Some observations on orbits of driftless bilinear systems*

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Abstract

The report explores questions of orbits and controllability in driftless bilinear systems. Motivated by the Lie algebra rank approach to controllability, several results pertaining to orbit size and structure are presented. In particular, the prominence of invariant subspaces to system matrices is examined and a necessary condition for controllability is given. The condition is shown to be sufficient if and only if the system is two-dimensional. Finally, a particular class of driftless systems is explored and a graph-theoretic criterion for determining controllability of the systems in that class is given. Such a criterion is easily implementable as an algorithm of quadratic complexity.

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1. Introduction

As a natural successor of linear systems, bilinear time-invariant control systems — differential equations of the form

$$\dot{x}(t) = Ax(t) + \sum_{i=1}^{m} u_i(t) B_i x(t)$$

— feature prominently in the work of Sontag [44, 45], Sussmann [47] and Elliott [8, 15, 23, 42], the last providing a particularly detailed treatise of the area. Furthermore, systems of this type appear naturally in a wide variety of applications, notably in medicine, ecology and engineering [14, 20, 35, 43]. However, these systems are generally difficult (albeit not impossible) to solve and, moreover, their explicit solutions do not necessarily provide the needed level of understanding of what is really happening with the system (e.g., relations between output components).

Among other things that we would like to know about the system, looking at things from more of an engineering perspective gives rise to the question of bringing the system to a certain state. In other words, we are interested in the notion of controllability. Indeed, this is one of the focal points of research in control theory. However, answering whether it is possible or not to bring the system to a given state is difficult. Lewis, for example, notes several problems and open questions in [33].

Bringing the above two paragraphs together, we would like to get as close as possible to answering the following question: For a bilinear time-invariant system, given the initial state at time $t = 0$, is it possible to easily determine the set of points that one can, using controls, guide the system to, in finite time? Obviously, the question is not the best-posed one, but provides an intuitive idea of what is being researched.

In this paper we will review the present state of the art for driftless bilinear systems, as well as discuss possible simplifications of the current results. The end goal would be to determine if the system is controllable (i.e., if it can be brought from any point to any other point), given matrices $B_1, B_2, \ldots, B_m$, and with as little calculations as possible. To that extent, an easily applicable criterion for controllability in the case of matrices where controls are coupled with states is proposed. Furthermore, the notion of orbits of bilinear systems, as a sort of generalization of the question of controllability, will also be discussed.

1.1. Preliminaries.

Definition. *A **control system** is an ordered quadruple $\Sigma = (M, C, f, \mathcal{U})$ such that*

(i) M is a smooth manifold (state space),

(ii) C is a set (control space),

(iii) $f : M \times C \to TM$ assigns to every pair $(x, u) \in M \times C$ a tangent vector $f(x, u)$ such that $f(x, u) \in T_x M$,

(iv) $\mathcal{U}$ is a class of functions defined on $[0, T]$, where T may depend on the function (space of admissible controls), and taking values in C.

The above definition roughly follows the work of Kalouptsidis, Elliott, Sussmann and Lewis [23, 33, 48]. Variations are possible[1] and further smoothness criteria on f are usually imposed. However, in our case, these variations are merely of philosophical importance, as functions governing bilinear systems are of particularly simple form.

Definition. *A **bilinear control system** is a control system for which $M = \mathbb{R}^n$ for some $n \in \mathbb{N}$, $C = \mathbb{R}^m$ for some $m \in \mathbb{N}$ and*

$$f(x, u) = Ax + \sum_{i=1}^{m} u_i B_i x,$$

where $A, B_1, B_2, \ldots, B_m \in \mathbb{R}^{n \times n}$.

Thus, a bilinear control system is in practice given by the equation

$$\dot{x}(t) = Ax(t) + \sum_{i=1}^{m} u_i(t) B_i x(t), \tag{1.1}$$

where $u_i : \mathbb{R} \to \mathbb{R}$ are required to be bounded measurable functions,[2] and solutions $x : \mathbb{R} \to \mathbb{R}^n$ are required to satisfy (1.1) almost everywhere. Without loss of generality, we may assume that $\{B_1, B_2, \ldots, B_m\}$ is a linearly independent set. Otherwise, it is trivially reduced to a linearly independent set by merely changing the controls u_i.

In our work, it will be useful to get rid of the "drift" component which does not depend on controls:

Definition. *A bilinear control system is **driftless** if $A = 0$.*

1.2. Organization of Thesis.

We will proceed by reviewing the current state of affairs in bilinear systems. This will be followed by introducing the notions of controllability and orbits and presenting several results on that topic given in [15]. Building upon those, we will explore the connections between the rank of the Lie algebra generated by system matrices and the dimensions of the system orbits, as well as between controllability and existence of common invariant subspaces of the system matrices. We provide a sufficient and necessary condition for controllability of driftless bilinear systems for $n = 2$, and show that such a condition is necessary, but not sufficient, in other dimensions. Using this condition, we further derive a graph-theoretic criterion for determining controllability of a very special class of "coupled" bilinear systems. In our concluding remarks we pose several open questions and finish with an outline of possible future work.

2. Controllability, Orbits and Lie Algebras

2.1. Controllability.

Definition. *A control system $(M, C, f, \mathcal{U})$ represented by the equation*

$$\dot{x}(t) = f(x(t), u(t))$$

[1] Indeed, the definitions in [48], [23] and [33] are not completely equivalent.

[2] Alternatively, as used later, they can be required to be piecewise constant functions.

*is **controllable** if, for all $x_B, x_E \in M$, there exist an admissible control function $\tilde{u} \in \mathcal{U}$ and time $T \geq 0$ such that*

$$\begin{array}{l} \tilde{x}(0) = x_B, \\ \dot{\tilde{x}}(t) = f(\tilde{x}(t), \tilde{u}(t)), \\ \tilde{x}(T) = x_E \end{array} \tag{2.1}$$

for some function $x : [0, T] \to M$.

As with control systems, there are several non-equivalent definitions of controllability. A short study of those and their relations is given in [33]. As for this paper, we will use the above definition: admissible functions will be, following [15], those such that $\tilde{u}|_{[0,T]}$ is piecewise constant.

In order to further explore controllability, it will be useful to define the set of reachable states from a certain point and the controllable set to a certain point.

Definition. *For a given system Σ, the **reachable set** from $x_B \in M$ is*

$$\mathcal{R}(x_B, \Sigma) = \{y \in M : (\exists u \in \mathcal{U}, x : [0, T] \to M) \\ (\dot{\tilde{x}}(t) = f(\tilde{x}(t), \tilde{u}(t)),\ x(0) = x_B,\ x(T) = y)\}.$$

Definition. *For a given system Σ, the **controllable set** to $x_E \in M$ is*

$$\mathcal{C}(x_E, \Sigma) = \{y \in M : (\exists u \in \mathcal{U}, x : [0, T] \to M) \\ (\dot{\tilde{x}}(t) = f(\tilde{x}(t), \tilde{u}(t)),\ x(0) = y,\ x(T) = x_E)\}.$$

The notion of controllability from $x_B \in M$ to $x_E \in M$ naturally follows from the above definitions.

Definition. *System Σ is **controllable from** x_B **to** x_E if $x_E \in \mathcal{R}(x_B, \Sigma)$ and $x_B \in \mathcal{C}(x_E, \Sigma)$.*

We make note of the following results:

Lemma 1. *A driftless bilinear system*

$$\dot{x} = \sum_{i=1}^{m} u_i B_i x \tag{2.2}$$

is controllable from $x_B \in M$ to $x_E \in M$ if and only if it is controllable from x_E to x_B.

Proof. Let (2.2) be controllable by $u|_{[0,T]}$ from $x_B \in M$ to $x_E \in M$. Thus, there exists a solution x to

$$\dot{x}(t) = \sum_{i=1}^{m} u_i(t) B_i x(t)$$

such that $x(0) = x_B$ and $x(T) = x_E$. Now, let us define $\tilde{u}$ by $\tilde{u}(t) = -u(T - t)$. We are now looking for a solution to

$$\begin{array}{l} x(0) = x_E, \\ \dot{x}(t) = -\sum_{i=1}^{m} u_i(T - t) B_i x(t), \\ x(T) = x_B. \end{array} \tag{2.3}$$

It can be trivially checked that

$$\tilde{x}(t) = x(T-t)$$

is indeed a solution to the above equation:

$$\frac{d\tilde{x}}{dt}(t) = -\frac{dx}{dt}(T-t) = -\sum_{i=1}^{m} u_i(T-t) B_i x(T-t) = \sum_{i=1}^{m} \tilde{u}_i(t) B_i \tilde{x}(t).$$

The other direction follows by symmetry. □

Corollary 1. *Given a driftless bilinear system Σ, $\mathcal{R}(x,\Sigma) = \mathcal{C}(x,\Sigma)$ for all $x \in \mathbb{R}$.*

Lemma 1 also proved the following.

