsf{T}_{v_x}\mathsf{TM}$ is spanned by vector fields of the form $X^T_1 + Y^V_1$ since vertical lifts obviously span the vertical space and since tangent lifts of nonzero vector fields are complementary to the vertical space. Therefore, for a fixed v_x, we can choose $X_1, \dots, X_n, Y_1, \dots, Y_n \in \Gamma^\infty(\mathsf{M})$ so that $(X^T_1(v_x), \dots, X^T_n(v_x))$ and $(Y^V_1(v_x), \dots, Y^V_n(v_x))$ comprise $\mathbb{G}^T$-orthonormal bases for the horizontal and vertical subspaces, respectively, of $\mathsf{T}_{v_x}\mathsf{TM}$. Note that these vector fields depend on v_x, but for the moment we will fix v_x. We use the following formula given by Barbero-Liñán and Lewis [2012, Lemma 4.5] for any vector field W of class C^ν on M:

$$W^T(v_x) = W^H(v_x) + \mathrm{vlft}(v_x, \nabla_{v_x} W(x)), \tag{9.5}$$

keeping in mind that we are supposing ∇ to be torsion-free.

By the sublemma, if $Z_a = X^T_{j_a}$, $a \in \{1, \dots, k\}$, then we have

$$\begin{aligned}(\nabla^T)^{(k-1)} X^T(v_x)(Z_1(v_x), \dots, Z_k(v_x)) = {} & (\nabla^{(k-1)} X(x)(X_{j_1}(x), \dots, X_{j_k}(x)))^H \\ & + \mathrm{vlft}(v_x, \nabla_{v_x}(\nabla^{(k-1)} X(X_{j_1}, \dots, X_{j_k}))(x)),\end{aligned} \tag{9.6}$$

using (9.5) with $W = \nabla^{(k-1)} X(X_{j_1}, \dots, X_{j_k})$. Again using (9.5), now with $W = X_{j_a}$, we have

$$X^T_{j_a}(v_x) = X^H_{j_a}(v_x) + \mathrm{vlft}(v_x, \nabla_{v_x} X_{j_a}(x)).$$

Since $X^T_{j_a}$ was specified so that it is horizontal at v_x, its vertical part must be zero, whence $\nabla_{v_x} X_{j_a}(x) = 0$. Therefore, expanding the second term on the right in (9.6), we get

$$\begin{aligned}(\nabla^T)^{(k-1)} X^T(v_x)(Z_1(v_x), \dots, Z_k(v_x)) = {} & (\nabla^{(k-1)} X(x)(X_{j_1}(x), \dots, X_{j_k}(x)))^H \\ & + \mathrm{vlft}(v_x, \nabla^{(k)} X(x)(X_{j_1}(x), \dots, X_{j_k}(x), v_x)).\end{aligned} \tag{9.7}$$

Symmetrising this formula with respect to $\{1, \dots, k\}$ gives

$$\begin{aligned}P^k_{\nabla^T}(X^T)(v_x)(Z_1(v_x), \dots, Z_k(v_x)) = {} & (P^k_\nabla(X)(x)(X_{j_1}(x), \dots, X_{j_k}(x)))^H \\ & + \mathrm{vlft}\Big(v_x, \nabla_{v_x} P^k_\nabla(X)(x)(X_{j_1}, \dots, X_{j_k})\Big),\end{aligned} \tag{9.8}$$

where, adopting the notation from Section 2.1, $P^k_\nabla(X) = \mathrm{Sym}_k \otimes \mathrm{id}_{\mathsf{TM}}(\nabla^{(k-1)} X)$. Now consider $Z_a \in \{X^T_{j_a}, Y^V_{j_a}\}$, $a \in \{1, \dots, k\}$, and suppose that at least one of these vector fields is vertical. Then, by the sublemma, we immediately have the estimate

$$P^k_{\nabla^T}(X^T)(v_x)(Z_1(v_x), \dots, Z_k(v_x)) = (P^k_\nabla(X_{j_1}(x), \dots, X_{j_k}(x)))^V, \tag{9.9}$$

where $\hat{X}_{j_1},\dots,\hat{X}_{j_k}$ are chosen from $X_1,\dots,X_n$ and $Y_1,\dots,Y_n$, corresponding to the way that $Z_1,\dots,Z_k$ are defined.

Now let us use these formulae in the various regularity classes to obtain the lemma.

$\nu=\infty$: Let $K\subseteq\mathsf{TM}$ be compact and let $m\in\mathbb{Z}_{\ge0}$. For the moment, suppose that X is time-independent. Combining (9.8) and (9.9), and noting that they hold as we evaluate $P^m_{\nabla^T}(X^T)(v_x)$ on a $\mathbb{G}^T$-orthonormal basis for $\mathsf{T}_{v_x}\mathsf{TM}$, we obtain the estimate

$$\|P^m_{\nabla^T}(X^T)(v_x)\|_{\mathbb{G}^T_m}\le C(\|P^m_{\nabla}(X)(x)\|_{\mathbb{G}_m}+\|P^{m+1}_{\nabla}(X)(x)\|_{\mathbb{G}_{m+1}}\|v_x\|_{\mathbb{G}}),\qquad v_x\in K,$$

for some $C\in\mathbb{R}_{>0}$. Now, if we make use of the fibre norms induced on jet bundles as in Section 2.2, we have

$$\|j_mX^T(v_x)\|_{\overline{\mathbb{G}^T}_m}\le C(\|j_mX(x)\|_{\overline{\mathbb{G}}_m}+\|j_{m+1}X(x)\|_{\overline{\mathbb{G}}_{m+1}}\|v_x\|_{\mathbb{G}}),\qquad v_x\in K,$$

for some possibly different $C\in\mathbb{R}_{>0}$. Since $v_x\mapsto\|v_x\|_{\mathbb{G}}$ is bounded on K, the previous estimate gives

$$\|j_mX^T_t(v_x)\|_{\overline{\mathbb{G}^T}_m}\le C\|j_{m+1}X_t(x)\|_{\overline{\mathbb{G}}_{m+1}},\qquad v_x\in K,\ t\in\mathbb{T},\tag{9.10}$$

for some appropriate $C\in\mathbb{R}_{>0}$.

Now we consider time-dependence, supposing that $X\in\mathrm{LI}\Gamma^\infty(\mathbb{T};\mathsf{TM})$. Then there exists $f\in\mathrm{L}^1_{\mathrm{loc}}(\mathbb{T};\mathbb{R}_{\ge0})$ such that

$$\|j_{m+1}X_t(x)\|_{\overline{\mathbb{G}}_{m+1}}\le f(t),\qquad x\in K,\ t\in\mathbb{T}.$$

We then immediately have

$$\|j_mX^T_t(v_x)\|_{\overline{\mathbb{G}^T}_m}\le Cf(t),\qquad x\in K,\ t\in\mathbb{T},$$

showing that $X^T\in\mathrm{LI}\Gamma^\infty(\mathbb{T};\mathsf{TTM})$, as desired.

$\nu=m$: This case follows directly from the computations in the smooth case.

$\nu=m+\mathrm{lip}$: Here we take $m=1$ as the general situation follows by combining this with the previous case. We consider X to be time-independent for the moment. We let $K\subseteq\mathsf{TM}$ be compact. By Lemma 3.12 we have

$$\begin{aligned}\operatorname{dil}X^T(v_x)=\inf\{\sup\{\|\nabla^T_{Y_{v_y}}X^T\|_{\mathbb{G}^T}\mid\ v_y\in\operatorname{cl}(\mathcal{W}),\ \|Y_{v_y}\|_{\mathbb{G}^T}=1,\ X^T\text{ differentiable at }v_y\}\mid\\ \mathcal{W}\text{ is a relatively compact neighbourhood of }v_x\}.\end{aligned}$$

Now we make use of Lemma 2.1, (9.10), and the fact that K is compact, to reduce this to an estimate

$$\begin{aligned}\operatorname{dil}X^T(v_x)\le C\inf\{\sup\{\|j_2X(y)\|_{\overline{\mathbb{G}}_1}\mid\ y\in\operatorname{cl}(\mathcal{U}),\ j_1X\text{ differentiable at }y\}\mid\\ \mathcal{U}\text{ a relatively compact neighbourhood of }x\}\end{aligned}$$

for some $C\in\mathbb{R}_{>0}$ and for every $x\in K$. By Lemma 3.12 then gives $\operatorname{dil}X^T(v_x)\le C\operatorname{dil}j_1X(x)$ for $x\in K$. From this we obtain the estimate

$$\lambda^{\mathrm{lip}}_K(X^T)\le Cp^{1+\mathrm{lip}}_{\pi_{\mathsf{TM}}(K)}(X).$$

From the proof above in the smooth case, we have

$$p^0_K(X^T) \le C' p^1_{\pi_{\mathsf{TM}}(K)}(X).$$

Combining these previous two estimates gives

$$p^{\mathrm{lip}}_K(X^T) \le C p^{1+\mathrm{lip}}_{\pi_{\mathsf{TM}}(K)}(X)$$

for some $C \in \mathbb{R}_{>0}$, and from this, this part of the result follows easily after adding the appropriate time-dependence.

$\nu = \omega$: For the moment, we take X to be time-independent. The following sublemma will allow us to estimate the last term in (9.8).

2 Sublemma: *Let M be a real analytic manifold, let ∇ be a real analytic affine connection on M, let $\mathbb{G}$ be a real analytic Riemannian metric on M, and let $K \subseteq \mathsf{M}$ be compact. Then there exist $C, \sigma \in \mathbb{R}_{>0}$ such that*

$$\|\nabla^k P^k_{\nabla}(X)(x)\|_{\overline{\mathbb{G}}_{k+1}} \le 2\|j_{k+1}X(x)\|_{\overline{\mathbb{G}}_{k+1}}$$

for every $x \in K$ and $k \in \mathbb{Z}_{\ge 0}$.

Proof: We use Lemma 2.1 to represent elements of $\mathsf{J}^k\mathsf{TM}$. Following [Kolář, Michor, and Slovák 1993, §17.1], we think of a connection $\tilde{\nabla}^k$ on $\mathsf{J}^k\mathsf{TM}$ as being defined by a vector bundle mapping

$$\begin{array}{ccc} \mathsf{J}^k\mathsf{TM} & \xrightarrow{\tilde{S}_k} & \mathsf{J}^1\mathsf{J}^k\mathsf{TM} \\ \downarrow & & \downarrow \\ \mathsf{M} & \longrightarrow & \mathsf{M} \end{array}$$

The connection $\nabla^{[k]}$, thought of in this way and using the decomposition of Lemma 2.1, gives the associated vector bundle mapping as zero. Now, with our identifications, we see that $P^k_{\nabla}(X) = j_kX - j_{k-1}X$, noting that $\mathsf{J}^{k-1}\mathsf{TM}$ is a subbundle of $\mathsf{J}^k\mathsf{TM}$ with our identification. Therefore, by definition of $\nabla^{[k]}$,

$$\nabla^k(P^k_{\nabla}(X)) = \nabla^{[k]}(j_kX - j_{k-1}X) = j_1(j_kX - j_{k-1}X).$$

As we pointed out in the proof of Lemma 9.1 above, the inclusion of $\mathsf{J}^{k+1}\mathsf{TM}$ in $\mathsf{J}^1\mathsf{J}^k\mathsf{TM}$ preserves the fibre metric. Therefore,

$$\|\nabla^k(P^k_{\nabla}(X))(x)\|_{\mathbb{G}_k} \le \|j_{k+1}X(x)\|_{\overline{\mathbb{G}}_{k+1}} + \|j_kX(x)\|_{\overline{\mathbb{G}}_k} \le 2\|j_{k+1}X(x)\|_{\overline{\mathbb{G}}_{k+1}},$$

as desired. ▼

Let $K \subseteq \mathsf{TM}$ be compact and let $\boldsymbol{a} \in \mathrm{c}_{\downarrow 0}(\mathbb{Z}_{\ge 0}; \mathbb{R}_{>0})$. As in the smooth case, but now using the preceding sublemma, we obtain an estimate

$$\|j_mX^T(v_x)\|_{\overline{\mathbb{G}^T}_m} \le C\|j_{m+1}X(x)\|_{\overline{\mathbb{G}}_{m+1}}, \qquad x \in K,\ m \in \mathbb{Z}_{\ge 0},$$

for some suitable $C \in \mathbb{R}_{>0}$.

Now, taking $X \in \mathrm{LI}\Gamma^\omega(\mathbb{T};\mathsf{TM})$, there exists $f \in \mathrm{L}^1_{\mathrm{loc}}(\mathbb{T};\mathbb{R}_{\geq 0})$ such that

$$a'_0 a'_1 \cdots a'_{m+1} \|j_{m+1} X_t(x)\|_{\overline{\mathbb{G}}_{m+1}} \leq f(t), \qquad x \in K,\ t \in \mathbb{T},\ m \in \mathbb{Z}_{\geq 0},$$

where $a'_{j+1} = a_j$, $j \in \{1, \dots, m\}$, and $a'_0 = C$. We then immediately have

$$a_0 a_1 \cdots a_m \|j_m X^T_t(v_x)\|_{\overline{\mathbb{G}^T}_m} \leq f(t), \qquad x \in K,\ t \in \mathbb{T},\ m \in \mathbb{Z}_{\geq 0},$$

showing that $X^T \in \mathrm{LI}\Gamma^\omega(\mathbb{T};\mathsf{TTM})$, as desired.