Corollary 2. *A driftless bilinear system (2.2) is not controllable to 0 from any other point, nor can any point $x \neq 0$ be reached from 0.*

Proof. The second statement is obvious: for any control function u, $\sum_{i=1}^{m} \tilde{u}_i B_i x$ always equals 0 for $x = 0$. We obtain a differential equation

$$\dot{x} = 0,$$
$$x(0) = 0.$$

Obviously, x can thus only stay constant at 0, if it starts at 0. The first statement now follows from Lemma 1. □

Low dimensions. Corollary 2 shows that it makes no sense to investigate controllability for driftless bilinear systems using $M = \mathbb{R}^n$. Thus, in future we explore controllability on $\mathbb{R}^n_* = \mathbb{R}^n \setminus \{0\}$. Even with such a relaxation, the above results also prove that one-variable driftless bilinear systems are necessarily not controllable. Further, it will be shown later that systems for which $m = 1$ cannot be controllable either.[3]

However, the question of controllability of a given system remains interesting and has been given much exposure, particularly in the cases where the dimension of state and/or control space is low. The case of $n = 2$ has been completely answered by Koditschek and Narendra in [26].

Theorem 1. *Let Σ be a driftless bilinear control system given by (2.2), with $n = 2$ and $2 \leq m \leq 3$. Then, Σ is controllable if and only if matrices $B_1, \ldots, B_m$ do not have any common real eigenvectors.*

The above theorem will naturally come up in Chapter 4 and will be covered in further detail there.

Continuing to the next dimension of state space, the case of $n = 3$ has been explored in [9] and [10]. However, the characterisations of controllability in this case are much less elegant. In particular, the following has been shown in [10, Ch. 12].

Theorem 2. *Let Σ be a general bilinear system given by (1.1). Σ is controllable if and only if*

[3]Nonetheless, the case of $m = 1$ is of interest in the theory of general bilinear systems. Some results are given in [7, 21, 34].

(i) 0 *is in the interior of the Lyapunov spectrum of* Σ *and*

(ii) system Σ' *given by*

$$\dot{q} = Aq + \sum_{i=1}^{m} u_i B_i q - \left(q^T A q + q^T \sum_{i=1}^{m} u_i B_i q \right) q,$$

with $M = S^{n-1}$*, is controllable.*

This does not help in as obvious a manner as Theorem 1 — indeed, it will be shown in Chapter 4 that a claim analogous to Theorem 1 for higher dimensions would not be true.

In the setting of small control spaces, we mention the following result of Kučera [31].

Proposition 1. *Let* Σ *be a driftless bilinear system given by (2.2), and let* $m = 2$*. If* Σ *is controllable, any state can be reached using controls only taking the values of* $u_i = \pm 1$*.*

At this point, we explore a simple system with $n = m = 2$. The proof of its controllability is given by Theorem 1. However, we will go through it manually as it gives an example of the technique used in Chapter 5.

Example 1. *Let*

$$B_1 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \ B_2 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}.$$

The bilinear system Σ *given by* $\dot{x} = u_1 B_1 x + u_2 B_2 x$ *is controllable.*

If $x = \binom{x_1}{x_2}$, we obtain the following system:

$$\begin{aligned} \dot{x_1} &= u_2 x_2, \\ \dot{x_2} &= u_1 x_1. \end{aligned} \tag{2.4}$$

We will first show that from the point $\left(x_1^B, x_2^B\right)$, $x_1^B, x_2^B \neq 0$, we can reach all the points in the same quadrant, including on the coordinate axes, except for $(0,0)$. Indeed, let us take any point $\left(x_1^E, x_2^E\right)$ in the same quadrant as $\left(x_1^B, x_2^B\right)$ and let us define

$$u_1 = \frac{x_2^E - x_2^B}{x_1^B} \chi_{[0,1]}$$

and

$$u_2 = \frac{x_1^E - x_1^B}{x_2^E} \chi_{[1,2]},$$

where χ is the characteristic function. Without loss of generality, we assume $x_2^E \neq 0$. Otherwise, we would take u_1 to involve x_1^E and be on $[1,2]$ and u_2 to involve x_2^B and be on $[0,1]$.

Now, we notice that on $[0,1]$ we are solving the differential equation

$$\begin{aligned} \dot{x_1}(t) &= 0, \\ \dot{x_2}(t) &= \tfrac{x_2^E - x_2^B}{x_1^B} x_1(t), \\ x_1(0) &= x_1^B, \\ x_2(0) &= x_2^B. \end{aligned} \tag{2.5}$$

Obviously, the solution to this is $x_1(t) \equiv x_1^B$, $x_2(t) = \left(x_2^E - x_2^B\right)t + x_2^B$. Analogously, on $[1,2]$ we are solving

$$\begin{array}{l} \dot{x_1}(t) = \frac{x_1^E - x_1^B}{x_2^E} x_2(t), \\ \dot{x_2}(t) = 0, \\ x_1(1) = x_1^B, \\ x_2(1) = x_2^E. \end{array} \tag{2.6}$$

We finally obtain $x_1(2) = x_1^E$, $x_2(2) = x_2^E$.

Invoking Lemma 1, we have shown that all the points in the same (closed) quadrant, except for origin, are controllable from and to one another. Now, let $\overset{\Sigma}{\sim}$ be the relation defined as follows: for $x, y \in \mathbb{R}^n$, $x \overset{\Sigma}{\sim} y$ if y can be reached from x.

Lemma 1 shows this relation is symmetric. It is obviously reflexive and, furthermore, it is transitive: if there exists a trajectory going from x to y, and from y to z, by concatenating these two (i.e., concatenating the controls used to obtain these trajectories) we obtain a trajectory going from x to z.

We have shown above that $\left(x_1^B, x_2^B\right) \overset{\Sigma}{\sim} y$ for all non-zero y which are in the same quadrant as $\left(x_1^B, x_2^B\right)$. Thus, for all x and y in, say, first and second quadrant, $x \overset{\Sigma}{\sim} y$ (if they are in different quadrants, we can "connect" them through some point on the vertical coordinate axis, which is a part of both quadrants). Analogously, we can say the same about first and fourth quadrant, and second and third. All quadrants are now connected and thus, for all $x, y \in \mathbb{R}^2 \setminus \{(0,0)\}$, $x \overset{\Sigma}{\sim} y$. Hence, considering the definition of $\overset{\Sigma}{\sim}$ and Lemma 1, Σ is controllable.

2.2. Orbits. As mentioned, the technique used in Example 1, as well as the class of systems where control u_i corresponds only to state x_i, will be interesting to us in Chapters 4 and 5. However, we note that, unlike Theorem 1, the above work was inelegant. Among other things, we physically came up with an appropriate control, which was again, even for an easy system such as this, inelegant. We want to know if we can generally be smarter than that and more easily see if a system is controllable, and, if not, where the problem lies — which points cannot be connected to the rest?

The former question was discussed above. The latter is formalized in the notion of orbits. The orbit of the system $(M, C, f, \mathcal{U})$ through a point $x \in M$ is normally [23, 33] defined as the set of all points that can be reached from x through concatenations of finitely many constant controls, and going forward or backwards in time for any of the controls.

However, Lemma 1 spares us from going back in time, as we have shown that for driftless bilinear systems, any point that can be reached going back in time, can be reached going forward as well. Furthermore, as we have allowed our controls to be only piecewise constant, we can produce a simpler definition:

Definition. *The* ***orbit*** *of a driftless bilinear system* $\Sigma = (M, C, f, \mathcal{U})$ *through point* $x \in M$ *is* $\operatorname{Orb}(x, \Sigma) = \{y \in M : y$ *is controllable from* $x\} = \mathcal{C}(x, \Sigma)$.

By Corollary 1, we also note $\operatorname{Orb}(x, \Sigma) = \mathcal{R}(x, \Sigma)$.

Proposition 2 in the next section connects the less restrictive case of controls being merely locally integrable to the above case and shows that the theory of controllability and orbits remains essentially the same. In general, we continue restricting ourselves to piecewise constant controls.

Corollary 2 shows that $\mathrm{Orb}\,(0,\Sigma)=\{0\}$ and we note that controllability, for driftless bilinear systems, equals having exactly two orbits: $\mathrm{Orb}\,(0,\Sigma)=\{0\}$ and $\mathrm{Orb}\,(x,\Sigma)=\mathbb{R}^n\backslash\{0\}$ for all $x\neq 0$. It is obvious, however, that we are really interested in orbits of uncontrollable systems.

Low Dimensions. Given an abundance of results obtained for controllability in low dimensions, we could hope for similar assistance in the question of determining orbits.

As above, the case of $n=1$ is uninteresting: orbits of such systems are trivially $\mathbb{R}_+$, $\mathbb{R}_-$ and $\{0\}$. The case of $m=1$, while less trivial, is also easily solvable manually: if $\dot{x}\,(t)=u_1B_1x\,(t)$, considering u_1 is piecewise constant,

$$x\,(t)=\prod_{i=1}^{k}\exp\left(B_1u_1^{(i)}\tau_i\right)x\,(0)=\exp\left(B_1\sum_{i=1}^{k}u_1^{(i)}\tau_i\right)x\,(0)\,.$$

Hence, $\mathrm{Orb}\,(x_0,\Sigma)$ is given by $\left\{e^{B_1s}x_0 : s\in\mathbb{R}\right\}$.

For $n\geq 2$, Elliott [15] provides an algorithm for determining orbits of a driftless bilinear system Σ given by (2.2), given that matrices $B_1,B_2,\ldots,B_m\in\mathbb{Q}^{n\times n}$. As he points out, however, due to its complexity, the algorithm is only useful for low dimensions of n.

The situation is even less helpful on the control space front: results from [15, 31], which we give at the end of this chapter as Proposition 3 and Theorem 3, indicate that the question of determining orbits is equally hard for any $m\geq 2$, no matter its size.

2.3. Lie Algebras.

In order to understand the basic results that we are building upon, we need to turn our attention for a moment into the concept of Lie algebras of vector fields. We will attempt to take the most straightforward approach, which will not reveal the full capacity of Lie theory, but it will keep us reasonably on topic. That said, Lie theory proves to be an indispensable tool in geometric control: for example, see [5, 6, 22, 24, 36].

Definition. *A **Lie group** is a smooth manifold which is endowed with group multiplication and inversion, and those operations are themselves smooth.*

Definition. *A **Lie algebra** is an ordered pair $(\mathfrak{g},[\cdot,\cdot])$, where $\mathfrak{g}$ is a vector space, and $[\cdot,\cdot]:\mathfrak{g}\times\mathfrak{g}\to\mathfrak{g}$ is a binary operation satisfying*

(i) $[ax+by,z]=a\,[x,z]+b\,[y,z]$ *and* $[x,ay+bz]=a\,[x,y]+b\,[x,z]$ *for all scalars a, b and all $x,y,z\in\mathfrak{g}$,*

(ii) $[x,x]=0$ *for all $x\in\mathfrak{g}$,*

(iii) $[x,[y,z]]+[y,[z,x]]+[z,[x,y]]=0$ *for all $x,y,z\in\mathfrak{g}$.*

$[\cdot,\cdot]$ from above is usually called the Lie bracket.

Furthermore, to every Lie group G one can, in a canonical way, associate a Lie algebra $\mathfrak{g}$ [25]. In the case of matrix Lie groups, such an algebra is particularly simple:

$$\mathfrak{g}=\{X:\exp\,(tX)\in G\text{ for all }t\in\mathbb{R}\}\,,$$

while the Lie bracket is simply given by the commutator operation:

$$[A, B] = AB - BA.$$

The above was studied at great length by Hall in [19].

At this point, what we are looking for is some kind of connection between Lie theory and control theory. Let us define the group of all transition matrices of a driftless bilinear system Σ represented by (2.2). It is clear that, for a constant u, the transition matrix (starting at $t_0 = 0$) would be

$$\Phi_{u,t} = \exp\left(t\sum_{i=1}^{m} u_i B_i\right).$$

Our case is not much more difficult: u is piecewise constant. Thus, all transition matrices are finite products of the above:

$$\prod_{j=1}^{k} \exp\left(t_j \sum_{i=1}^{m} u_{ij} B_i\right),$$

where all $t_j \geq 0$.

Thus, (following Elliott's notation in [15]),

$$\mathbf{\Phi} = \mathbf{\Phi}(B_1, B_2, \ldots, B_m) = \left\{\prod_{j=1}^{k} \exp\left(t_j \sum_{i=1}^{m} u_{ij} B_i\right) : u_{ij} \in \mathbb{R}, t_j \geq 0 \text{ for all } i, j\right\}$$

is the desired set of transition matrices of Σ.

It easily follows that $\mathbf{\Phi}$ is a subgroup of $GL(n, \mathbb{R})$: it contains the identity matrix, for every element contained, its inverse is contained as well (we just change the signs of u_{ij}'s), and a product of two elements in $\mathbf{\Phi}$ is clearly in $\mathbf{\Phi}$. Furthermore, as Elliott notes, it is clearly path-connected[4] and it is, by Abel's relation — $\det(\exp(A)) = \exp(\operatorname{tr}(A)) > 0$ — in fact a subgroup of $GL^+(n, \mathbb{R}) = \{X \in GL(n, \mathbb{R}) : \det X > 0\}$.

We note that, given a driftless system (2.2), the orbit of $x \in \mathbb{R}^n$ is $\mathbf{\Phi}x$; that is exactly what transition matrices do. Thus, for solving the question of controllability of a driftless bilinear system, we need to understand when $\mathbf{\Phi}$ is, as a group, transitive on $\mathbb{R}^n_*$. Through the following result [15, Ch. 2], the same criterion applies to systems with unrestricted locally integrable controls.

Proposition 2. *The group of transition matrices $\mathbf{\Phi}$ of system (2.2) for the set of locally integrable controls is the same as $\mathbf{\Phi}$ of the same system for the set of piecewise constant controls.*

At first sight, the above characterisation of controllability as transitivity of the group of transition matrices does not appear to help us much. That is wrong. First, Tits, Boothby, Wilson, and finally Kramer, have identified the complete list of transitive groups acting on

[4]When going from I to $X \in \mathbf{\Phi}$ one can just "travel" on the exponential arcs $\exp\left(t_j \sum_{i=1}^{m} u_{ij} B_i\right)$.

$\mathbb{R}^n_*$ [2, 4, 29]. In theory, this enables us to explicitly determine if a system is controllable and that classification is given in full in [29] and [15]. However, it is highly impractical and provides little control theoretic intuition.

On the other hand, in [15], Elliott provides a Lie algebraic approach to solving problems of controllability. This gives an explanation for this section's interest in Lie theory. First, a long and difficult proof of the following is given:

Lemma 2. *$\mathbf{\Phi}$ is a Lie group.*

Moreover, if $\mathfrak{g}$ is the Lie algebra corresponding to the $\mathbf{\Phi}$, for every $x \in \mathbb{R}^n$, the vector subspace $\mathfrak{g}x$ is the tangent space to the orbit at point x [18].

Finally, the following is proved:

Lemma 3. *$\mathfrak{g}$ is the smallest Lie algebra containing $\{B_1, B_2, \ldots, B_m\}$.*

In other words,

$$\mathfrak{g} = \operatorname{span}\{B_1, B_2, \ldots, B_l\}$$

for some matrices B_{m+1}, B_{m+2}, $\ldots$, B_l. More intuitively, B_{m+1}, B_{m+2}, $\ldots$, B_l are Lie brackets involving elements of $\{B_1, B_2, \ldots, B_m\}$ such that $\{B_1, B_2, \ldots, B_l\}$ is linearly independent and $\operatorname{span}\{B_1, B_2, \ldots, B_l\}$ is closed under taking Lie brackets.

We finally note that, by the Orbit theorem, every orbit is an immersed submanifold. Hence, it makes sense to discuss orbit dimensions. Combining all of the above, we finally reach the following results. These provide a foundation for the work in future chapters.

Proposition 3. *For any $x \in \mathbb{R}^n$, let*

$$B(x) = \begin{bmatrix} B_1 x & B_2 x & \cdots & B_l x \end{bmatrix}, \tag{2.7}$$

where B_1, B_2, $\ldots$, B_m are matrices appearing in (2.2), and B_{m+1}, B_{m+2}, $\ldots$, B_l are as above. Rank of $B(x)$ is the dimension of the orbit through point x.