(iii) We now prove the third assertion. It is local, so we work in a chart. Thus we assume that we are working in an open subset $\mathcal{U} \subseteq \mathbb{R}^n$. We let $\boldsymbol{X}\colon \mathbb{T} \times \mathcal{U} \to \mathbb{R}^n$ be the principal part of the vector field so that a trajectory for $\boldsymbol{X}$ is a curve $\boldsymbol{\xi}\colon \mathbb{T} \to \mathcal{U}$ satisfying

$$\frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{\xi}(t) = \boldsymbol{X}(t, \boldsymbol{\xi}(t)), \qquad \text{a.e. } t \in \mathbb{T}.$$

The solution with initial condition $\boldsymbol{x}_0$ and t_0 we denote by $t \mapsto \Phi^{\boldsymbol{X}}(t, t_0, \boldsymbol{x}_0)$. For fixed $(t_0, \boldsymbol{x}_0) \in \mathbb{T} \times \mathcal{U}$ and for t sufficiently close to t_0, let us define a linear map $\boldsymbol{\Psi}(t) \in \mathrm{Hom}_{\mathbb{R}}(\mathbb{R}^n;\mathbb{R}^n)$ by

$$\boldsymbol{\Psi}(t) \cdot \boldsymbol{w} = \boldsymbol{D}_3 \Phi^{\boldsymbol{X}}(t, t_0, \boldsymbol{x}_0) \cdot \boldsymbol{w}.$$

We have

$$\frac{\mathrm{d}}{\mathrm{d}t}\Phi^{\boldsymbol{X}}(t, t_0, \boldsymbol{x}_0) = \boldsymbol{X}(t, \Phi^{\boldsymbol{X}}(t, t_0, \boldsymbol{x}_0)), \qquad \text{a.e. } t,$$

for t sufficiently close to t_0. Therefore,

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{D}_3 \Phi^{\boldsymbol{X}}(t, t_0, \boldsymbol{x}_0) &= \boldsymbol{D}_3(\tfrac{\mathrm{d}}{\mathrm{d}t}\Phi^{\boldsymbol{X}}(t, t_0, \boldsymbol{x}_0)) \\ &= \boldsymbol{D}_2 \boldsymbol{X}(t, \Phi^{\boldsymbol{X}}(t, t_0, \boldsymbol{x}_0)) \cdot \boldsymbol{D}_3 \Phi^{\boldsymbol{X}}(t, t_0, \boldsymbol{x}_0).\end{aligned}$$

In the preceding expression, we have used [Schuricht and von der Mosel 2000, Corollary 2.2] to swap the time and spatial derivatives. This shows that $t \mapsto \boldsymbol{\Psi}(t)$ satisfies the initial value problem

$$\frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{\Psi}(t) = \boldsymbol{D}_2 \boldsymbol{X}(t, \Phi^{\boldsymbol{X}}(t, t_0, \boldsymbol{x}_0)) \cdot \boldsymbol{\Psi}(t), \qquad \boldsymbol{\Psi}(t_0) = \boldsymbol{I}_n.$$

By [Sontag 1998, Proposition C.3.8], $t \mapsto \boldsymbol{\Psi}(t)$ can be defined for all t such that $(t, t_0, \boldsymbol{x}_0) \in D_{\boldsymbol{X}}$. Moreover, for $\boldsymbol{v}_0 \in \mathbb{R}^n$ (which we think of as being the tangent space at $\boldsymbol{x}_0$), the curve $t \mapsto \boldsymbol{v}(t) \triangleq \boldsymbol{\Psi}(t) \cdot \boldsymbol{v}_0$ satisfies

$$\frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{v}(t) = \boldsymbol{D}_2 \boldsymbol{X}(t, \Phi^{\boldsymbol{X}}(t, t_0, \boldsymbol{x}_0)) \cdot \boldsymbol{v}(t).$$

Returning now to geometric notation, the preceding chart computations, after sifting through the notation, show that

$$\Phi^{X^T}(t, t_0, v_{x_0}) = T_x \Phi^X(t, t_0, x_0)(v_{x_0}),$$

and differentiation with respect to t at t_0 gives this part of the lemma.

(ii) This was proved along the way to proving (iii). ■

We will also use some features of the geometry of the double tangent bundle, i.e., TTM. This is an example of what is known as a "double vector bundle," and we refer to [Mackenzie 2005, Chapter 9] as a comprehensive reference. A review of the structure we describe here can be found [Barbero-Liñán and Lewis 2012], along with an interesting application of this structure. We begin by noting that the double tangent bundle possesses two natural vector bundle structures over $\pi_{\mathsf{TM}}\colon \mathsf{TM} \to \mathsf{M}$:

$$\begin{tikzcd} \mathsf{TTM} \arrow[r, "\pi_{\mathsf{TTM}}"] \arrow[d, "T\pi_{\mathsf{TM}}"'] & \mathsf{TM} \arrow[d, "\pi_{\mathsf{TM}}"] \\ \mathsf{TM} \arrow[r, "\pi_{\mathsf{TM}}"'] & \mathsf{M} \end{tikzcd} \qquad \begin{tikzcd} \mathsf{TTM} \arrow[r, "T\pi_{\mathsf{TM}}"] \arrow[d, "\pi_{\mathsf{TTM}}"'] & \mathsf{TM} \arrow[d, "\pi_{\mathsf{TM}}"] \\ \mathsf{TM} \arrow[r, "\pi_{\mathsf{TM}}"'] & \mathsf{M} \end{tikzcd}$$

The left vector bundle structure is called the ***primary vector bundle*** and the right the ***secondary vector bundle***. We shall denote vector addition in the vector bundles as follows. If $u, v \in \mathsf{TTM}$ satisfy $\pi_{\mathsf{TTM}}(u) = \pi_{\mathsf{TTM}}(v)$, then the sum of u and v in the primary vector bundle is denoted by $u +_1 v$. If $u, v \in \mathsf{TTM}$ satisfy $T\pi_{\mathsf{TM}}(u) = T\pi_{\mathsf{TM}}(v)$, then the sum of u and v in the secondary vector bundle is denoted by $u +_2 v$.

The two vector bundle structures admit a naturally defined isomorphism between them, described as follows. Let ρ be a smooth map from a neighbourhood of $(0,0) \in \mathbb{R}^2$ to M. We shall use coordinates (s,t) for $\mathbb{R}^2$. For fixed s and t define $\rho_s(t) = \rho^t(s) = \rho(s,t)$. We then denote

$$\frac{\partial}{\partial t}\rho(s,t) = \frac{\mathrm{d}}{\mathrm{d}t}\rho_s(t) \in \mathsf{T}_{\rho(s,t)}\mathsf{M}, \quad \frac{\partial}{\partial s}\rho(s,t) = \frac{\mathrm{d}}{\mathrm{d}s}\rho^t(s) \in \mathsf{T}_{\rho(s,t)}\mathsf{M}.$$

Note that $s \mapsto \frac{\partial}{\partial t}\rho(s,t)$ is a curve in TM for fixed t. The tangent vector field to this curve we denote by

$$s \mapsto \frac{\partial}{\partial s}\frac{\partial}{\partial t}\rho(s,t) \in \mathsf{T}_{\frac{\partial}{\partial t}\rho(s,t)}\mathsf{TM}.$$

We belabour the development of the notation somewhat since these partial derivatives are not the usual partial derivatives from calculus, although the notation might make one think they are. For example, we do not generally have equality of mixed partials, i.e., generally we have

$$\frac{\partial}{\partial s}\frac{\partial}{\partial t}\rho(s,t) \neq \frac{\partial}{\partial t}\frac{\partial}{\partial s}\rho(s,t).$$

Now let ρ_1 and ρ_2 be smooth maps from a neighbourhood of $(0,0) \in \mathbb{R}^2$ to M. We say two such maps are **equivalent** if

$$\frac{\partial}{\partial s}\frac{\partial}{\partial t}\rho_1(0,0) = \frac{\partial}{\partial s}\frac{\partial}{\partial t}\rho_2(0,0).$$

To the equivalence classes of this equivalence relation, we associate points in TTM by

$$[\rho] \mapsto \frac{\partial}{\partial s}\frac{\partial}{\partial t}\rho(0,0).$$

The set of equivalence classes is easily seen to be exactly the double tangent bundle TTM. We easily verify that

$$\pi_{\mathsf{TTM}}([\rho]) = \frac{\partial}{\partial t}\rho(0,0), \quad T\pi_{\mathsf{TM}}([\rho]) = \frac{\partial}{\partial s}\rho(0,0). \tag{9.11}$$

Next, using the preceding representation of points in TTM, we relate the two vector bundle structures for TTM by defining a canonical involution of TTM. If ρ is a smooth map from a neighbourhood of $(0,0) \in \mathbb{R}^2$ into M, define another such map by $\bar{\rho}(s,t) = \rho(t,s)$. We then define the ***canonical tangent bundle involution*** as the map $I_{\mathsf{M}} \colon \mathsf{TTM} \to \mathsf{TTM}$ given by $I_{\mathsf{M}}([\rho]) = [\bar{\rho}]$. Clearly $I_{\mathsf{M}} \circ I_{\mathsf{M}} = \mathrm{id}_{\mathsf{TTM}}$. In a natural coordinate chart for TTM associated to a natural coordinate chart for TM, the local representative of I_{M} is

$$((\boldsymbol{x},\boldsymbol{v}),(\boldsymbol{u},\boldsymbol{w})) \mapsto ((\boldsymbol{x},\boldsymbol{u}),(\boldsymbol{v},\boldsymbol{w})).$$

One readily verifies that I_{M} is a vector bundle isomorphism from TTM with the primary (resp. secondary) vector bundle structure to TTM with the secondary (resp. primary) vector bundle structure [Barbero-Liñán and Lewis 2012, Lemma A.4].

The following technical lemma is Lemma A.5 from [Barbero-Liñán and Lewis 2012].

9.3 Lemma: (A property of vertical lifts) *If $w \in \mathsf{TTM}$ satisfies $\pi_{\mathsf{TTM}}(w) = v$ and $T\pi_{\mathsf{TM}} = u$ and if $z \in \mathsf{T}_x\mathsf{M}$, then*

$$w +_2 I_{\mathsf{M}} \circ \mathrm{vlft}(u,z) = w +_1 \mathrm{vlft}(v,z).$$

The final piece of tangent bundle geometry we will consider concerns presheaves and sheaves of sets of vector fields on tangent bundles. We shall need the following natural notion of such a presheaf.

9.4 Definition: (Projectable presheaf) Let $m \in \mathbb{Z}_{\geq 0}$ and $m' \in \{0, \mathrm{lip}\}$, let $\nu \in \{m + m', \infty, \omega\}$, and let $r \in \{\infty, \omega\}$, as required. Let M be a C^r-manifold and let $\mathscr{G}$ be a presheaf of sets of vector fields of class C^ν on TM. The presheaf $\mathscr{G}$ is ***projectable*** if

$$\mathscr{G}(\mathcal{W}) = \{Z|\mathcal{W} \mid \; Z \in \mathscr{G}(\pi_{\mathsf{TM}}^{-1}(\pi_{\mathsf{TM}}(\mathcal{W})))\}. \qquad \bullet$$

The idea is that a projectable sheaf is determined by the local sections over the open sets $\pi_{\mathsf{TM}}^{-1}(\mathcal{U})$ for $\mathcal{U} \subseteq \mathsf{M}$ open.

9.2. Linearisation of systems. Throughout this section, unless stated otherwise, we let $m \in \mathbb{Z}_{>0}$, $m' \in \{0, \mathrm{lip}\}$, and let $\nu \in \{m + m', \infty, \omega\}$. We take $r \in \{\infty, \omega\}$, as required.

When linearising, one typically does so about a trajectory. We will do this also. But before we do so, let us provide the notion of the linearisation of a *system*. The result, gratifyingly, is a system on the tangent bundle. Before we produce the definition, let us make a motivating computation. We let $\mathfrak{G} = (\mathsf{M}, \mathscr{F})$ be a globally generated tautological control system of class C^ν. By Example 8.10–2, we have the corresponding C^ν-control system $\Sigma_{\mathfrak{G}} = (\mathsf{M}, F_{\mathscr{F}}, \mathcal{C}_{\mathscr{F}})$ with $\mathcal{C}_{\mathscr{F}} = \mathscr{F}(\mathsf{M})$ and $F_{\mathscr{F}}(x, X) = X(x)$. This is a control system whose control set is a vector space, and so is a candidate for classical Jacobian linearisation, provided one is prepared to overlook technicalities of differentiation in locally convex spaces... and we are for the purposes of this motivational computation. In Jacobian linearisation one considers perturbations of state and control. In our framework, we linearise about a state/control (x, X). We perturb the state by considering a C^1-curve $\gamma \colon J \to \mathsf{M}$ defined on an interval J for which $0 \in \mathrm{int}(J)$ and with $\gamma'(0) = v_x$. Thus we perturb the state in the direction of v_x. We perturb the control from X in the direction of $Y \in \mathscr{F}(\mathsf{M})$

by considering a curve of controls $s \mapsto X + sY$. Let us then define $\sigma\colon \mathcal{N} \to \mathsf{M}$ on a neighbourhood $\mathcal{N}$ of $(0,0) \in \mathbb{R}^2$ by

$$\sigma(t,s) = \Phi_t^{X+sY}(\gamma(s));$$

thus $\sigma(t,s)$ gives the flow at time t corresponding to the perturbation at parameter s. Now we compute

$$\begin{aligned}
\frac{\partial}{\partial t}\frac{\partial}{\partial s}\sigma(t,s) &= \frac{\partial}{\partial t}\frac{\partial}{\partial s}\Phi_t^{X+sY}(\gamma(s)) \\
&= \frac{\partial}{\partial t}\frac{\partial}{\partial s}\Phi_t^{X}(\gamma(s)) + \frac{\partial}{\partial t}\frac{\partial}{\partial s}\Phi_t^{X+sY}(x) \\
&= \frac{\partial}{\partial t}T_x\Phi_t^{X}(\gamma'(s)) + I_{\mathsf{M}}\Big(\frac{\partial}{\partial s}\frac{\partial}{\partial t}\Phi_t^{X+sY}(x)\Big) \\
&= \frac{\partial}{\partial t}T_x\Phi_t^{X}(\gamma'(s)) + I_{\mathsf{M}}\Big(\frac{\partial}{\partial s}(X+sY)(\Phi_t^{X+sY}(x))\Big),
\end{aligned}$$

from which we have

$$\frac{\partial}{\partial t}\frac{\partial}{\partial s}\sigma(0,0) = X^T(v_x) + I_{\mathsf{M}}(\mathrm{vlft}(X(x),Y(x))) = X^T(v_x) + Y^V(v_x), \tag{9.12}$$

using Lemma 9.3.

The formula clearly suggests what the linearisation of a tautological control system should be. However, we need the following lemma to make a sensible definition in our sheaf framework.