Corollary 3 (Lie Algebra Rank Condition (LARC))**.** *With B defined as above, the driftless bilinear system (2.2) is controllable if and only if* $\operatorname{rank}(B(x)) = n$ *for all $x \in \mathbb{R}^n_*$.*

A further result is given by Boothby in [2]. Taking into account Corollary 3, it shows that, to some degree, the only driftless bilinear systems we should care about are those with only two controls:

Theorem 3. *Every Lie algebra transitive on R^n_* can be generated by two matrices.*

This concludes the review chapter of this paper. We continue by investigating orbits and controllability of driftless bilinear systems, primarily using Proposition 3 and Corollary 3.

3. Matters of Rank

Considering Proposition 3 and Corollary 3, it is of obvious importance to know more about the matrix $B(x)$, as defined in (2.7). In particular, we would ideally like to find a necessary and sufficient condition for $B(x) \in \mathbb{R}^{n \times l}$ to be of rank n. Furthermore, we would like to be able to find out more about the orbits of (2.2) from knowing the ranks of $B(x)$ for all x.

Let us first note the following.

Lemma 4. *Given a driftless bilinear system (2.2), B_1x_0, B_2x_0, $\ldots$, B_mx_0 span the space of directions $\dot{x}(0)$ for controlled trajectories x such that $x(0) = x_0$.*

Proof. The possible directions are given by $\sum_{i=1}^{m} u_i (B_i x_0)$, i.e. linear combinations of B_1x_0, B_2x_0, $\ldots$, B_mx_0. $\square$

Considering the above, the following intuitive notion comes to mind: Since we assumed without loss of generality that $\{B_1, B_2, \ldots, B_m\}$ is a linearly independent set, we might expect that, as long as $m \leq n$, $\operatorname{rank}(B(x)) \geq m$ for at least one $x \in \mathbb{R}^n$. However, as the next example demonstrates, this can easily fail to hold:

Example 2. *Let*

$$B_1 = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \; B_2 = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \; B_3 = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

It is clear that these matrices are linearly independent. However, if B is the matrix associated with the system

$$\dot{x} = u_1 B_1 x + u_2 B_2 x + u_3 B_3 x$$

in the sense of (2.7), $\operatorname{rank}(B(x)) \leq 2$ for all $x \in \mathbb{R}^3$.

We note that all three of the above matrices have an empty third row. Thus, their commutators will have their third rows empty, and so will all further Lie brackets involving just B_1, B_2 and B_3. Thus,

$$B(x) = \begin{pmatrix} * & * & * & \cdots & * \\ * & * & * & \cdots & * \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix}.$$

In other words, the rank of $B(x)$ cannot be higher than 2.

A further set of examples in the following chapter will, while showing something else, prove the same point with a slightly more difficult and interesting derivation, without a dummy variable such as we have here.

3.1. Cones.

Beaten down (for now) by our defeat in the matter of easily generating systems with large enough orbits, we turn our attention to trying to figure out what the orbits of a given system look like, knowing the values of $\operatorname{rank}(B(x))$ at every point $x \in \mathbb{R}^n$. We definitely know what happens when $\operatorname{rank}(B(x))$ equals 0 or n, but have no intuition in between.

Example 3. *Let*

$$B_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \; B_2 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}.$$

In other words, we consider the system

$$\begin{aligned} \dot{x}_1 &= u_1 x_1, \\ \dot{x}_2 &= u_2 x_2. \end{aligned} \tag{3.1}$$

This example is strikingly similar to Example 1 and will also be useful to us in Chapter 5. However, it produces completely different results. In particular, it is clear that one can look at this system as two separate equations. The orbits of a one-dimensional system

$$\dot{x}_1 = u_1 x_1$$

are, as mentioned previously, $\{0\}$, $\mathbb{R}_+$ and $\mathbb{R}_-$.[5] Thus, the orbits of (3.1) are $\{(0,0)\}$, $\mathbb{R}_+ \times \{0\}$, $\mathbb{R}_- \times \{0\}$, $\{0\} \times \mathbb{R}_+$, $\{0\} \times \mathbb{R}_-$, $\mathbb{R}_+ \times \mathbb{R}_+$, $\mathbb{R}_+ \times \mathbb{R}_-$, $\mathbb{R}_- \times \mathbb{R}_+$ and $\mathbb{R}_- \times \mathbb{R}_-$.

In other words, all the orbits in (3.1) are cones: $\alpha(x_1, x_2)$ is in the same orbit as (x_1, x_2) for all $\alpha > 0$. Seemingly, this is given by Proposition 3:

$$\operatorname{rank}(B(\alpha x)) = \operatorname{rank}(\alpha B(x)) = \operatorname{rank}(B(x)) \tag{3.2}$$

for all $\alpha \neq 0$. However, one must be careful: Proposition 3, along with the above calculation, only implies that αx and x have orbits of same dimensions for all $\alpha \neq 0$. However, they may not be in the same orbit.

Example 4. *Let $m = 1$ and*

$$B_1 = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}.$$

In other words, we consider the following system of equations:

$$\begin{aligned} \dot{x}_1 &= u_1 x_1, \\ \dot{x}_2 &= u_1 x_1 + u_1 x_2. \end{aligned} \tag{3.3}$$

Now, let us look at solutions to this system, given an initial point (x_{01}, x_{02}) at time $t = 0$. Clearly,

$$x_1(t) = e^{\int_0^t u_1(\tau) d\tau} x_{01},$$

while solving for x_2 we obtain

$$x_2(t) = e^{\int_0^t u_1(\tau) d\tau} \left(x_{01} \int_0^t u_1(\tau)\, d\tau + x_{02} \right).$$

In particular, let us show that $(2,0) \notin \operatorname{Orb}((1,0), \Sigma)$. Let us assume otherwise. Then we have $2 = e^{\int_0^T u_1(\tau) d\tau}$ for some $T > 0$, along with $0 = e^{\int_0^T u_1(\tau) d\tau} \int_0^T u_1(\tau)\, d\tau$. Thus, $\int_0^T u_1(\tau)\, d\tau = 0$. However, that means $e^{\int_0^T u_1(\tau) d\tau} = 1$, leading to a contradiction.

So, we have shown that, unlike in Example 3, $\operatorname{Orb}((1,0), \Sigma)$ is not a cone in Example 4. We can even explicitly find that orbit: it equals $\{(x_1, x_1 \ln x_1) : x_1 > 0\}$ and is thus one-dimensional. However, unlike Examples 1 and 3, it is not embedded in any one-dimensional vector subspace. This will be the starting point of Chapter 4.

A more involved study of (convex) cones was done for systems with drift in [13] and [11]. However, while the latter gives a sufficient condition for a particular class of bilinear systems and a particular class of cones, the general question of when orbits are contained in cones remains largely open. In Chapter 5, orthant containment — a particular version of cone containment — shall be discussed.

[5] For a constant u_1, the solution to the above equation is $x_1(t) = e^{u_1 t} x_1(0)$, which shows one can move anywhere within the same sign. It is, however, as we have shown in Corollary 2, impossible to cross 0.

4. Common Invariant Subspaces

Chapter 3 dealt with attempting to determine a nice structure (cone) for orbits of a driftless bilinear system. As an extension of that idea, we now attempt to fit orbits into nice structures. To that end, we try to connect orbits with vector subspaces. Examples 1 and 3, in particular, provide motivation for this: the orbits in the former are a 0-dimensional subspace and the whole space without one point, while the orbits in the latter are a 0-dimensional subspace, "halves" of two 1-dimensional subspaces and quarters of the whole space without the coordinate axes.

Let us see what is required for an orbit to stay in a vector subspace $V \subseteq \mathbb{R}^n$.

Proposition 4. *Let Σ be a driftless bilinear system with matrices B_1, B_2, ..., B_m, and let $x_0 \in \mathbb{R}^n$. If $V \subseteq \mathbb{R}^n$ is a vector subspace,* $\operatorname{Orb}(x_0, \Sigma) \subseteq V$ *if and only if $x_0 \in V$ and* $\operatorname{span}\{B_1 y, B_2 y, \ldots, B_m y\} \subseteq V$ *for every $y \in \operatorname{Orb}(x_0, \Sigma)$.*

Proof. In one direction, if $\operatorname{Orb}(x_0, \Sigma) \subseteq V$, obviously $x_0 \in V$. Furthermore, let us assume $\operatorname{span}\{B_1 y, B_2 y, \ldots, B_m y\} \nsubseteq V$ for some $y \in \operatorname{Orb}(x_0, \Sigma)$. Then, by Lemma 4, there exists $\tilde{u}$ such that $\sum_{i=1}^m \tilde{u}_i B_i y \notin V$. Let us indeed use such $\tilde{u}$ as a control $u|_{[0,\varepsilon]} \equiv \tilde{u}$ for trajectory x, with $x(0) = y$. This is now a linear system

$$\dot{x}(t) = \sum_{i=1}^m \tilde{u}_i B_i x(t)$$

and we trivially obtain

$$x(t) = \exp\left(t \sum_{i=1}^m \tilde{u}_i B_i\right) y.$$

Using the Taylor expansion of the exponential function, we obtain

$$x(t) = y + \left(\sum_{i=1}^m \tilde{u}_i B_i\right) yt + h_1(t)\, yt,$$

where $h_1(t) \to 0$ as $t \to 0$. Since we know $y \in \operatorname{Orb}(x_0, \Sigma) \subseteq V$, and $x(t) \in \operatorname{Orb}(y, \Sigma) = \operatorname{Orb}(x_0, \Sigma) \subseteq V$, we have $\left(\sum_{i=1}^m \tilde{u}_i B_i\right) yt + h_1(t)\, yt \in V$ for small enough t. Thus, as V is a vector subspace, $\sum_{i=1}^m (\tilde{u}_i B_i)\, y + h_1(t)\, y \in V$. Since V is closed, this implies[6]

$$\sum_{i=1}^m (\tilde{u}_i B_i)\, y = \lim_{t \to 0+} \sum_{i=1}^m (\tilde{u}_i B_i)\, y + h_1(t)\, y \in V,$$

which is in contradiction to our assumption. We are done.

In the other direction, let us assume that $\operatorname{Orb}(x_0, \Sigma) \nsubseteq V$, i.e., there exists $z \in \operatorname{Orb}(x_0, \Sigma) \setminus V$. We also assume $x_0 \in V$ and $\operatorname{span}\{B_1 y, B_2 y, \ldots, B_m y\} \subseteq V$ for every $y \in \operatorname{Orb}(x_0, \Sigma)$. Since $z \in \operatorname{Orb}(x_0, \Sigma)$, there exists a controlled trajectory $x(t)$ connecting $x(0) = x_0$ and $x(T) = z$, using a piecewise constant function $\tilde{u}$.

If we look at the set $S = \{t \in [0, T] : d(x(t), V) = 0\}$, we note, by a known theorem proved in, for example, [28], that this is a closed set (as a pre-image of a closed set under a

[6] We know that all limit points of sequences of points in V are again in V [46].

continuous function). So, since S is bounded and does not include T, there exists $y = x(t')$ such that $y \in V$ and $x(t) \notin V$ for all $t' < t \leq T$.

Since u is piecewise constant, there exists $\varepsilon > 0$ such that $u|_{(t',t'+\varepsilon]} \equiv \tilde{u}$. Thus, on $[t', t'+\varepsilon]$,

$$\dot{x}(t) = \sum_{i=1}^{m} \tilde{u}_i B_i x(t).$$

As in the previous direction, we obtain

$$x(t) = \exp\left(\sum_{i=1}^{m} \left(t - t'\right) \tilde{u}_i B_i\right) y,$$

which, by Taylor's theorem, equals $\sum_{n=0}^{\infty} \left(\sum_{i=1}^{m} (t-t') \tilde{u}_i B_i\right)^n \frac{1}{n!} y$. Since y is the last point on this trajectory that is in V, we know that

$$x(t) = \sum_{n=0}^{\infty} \left(\sum_{i=1}^{m} \left(t - t'\right) \tilde{u}_i B_i\right)^n \frac{1}{n!} y \notin V$$

for any $t \in (t', t'+\varepsilon]$.

On the other hand, we know that $\operatorname{span}\{B_1 x(t), B_2 x(t), \ldots, B_m x(t)\} \subseteq V$ for every $t \in (t', t'+\varepsilon]$. Thus, if we denote $\sum_{i=1}^{m} \tilde{u}_i B_i$ by C, $Cx(t) \in V$ for every $t \in (t', t'+\varepsilon]$. Let us consider

$$\left(t - t'\right) C x(t) = \sum_{n=0}^{\infty} \left(\left(t - t'\right) C\right)^{n+1} \frac{1}{n!} y$$

and let us assume there exists a $k \in \mathbb{N}_0$ such that $C^k y \notin V$. Furthermore, let k be the smallest such number. We know from our assumptions that $k \geq 2$. Now, as $(t-t') Cx(t) \in V$,

$$\left(t - t'\right) C x(t) - \sum_{n=0}^{k-2} \left(\left(t - t'\right) C\right)^{n+1} \frac{1}{n!} y = \sum_{n=k-1}^{\infty} \left(\left(t - t'\right) C\right)^{n+1} \frac{1}{n!} y \in V.$$

As this holds for all $t \in (t', t'+\varepsilon]$, by dividing by $(t-t')^k / (k-1)!$ we obtain

$$C^k y + \left(t - t'\right) \sum_{n=k}^{\infty} C^{n-k+1} \frac{\left(t - t'\right)^{n-k} (k-1)!}{n!} C^k y \in V.$$

We are now in the same situation as in the previous direction: $C^k y \notin V$, while $C^k y + h(t-t') C^k y \in V$, where $h(t-t') \to 0$ as $t - t'$ approaches 0 from above. Thus, using the same method as in the previous direction, we note that

$$\lim_{t \to t'+} C^k y + h\left(t - t'\right) y \in V$$

(as V is closed as a finite-dimensional vector subspace) and thus $C^k y \in V$. That is a contradiction. Hence, $C^k y = \left(\sum_{i=1}^{m} \tilde{u}_i B_i\right)^k y \in V$ for all $k \in \mathbb{N}_0$.

So, $\left(\sum_{i=1}^{m}(t-t')\,\tilde{u}_i B_i\right)^n \frac{1}{n!}y \in V$ for every $n \in \mathbb{N}_0$ and every $t \in (t', t'+\varepsilon]$. Thus, since V is closed, we conclude

$$
\begin{aligned}
x(t) &= \sum_{n=0}^{\infty}\left(\sum_{i=1}^{m}(t-t')\,\tilde{u}_i B_i\right)^n \frac{1}{n!}y \\
&= \lim_{N\to\infty}\sum_{n=0}^{N}\left(\sum_{i=1}^{m}(t-t')\,\tilde{u}_i B_i\right)^n \frac{1}{n!}y \in V
\end{aligned}
$$

for all $t \in (t', t'+\varepsilon]$.

This is now in contradiction to $x(t) \notin V$ for all $t \in (t', T]$. We are done. $\square$

4.1. Invariant Subspaces Imply Uncontrollability. The above result, while pleasant, is not very useful. For one, we intended to find out more about an orbit by embedding it in a vector subspace. However, the proposition requires us to know the orbit before we can use it. We can, however, come up with a weaker result which does not require us to a priori know the orbit we are looking for.

Corollary 4. *Let Σ be a driftless bilinear system with matrices B_1, B_2, …, B_m. If $V \subseteq \mathbb{R}^n$ is a vector subspace such that* $\operatorname{span}\{B_1y, B_2y, \dots, B_my\} \subseteq V$ *for every $y \in V$, then* $\operatorname{Orb}(y,\Sigma) \subseteq V$ *for every $y \in V$.*

Now, let $V \subseteq \mathbb{R}^n$ be a subspace. The condition that

$$\operatorname{span}\{B_1y, B_2y, \dots, B_my\} \subseteq V,$$

as a span is also a linear subspace, boils down to $B_iy \in V$ for every $i \in \{1, 2, \dots, m\}$. On the other hand, as $y \in V$, this is equivalent to saying it suffices to show that $B_iV \subseteq V$ for every $i \in \{1, 2, \dots, m\}$. Thus, the Corollary 4 is equivalent to the following.