9.5 Lemma: (Presheaves for linearisation) *Let $m \in \mathbb{Z}_{>0}$ and $m' \in \{0, \mathrm{lip}\}$, let $\nu \in \{m+m', \infty, \omega\}$, and let $r \in \{\infty, \omega\}$, as required. Let $\mathscr{F}$ be a presheaf of sets of C^ν-vector fields on a C^r-manifold M. Then there exist unique projectable presheaves $\mathscr{F}^T$ and $\mathscr{F}^V$ of $\mathrm{C}^{\nu-1}$-vector fields and C^ν-vector fields on TM with the property that*

$$\mathscr{F}^T(\pi_{\mathsf{TM}}^{-1}(\mathcal{U})) = \{X^T \mid X \in \mathscr{F}(\mathcal{U})\}$$

and

$$\mathscr{F}^V(\pi_{\mathsf{TM}}^{-1}(\mathcal{U})) = \{X^V \mid X \in \mathscr{F}(\mathcal{U})\}$$

for every open set $\mathcal{U} \subseteq \mathsf{M}$. Moreover,

(i) $\mathscr{F}^T$ is a sheaf if and only if $\mathscr{F}$ is a sheaf,

(ii) $\mathscr{F}^V$ is a sheaf if and only if $\mathscr{F}$ is a sheaf,

(iii) $\mathrm{Sh}(\mathscr{F}^T) = \mathrm{Sh}(\mathscr{F})^T$, and

(iv) $\mathrm{Sh}(\mathscr{F}^V) = \mathrm{Sh}(\mathscr{F})^V$.

Proof: Let $\mathcal{W} \subseteq \mathsf{TM}$ be open and note that $\mathcal{U}_{\mathcal{W}} = \pi_{\mathsf{TM}}(\mathcal{W})$ is open. For $\mathcal{W} \subseteq \mathsf{TM}$ open we define

$$\mathscr{F}^T(\mathcal{W}) = \{X^T|\mathcal{W} \mid X \in \mathscr{F}(\mathcal{U}_{\mathcal{W}})\}$$

and

$$\mathscr{F}^V(\mathcal{W}) = \{X^V|\mathcal{W} \mid X \in \mathscr{F}(\mathcal{U}_{\mathcal{W}})\}.$$

If $\mathcal{W}, \mathcal{W}' \subseteq \mathsf{TM}$ are open with $\mathcal{W}' \subseteq \mathcal{W}$ and if $X^T|\mathcal{W} \in \mathscr{F}^T(\mathcal{W})$, then, for $v_x \in \mathcal{W}'$, we have

$$(X^T(v_x)|\mathcal{W}')(v_x) = ((X|\mathcal{U}_{\mathcal{W}'})^T)(v_x),$$

this making sense since $X^T(v_x)$ depends only on the values of X in a neighbourhood of x, and since $\mathcal{U}_{\mathcal{W}'}$ contains a neighbourhood of x if $v_x \in \mathcal{W}'$. In any case, we have that

$$X^T|\mathcal{W}' \in \mathscr{F}^T(\mathcal{W}'),$$

which shows that $\mathscr{F}^T$ is a presheaf. A similar argument, of course, works for $\mathscr{F}^V$. This gives the existence assertion of the lemma. Uniqueness follows immediately from the requirement that $\mathscr{F}^T$ and $\mathscr{F}^V$ be projectable.

(i) Suppose that $\mathscr{F}$ is a sheaf. We shall first show that $\mathscr{F}^T$ is a sheaf. Let $\mathcal{W} \subseteq \mathsf{TM}$ be open, and let $(\mathcal{W}_a)_{a\in A}$ be an open cover of $\mathcal{W}$. Let $Z_a \in \mathscr{F}^T(\mathcal{W}_a)$, supposing that

$$Z_a|\mathcal{W}_a \cap \mathcal{W}_b = Z_b|\mathcal{W}_a \cap \mathcal{W}_b$$

for $a, b \in A$. For each $a \in A$, we have, by our definition of $\mathscr{F}^T$ above, $Z_a = X_a^T|\mathcal{W}_a$ for $X_a \in \mathscr{F}(\mathcal{U}_{\mathcal{W}_a})$. Using the fact that $\Gamma^{\nu-1}(\mathsf{TTM})$ is a sheaf, we infer that there exists $Z \in \Gamma^{\nu-1}(\mathsf{TTM})$ such that $Z|\mathcal{W}_a = X_a^T|\mathcal{W}_a$ for each $a \in A$. Now, for each $x \in \mathcal{U}_{\mathcal{W}}$, let us fix $a_x \in A$ such that $x \in \pi_{\mathsf{TM}}(\mathcal{W}_a)$. Note that $Z|\mathcal{W}_{a_x} = X_{a_x}^T|\mathcal{W}_{a_x}$ and so there is a neighbourhood $\mathcal{U}_x \subseteq \mathcal{U}_{\mathcal{W}_{a_x}}$ of x and $X_x \in \Gamma^{\nu-1}(\mathsf{T}\mathcal{U}_x)$ such that $X_x = X_{a_x}|\mathcal{U}_x$. In particular, $X_x \in \mathscr{F}(\mathcal{U}_x)$. Moreover, since $\mathscr{F}^T$ is projectable, we can easily see that $[X_x]_x$ is independent of the rule for choosing a_x. Now let $x_1, x_2 \in \mathsf{M}$ and let $x \in \mathcal{U}_{x_1} \cap \mathcal{U}_{x_2}$. By projectability of $\mathscr{F}^T$, there exist a neighbourhood $\mathcal{V}_x \subseteq \mathcal{U}_{x_1} \cap \mathcal{U}_{x_2}$ and $X'_x \in \mathscr{F}(\mathcal{V}_x)$ such that

$$X_{a_{x_j}}^T|\mathcal{W}_{a_{x_j}} \cap \pi_{\mathsf{TM}}^{-1}(\mathcal{V}_x) = (X'_x)^T|\mathcal{W}_{a_{x_j}}, \qquad j \in \{1, 2\}.$$

We conclude, therefore, that $X_{x_1}(x) = X_{x_2}(x)$. Thus we have an open covering $(\mathcal{U}_x)_{x\in\mathcal{U}_{\mathcal{W}}}$ of $\mathcal{U}_{\mathcal{W}}$ and local sections $X_x \in \mathscr{F}(\mathcal{U}_x)$ pairwise agreeing on intersections. Since $\mathscr{F}$ is a sheaf, there exists $X \in \mathscr{F}(\mathcal{U}_{\mathcal{W}})$ such that $X|\mathcal{U}_x = X_x$ for each $x \in \mathcal{U}_{\mathcal{W}}$. Since

$$X^T|\mathcal{W}_{a_x} \cap \pi_{\mathsf{TM}}^{-1}(\mathcal{U}_x) = X_x^T|\mathcal{W}_{a_x} \cap \pi_{\mathsf{TM}}^{-1}(\mathcal{U}_x) = X_{a_x}^T|\mathcal{W}_{a_x} \cap \pi_{\mathsf{TM}}^{-1}(\mathcal{U}_x),$$

projectability of $\mathscr{F}^T$ allows us to conclude that $Z = X^T|\mathcal{W}$.

Now suppose that $\mathscr{F}^T$ is a sheaf and let $\mathcal{U} \subseteq \mathsf{M}$ be open, let $(\mathcal{U}_a)_{a\in A}$ be an open covering of $\mathcal{U}$, and let $X_a \in \mathscr{F}(\mathcal{U}_a)$, $a \in A$ be such that $X_a|\mathcal{U}_a \cap \mathcal{U}_b = X_b|\mathcal{U}_a \cap \mathcal{U}_b$. This implies that

$$X_a^T|\pi_{\mathsf{TM}}^{-1}(\mathcal{U}_a \cap \mathcal{U}_b) = X_b^T|\pi_{\mathsf{TM}}^{-1}(\mathcal{U}_a \cap \mathcal{U}_b).$$

Therefore, by hypothesis, there exists $X \in \mathscr{F}(\mathcal{U})$ such that $X^T|\pi_{\mathsf{TM}}^{-1}(\mathcal{U}_a) = X_a^T$ for each $a \in A$. Projecting to M gives $X|\mathcal{U}_a = X_a$ for each $a \in A$, showing that $\mathscr{F}$ is a sheaf.

(ii) To show that $\mathscr{F}^V$ is a sheaf can be made with an identically styled argument as above in showing that $\mathscr{F}^T$ is a sheaf. The argument, indeed, is even easier since vertical lifts do not depend on the value of their projections in a neighbourhood of a point in TM, only on the projection at the point.

(iii) Let $\mathcal{W} \subseteq \mathsf{TM}$ be open and let $Z \in \mathrm{Sh}(\mathscr{F}^T)(\mathcal{W})$. This means that, for each $v_x \in \mathcal{W}$, $[Z]_{v_x} \in \mathscr{F}_{0,v}^T$. Therefore, there exist a neighbourhood $\mathcal{W}_{v_x}$ of v_x and $X_x \in \mathscr{F}(\mathcal{U}_{\mathcal{W}_{v_x}})$ such that $Z|\mathcal{W}_{v_x} = X_x^T|\mathcal{W}_{v_x}$. We now proceed as in the preceding part of the proof. Thus, for each $x \in \mathcal{U}_{\mathcal{W}}$ let us fix $v_x \in \mathcal{W}$. Note that $Z|\mathcal{W}_{v_x} = X^T_{v_x}|\mathcal{W}_{v_x}$ and so there is a neighbourhood $\mathcal{U}_x \subseteq \mathcal{U}_{\mathcal{W}_{v_x}}$ of x and $X_x \in \Gamma^{\nu-1}(\mathsf{T}\mathcal{U}_x)$ such that $X_x = X_{v_x}|\mathcal{U}_x$. In particular, $X_x \in \mathscr{F}(\mathcal{U}_x)$. Moreover, since $\mathscr{F}^T$ is projectable, we can easily see that $[X_x]_x$ is independent of the rule for choosing $v_x \in \mathcal{W}$. Now let $x_1, x_2 \in \mathsf{M}$ and let $x \in \mathcal{U}_{x_1} \cap \mathcal{U}_{x_2}$. By projectability of $\mathscr{F}^T$, there exist a neighbourhood $\mathcal{V}_x \subseteq \mathcal{U}_{x_1} \cap \mathcal{U}_{x_2}$ and $X'_x \in \mathscr{F}(\mathcal{V}_x)$ such that

$$X^T_{v_{x_j}}|\mathcal{W}_{v_{x_j}} \cap \pi^{-1}_{\mathsf{TM}}(\mathcal{V}_x) = (X'_x)^T|\mathcal{W}_{v_{x_j}}, \qquad j \in \{1,2\}.$$

We conclude, therefore, that $X_{x_1}(x) = X_{x_2}(x)$. Thus we have an open covering $(\mathcal{U}_x)_{x\in\mathcal{U}_{\mathcal{W}}}$ and local sections $X_x \in \mathscr{F}(\mathcal{U}_x)$ pairwise agreeing on intersections. Thus there exists $X \in \mathrm{Sh}(\mathscr{F}(\mathcal{U}_{\mathcal{W}}))$ such that $X|\mathcal{U}_x = X_x$ for each $x \in \mathcal{U}_{\mathcal{W}}$. Since

$$X^T|\mathcal{W}_{v_x} \cap \pi^{-1}_{\mathsf{TM}}(\mathcal{U}_x) = X^T_x|\mathcal{W}_{v_x} \cap \pi^{-1}_{\mathsf{TM}}(\mathcal{U}_x) = X^T_{v_x}|\mathcal{W}_{v_x} \cap \pi^{-1}_{\mathsf{TM}}(\mathcal{U}_x),$$

projectability of $\mathrm{Sh}(\mathscr{F}^T)$ allows us to conclude that $Z = X^T|\mathcal{W}$, i.e., $Z \in \mathrm{Sh}(\mathscr{F})^T(\mathcal{W})$.

(iv) A similar argument as in the preceding part of the proof works to give this part of the proof as well. ■

With the preceding computations and sheaf lemma as motivation, we make the following definition.

9.6 Definition: (Linearisation of a tautological control system) Let $m \in \mathbb{Z}_{>0}$ and $m' \in \{0, \mathrm{lip}\}$, let $\nu \in \{m+m', \infty, \omega\}$, and let $r \in \{\infty, \omega\}$, as required. Let $\mathfrak{G} = (\mathsf{M}, \mathscr{F})$ be a C^ν-tautological control system. The ***linearisation*** of $\mathfrak{G}$ is the $\mathrm{C}^{\nu-1}$-tautological control system $T\mathfrak{G} = (\mathsf{TM}, T\mathscr{F})$, where the projectable presheaf of sets of vector fields $T\mathscr{F}$ is characterised uniquely by the requirement that, for every open subset $\mathcal{U} \subseteq \mathsf{M}$,

$$T\mathscr{F}(\pi^{-1}_{\mathsf{TM}}(\mathcal{U})) = \{X^T + Y^V \mid X, Y \in \mathscr{F}(\mathcal{U})\}. \qquad \bullet$$

This definition may look a little strange at a first glance. However, as we go along, we shall use the definition in more commonplace settings, and we will see then that it connects to more familiar constructions.

9.3. Trajectories for linearisations. As a tautological control system, $T\mathfrak{G}$ provides a forum for all of the constructions of Sections 8.2, 8.3, and 8.4 concerning such systems. In particular, the linearisation has trajectories, so let us look at these.

Let us first think about open-loop systems. By definition, an open-loop system for $T\mathfrak{G}$ is a triple $(Z, \mathbb{T}, \mathcal{W})$ with $\mathbb{T} \subseteq \mathbb{R}$ an interval, $\mathcal{W} \subseteq \mathsf{TM}$ an open set, and $Z \in \mathrm{LI}\Gamma^{\nu-1}(\mathbb{T}; T\mathscr{F}(\mathcal{W}))$. Thus $Z(t) = X(t)^T + Y(t)^V$ for $X, Y \colon \mathbb{T} \to \mathscr{F}(\pi_{\mathsf{TM}}(\mathcal{W}))$. We will write $Z = X^T + Y^V$ with the understanding that this means precisely what we have just written. We should, however, verify that X and Y have useful properties.

9.7 Lemma: (Property of open-loop systems for linearisation) *Let $m \in \mathbb{Z}_{>0}$ and $m' \in \{0, \mathrm{lip}\}$, let $\nu \in \{m+m', \infty, \omega\}$, and let $r \in \{\infty, \omega\}$, as required. Let $\mathfrak{G} = (\mathsf{M}, \mathscr{F})$ be a C^ν-tautological control system with linearisation $T\mathfrak{G} = (\mathsf{TM}, T\mathscr{F})$. Let $\mathbb{T}$ be a time-domain and let $\mathcal{W} \subseteq \mathsf{TM}$ be open. If $Z \in \mathrm{LI}\Gamma^{\nu-1}(\mathbb{T}; T\mathscr{F}(\mathcal{W}))$ is given by*

$$Z(t, v_x) = X^T(t, v_x) + Y^V(t, v_x)$$

for maps $X, Y \colon \mathbb{T} \times \pi_{\mathsf{TM}}(\mathcal{W}) \to \mathsf{TM}$ for which $X_t, Y_t \in \Gamma^\nu(\mathbb{T}; \pi_{\mathsf{TM}}(\mathcal{W}))$ for every $t \in \mathbb{T}$, then $X \in \mathrm{LI}\Gamma^\nu(\mathbb{T}; \mathscr{F}(\pi_{\mathsf{TM}}(\mathcal{W})))$ and $Y \in \mathrm{LI}\Gamma^{\nu-1}(\mathbb{T}; \mathscr{F}(\pi_{\mathsf{TM}}(\mathcal{W})))$.

Proof: It is possible to make oneself believe the lemma by a coordinate computation. However, we shall give a coordinate-free proof. To do this, we will use the Riemannian metric $\mathbb{G}^T$ and the affine connection ∇^T on TM defined by a Riemannian metric $\mathbb{G}$ and affine connection ∇ on M, as described in the proof of Lemma 9.2. For simplicity, and since we will make use of some formulae derived in the proof of Lemma 9.2 where this assumption was made, we suppose that ∇ is torsion-free.

Since we will be calculating iterated covariant differentials as in Section 3.1, only now using the affine connection ∇^T on TM, we should also think about the character of $\mathrm{T}^k(\mathsf{T}^*\mathsf{TM})$. For $v_x \in \mathsf{T}_x\mathsf{M}$, $T_{v_x}\pi_{\mathsf{TM}}$ is a surjective linear mapping from $\mathsf{T}_{v_x}\mathsf{TM}$ to $\mathsf{T}_x\mathsf{M}$. Thus its dual, $(T_{v_x}\pi_{\mathsf{TM}})^*$, is an injective linear mapping from $\mathsf{T}^*_x\mathsf{M}$ to $\mathsf{T}^*_{v_x}\mathsf{TM}$. It induces, therefore, an injective linear mapping from $\mathrm{T}^k(\mathsf{T}^*_x\mathsf{M})$ to $\mathrm{T}^k(\mathsf{T}^*_{v_x}\mathsf{TM})$ [Bourbaki 1989a, Proposition III.5.2.2]. Yano and Kobayashi [1966] call this the ***vertical lift*** of $\mathrm{T}^k(\mathsf{T}^*\mathsf{M})$ into $\mathrm{T}^k(\mathsf{T}^*\mathsf{TM})$. Note that vertically lifted tensors, thought of as multilinear maps, vanish if they are given a vertical vector as one of their arguments, i.e., they are "semi-basic" (in fact, they are even "basic"). Note that $\mathsf{T}^*_{v_x}\mathsf{TM} \simeq \mathsf{T}^*_x\mathsf{M} \oplus \mathsf{T}^*_x\mathsf{M}$ by dualising the splitting of the tangent bundle. So as to notationally distinguish between the two components of the direct sum, let us denote the first component by $(\mathsf{T}^*_x\mathsf{M})_1$ and the second component by $(\mathsf{T}^*_x\mathsf{M})_2$, noting that the first component is defined to be the image of the canonical injection from $\mathsf{T}^*_x\mathsf{M}$ to $\mathsf{T}^*_{v_x}\mathsf{TM}$. We then have

$$\mathrm{T}^k((\mathsf{T}^*_x\mathsf{M})_1 \oplus (\mathsf{T}^*_x\mathsf{M})_2) \simeq \bigoplus_{a_1,\dots,a_k\in\{1,2\}} (\mathsf{T}^*_x\mathsf{M})_{a_1} \otimes \dots \otimes (\mathsf{T}^*_x\mathsf{M})_{a_k}$$

by [Bourbaki 1989a, §III.5.5]. Let

$$\pi_k \colon \mathrm{T}^k(\mathsf{T}^*_{v_x}\mathsf{TM}) \to (\mathsf{T}^*_x\mathsf{M})_1 \otimes \dots \otimes (\mathsf{T}^*_x\mathsf{M})_1$$

be the projection onto the component of the direct sum decomposition.

With all of the preceding, we can now make sense of the following sublemma. We adopt the notation (2.1) introduced in the proof of Theorem 3.5.

1 Sublemma: *If, for $X, Y \in \Gamma^\nu(\mathsf{TM})$, we have $Z = X^T + Y^V$, then we have*

$$\pi_k \otimes \mathrm{id}_{\mathsf{TTM}}((\nabla^T)^{(k)} Z(0_x)) = \nabla^{(k)}X(x) \oplus (\nabla^{(k)}Y(x))$$

for $k \in \mathbb{Z}_{\geq 0}$ satisfying $k \leq \nu$.

Proof: Obviously we can consider two special cases, the first where $Y = 0$ and the second where $X = 0$. When $Y = 0$, the result follows from Sublemma 1 from the proof of Lemma 9.2, especially the formula (9.7) we derived from the sublemma. When $X = 0$ the result immediately follows from the same sublemma. ▼

By the preceding sublemma, $Z(t, 0_x) = X(t, x) \oplus Y(t, x)$. Since the projections onto the first and second component of the direct sum decomposition of TTM are continuous, we immediately conclude that $X, Y \in \mathrm{CF}\Gamma^\nu(\mathbb{T}; \mathsf{T}(\pi_{\mathsf{TM}}(\mathcal{W})))$.

The remainder of the proof breaks into the various cases of regularity.

$\nu = \infty$: Let $K \subseteq \mathsf{M}$ be compact and let $m \in \mathbb{Z}_{\geq 0}$. Since K is also a compact subset of TM, there exists $g \in \mathrm{L}^1_{\mathrm{loc}}(\mathbb{T}; \mathbb{R}_{\geq 0})$ such that

$$\|j_m Z(t, 0_x)\|_{\overline{\mathbb{G}^T}_m} \leq g(t), \qquad t \in \mathbb{T},\ x \in K.$$

Let $\overline{\pi}_m \colon \mathsf{J}^m\mathsf{TTM} \to \oplus_{j=0}^m \mathrm{T}^j(\pi_{\mathsf{TM}}^*\mathsf{TM}) \otimes \mathsf{TTM}$ be defined by

$$\overline{\pi}_m(j_m Z'(v_x)) = \sum_{j=0}^m \pi_j \otimes \mathrm{id}_{\mathsf{TTM}}((\nabla^T)^{(j-1)} Z'(v_x)),$$

this making sense by virtue of Lemma 2.1. By the sublemma, by the definition of $\mathbb{G}^T$, and by the definition of the fibre metrics on $\mathsf{J}^m\mathsf{TM}$ and $\mathsf{J}^m\mathsf{TTM}$ induced by the decomposition of Lemma 2.1, we have

$$\|\overline{\pi}_m(j_m Z(t, 0_x))\|^2_{\overline{\mathbb{G}^T}_m} = \|j_m X(t,x)\|^2_{\overline{\mathbb{G}}_m} + \|j_m Y(t,x)\|^2_{\overline{\mathbb{G}}_m}.$$

This gives

$$\|j_m X(t,x)\|_{\overline{\mathbb{G}}_m} \le g(t), \ \ \|j_m Y(t,x)\|_{\overline{\mathbb{G}}_m} \le g(t), \qquad t \in \mathbb{T},\ x \in K,$$

which gives the lemma in this case.

$\nu = m$: From the computations above in the smooth case we have that X and Y are locally integrally C^{m-1}-bounded. To show X is, in fact, locally integrally C^m-bounded, we will use the computations from the proof of Lemma 9.2. Let $K \subseteq \mathsf{M}$ and let

$$K_1 = \{v_x \in \mathsf{TM} \mid \ x \in K,\ \|v_x\|_{\mathbb{G}} \le 1\}$$

so K_1 is a compact subset of TM. For the moment, let us fix $t \in \mathbb{T}$. We now recall equation (9.8) which gives a formula for $P^m_{\nabla^T}(X^T_t)$ when all arguments are horizontal. Since, in the expression (9.8), v_x is arbitrary, by letting it vary over vectors of unit length we get an estimate

$$\|P^m_{\nabla}(X_t)(x)\|_{\mathbb{G}_m} \le C(p^{m-1}_K(X_t) + p^{m-1}_{K_1}(X^T_t))$$

for some $C \in \mathbb{R}_{>0}$. Since $X, Y \in \mathrm{LI}\Gamma^{m-1}(\mathbb{T};\mathsf{M})$ and since $X^T = Z - Y^V \in \mathrm{LI}\Gamma^{m-1}(\mathbb{T};\mathsf{TM})$, by Lemma 2.1 there exists $g \in \mathrm{L}^1_{\mathrm{loc}}(\mathbb{T};\mathbb{R}_{\ge 0})$ such that

$$\|j_m X_t(x)\| \le g(t), \qquad (t,x) \in \mathbb{T} \times K,$$

which gives $X \in \mathrm{LI}\Gamma^m(\mathsf{TM})$.

$\nu = m+\mathrm{lip}$: This follows from the computations above, using Lemma 3.12, cf. the proof of the Lipschitz part of the proof of Lemma 9.2.

$\nu = \omega$: Let $K \subseteq \mathsf{M}$ be compact and let $\boldsymbol{a} \in \mathrm{c}_{\downarrow 0}(\mathbb{Z}_{\ge 0};\mathbb{R}_{>0})$. Since K is also a compact subset of TM, there exists $g \in \mathrm{L}^1_{\mathrm{loc}}(\mathbb{T};\mathbb{R}_{\ge 0})$ such that

$$a_0 a_1 \cdots a_m \|j_m Z(t, 0_x)\|_{\overline{\mathbb{G}^T}_m} \le g(t), \qquad t \in \mathbb{T},\ x \in K,\ m \in \mathbb{Z}_{\ge 0}.$$

As in the smooth case we have

$$\|\overline{\pi}_m(j_m Z(t, 0_x))\|^2_{\overline{\mathbb{G}^T}_m} = \|j_m X(t,x)\|^2_{\overline{\mathbb{G}}_m} + \|j_m Y(t,x)\|^2_{\overline{\mathbb{G}}_m}.$$

This gives

$$a_0 a_1 \cdots a_m \|j_m X(t,x)\|_{\overline{\mathbb{G}}_m} \le g(t), \quad a_0 a_1 \cdots a_m \|j_m Y(t,x)\|_{\overline{\mathbb{G}}_m} \le g(t),$$

for $t \in \mathbb{T}$, $x \in K$, and $m \in \mathbb{Z}_{\ge 0}$, which gives the lemma. ■

Next let us think about open-loop subfamilies for linearisations. Generally speaking, one may wish to consider different classes of open-loop systems for the "tangent lift part" and the "vertical lift part" of a linearised system. The open-loop systems for the tangent lift part will be those giving rise to reference trajectories and reference flows. On the other hand, the open-loop systems for the vertical lift part will be those that we will allow as perturbing the reference flow. There is no reason that these should be the same. While this proliferation of open-loop subfamilies will lead to some notational complexity, the freedom to carefully account for these possibilities is one of the strengths of our theory. Indeed, in standard Jacobian linearisation, it is difficult to keep track of how the controls—constraints on them and attributes of them—are carried over to the linearisation. In our theory, this is natural.

We first make tangent and vertical lift constructions for open-loop subfamilies.

9.8 Definition: (Tangent and vertical lifts of open-loop subfamilies) Let $m \in \mathbb{Z}_{>0}$ and $m' \in \{0, \mathrm{lip}\}$, let $\nu \in \{m+m', \infty, \omega\}$, and let $r \in \{\infty, \omega\}$, as required. Let $\mathfrak{G} = (\mathsf{M}, \mathscr{F})$ be a C^ν-tautological control system with linearisation $T\mathfrak{G} = (\mathsf{TM}, T\mathscr{F})$, and let $\mathscr{O}_\mathfrak{G}$ be an open-loop subfamily for $\mathfrak{G}$.

(i) The ***tangent lift*** of $\mathscr{O}_\mathfrak{G}$ is the open-loop subfamily $\mathscr{O}_\mathfrak{G}^T$ for $(\mathsf{TM}, \mathscr{F}^T)$ defined by

$$\mathscr{O}_\mathfrak{G}^T(\mathbb{T}, \mathcal{W}) = \{X^T|\mathcal{W} \mid X \in \mathscr{O}_\mathfrak{G}(\mathbb{T}, \pi_{\mathsf{TM}}(\mathcal{W}))\}$$

for a time-domain $\mathbb{T}$ and for $\mathcal{W} \subseteq \mathsf{TM}$ open.