Corollary 5. *Let Σ be a driftless bilinear system with matrices B_1, B_2, …, B_m. If V is a common invariant subspace of those matrices, then* $\operatorname{Orb}(y,\Sigma) \subseteq V$ *for every $y \in V$.*

Let us follow Examples 3 and 4 again. In Example 3, B_1 and B_2 were as follows:

$$B_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix},\ B_2 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}.$$

As we are not interested in the trivial subspace (nor in the whole $\mathbb{R}^2$), we are left with looking for 1-dimensional invariant subspaces. In other words, subspaces generated by eigenvectors. For B_1, those are $\operatorname{span}\{\left(\begin{smallmatrix}0\\1\end{smallmatrix}\right)\}$ and $\operatorname{span}\{\left(\begin{smallmatrix}1\\0\end{smallmatrix}\right)\}$, and the same holds for B_2. Thus, these two spaces satisfy the condition of Corollary 5 and so, we conclude that the orbits of points on coordinate axes are subsets of those axes. Indeed, as, we saw in Chapter 3, the orbits in Example 3 are, among others, $\mathbb{R}_+ \times \{0\}$, $\mathbb{R}_- \times \{0\}$, $\{0\} \times \mathbb{R}_+$ and $\{0\} \times \mathbb{R}_-$.

In Example 4, we note that such an invariant space cannot contain $(1,0)$ (nor, by similar calculation, any other point on the x-axis). Let us examine what the invariant spaces of $B_1 = \left(\begin{smallmatrix}1&0\\1&1\end{smallmatrix}\right)$ are. Again, apart from the trivial space and the whole space, we are left with eigenvectors. In this case, the only one gives $\operatorname{span}\{\left(\begin{smallmatrix}0\\1\end{smallmatrix}\right)\}$. So, the orbits of points on y-axis should stay within it. Let us see. From calculations in Chapter 3, we note that $\operatorname{Orb}((0,x_{02}),\Sigma) = \{(0,\alpha x_{02}) : \alpha > 0\}$. Thus, the orbits on the y-axis are $\{0\}\times\mathbb{R}_+$, $\{(0,0)\}$ and $\{0\}\times\mathbb{R}_-$.

4.2. Lack of Invariant Subspaces. Having proved above that common nontrivial invariant subspaces of B_1, B_2, …, B_m imply that (2.2) is not controllable on $\mathbb{R}^n_*$, one naturally wonders if the converse is true. Indeed, in (controllable) Example 1, the only 1-dimensional invariant subspace of B_1 is $\operatorname{span}\{\left(\begin{smallmatrix}0\\1\end{smallmatrix}\right)\}$, while the only 1-dimensional invariant subspace of B_2 is $\operatorname{span}\{\left(\begin{smallmatrix}1\\0\end{smallmatrix}\right)\}$.

The general claim, however, is sadly not true.

Proposition 5. *Let $m = 2$, $n \geq 3$ and*

$$B_1 = \begin{pmatrix} 0 & n-1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & n-2 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & 1 \\ 0 & 0 & 0 & \cdots & 0 & 0 \end{pmatrix}, \; B_2 = \begin{pmatrix} 0 & 0 & 0 & \cdots & 0 & 0 \\ 1 & 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & \cdots & n-1 & 0 \end{pmatrix}.$$

These two matrices have no common invariant subspaces, and if $B(x)$ is a matrix associated with the driftless bilinear system generated by these two matrices in the sense of (2.7), then $\operatorname{rank}(B(x)) \leq 3$ *for every $x \in \mathbb{R}^n$. Furthermore, if $n = 3$,* $\operatorname{rank}(B(x)) \leq 2$ *for every $x \in \mathbb{R}^n$.*

Proof. Let us first find common nontrivial invariant subspaces of B_1 and B_2. Let V be such an invariant subspace and let $a^{(1)} \in V$ be a non-zero vector. Let k be such that $a^{(1)}_k \neq 0$, but $a^{(1)}_i = 0$ for all $k < i \leq n$. By iterating $B_1^{k-1}a^{(1)}$ we obtain the vector

$$\begin{pmatrix} (n-1)(n-2)\cdots(n-k+1)\,a^{(1)}_k \\ 0 \\ \vdots \\ 0 \end{pmatrix} \in V.$$

Thus, $e_1 \in V$. However, by now iterating $B_2 e_1$, $B_2^2 e_1$, …, $B_2^{n-1} e_1$, we obtain that $e_i \in V$ for every $1 \leq i \leq n$. So, $V = \mathbb{R}^n$ and thus B_1 and B_2 do not have any proper common nontrivial invariant subspaces.

Let us now look at Lie brackets of B_1 and B_2. By calculating $B_3 = [B_1, B_2] = B_1 B_2 - B_2 B_1$, we obtain

$$B_3 = \begin{pmatrix} n-1 & 0 & \cdots & 0 & 0 \\ 0 & 2(n-2)-(n-1) & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & (n-1)-2(n-2) & 0 \\ 0 & 0 & \cdots & 0 & -(n-1) \end{pmatrix}.$$

However, since $[B_3, B_1] = 2B_1$ and $[B_3, B_2] = -2B_2$, all other Lie brackets containing only B_1, B_2 and B_3 will be spanned by those three matrices. Thus, $\operatorname{span}\{B_1, B_2, B_3\}$ is the Lie algebra $\mathfrak{g}$ of Lemma 3.

So, $B(x) = \left[\begin{smallmatrix} B_1 x & B_2 x & B_3 x \end{smallmatrix}\right]$, and thus we have proved $\operatorname{rank}(B(x)) \leq 3$ for all $x \in \mathbb{R}^n$.

In the special case of $n = 3$, we observe that

$$B_3 = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -2 \end{pmatrix}$$

and thus

$$B(x) = \begin{pmatrix} 2x_2 & 0 & 2x_1 \\ x_3 & x_1 & 0 \\ 0 & 2x_2 & -2x_3 \end{pmatrix}.$$

The determinant of this matrix is always 0 and thus $\operatorname{rank}(B(x)) \leq 2$ for all $x \in \mathbb{R}^3$. □

As the example above shows, none of the systems generated by the matrices above are controllable. Yet, the matrices have no common proper nontrivial invariatn subspaces. We note, however, that Example 1, the 2-dimensional analogue of Proposition 5, is controllable.

In fact, as we mentioned previously, Theorem 1 [26, 27], based on Boothby's and Wilson's classification in [4], proves our desired claim for $n = 2$. We note that in the case of two-dimensional systems, eigenvectors are the only proper nontrivial invariant subspaces. Furthermore, the cases of $m = 1$ and $m = 4$ (as we require B_i's to be linearly independent, $m > 4$ is impossible) are uninteresting when $n = 2$. From Corollary 3, we know $m = 1$ results in $B(x) = B_1 x$, which is a matrix of rank not more than 1. Thus, the system

$$\dot{x} = u_1 B_1 x$$

is not controllable.

On the other hand, $m = 4$ implies that $\{B_1, B_2, B_3, B_4\}$ is a basis for $\mathbb{R}^{2\times 2}$. So, through a linear change in controls ($v = Du$ for some $D \in GL(2, \mathbb{R})$, we can write (2.2) as

$$\dot{x} = v_1 \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} x + v_2 \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} x + v_3 \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} x + v_4 \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} x. \tag{4.1}$$

By Example 1, the system obtained from (4.1) by restricting $v_3, v_4 \equiv 0$ is controllable. Hence, system (4.1) is controllable as well.

5. Coupled State–Control Systems

Following the semi-conclusive observations of Chapter 4, we turn our attention back to Examples 1 and 3. We notice that, in both of those systems, it was relatively straightforward to determine their orbits, and the only problems arose when dealing with coordinate axes.

We notice that both of those systems had the following property: on the right hand sides in the system equations, control u_i only appeared coupled with state x_i. In other words, we have systems of the form

$$\dot{x} = \sum_{i=1}^{n} u_i x_i b_i, \tag{5.1}$$

where $b_1, b_2, \ldots, b_n \in \mathbb{R}^n$.

We note that equation (5.1) corresponds to (2.2) with $B_i = b_i e_i^T$. Indeed,

$$u_i B_i x = u_i \left(b_i e_i^T\right) x = u_i b_i \left(e_i^T x\right) = u_i b_i x_i = \left(u_i x_i\right) b_i. \tag{5.2}$$

The ease with which we dealt with Examples 1 and 3 motivates us to further examine such systems. In the aforementioned cases, b_i's were linearly independent. If they are not, the matter of controllability is easily sorted out.