(ii) The ***vertical lift*** of $\mathscr{O}_\mathfrak{G}$ is the open-loop subfamily $\mathscr{O}_\mathfrak{G}^V$ for $(\mathsf{TM}, \mathscr{F}^V)$ defined by

$$\mathscr{O}_\mathfrak{G}^V(\mathbb{T}, \mathcal{W}) = \{Y^V|\mathcal{W} \mid Y \in \mathscr{O}_\mathfrak{G}(\mathbb{T}, \pi_{\mathsf{TM}}(\mathcal{W}))\}$$

for a time-domain $\mathbb{T}$ and for $\mathcal{W} \subseteq \mathsf{TM}$ open. •

9.9 Definition: (Open-loop subfamily for linearisation) Let $m \in \mathbb{Z}_{>0}$ and $m' \in \{0, \mathrm{lip}\}$, let $\nu \in \{m+m', \infty, \omega\}$, and let $r \in \{\infty, \omega\}$, as required. Let $\mathfrak{G} = (\mathsf{M}, \mathscr{F})$ be a C^ν-tautological control system with linearisation $T\mathfrak{G} = (\mathsf{TM}, T\mathscr{F})$. An ***open-loop subfamily*** for $T\mathfrak{G}$ defined by a pair $(\mathscr{O}_{\mathfrak{G},0}, \mathscr{O}_{\mathfrak{G},1})$ of open-loop subfamilies for $\mathfrak{G}$ is the open-loop subfamily $\mathscr{O}_{\mathfrak{G},0}^T + \mathscr{O}_{\mathfrak{G},1}^V$ defined by:

$$X^T + Y^V \in (\mathscr{O}_{\mathfrak{G},0}^T + \mathscr{O}_{\mathfrak{G},1}^V)(\mathbb{T}, \mathcal{W}) \iff X^T \in \mathscr{O}_{\mathfrak{G},0}^T(\mathbb{T}, \pi_{\mathsf{TM}}(\mathcal{W})),\ Y^V \in \mathscr{O}_{\mathfrak{G},1}^V(\mathbb{T}, \pi_{\mathsf{TM}}(\mathcal{W})).$$

•

Note that the restriction properties of open-loop subfamilies as per Definition 8.17 are satisfied by our construction above, so the result is indeed an open-loop subfamily for $T\mathfrak{G}$.

Next we can define what we mean by trajectories for the linearisation in the more or less obvious way.

9.10 Definition: (Trajectory for linearisation of tautological control system) Let $m \in \mathbb{Z}_{>0}$ and $m' \in \{0, \mathrm{lip}\}$, let $\nu \in \{m+m', \infty, \omega\}$, and let $r \in \{\infty, \omega\}$, as required. Let $\mathfrak{G} = (\mathsf{M}, \mathscr{F})$ be a C^ν-tautological control system with linearisation $T\mathfrak{G} = (\mathsf{TM}, T\mathscr{F})$. Let $\mathscr{O}_{\mathfrak{G},0}$ and $\mathscr{O}_{\mathfrak{G},1}$ be open-loop subfamilies for $\mathfrak{G}$.

(i) For a time-domain $\mathbb{T}$, an open set $\mathcal{W} \subseteq \mathsf{TM}$, and for $X \in \mathscr{O}_{\mathfrak{G},0}(\mathbb{T}, \mathcal{U})$ and $Y \in \mathscr{O}_{\mathfrak{G},1}$, an $\boldsymbol{(X, Y, \mathbb{T}, \mathcal{W})}$***-trajectory*** for $(\mathscr{O}_{\mathfrak{G},0}, \mathscr{O}_{\mathfrak{G},1})$ is a curve $\Upsilon \colon \mathbb{T} \to \mathcal{W}$ such that $\Upsilon'(t) = X^T(t, \Upsilon(t)) + Y^V(t, \Upsilon(t))$.

(ii) For a time-domain $\mathbb{T}$ and an open set $\mathcal{W} \subseteq \mathsf{TM}$, a ***$(\mathbb{T}, \mathcal{W})$-trajectory*** for the pair $(\mathscr{O}_{\mathfrak{G},0}, \mathscr{O}_{\mathfrak{G},1})$ is a $(\mathbb{T}, \mathcal{W})$-trajectory for $\mathscr{O}^T_{\mathfrak{G},0} + \mathscr{O}^V_{\mathfrak{G},1}$.

(iii) A plain ***trajectory*** for the pair $(\mathscr{O}_{\mathfrak{G},0}, \mathscr{O}_{\mathfrak{G},1})$ is a curve that is a $(\mathbb{T}, \mathcal{W})$-trajectory for $(\mathscr{O}_{\mathfrak{G},0}, \mathscr{O}_{\mathfrak{G},1})$ for some time-domain $\mathbb{T}$ and some open $\mathcal{W} \subseteq \mathsf{TM}$.

We denote by:

(iv) $\mathrm{Traj}(X, Y, \mathbb{T}; \mathcal{W})$ the set of $(X, Y, \mathbb{T}, \mathcal{U})$-trajectories for $(\mathscr{O}_{\mathfrak{G},0}, \mathscr{O}_{\mathfrak{G},1})$;

(v) $\mathrm{Traj}(\mathbb{T}, \mathcal{W}, (\mathscr{O}_{\mathfrak{G},0}, \mathscr{O}_{\mathfrak{G},1}))$ the set of $(\mathbb{T}, \mathcal{U})$-trajectories for $(\mathscr{O}_{\mathfrak{G},0}, \mathscr{O}_{\mathfrak{G},1})$;

(vi) $\mathrm{Traj}(\mathscr{O}_{\mathfrak{G},0}, \mathscr{O}_{\mathfrak{G},1})$ the set of trajectories for $(\mathscr{O}_{\mathfrak{G},0}, \mathscr{O}_{\mathfrak{G},1})$.

We shall abbreviate

$$\mathrm{Traj}(\mathbb{T}, \mathcal{W}, (\mathscr{O}_{\mathfrak{G},\mathrm{full}}, \mathscr{O}_{\mathfrak{G},\mathrm{full}})) = \mathrm{Traj}(\mathbb{T}, \mathcal{W}, T\mathfrak{G})$$

and $\mathrm{Traj}(\mathscr{O}_{\mathfrak{G},\mathrm{full}}, \mathscr{O}_{\mathfrak{G},\mathrm{full}}) = \mathrm{Traj}(T\mathfrak{G})$. •

Now that we have been clear about what we mean by the trajectory of a linearised system, let us say some things about these trajectories.

9.11 Proposition: (Trajectories for the linearisation of a tautological control system) *Let $m \in \mathbb{Z}_{>0}$ and $m' \in \{0, \mathrm{lip}\}$, let $\nu \in \{m + m', \infty, \omega\}$, and let $r \in \{\infty, \omega\}$, as required. Let $\mathfrak{G} = (\mathsf{M}, \mathscr{F})$ be a C^ν-tautological control system with linearisation $T\mathfrak{G}$, and let $\mathscr{O}_{\mathfrak{G},0}$ and $\mathscr{O}_{\mathfrak{G},1}$ be open-loop subfamilies for $\mathfrak{G}$. Let $\mathbb{T} \subseteq \mathbb{R}$ be a time-domain and let $\mathcal{W} \subseteq \mathsf{TM}$ be open. If $\xi^T \in \mathrm{Traj}(\mathbb{T}, \mathcal{W}, (\mathscr{O}_{\mathfrak{G},0}, \mathscr{O}_{\mathfrak{G},1}))$ then the following statements hold:*

(i) there exist $X \in \mathscr{O}_{\mathfrak{G},0}(\mathbb{T}, \pi_{\mathsf{TM}}(\mathcal{W}))$ and $Y \in \mathscr{O}_{\mathfrak{G},1}(\mathbb{T}, \pi_{\mathsf{TM}}(\mathcal{W}))$ such that

$$(\xi^T)'(t) = X^T(t, \xi^T(t)) + Y^V(t, \xi^T(t));$$

(ii) there exists $\xi \in \mathrm{Traj}(\mathbb{T}, \pi_{\mathsf{TM}}(\mathcal{W}), \mathscr{O}_{\mathfrak{G},0})$ such that the diagram

$$\begin{tikzcd}
\mathbb{T} \arrow[r, "\xi^T"] \arrow[dr, "\xi"'] & \mathsf{TM} \arrow[d, "\pi_{\mathsf{T}\mathcal{U}}"] \\
& \mathsf{M}
\end{tikzcd}$$

commutes, i.e., ξ^T is a vector field along ξ.

Proof: The first assertion follows from Lemma 9.7. The second assertion follows by taking $\xi = \pi_{\mathsf{TM}} \circ \xi^T$, and noting that

$$\xi'(t) = T_{\xi^T(t)}\pi_{\mathsf{TM}}((\xi^T)'(t)) = T_{\xi^T(t)}\pi_{\mathsf{TM}}(X^T(t, \xi^T(t)) + Y^V(t, \xi^T(t))) = X(t, \xi(t))$$

and X is an open-loop system for $\mathscr{O}_{\mathfrak{G},0}$. ■

9.4. Linearisation about reference trajectories and reference flows. Let us now slowly begin to pull back our general notion of linearisation to something more familiar. In this section we will linearise about two sorts of things, trajectories and flows. We will see in the next section that it is the distinction between these two things that accounts for the problems observed in Example 1.1.

But for now, we proceed in general. We let $\mathfrak{G}$ be a tautological control system and $\mathscr{O}_{\mathfrak{G}}$ an open-loop subfamily. We recall from Example 8.24 that, if $\mathbb{T}$ is a time-domain, if $\mathcal{U} \subseteq \mathsf{M}$ is open, and if $\xi \in \mathrm{Traj}(\mathbb{T}, \mathcal{U}, \mathscr{O}_{\mathfrak{G}})$, then $\mathscr{O}_{\mathfrak{G},\xi}$ is the open-loop subfamily associated to the trajectory ξ, i.e., all open-loop systems from $\mathscr{O}_{\mathfrak{G}}$ possessing ξ as a trajectory. Having made this recollection, we make the following definition.

9.12 Definition: (Linearisation of a tautological control system about a trajectory) Let $m \in \mathbb{Z}_{>0}$ and $m' \in \{0, \mathrm{lip}\}$, let $\nu \in \{m + m', \infty, \omega\}$, and let $r \in \{\infty, \omega\}$, as required. Let $\mathfrak{G} = (\mathsf{M}, \mathscr{F})$ be a C^{ν}-tautological control system with linearisation $T\mathfrak{G}$. Let $\mathscr{O}_{\mathfrak{G},0}$ and $\mathscr{O}_{\mathfrak{G},1}$ be open-loop subfamilies for $\mathfrak{G}$, let $\mathbb{T}$ be a time-domain, let $\mathcal{U} \subseteq \mathsf{M}$ be open, and let $\xi_{\mathrm{ref}} \in \mathrm{Traj}(\mathbb{T}, \mathcal{U}, \mathscr{O}_{\mathfrak{G},0})$. The ***($\mathscr{O}_{\mathfrak{G},0}, \mathscr{O}_{\mathfrak{G},1}$)-linearisation of $\mathfrak{G}$ about ξ_{ref}*** is the open-loop subfamily $\mathscr{O}^{T}_{\mathfrak{G},0,\xi_{\mathrm{ref}}} + \mathscr{O}^{V}_{\mathfrak{G},1}$ for $T\mathfrak{G}$. A ***trajectory*** for this linearisation is a $(\mathbb{T}', \mathcal{W})$-trajectory Υ for $(\mathscr{O}_{\mathfrak{G},0,\xi_{\mathrm{ref}}}, \mathscr{O}_{\mathfrak{G},1})$ satisfying $\pi_{\mathsf{TM}} \circ \Upsilon = \xi_{\mathrm{ref}}$, and where $\mathbb{T}' \subseteq \mathbb{T}$ and $\mathcal{W} \subseteq \pi^{-1}_{\mathsf{TM}}(\mathcal{U})$. •

By definition, a trajectory for the linearisation about the reference trajectory ξ_{ref} is a curve $\Upsilon\colon \mathbb{T}' \to \mathcal{W}$ satisfying

$$\Upsilon'(t) = X^{T}(t, \Upsilon(t)) + Y^{V}(t, \Upsilon(t)),$$

for $X \in \mathscr{O}_{\mathfrak{G},0,\xi_{\mathrm{ref}}}(\mathbb{T}', \pi_{\mathsf{TM}}(\mathcal{W}))$ and for $Y \in \mathscr{O}_{\mathfrak{G},1}(\mathbb{T}', \pi_{\mathsf{TM}}(\mathcal{W}))$, and where Υ is a tangent vector field along ξ_{ref}. Note that there may well be trajectories for $(\mathscr{O}_{\mathfrak{G},0,\xi_{\mathrm{ref}}}, \mathscr{O}_{\mathfrak{G},1})$ that are *not* vector fields along ξ_{ref}; we just do not call these trajectories for the linearisation *about* ξ_{ref}.

Let us now talk about linearisation, not about a trajectory, but about a flow. Here we recall the notion of the open-loop subfamily associated to an open-loop system in Example 8.18–5.

9.13 Definition: (Linearisation of a tautological control system about a flow) Let $m \in \mathbb{Z}_{>0}$ and $m' \in \{0, \mathrm{lip}\}$, let $\nu \in \{m + m', \infty, \omega\}$, and let $r \in \{\infty, \omega\}$, as required. Let $\mathfrak{G} = (\mathsf{M}, \mathscr{F})$ be a C^{ν}-tautological control system with linearisation $T\mathfrak{G}$. Let $\mathscr{O}_{\mathfrak{G},0}$ and $\mathscr{O}_{\mathfrak{G},1}$ be open-loop subfamilies for $\mathfrak{G}$, let $\mathbb{T}$ be a time-domain, let $\mathcal{U} \subseteq \mathsf{M}$ be open, and let $X_{\mathrm{ref}} \in \mathscr{O}_{\mathfrak{G},0}(\mathbb{T}, \mathcal{U})$. The ***$\mathscr{O}_{\mathfrak{G},1}$-linearisation of $\mathfrak{G}$ about X_{ref}*** is the open-loop subfamily $\mathscr{O}^{T}_{\mathfrak{G},0,X_{\mathrm{ref}}} + \mathscr{O}^{V}_{\mathfrak{G},1}$ for $T\mathfrak{G}$. A ***trajectory*** for this linearisation is a $(\mathbb{T}', \mathcal{W})$-trajectory for $(\mathscr{O}_{\mathfrak{G},0,X_{\mathrm{ref}}}, \mathscr{O}_{\mathfrak{G},1})$, where $\mathbb{T}' \subseteq \mathbb{T}$ and where $\mathcal{W} \subseteq \pi^{-1}_{\mathsf{TM}}(\mathcal{U})$. •

By definition, a trajectory for the linearisation about the reference flow X_{ref} is a curve $\Upsilon\colon \mathbb{T}' \to \mathcal{W}$ satisfying

$$\Upsilon'(t) = X^{T}_{\mathrm{ref}}(t, \Upsilon(t)) + Y^{V}(t, \Upsilon(t)),$$

for $Y \in \mathscr{O}_{\mathfrak{G},1}(\mathbb{T}', \pi_{\mathsf{TM}}(\mathcal{W}))$. Note that the definition of $\mathscr{O}_{\mathfrak{G},0,X_{\mathrm{ref}}}$ necessarily implies that $\pi_{\mathsf{TM}} \circ \Upsilon$ is an integral curve for X_{ref}. Unlike the case of linearisation about a reference trajectory, we do not specify that the trajectories for the linearisation about a reference flow follow a specific trajectory for $\mathfrak{G}$, although one can certainly do this as well.