Proposition 6. *Let Σ represented by (5.1) be a control system. If $\{b_1, b_2, \dots, b_n\}$ is a linearly dependent set, Σ is not controllable.*

Proof. Since $\{b_1, b_2, \dots, b_n\}$ is not linearly independent, the square matrix

$$B' = \begin{bmatrix} b_1 & b_2 & \cdots & b_n \end{bmatrix} \tag{5.3}$$

is not of full rank. Thus, there exist $\alpha_1, \alpha_2, \dots, \alpha_n \in \mathbb{R}$, which are not all 0, such that $\sum_{i=1}^n \alpha_i \left[B'\right]_{ij} = 0$ for all $1 \le j \le n$. Hence, $\sum_{i=1}^n \alpha_i \left[B'\right]_{ij} x_j u_j = 0$ for all j and so

$$\sum_{i=1}^n \alpha_i \left(\sum_{j=1}^n \left[B'\right]_{ij} x_j u_j \right) = 0.$$

We note from (5.1) that $\dot{x}_i = \sum_{j=1}^n \left[B'\right]_{ij} x_j u_j$ for all $1 \le i \le n$. Thus, $\sum_{i=1}^n \alpha_i \dot{x}_i = 0$. In other words, $\sum_{i=1}^n \alpha_i x_i(t)$ stays constant regardless of the time and controls. Such a system can obviously not be controllable. $\square$

Having solved the case of linear dependence, in the rest of the chapter we take $\{b_1, b_2, \dots, b_n\}$ to be linearly independent.

Before arriving at the main result, we remind ourselves of a definition of a basic Cartesian object and prove an interesting lemma.

Definition. *An* ***open orthant (open hyperoctant)*** *is a set*

$$\left\{(x_1, x_2, \dots, x_n) \in \mathbb{R}^n : \varepsilon_1 x_1 > 0, \varepsilon_2 x_2 > 0, \dots, \varepsilon_n x_n > 0\right\},$$

where $\varepsilon_1, \varepsilon_2, \dots, \varepsilon_n \in \{-1, 1\}$.

Thus, orthants are generalizations of quadrants and octants in two and three dimensions. A closed orthant is defined analogously.

Lemma 5. *Let Σ represented by (5.1) be a control system, $\{b_1, b_2, \dots, b_n\}$ being a linearly independent set, and let O be an open orthant in $\mathbb{R}^n$. Then, for every $x \in O$, $O \subseteq \mathrm{Orb}\,(x, \Sigma)$.*

Proof. Let $B\,(x)$ be the matrix associated with Σ in the sense of Proposition 3. In other words,

$$B\,(x) = \begin{bmatrix} B_1 x & B_2 x & \cdots & B_l x \end{bmatrix}.$$

Hence,

$$\operatorname{rank}\left(B\,(x)\right) \ge \operatorname{rank}\left(\begin{bmatrix} B_1 x & B_2 x & \cdots & B_n x \end{bmatrix}\right)$$

for all $x \in \mathbb{R}^n$. We noted in (5.2) that $B_i x = x_i b_i$ for all $i \in \{1, 2, \dots, n\}$. Thus,

$$\operatorname{rank}\left(B\,(x)\right) \ge \operatorname{rank}\left(x_1 x_2 \cdots x_n \begin{bmatrix} b_1 & b_2 & \cdots & b_n \end{bmatrix}\right).$$

As we required $x \in O$, $x_1 x_2 \cdots x_n \neq 0$, and since we required $\{b_1, b_2, \ldots, b_n\}$ to be linearly independent,

$$\operatorname{rank}\left(x_1 x_2 \cdots x_n \begin{bmatrix} b_1 & b_2 & \cdots & b_n \end{bmatrix}\right) = n.$$

Thus, all points in O are, considering results at the end of Chapter 2, in orbits of dimension n. It is intuitively clear, then, that for each O, there only must exist one orbit covering all its points. The formal proof is as follows.

Let us take any two points $x, y \in O$. Now let us consider the line segment $[x, y] \subseteq O$. As every point of that line segment is in O, around every $z \in [x, y]$ there exists[7] an open (in the subspace topology on $[x, y]$) interval $U_z \in [x, y]$ such that $\operatorname{Orb}(z', \Sigma) = \operatorname{Orb}(z, \Sigma)$ for all $z' \in U_z$. As $[x, y]$ is compact, it can be covered by finitely many such neighbourhoods: U_{z_1}, U_{z_2}, ..., U_{z_k}. As these sets are open intervals, we may without loss of generality assume that $U_{z_i} \cap U_{z_{i+1}} \neq \emptyset$ for all $i \in \{1, 2, \ldots, k-1\}$. However, that means that $\operatorname{Orb}(z_i, \Sigma) = \operatorname{Orb}(z_{i+1}, \Sigma)$ for all i, i.e., all the points in those intervals have the same orbits. As

$$\bigcup_{i=1}^{k} U_{z_i} = [x, y],$$

we conclude that $\operatorname{Orb}(x, \Sigma) = \operatorname{Orb}(y, \Sigma)$. □

Considering the above, we will denote the orbit of all the points in the orthant O by $\operatorname{Orb}(O, \Sigma)$.

The equivalent of Lemma 5 is not true for general bilinear control systems: in particular, [1, 3, 39, 40] deal with controllability in the orthant where all coordinates are positive, bringing back the notion of cones from 3. For a discussion of orbits of orthants on a slightly different class of bilinear control systems, see [41].

We note that the second part of the above proof is also the formal proof[8] for the claim that dimension of orbits of all points in $\mathbb{R}^n_*$ being n implies controllability, left out in [15] and in our previous chapters.

So, we have shown that, generically, points in $\mathbb{R}^n$ are controllable, given the system (5.1). What is left are, as in Examples 1 and 3, the coordinate planes.

Surprisingly enough, graph theory comes into play here. To see this, we recall the matrix $B' \in \mathbb{R}^{n \times n}$ defined in (5.3). As we mentioned above, we require B' to be of rank n. Furthermore, let us define $\bar{B}' \in \mathbb{R}^{n \times n}$ as follows:

$$\left[\bar{B}'\right]_{ij} = \begin{cases} 0, & \text{if } B'_{ij} = 0, \\ 1, & \text{otherwise}, \end{cases} \tag{5.4}$$

for all $i, j \in \{1, 2, \ldots, n\}$.

We note that $\bar{B}'$ is an adjacency matrix of sorts: $\left[\bar{B}'\right]_{ij} = 1$ if and only if $x_j u_j$ appears in the differential equation describing $\dot{x}_i$. We will say that in that case x_j *(directly) influences* x_i. Now, let G be a directed graph defined by the adjacency matrix $\bar{B}'^T$: there is an edge going from i to j if and only if i influences j. We remind ourselves of the following definition.

[7] Since the dimension of $\operatorname{Orb}(z, \Sigma)$ is n, it is open [32].

[8] If needed, one can make a small arc around 0 instead of going through it.

Definition. *A directed graph G is said to be* ***strongly connected*** *if, for every two vertices u and v in G, there exists a path going from u to v and a path going from v to u.*

The following result holds.

Theorem 4. *Let $n \geq 2$, let Σ be an n-variable control system represented by (5.1) and let G be defined as above, with $\operatorname{rank}(B') = n$. Then, Σ is controllable if and only if G is strongly connected.*

Proof. Let us first prove the direction where we assume G is not strongly connected. As the condition for strong connectedness is an obvious equivalence relation, we may break G up into strongly connected components. Without loss of generality, we assume that $\{1, 2, \ldots, k\}$, $1 \leq k < n$, is one of those components. We can further assume, without loss of generality, that $\{1, 2, \ldots, k\}$ cannot be reached from $k + 1$.

We now claim that $V = \{0\}^k \times \mathbb{R}^{n-k}$ is a common invariant subspace for matrices $B_1 = b_1 e_1^T$, $B_2 = b_2 e_2^T$, $\ldots$, $B_n = b_n e_n^T$. As $e_i^T e_j = \delta_{ij}$ and $\{e_{k+1}, e_{k+2}, \ldots, e_n\}$ is the basis for V, this reduces to claiming that $b_{k+1}, b_{k+2}, \ldots, b_n \in \{0\}^k \times \mathbb{R}^{n-k}$. However, that is exactly what we have: $b_{k+1}, b_{k+2}, \ldots, b_n \in \{0\}^k \times \mathbb{R}^{n-k}$ means that from $i > k$ one can only reach (in one step, and so in any number of steps) another vertex with index strictly greater than k. If that were not true, there would exist a vertex $i > k$ with an edge going from it to $j \leq k$, and so $\{1, 2, \ldots, k\}$ could be reached (over i) from $k + 1$.