9.5. Linearisation about an equilibrium point. Continuing to make things concrete, let us consider linearising about trivial reference trajectories and reference flows. We begin by considering what an equilibrium point is in our framework.

9.14 Definition: (Tautological control system associated to an equilibrium point) Let $m \in \mathbb{Z}_{>0}$ and $m' \in \{0, \mathrm{lip}\}$, let $\nu \in \{m + m', \infty, \omega\}$, and let $r \in \{\infty, \omega\}$, as required. Let $\mathfrak{G} = (\mathsf{M}, \mathscr{F})$ be a C^ν-tautological control system and let $x_0 \in \mathsf{M}$.

(i) The ***tautological control system for*** $\mathfrak{G}$ ***at*** x_0 is the C^ν-tautological control system $\mathfrak{G}_{x_0} = (\mathsf{M}, \mathrm{Eq}_{\mathscr{F},x_0})$, where

$$\mathrm{Eq}_{\mathscr{F},x_0}(\mathcal{U}) = \{X \in \mathscr{F}(\mathcal{U}) \mid X(x_0) = 0_{x_0}\}.$$

(ii) If there exists an open set $\mathcal{U} \subseteq \mathsf{M}$ for which $\mathrm{Eq}_{\mathscr{F},x_0}(\mathcal{U}) \neq \varnothing$, then x_0 is an ***equilibrium point*** for $\mathfrak{G}$. •

Of course, by properties of presheaves, if $X \in \mathrm{Eq}_{\mathscr{F},x_0}(\mathcal{U})$, then $X|\mathcal{V} \in \mathrm{Eq}_{\mathscr{F},x_0}(\mathcal{V})$ for every open set $\mathcal{V} \subseteq \mathcal{U}$. Thus $\mathfrak{G}_{x_0}$ is indeed a tautological control system.

Let us examine the nature of tautological control systems at x_0. This amounts to understanding any particular structure that one can associate to vector fields that vanish at a point. This is the content of the following lemma.

9.15 Lemma: (Properties of vector fields vanishing at a point) *Let* M *be a smooth manifold, let* $x_0 \in \mathsf{M}$*, and let* $X \in \Gamma^1(\mathsf{M})$*. If* $X(x_0) = 0_{x_0}$*, then there exists a unique* $A_{X,x_0} \in \mathrm{End}_{\mathbb{R}}(\mathsf{T}_{x_0}\mathsf{M})$ *satisfying either of the following equivalent characterisations:*

(i) noting that $X^T|\mathsf{T}_{x_0}\mathsf{M} \colon \mathsf{T}_{x_0}\mathsf{M} \to \mathsf{V}_{0_{x_0}}\mathsf{TM} \simeq \mathsf{T}_{x_0}\mathsf{M}$*,* $A_{X,x_0} = X^T|\mathsf{T}_{x_0}\mathsf{M}$*;*

(ii) $A_{X,x_0}(v_{x_0}) = [V, X](x_0)$ *where* $V \in \Gamma^\infty(\mathsf{M})$ *satisfies* $V(x_0) = v_{x_0}$*.*

Proof: We will show that the characterisation from part (i) makes sense, and that it agrees with the second characterisation.

First, note that, since $X(x_0) = 0_{x_0}$, $T_{v_{x_0}}\pi_{\mathsf{TM}}(X^T(v_{x_0})) = 0_{x_0}$ for every $v_{x_0} \in \mathsf{T}_{x_0}\mathsf{M}$. Thus $X^T(v_{x_0}) \in \mathsf{V}_{0_{x_0}}\mathsf{TM}$, as claimed. That $X^T|\mathsf{T}_{x_0}\mathsf{M}$ is linear is a consequence of the fact that X^T is a linear vector field, i.e., that the diagram (9.3) commutes. In the particular case that $X(x_0) = 0_{x_0}$, the diagram implies that X^T is a linear map from $\mathsf{T}_{x_0}\mathsf{M}$ to $\mathsf{T}_{0_{x_0}}\mathsf{TM}$. As we already know that $X^T|\mathsf{T}_{x_0}\mathsf{M}$ is $\mathsf{V}_{0_{x_0}}\mathsf{TM}$-valued, the characterisation from part (i) does indeed uniquely define an endomorphism of $\mathsf{T}_{x_0}\mathsf{M}$.

Let us now show that the characterisation of part (ii) agrees with that of part (i). By [Abraham, Marsden, and Ratiu 1988, Theorem 4.2.19], we have

$$\begin{aligned}\mathrm{vlft}(0_{x_0}, [V, X](x_0)) &= \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} T_{\Phi^X_{-t}(x_0)}\Phi^X_t \circ V \circ \Phi^X_{-t}(x_0) \\ &= \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} T_{x_0}\Phi^X_t \circ V(x_0) = X^T(V(x_0)),\end{aligned}$$

as desired. ■

According to the lemma, we can make the following definitions.

9.16 Definition: (Data associated with linearisation about an equilibrium point) Let $m \in \mathbb{Z}_{>0}$ and $m' \in \{0, \mathrm{lip}\}$, let $\nu \in \{m + m', \infty, \omega\}$, and let $r \in \{\infty, \omega\}$, as required. Let $\mathfrak{G} = (\mathsf{M}, \mathscr{F})$ be a C^ν-tautological control system. For an equilibrium point $x_0 \in \mathsf{M}$ for $\mathfrak{G}$, we define

$$\mathscr{L}_{\mathscr{F},x_0} = \{A_{X,x_0} \mid [X]_{x_0} \in (\mathrm{Eq}_{\mathscr{F},x_0})_{x_0}\}$$

(where $(\mathrm{Eq}_{\mathscr{F},x_0})_{x_0}$ denotes the stalk of the presheaf $\mathrm{Eq}_{\mathscr{F},x_0}$ at x_0) and

$$\mathscr{F}(x_0) = \{X(x_0) \mid [X]_{x_0} \in \mathscr{F}_{x_0}\}. \qquad \bullet$$

Associated to an equilibrium point are natural notions of open-loop systems that preserve the equilibrium point.

9.17 Definition: (Open-loop subfamilies and equilibrium points) Let $m \in \mathbb{Z}_{>0}$ and $m' \in \{0, \mathrm{lip}\}$, let $\nu \in \{m+m', \infty, \omega\}$, and let $r \in \{\infty, \omega\}$, as required. Let $\mathfrak{G} = (\mathsf{M}, \mathscr{F})$ be a C^ν-tautological control system. If $x_0 \in \mathsf{M}$ and if $\mathscr{O}_{\mathfrak{G}}$ is an open-loop subfamily for $\mathfrak{G}$, the open-loop subfamily $\mathscr{O}_{\mathfrak{G},x_0}$ is defined by specifying that, for a time-domain $\mathbb{T}$ and an open set $\mathcal{U} \subseteq \mathsf{M}$,

$$\mathscr{O}_{\mathfrak{G},x_0}(\mathbb{T}, \mathcal{U}) = \{X \in \mathscr{O}_{\mathfrak{G}}(\mathbb{T}; \mathcal{U}) \mid X(t) \in \mathrm{Eq}_{\mathscr{F},x_0}(\mathcal{U}),\ t \in \mathbb{T}\}. \qquad \bullet$$

Note that the only trajectory of $\mathscr{O}_{\mathfrak{G},x_0}$ passing through x_0 is the constant trajectory $t \mapsto x_0$, as it should be.

It is now more or less obvious how one should define linearisations about an equilibrium point. This can be done for trajectories and flows. We start with trajectories.

9.18 Definition: (Linearisation of a tautological control system about an equilibrium trajectory) Let $m \in \mathbb{Z}_{>0}$ and $m' \in \{0, \mathrm{lip}\}$, let $\nu \in \{m+m', \infty, \omega\}$, and let $r \in \{\infty, \omega\}$, as required. Let $\mathfrak{G} = (\mathsf{M}, \mathscr{F})$ be a C^ν-tautological control system with linearisation $T\mathfrak{G}$. Let $\mathscr{O}_{\mathfrak{G},0}$ and $\mathscr{O}_{\mathfrak{G},1}$ be open-loop subfamilies for $\mathfrak{G}$ and let $x_0 \in \mathsf{M}$. The ***$(\mathscr{O}_{\mathfrak{G},0}, \mathscr{O}_{\mathfrak{G},1})$-linearisation of $\mathfrak{G}$ about x_0*** is the open-loop subfamily $\mathscr{O}^T_{\mathfrak{G},0,x_0} + \mathscr{O}^V_{\mathfrak{G},1}$ for $T\mathfrak{G}$. A ***trajectory*** for this linearisation is a $(\mathbb{T}, \mathcal{W})$-trajectory for the $(\mathscr{O}_{\mathfrak{G},0,x_0}, \mathscr{O}_{\mathfrak{G},1})$-linearisation about the trivial reference trajectory $t \mapsto x_0$, where $\mathbb{T}$ is a time-domain and where $\mathcal{W}$ is a neighbourhood of $\mathsf{T}_{x_0}\mathsf{M}$. $\bullet$

By definition and by the characterisation of X^T at equilibria, a trajectory for the linearisation about x_0 will be a curve $\Upsilon \colon \mathbb{T} \to \mathsf{T}_{x_0}\mathsf{M}$ satisfying

$$\Upsilon'(t) = A_{X(t),x_0}(\Upsilon(t)) + b(t),$$

where $t \mapsto X(t)$ is a curve in $\mathscr{L}_{\mathscr{F},x_0}$ whose nature is determined by the open-loop subfamily $\mathscr{O}_{\mathfrak{G},0}$, e.g., it may be locally integrable, locally essentially bounded, piecewise constant, etc., and where $t \mapsto b(t)$ is a curve in $\mathscr{F}(x_0) \subseteq \mathsf{T}_{x_0}\mathsf{M}$, again whose nature is determined by the open-loop subfamily $\mathscr{O}_{\mathfrak{G},1}$. Note that the linearisation about x_0 will, therefore, generally be a family of *time-dependent* linear systems on $\mathsf{T}_{x_0}\mathsf{M}$. This may come as a surprise to those used to Jacobian linearisation, but we will see in Example 9.25 below how this arises in practice.

Let us now talk about linearisation about an equilibrium point, not about a trajectory, but about a flow.

9.19 Definition: (Linearisation of a tautological control system about an equilibrium flow) Let $m \in \mathbb{Z}_{>0}$ and $m' \in \{0, \mathrm{lip}\}$, let $\nu \in \{m+m', \infty, \omega\}$, and let $r \in \{\infty, \omega\}$, as required. Let $\mathfrak{G} = (\mathsf{M}, \mathscr{F})$ be a C^ν-tautological control system with linearisation $T\mathfrak{G}$. Let $\mathscr{O}_{\mathfrak{G},0}$ and $\mathscr{O}_{\mathfrak{G},1}$ be open-loop subfamilies for $\mathfrak{G}$, let $\mathbb{T}$ be a time-domain, let $x_0 \in \mathsf{M}$, let $\mathcal{U} \subseteq \mathsf{M}$ be a neighbourhood of x_0, and let $X_{\mathrm{ref}} \in \mathscr{O}_{\mathfrak{G},0,x_0}(\mathbb{T}, \mathcal{U})$. The ***$\mathscr{O}_{\mathfrak{G},1}$-linearisation of $\mathfrak{G}$ about (X_{ref}, x_0)*** is the open-loop subfamily $\mathscr{O}^T_{\mathfrak{G},0,X_{\mathrm{ref}}} + \mathscr{O}^V_{\mathfrak{G},1}$ for $T\mathfrak{G}$. A ***trajectory*** for this linearisation is a $(\mathbb{T}', \mathcal{W})$-trajectory for $(\mathscr{O}_{\mathfrak{G},0,X_{\mathrm{ref}}}, \mathscr{O}_{\mathfrak{G},1})$, where $\mathbb{T}' \subseteq \mathbb{T}$ and where $\mathcal{W} \subseteq \pi^{-1}_{\mathsf{TM}}(\mathcal{U})$. $\bullet$

In this case, we have a prescribed curve $t \mapsto X_{\mathrm{ref}}(t)$ such that $X_{\mathrm{ref}}(t, x_0) = 0_{x_0}$ for every t. Thus this defines a curve $A_{X_{\mathrm{ref}}(t),x_0}$ in $\mathscr{L}_{\mathscr{F},x_0}$. By definition, a trajectory for the linearisation about the pair (X_{ref}, x_0) is a curve $\Upsilon\colon \mathbb{T}' \to \mathsf{T}_{x_0}\mathsf{M}$ satisfying

$$\Upsilon'(t) = A_{X_{\mathrm{ref}}(t),x_0}(\Upsilon(t)) + b(t),$$

where $t \mapsto b(t)$ is a curve in $\mathscr{F}(x_0)$ having properties determined by the open-loop subfamily $\mathscr{O}_{\mathfrak{G},1}$. Note that this linearisation will still generally be time-dependent, but it is now a *single* time-dependent linear system, not a family of them, as with linearisation about a trajectory. Moreover, if X_{ref} is chosen to be time-independent, then the linearisation will also be time-invariant. But there is no reason in the general theory to do this.

The above comments about the possibility of time-varying linearisations notwithstanding, there is one special case where we can be sure that linearisations will be time-independent, and this is when $\mathscr{L}_{\mathscr{F},x_0}$ consists of a single vector field. The following result gives a common case where this happens. Indeed, the ubiquity of this situation perhaps explains the neglect of the general situation that has led to the seeming contradictions in the standard treatments, such as are seen in Example 1.1.

9.20 Proposition: (Time-independent linearisations for certain control-affine systems) *Let $\Sigma = (\mathsf{M}, F, \mathcal{C})$ be a C^1-control-affine system with $\mathcal{C} \subseteq \mathbb{R}^k$ and*

$$F(x, \boldsymbol{u}) = f_0(x) + \sum_{a=1}^{k} u^a f_a(x).$$

For $x_0 \in \mathsf{M}$, suppose that

(i) there exists $\boldsymbol{u}_0 \in \mathcal{C}$ such that

$$f_0(x_0) = \sum_{a=1}^{k} u_0^a f_a(x_0)$$

and

(ii) $(f_1(x_0), \dots, f_k(x_0))$ is linearly independent.