Thus, we have shown that B_i's used to define Σ have a nontrivial proper common invariant subspace. So, by Corollary 5, Σ is not controllable.

In the other direction, where we assume G is strongly connected, the claim will clearly follow if we prove the following two assertions:

1) Let O and O' be neighbouring open orthants, in the sense of the boundary between them being $n-1$-dimensional. Then, reminding ourselves of Lemma 5, $\operatorname{Orb}(O, \Sigma) = \operatorname{Orb}(O', \Sigma)$.

2) Let $x^0 \in \mathbb{R}^n_*$. One can, from x^0 and through a sequence of piecewise constant controls, reach some point $y \in O$ in some open orthant O.

The general technique used is, rightly so, reminiscent of the method employed in Example 1. We proceed along those lines:

1) Without loss of generality, we may assume that

$$O = \{(x_1, x_2, \ldots, x_n) : x_1 < 0, x_2, x_3, \ldots, x_n > 0\}$$

and

$$O' = \{(x_1, x_2, \ldots, x_n) : x_1, x_2, x_3, \ldots, x_n > 0\}.$$

Their boundary is then $P = \{0\} \times \mathbb{R}^n_+$. In particular,

$$x^0 = (0, 1, 1, \ldots, 1) \in P.$$

We will show that $\operatorname{Orb}(x^0, \Sigma) = \operatorname{Orb}(O, \Sigma)$ and $\operatorname{Orb}(x^0, \Sigma) = \operatorname{Orb}(O', \Sigma)$.

Thus we need to show that, from point x^0, we can reach some point in O and some point in O' through a series of piecewise constant controls.

Now let us look at the differential equation describing $\dot{x}_1$. As G is strongly connected, there must be a state other than x_1 directly influencing that component. Without loss of generality, we may assume that this is x_2. In other words, au_2x_2 appears somewhere in equation describing $\dot{x}_1$, for some $a \neq 0$. Let now $u|_{[0,T)} \equiv e_2$, for some small T. In other words, $u_i = 0$ for all $i \neq 2$, while $u_2 = 1$. In that case, the direction in which the system is heading at time $t = 0$ is given by

$$\sum_{i=1}^{n} u_i x_i b_i = b_2.$$

However, we know that the first component of b_2 is $a \neq 0$. Thus, the state is heading out of $P = \{0\} \times \mathbb{R}_+^n$ and into either O or O', depending on the sign of a.[9] So, from x^0 the system we will reach one of the neighbouring orthants. For the other neighbor, we merely make $u|_{[0,T)} \equiv -e_2$, in which case the state will be heading out in the direction of $-b_2$. This finished the first step.

2) If such a point x^0 is already in some open orthant, we are done. Let us thus assume, without loss of generality, that $x^0 = \left(0, 0, \ldots, 0, x_{k+1}^0, x_{k+2}^0, \ldots, x_n^0\right)$, where $x_{k+1}^0 x_{k+2}^0 \cdots x_n^0 \neq 0$. We proceed similarly to the first step. First, as G is strongly connected, there must be a state x_p, with $p \in \{k+1, k+2, \ldots, n\}$, directly influencing one of the first k states. Without loss of generality, let us assume that the state influenced is x_k. Then, by making $u|_{[0,T_1)} \equiv e_p$ for some small $T_1 > 0$, we are able, exactly as in the first step, to move the system into a state where $x_k \neq 0$, with states x_{k+1}, x_{k+2}, $\ldots$, x_n still all being different from 0.

   States x_1, x_2, $\ldots$, x_{k-1} may or may not have stayed at 0 after this manoeuvre. In any case, we have reduced the number of zeros in the system by at least 1. We proceed analogously — in the remaining states that are still at 0, strong connectedness of G implies there is a state x_r currently not at 0 directly influencing it. By making $u_r|_{[T_1,T_1+T_2)} \equiv e_r$ for some small $T_2 > 0$, we can push that state out of zero without making the signs of non-zero states change. We proceed as long as there are states still at 0. When there are no such states left, we have reached a point in an open orthant.

Thus, we have shown that all the open orthants have the same orbit, and all the points not on open orthants have the same orbit as one of the open orthants. So, there indeed only exists one orbit on $\mathbb{R}_*^n$. □

The above result proves that a system given by (5.1) for $n \geq 2$ is controllable if and only if every state influences (not necessarily directly) every other state. Algorithmically, the problem of strong connectedness, and thus controllability of a given system can be solved in $\mathcal{O}\left(n^2\right)$ by a path-based strong component algorithms. A short study of those is made in [17] and a particularly efficient version has been given in [12, Ch. 25].

The above complexity is of particular interest in comparison with the study of controllability of bilinear systems in [45], where it was proved that determining controllability for subsystems of bilinear systems is in general NP-hard.

[9] For a small enough T, other components of P will not change signs.

To translate Theorem 4 from graph-theoretic terms into linear algebra, we remember the definition of an irreducible matrix and use the following theorem translating strongly-connected graphs into irreducible matrices [16, 30].

Definition. *A $n \times n$ matrix A is **irreducible** if there does not exist a permutation matrix P such that*

$$P^T AP = \begin{bmatrix} A_{11} & 0 \\ A_{21} & A_{22} \end{bmatrix},$$

where A_{11} and A_{22} are square matrices of dimension at least 1.

Theorem 5. *A directed graph is strongly connected if and only if its adjacency matrix is irreducible.*

Thus, noting that the matrices B', $\bar{B}'$ and $\bar{B}'^T$ are all irreducible if and only if one of them is irreducible, and taking into account Proposition 6, Theorem 4 can be alternatively stated as follows.

Corollary 6. *Let $n \geq 2$, and let Σ be a n-variable control system represented by (5.1). Σ is controllable if and only if the matrix $B' = \left[\begin{smallmatrix} b_1 & b_2 & \cdots & b_n \end{smallmatrix}\right]$ is of full rank and irreducible.*

6. Summary and Conclusion

Motivated by expositions of Mohler, Pardalos, Yatsenko and Elliott [15, 35, 37], as well as the work of Rink and Boothby [38, 3] this report sought to make progress towards find an easy and intuitive way of answering the question of controllability of a driftless bilinear systems and describing the orbits of such systems. Current methods, as given in [15], rely on Kramer's [29] classification of transitive Lie algebras and work on systems of small dimensions. Thus, they are largely computationally difficult and, especially in terms of finding the orbits of a system, provide little to no intuition in terms of control theory. Indeed, after reviewing current work in Chapter 2, in Chapter 3 we have given several examples of failures of usual thinking. Chapter 3 also provided us with a notion of cone invariance, which makes a further appearance in Chapter 5.

Chapter 4 largely dealt with a relaxation of the structure we attempted to impose in Chapter 3. A necessary condition for controllability was identified in terms of common invariant subspaces of system matrices, and this condition was be further used in Chapter 5. For two-dimensional systems, we have proved that such a condition is also sufficient. However, as the matrices of Proposition 5 portray, the claim analogue to Theorem 1 does not hold in other dimensions.

In Chapter 5 we turn our attention to a special class of systems we identified as having particularly nice properties. For those systems, we have identified a necessary and sufficient condition for controllability in graph-theoretic terms.

In general, the price paid for attempting to approach the problem with lighter machinery than usual is, as expected, inconclusiveness: results obtained are largely sufficient or necessary conditions, as well as claims valid for specific classes of systems. These, however, point in the possible directions of where to look next. In particular, as Corollary 5 in Chapter 4 gives a weak, but positive, result in the cases of common invariant subspaces, it might be interesting to investigate orbits of systems without common invariant subspaces

— for example, those in Proposition 5. In the big picture, the question of influence of invariant subspaces on orbits and controllability, as given indication of in Chapter 4, would be interesting to be fully resolved.

On a more local scale, Chapter 5 leaves open the matter of orbits of coupled bilinear systems in the cases when the associated graph is not strongly connected. This does not seem too far out of reach and one would expect to obtain results similar to those in Example 3. The underlying question of that chapter, of course, is what makes coupled systems so special and easy to deal with, and how do other systems behave with regard to orthant invariance. Unlike systems mentioned in [41], systems satisfying (5.1) are not generally separable into n one-variable equations, although their orbits do share the some of the properties of such trivial systems, as shown by Lemma 5. The answer might be in the apparent simplicity of their invariant subspaces, bringing us back to matters of Chapter 4.

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