Then x_0 is an equilibrium point for $\mathfrak{G}_\Sigma$ and $\mathscr{L}_{\mathscr{F}_\Sigma,x_0}$ consists of a single linear map.

Proof: Let us define

$$f_0' = f_0 - \sum_{a=1}^{k} u_0^a f_a,$$

noting that $f_0' \in \mathscr{F}_\Sigma$. Since $f_0'(x_0) = 0_{x_0}$, we conclude that x_0 is an equilibrium point. Now suppose that $F(x_0, \boldsymbol{u}) = 0_{x_0}$. Thus

$$f_0(x_0) + \sum_{a=1}^{k} u^a f_a(x_0) = 0_{x_0} \quad \Longrightarrow \quad f_0(x_0) = -\sum_{a=1}^{k} u^a f_a(x_0).$$

This last equation has a solution for $\boldsymbol{u}$, namely $\boldsymbol{u} = -\boldsymbol{u}_0$, and since $(f_1(x_0), \dots, f_m(x_0))$ is linearly independent, this solution is unique. Thus, for any neighbourhood $\mathcal{U}$ of x_0,

$$\mathrm{Eq}_{\mathscr{F}_\Sigma,x_0}(\mathcal{U}) = \Big\{ f_0 - \sum_{a=1}^{k} u_0^a f_a(x_0) \Big\} = \{f_0'(x_0)\}.$$

This shows that $\mathscr{L}_{\mathscr{F}_\Sigma,x_0} = \{A_{f_0',x_0}\}$, as claimed. ■

While we are definitely not giving a comprehensive account of controllability in this paper—see Section 10.1 for a discussion of controllability—in order to "close the loop" on Example 1.1, let us consider here how one talks about linear controllability in our framework. First we introduce some general notation.

9.21 Definition: (Subspaces invariant under families of linear maps) Let F be a field, let V be an F-vector space, let $\mathscr{L} \subseteq \mathrm{End}_{\mathsf{F}}(\mathsf{V})$, and let $S \subseteq \mathsf{V}$. By $\langle \mathscr{L}, S\rangle$ we denote the smallest subspace of V that (i) contains S and (ii) is invariant under L for every $L \in \mathscr{L}$. •

We can give a simple description of this subspace.

9.22 Lemma: (Characterisation of smallest invariant subspace) *If F is a field, if V is an F-vector space, if $\mathscr{L} \subseteq \mathrm{End}_{\mathsf{F}}(\mathsf{V})$, and if $S \subseteq \mathsf{V}$, then $\langle \mathscr{L}, S\rangle$ is spanned by elements of V of the form*

$$L_1 \circ \cdots \circ L_k(v), \qquad k \in \mathbb{Z}_{\geq 0},\ L_1, \dots, L_k \in \mathscr{L},\ v \in S. \tag{9.13}$$

Proof: Let $\mathsf{U}_{\mathscr{L},S}$ be the subspace spanned by elements of the form (9.13). Clearly $S \subseteq \mathsf{U}_{\mathscr{L},S}$ (taking the convention that $L_1 \circ \cdots \circ L_k(v) = v$ if $k = 0$) and, if $L \in \mathscr{L}$, then $L(\mathsf{U}_{\mathscr{L},S}) \subseteq \mathsf{U}_{\mathscr{L},S}$ since an endomorphism from $\mathscr{L}$ maps a generator of the form (9.13) to another generator of this form. Therefore, $\langle \mathscr{L}, S\rangle \subseteq \mathsf{U}_{\mathscr{L},S}$. Now, if $v \in S$, then clearly $v \in \langle \mathscr{L}, S\rangle$. Since $\langle \mathscr{L}, S\rangle$ is invariant under endomorphisms from $\mathscr{L}$, $L(v) \in \langle \mathscr{L}, S\rangle$ for every $v \in S$ and $L \in \mathscr{L}$. Recursively, we see that all generators of the form (9.13) are in $\langle \mathscr{L}, S\rangle$, whence $\mathsf{U}_{\mathscr{L},S} \subseteq \langle \mathscr{L}, S\rangle$ since $\mathsf{U}_{\mathscr{L},S}$ is a subspace. ■

With the preceding as setup, let us make the following definition.

9.23 Definition: (Linear controllability) Let $m \in \mathbb{Z}_{>0}$ and $m' \in \{0, \mathrm{lip}\}$, let $\nu \in \{m + m', \infty, \omega\}$, and let $r \in \{\infty, \omega\}$, as required. Let $\mathfrak{G} = (\mathsf{M}, \mathscr{F})$ be a C^ν-tautological control system with linearisation $T\mathfrak{G}$, and let $x_0 \in \mathsf{M}$ be an equilibrium point for $\mathfrak{G}$. The system $\mathfrak{G}$ is ***linearly controllable*** at x_0 if there exists $S \subseteq \mathscr{F}(x_0)$ such that (i) $0_{x_0} \in \mathrm{conv}(S)$ and (ii) $\langle \mathscr{L}_{\mathscr{F},x_0}, S\rangle = \mathsf{T}_{x_0}\mathsf{M}$. •

9.24 Remark: (Relationship to rank test) For readers who may not recognise the relationship between our definition of linear controllability and the classical Kalman rank test [Brockett 1970, Theorem 13.3], we make the following comments. Consider the linear system

$$\dot{\boldsymbol{x}}(t) = \boldsymbol{A}\boldsymbol{x}(t) + \boldsymbol{B}\boldsymbol{u}(t),$$

with $\boldsymbol{x} \in \mathbb{R}^n$, $\boldsymbol{u} \in \mathbb{R}^m$, and for appropriately sized matrices $\boldsymbol{A}$ and $\boldsymbol{B}$. Using Lemma 9.22 and the Cayley–Hamilton Theorem, it is easy to check that the smallest $\boldsymbol{A}$-invariant subspace containing $\mathrm{image}(\boldsymbol{B})$ is exactly the columnspace of the Kalman controllability matrix,

$$\left[\ \boldsymbol{B} \mid \boldsymbol{A}\boldsymbol{B} \mid \cdots \mid \boldsymbol{A}^{n-1}\boldsymbol{B}\ \right].$$

For the more geometric approach to topics in linear system theory, we refer to the excellent book of Wonham [1985]. •

We state linear controllability as a definition, not a theorem, because we do not want to develop the definitions required to state a theorem. However, it *is* true that a system that is linearly controllable according to our definition is small-time locally controllable in the usual sense of the word. This is proved by Aguilar [2010, Theorem 5.14]. The setting of Aguilar is not exactly that of our paper. However, it is easy to see that this part of Aguilar's development easily translates to what we are doing here.

Let us close this section, and the technical part of the paper, by revisiting Example 1.1 where we saw that the classical picture of Jacobian linearisation presents some problems.

9.25 Example: (Revisiting Example 1.1) We work with the system

$$\begin{aligned}\dot{x}_1(t) &= x_2(t),\\ \dot{x}_2(t) &= x_3(t)u_1(t),\\ \dot{x}_3(t) &= u_2(t).\end{aligned}$$

We could as well work with the other representation for the system from Example 1.1, but since the family of vector fields is the same (what changes between the two representations is the parameterisation of the set of vector fields!), we will get the same conclusions; this, after all, is the point of our feedback-invariant approach.

This, of course, is a control-affine system, and the resulting tautological control system is $\mathfrak{G} = (\mathbb{R}^3, \mathscr{F})$ where $\mathscr{F}$ is the globally generated presheaf with

$$\mathscr{F}(\mathbb{R}^3) = \{f_0 + u^1 f_1 + u^2 f_2 \mid (u^1, u^2) \in \mathbb{R}^2\},$$

with

$$f_0 = x_2 \frac{\partial}{\partial x_1}, \quad f_1 = x_3 \frac{\partial}{\partial x_2}, \quad f_2 = \frac{\partial}{\partial x_3}.$$

We have an equilibrium point at $(0, 0, 0)$.

1 Lemma: $\mathrm{Eq}_{\mathscr{F},(0,0,0)}(\mathbb{R}^3) = f_0 + \mathrm{span}_{\mathbb{R}}(f_1)$.

Proof: It is clear that $f_0(0, 0, 0) = f_1(0, 0, 0) = 0$, and, therefore, any linear combination of f_0 and f_1 will also vanish at $(0, 0, 0)$, and particularly those from the affine subspace $f_0 + \mathrm{span}_{\mathbb{R}}(f_1)$. Conversely, if

$$f_0(0, 0, 0) + u^1 f_1(0, 0, 0) + u^2 f_2(0, 0, 0) = 0,$$

then $u^2 = 0$ and so the resulting vector field is in the asserted affine subspace. ▼

We, therefore, have

$$\mathscr{L}_{\mathscr{F},(0,0,0)} = \left\{ \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & a \\ 0 & 0 & 0 \end{bmatrix} \;\middle|\; a \in \mathbb{R} \right\}.$$

We also have

$$\mathscr{F}((0, 0, 0)) = \{b f_2(0, 0, 0) \mid b \in \mathbb{R}\} = \left\{ \begin{bmatrix} 0 \\ 0 \\ b \end{bmatrix} \;\middle|\; b \in \mathbb{R} \right\}.$$

Thus a curve in $\mathscr{L}_{\mathscr{F},(0,0,0)}$ has the form

$$t \mapsto \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & a(t) \\ 0 & 0 & 0 \end{bmatrix}$$

for a function a having whatever properties might be induced from the open-loop subfamily $\mathscr{O}_{\mathfrak{G},0}$ one is using, e.g., locally integrable, locally essentially bounded. A curve in $\mathscr{F}((0,0,0))$ has the form

$$t \mapsto \begin{bmatrix} 0 \\ 0 \\ b(t) \end{bmatrix}$$

for a function b having whatever properties might be induced from the open-loop subfamily $\mathscr{O}_{\mathfrak{G},1}$ one is using. Trajectories for the linearisation about $(0,0,0)$ then satisfy

$$\begin{bmatrix} \dot{v}_1(t) \\ \dot{v}_2(t) \\ \dot{v}_3(t) \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & a(t) \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} v_1(t) \\ v_2(t) \\ v_3(t) \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ b(t) \end{bmatrix}.$$

Note that this is not a fixed time-varying linear system, but a family of these, since the function a is not *a priori* specified, but is variable.

Next let us look at two instances of linearisation about a reference flow by choosing the two reference flows $X_1 = f_0$ and $X_2 = f_0 + f_1$. We use coordinates $((x_1,x_2,x_3),(v_1,v_2,v_3))$ for $\mathsf{T}\mathbb{R}^3$ and we compute

$$X_1^T = x_2\frac{\partial}{\partial x_1} + v_2\frac{\partial}{\partial v_1}, \qquad X_2^T = x_2\frac{\partial}{\partial x_1} + x_3\frac{\partial}{\partial x_2} + v_2\frac{\partial}{\partial v_1} + v_3\frac{\partial}{\partial v_2}.$$

If $t \mapsto Y(t)$ is a time-dependent vector field with values in $\mathscr{F}(\mathbb{R}^3)$, then

$$Y_t = f_0 + \nu_1(t)f_1 + \nu_2(t)f_2 = x_2\frac{\partial}{\partial x_1} + \nu_1(t)x_3\frac{\partial}{\partial x_2} + \nu_2(t)\frac{\partial}{\partial x_3},$$

for functions ν_1 and ν_2 whose character is determined by the open-loop subfamily $\mathscr{O}_{\mathfrak{G},1}$. The linearisation about the two reference flows are described by the differential equations

$$\begin{aligned} \dot{x}_1(t) &= x_2(t), & \dot{x}_1(t) &= x_2(t), \\ \dot{x}_2(t) &= 0; & \dot{x}_2(t) &= x_3(t); \\ \dot{x}_3(t) &= 0, & \dot{x}_3(t) &= 0, \\ \dot{v}_1(t) &= v_2(t) + x_2(t), & \dot{v}_1(t) &= v_2(t) + x_2(t), \\ \dot{v}_2(t) &= \nu_1(t)x_3(t), & \dot{v}_2(t) &= v_3(t) + \nu_1(t)x_3(t), \\ \dot{v}_3(t) &= \nu_2(t), & \dot{v}_3(t) &= \nu_2(t), \end{aligned}$$

respectively. The linearisations about $(X_1,(0,0,0))$ and $(X_2,(0,0,0))$ will be time-independent since the vector fields X_1 and X_2 are time-independent, and we easily determine that these linearisations are given by

$$\begin{bmatrix} \dot{v}_1(t) \\ \dot{v}_2(t) \\ \dot{v}_3(t) \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} v_1(t) \\ v_2(t) \\ v_3(t) \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \nu_2(t) \end{bmatrix}$$

and

$$\begin{bmatrix}\dot{v}_1(t)\\\dot{v}_2(t)\\\dot{v}_3(t)\end{bmatrix}=\begin{bmatrix}0&1&0\\0&0&1\\0&0&0\end{bmatrix}\begin{bmatrix}v_1(t)\\v_2(t)\\v_3(t)\end{bmatrix}+\begin{bmatrix}0\\0\\\nu_2(t)\end{bmatrix},$$

respectively. These are exactly the two distinct linearisations we encountered in Example 1.1. Thus we can see here what was going on in Example 1.1: we were linearising about two different reference flows. This also highlights the dangers of explicit and fixed parameterisations by control: one can unknowingly make choices that affect conclusions.

We comment that the reason this example does not meet the conditions of Proposition 9.20 is that the vector fields f_1 and f_2 are not linearly independent at $(0,0,0)$. The distribution generated by these vector fields has $(0,0,0)$ as a singular point. These sorts of matters will doubtless be interesting in subsequent studies of geometric control systems in our framework.

Finally, using Lemma 9.22, we can easily conclude that this system is linearly controllable. •

10. Future work

There is a lot of control theory that has yet to be done in our framework of tautological control systems. In this closing section, we discuss a few avenues for future work, and provide a few preliminary ideas related to these directions.

10.1. Controllability. The controllability of nonlinear systems comprises a vast and difficult component of the geometric control theory literature. A number of papers have been published addressing the seemingly impenetrable nature of the problems of controllability [Agrachev 1999, Bianchini and Kawski 2003, Kawski 1990a, Kawski 1990b, Kawski 2006, Sontag 1988]. Despite this, there has been substantial effort dedicated to determining sufficient or necessary conditions for controllability [Agrachev and Gamkrelidze 1993, Bacciotti and Stefani 1983, Bianchini and Stefani 1984, Bianchini and Stefani 1986, Bianchini and Stefani 1993, Haynes and Hermes 1970, Hermes 1974, Hermes 1976a, Hermes 1976b, Hermes 1977, Hermes 1982, Hermes and Kawski 1987, Kawski 1987, Kawski 1988, Kawski 1991, Kawski 1998, Kawski 1999, Stefani 1986, Sussmann 1973, Sussmann 1978, Sussmann 1983, Sussmann 1987, Sussmann and Jurdjevic 1972]. The problem of controllability has a certain lure that attracts researchers in geometric control theory. The problem is such a natural one that it feels as if it should be possible to obtain complete results, at least in some quite general situations. However, this objective remains to be fulfilled.

Our view is that one of the reasons for this is that many of the approaches to controllability are not feedback-invariant. An extreme example of this are methods for studying controllability of control-affine systems, *fixing* a drift vector field f_0 and control vector fields $f_1,\dots,f_m$, and using these as generators of a free Lie algebra. In this sort of analysis, Lie series are truncated, leading to the notion of "nilpotent approximation" of control systems. These ideas are reflected in a great many of the papers cited above. The difficulty with this approach is that it will behave very badly under feedback transformations, cf. Example 1.1. This is discussed by Lewis [2012].

One approach is then to attempt to find feedback-invariant conditions for local controllability. In the first-order case, i.e., the more or less linear case, this leads to Definition 9.23;

see also [Bianchini and Stefani 1984]. Second-order feedback-invariant conditions are considered in [Basto-Gonccalves 1998, Hirschorn and Lewis 2002]. Any attempts to determine higher-order feedback-invariant controllability conditions have, as far as we know, met with no success. Indeed, the likelihood of this approach leading anywhere seems very small, given the extremely complicated manner in which feedback transformations interact with controllability conditions.

Therefore, the most promising idea would appear to be to develop a framework for control theory that has feedback-invariance "built in." It is this that we have done in this paper. In his PhD thesis, Aguilar [2010] provides a class of control variations that is well-suited to our feedback-invariant approach. Aguilar and Lewis [2012] have used these control variations to completely characterise controllability of a class of homogeneous systems. It will be an interesting project to apply the variations of Aguilar in our framework to see what sorts of conditions for controllability naturally arise.

10.2. Optimal control and the Maximum Principle. It should be a fairly straightforward exercise to formulate optimal control problems in our framework. Also, our approach to linearisation in Section 9 already provides us with the natural means by which needle variations can be transported along reference trajectories, and so one expects that an elegant version of the Maximum Principle of Pontryagin, Boltyanskiĭ, Gamkrelidze, and Mishchenko [1961] will be possible. There will be a resemblance in this to the work of Sussmann [1998], who provides already a satisfying formulation of the Maximum Principle on manifolds. In the same way as the natural feedback-invariance of our formulation should aid in the study of controllability, it should also aid in the study of higher-order conditions for optimality. In geometric control theory, the study of so-called singular extremals (those not characterised by the Maximum Principle) is problematic for multi-input systems, so hopefully our approach can shed light on this.

As outlined in Section 8.8, problems in sub-Riemannian geometry fit naturally into the tautological control system framework, and can likely be handled well by a theory of optimal control for tautological control systems.

10.3. Feedback and stabilisation theory. There are, one could argue, three big problems in control theory. Two, controllability and optimal control, are discussed above. The third is stabilisation. This problem, being one of enormous practical importance, has been comprehensively studied, mainly from the point of view of Lyapunov theory, where the notion of a "control-Lyapunov function" provides a useful device for characterising when a system is stabilisable [Clarke, Ledyaev, Sontag, and Subotin 1997] and for stabilisation if one is known [Sontag 1989]. Our view is that Lyapunov characterisations for stabilisability are important from a practical point of view, but, from a fundamental point of view, merely replace one impenetrable notion, "stabilisability," with another, "existence of a control-Lyapunov function." This is expressed succinctly by Sontag.

> *In any case, all converse Lyapunov results are purely existential, and are of no use in guiding the search for a Lyapunov function. The search for such functions is more of an art than a science, and good physical insight into a given system plus a good amount of trial and error is typically the only way to proceed.—Sontag [1998, page 259]*

As Sontag goes on to explain, there are many heuristics for guessing control-Lyapunov functions. However, this is unsatisfying if one is seeking a general understanding of the problem of stabilisability, and not just a means of designing stabilising controllers for classes of systems.

It is also the case that there has been virtually no work on stabilisability from a geometric perspective. Topological characterisations of stabilisability such as those of Brockett [1983] (refined by Orsi, Praly, and Mareels [2003] and Zabczyk [1989]) and Coron [1990] are gratifying when they are applicable, but they are far too coarse to provide anything even close to a complete characterisation of the problem. Indeed, the extremely detailed and intricate analysis of controllability, as reflected by the work we cite above, is simply not present for stabilisability. It is fair to say that, outside the control-Lyapunov framework, very little work has been done in terms of really understanding the structural obstructions to stabilisability. Moreover, it is also fair to say that almost none of the published literature on stabilisation and stabilisability passes the "acid test" for feedback invariance that we discuss in Section 1.1. For researchers such as ourselves interested in structure, this in an unsatisfying state of affairs.

Our framework provides a natural means of addressing problems like this, just as with controllability and optimal control, because of the feedback-invariance of the framework. Indeed, upon reflection, one sees that the problem of stabilisability should have some relationships with that of controllability, although little work has been done along these lines (but see the PhD thesis of Isaiah [2012]). This area of research is wide open [Lewis 2012].

10.4. Linear system theory. Our definition of linearisation suggests an immediate generalisation from tangent bundles to vector bundles. Let us quickly see how it will work, making no pretence to the level of generality of the main body of the paper.

10.1 Definition: (Linear vector field) Let $r \in \{\infty, \omega\}$ and let $\pi\colon \mathsf{E} \to \mathsf{M}$ be vector bundle of class C^r. A vector field $X \in \Gamma^r(\mathsf{TE})$ is ***linear*** if

(i) X is π-projectable, i.e., there exists a vector field $\pi X \in \Gamma^r(\mathsf{TM})$ such that $T_{e_x}\pi(X(e_x)) = \pi X(x)$ for every $x \in \mathsf{M}$ and $e_x \in \mathsf{E}_x$, and

(ii) X is a vector bundle mapping for which the diagram

$$\begin{tikzcd} \mathsf{E} \arrow[r, "X"] \arrow[d, "\pi"'] & \mathsf{TE} \arrow[d, "T\pi"] \\ \mathsf{M} \arrow[r, "\pi X"'] & \mathsf{TM} \end{tikzcd}$$

commutes. •

The prototypical linear vector field is the tangent lift X^T, which is a linear vector field on the vector bundle $\pi_{\mathsf{TM}}\colon \mathsf{TM} \to \mathsf{M}$ according to the preceding definition. One may show that flows of linear vector fields are such that the diagram

$$\begin{tikzcd} \mathsf{E} \arrow[r, "\Phi_t^X"] \arrow[d, "\pi"'] & \mathsf{E} \arrow[d, "\pi"] \\ \mathsf{M} \arrow[r, "\Phi_t^{\pi X}"'] & \mathsf{M} \end{tikzcd}$$

commutes and $\Phi_t^X|\mathsf{E}_x$ is an isomorphism of E_x with $\mathsf{E}_{\Phi_t^{\pi X}(x)}$ [Kolář, Michor, and Slovák 1993, Proposition 47.9].

Vertical lifts are also easily defined for vector bundles. We first define the vector bundle map vlft: $\pi^*\mathsf{E} \to \mathsf{TE}$ as follows. Let $x \in \mathsf{M}$ and let $e_x, f_x \in \mathsf{E}_x$. The **vertical lift** of f_x to e_x is given by

$$\mathrm{vlft}(e_x, f_x) = \left.\tfrac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} (e_x + t f_x).$$

Now, if $\xi \in \Gamma^\infty(\mathsf{E})$, we define $\xi^V \in \Gamma^\infty(\mathsf{TE})$ by $\xi^V(e_x) = \mathrm{vlft}(e_x, \xi(x))$.

One also has the notion of a projectable presheaf of vector fields on a vector bundle.

10.2 Definition: (Projectable presheaf on a vector bundle) Let $r \in \{\infty, \omega\}$, let $\pi\colon \mathsf{E} \to \mathsf{M}$ be a vector bundle of class C^r, and let $\mathscr{G}$ be a presheaf of sets of vector fields of class C^r on E. The presheaf $\mathscr{G}$ is ***projectable*** if

$$\mathscr{G}(\mathcal{W}) = \{Z|\mathcal{W} \mid\ Z \in \mathscr{G}(\pi^{-1}(\pi(\mathcal{W})))\}. \qquad \bullet$$

One also has the more or less obvious notion of presheaves of sets of sections of E.

10.3 Definition: (Presheaf of sets of sections) Let $r \in \{\infty, \omega\}$ and let $\pi\colon \mathsf{E} \to \mathsf{M}$ be a vector bundle of class C^r. A ***presheaf of sets of $\mathbf{C}^r$-sections*** of E is an assignment, to each open set $\mathcal{U} \subseteq \mathsf{M}$, a subset $\mathscr{F}(\mathcal{U})$ of $\Gamma^r(\mathsf{E}|\mathcal{U})$ with the property that, for open sets $\mathcal{U}, \mathcal{V} \subseteq \mathsf{M}$ with $\mathcal{V} \subseteq \mathcal{U}$, the map

$$\begin{aligned} r_{\mathcal{U},\mathcal{V}}\colon \mathscr{F}(\mathcal{U}) &\to \Gamma^r(\mathsf{T}\mathcal{V}) \\ \xi &\mapsto \xi|\mathcal{V} \end{aligned}$$

takes values in $\mathscr{F}(\mathcal{V})$. Elements of $\mathscr{F}(\mathcal{U})$ are called ***local sections*** over $\mathcal{U}$. •

One also has an analogue of Lemma 9.5 for vector bundles, which makes sense of the following, final, definition.

10.4 Definition: (Linear system) Let $r \in \{\infty, \omega\}$ and let $\pi\colon \mathsf{E} \to \mathsf{M}$ be a vector bundle of class C^r. A ***$\mathbf{C}^r$-linear system*** on E is a C^r-tautological control system $\mathfrak{G} = (\mathsf{E}, \mathscr{F})$, where the projectable presheaf of sets of vector fields $\mathscr{F}$ is characterised uniquely by the requirement that, for every open subset $\mathcal{U} \subseteq \mathsf{M}$,

$$\mathscr{F}(\pi^{-1}(\mathcal{U})) = \{X + Y^V \mid\ X \in \mathscr{F}_0(\pi^{-1}(\mathcal{U}),\ Y \in \mathscr{F}_1(\mathcal{U})\},$$

where $\mathscr{F}_0$ is a projectable presheaf of sets of linear vector fields on E and $\mathscr{F}_1$ is a presheaf of sets of sections of E. •

This is then a class of tautological control systems containing linearisations of tautological control systems as a special case. One is then interested in what one can say about problems of control—controllability, optimal control theory, stabilisation—for these systems. An approach to this is presented in [Lewis and Tyner 2010] for control-affine systems. In [Colonius and Kliemann 2000, Chapter 5] one can find a setup along these lines, but with a decidedly different perspective.

10.5. The category of tautological control systems. In Section 8.7 we introduced morphisms between tautological control systems with the objective of showing that our framework is feedback-invariant. The notion of morphism we present is one that is natural and possibly easy to work with. It would be, therefore, interesting to do all of the exercises of category theory with the category of tautological control systems. That is, one would like to study epimorphisms, monomorphisms, subobjects, quotient objects, products, coproducts, pull-backs, push-outs, and various functorial operations in this category. Many of these may not be interesting or useful, or even exist. But probably some of it would be of interest. For example, Tabuada and Pappas [2005] study quotients of control systems, and Elkin [1999] studies various categorical constructions for control-affine systems.

10.6. Real analytic chronological calculus. As we have mentioned a few times, the treatment of real analytic time-varying vector fields by Agrachev and Gamkrelidze [1978] is carried out under very restrictive hypotheses, namely that the real analytic vector fields are required to admit bounded holomorphic extensions to a fixed neighbourhood in the complexification whose width is bounded uniformly from below. Even in the case of compact real analytic manifolds, this is a severe restriction. With the theory of real analytic time-varying vector fields presented in this paper, a fully functioning real analytic chronological calculus ought to be feasible.

Moreover, the results that we have proved above allow a strengthening of the existing results of Agrachev and Gamkrelidze [1978], even in the smooth case, in the following way. Agrachev and Gamkrelidze do everything "weakly." By this we mean the following. Vector fields are characterised by Agrachev and Gamkrelidze by what they do to functions, i.e., they use what we call the weak-$\mathscr{L}$ topology. In Theorems 3.5, 3.14, and 5.8 we see that this is equivalent to working directly with the appropriate topologies for vector fields. Probably this is well understood in the finitely differentiable and smooth cases, but in this paper we have understood that this is also true in the real analytic case. Also, when dealing with matters such as measurability, integrability, and absolute continuity, Agrachev and Gamkrelidze reduce to the scalar case by first composing all objects with the evaluation functionals ev_x as in the proof of Theorem 6.3 (and by implication, in the proofs of Theorems 6.9 and 6.21), and defining and computing with the scalar versions of these notions. However, Theorems 6.4, 6.10, and 6.22 ensure that this is equivalent to doing computations in the spaces of finitely differentiable, smooth, or real analytic vector fields. Again, perhaps this is understood in the finitely differentiable and smooth cases, but we have shown that this is also true in the real analytic case.

Thus, combining the preceding two paragraphs, one should be able to develop the chronological calculus of Agrachev and Gamkrelidze [1978] into a more powerful and broadly applicable tool.

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